Problem 1.1. (a) For a smooth function \( f \), show that the Gaussian curvature of the level surface \( \Sigma = \{(x, y, z) \in \mathbb{R}^3 \mid z = f(x, y)\} \) is
\[
\kappa(x, y) = \frac{f_{xx}f_{yy} - f_{xy}^2}{(1 + f_x^2 + f_y^2)^2}
\]
where \( f_x \) denotes \( \frac{\partial f}{\partial x} \), \( f_{xx} \) denotes \( \frac{\partial^2 f}{\partial x^2} \), etc. Follow these steps:

(i) \( \Sigma \) is the level set of \( F(x, y) = f(x, y) - z \); calculate \( \nu = \nabla F / |\nabla F| \) and the tangent vectors \( u = \frac{\partial F}{\partial x} \) and \( v = \frac{\partial F}{\partial y} \) to \( \Sigma \).

(ii) Find the metric \( g_{ij}(x, y) \) by calculating \( g_{11} = u \cdot u \), \( g_{12} = g_{21} = u \cdot v \), and \( g_{22} = v \cdot v \).

(iii) Find the second fundamental form \( h_{ij} \) by
\[
h_{11} = -\frac{\partial \nu}{\partial x} \cdot u \quad h_{12} = -\frac{\partial \nu}{\partial x} \cdot v = -\frac{\partial \nu}{\partial y} \cdot u \quad h_{22} = -\frac{\partial \nu}{\partial y} \cdot v.
\]

(iv) The formula \( h(X, Y) = g(SX, Y) \) shows that \( h = gS \) as \( 2 \times 2 \) matrices, so the Gaussian curvature is
\[
K = \det S = \frac{\det h}{\det g}.
\]

(b) Use the above formula to show that the Gaussian curvature of the saddle surface \( \Sigma = \{z = x^2 - y^2\} \) is everywhere negative, and decays to 0 asymptotically like \( \frac{1}{r^4} \) as \( r = \sqrt{x^2 + y^2} \to \infty \).

Problem 1.2. This exercise gives practice with the index notation used in classical differential geometry. Treat it as formal algebra.

In a local coordinate system \( \{x^i\} \) the Riemannian metric can be written as \( g = g_{ij} \, dx^i \, dx^j \), where it is understood that we are summing over repeated indices ("Einstein summation notation"), that \( g_{ij} = g_{ji} \), and that \( dx^i \, dx^j \) means \( dx^i \otimes dx^j \). In this setting, the Christoffel symbols are
\[
\Gamma^k_{ij} = \frac{1}{2}g^{kl} \left( \frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right)
\]
(again using the summation convention, and where \( g^{kl} \) denotes the entries to the inverse of the matrix whose entries are \( g_{kl} \)).

(a) Show that \( \Gamma^k_{ij} = \Gamma^k_{ji} \), i.e. the Christoffel symbols are symmetric in their lower indices.

(b) Write the expression \( A^k_{ij} \) for \( \Gamma^k_{ij} \) at a point where \( g_{ij} = \delta_{ij} \) (similar to the above, but a little simpler).
(c) Show that the $A^k_{ij}$ from (b) are a solution to the equation

$$A^i_{kj} + A^j_{ki} - \frac{\partial g_{ij}}{\partial x^k} = 0.$$