INTERSCIENCE TRACTS
IN PURE AND APPLIED MATHEMATICS

Number 15, Volume I
FOUNDATIONS OF DIFFERENTIAL GEOMETRY
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VOLUME I

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1963
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neighborhood U_i of a_i in M_i with compact closure. Then $W(a_1, \ldots, a_s; U_1, \ldots, U_s) = \{\varphi \in G; \varphi(a_i) \in U_i \text{ for } i = 1, \ldots, s\}$ is a neighborhood of the identity of G with compact closure. QED.

COROLLARY 4.10. *If M is compact in addition to the assumption of Corollary 4.9, then G is compact.*

Proof. Let $G^* = \{\varphi \in G; \varphi(M_i) = M_i \text{ for } i = 1, \ldots, s\}$. Then G^* is a subgroup of G of finite index. In the proof of Corollary 4.9, let $U_i = M_i$. Then G^* is compact. Hence, G is compact. QED.

5. *Fibre bundles*

Let M be a manifold and G a Lie group. A *(differentiable) principal fibre bundle over M with group G* consists of a manifold P and an action of G on P satisfying the following conditions:

(1) G acts freely on P on the right: $(u, a) \in P \times G \rightarrow ua = R_a u \in P$;

(2) M is the quotient space of P by the equivalence relation induced by G, $M = P/G$, and the canonical projection $\pi: P \rightarrow M$ is differentiable;

(3) P is locally trivial, that is, every point x of M has a neighborhood U such that $\pi^{-1}(U)$ is isomorphic with $U \times G$ in the sense that there is a diffeomorphism $\psi: \pi^{-1}(U) \rightarrow U \times G$ such that $\psi(u) = (\pi(u), \varphi(u))$ where φ is a mapping of $\pi^{-1}(U)$ into G satisfying $\varphi(ua) = (\varphi(u))a$ for all $u \in \pi^{-1}(U)$ and $a \in G$.

A principal fibre bundle will be denoted by $P(M, G, \pi)$, $P(M, G)$ or simply P. We call P the *total space* or the *bundle space*, M the *base space*, G the *structure group* and π the *projection*. For each $x \in M$, $\pi^{-1}(x)$ is a closed submanifold of P, called the *fibre* over x. If u is a point of $\pi^{-1}(x)$, then $\pi^{-1}(x)$ is the set of points ua, $a \in G$, and is called the fibre through u. Every fibre is diffeomorphic to G.

Given a Lie group G and a manifold M, G acts freely on $P = M \times G$ on the right as follows. For each $b \in G$, R_b maps $(x, a) \in M \times G$ into $(x, ab) \in M \times G$. The principal fibre bundle $P(M, G)$ thus obtained is called *trivial*.

From local triviality of $P(M, G)$ we see that if W is a submanifold of M then $\pi^{-1}(W)(W, G)$ is a principal fibre bundle. We call it the *portion of P over W* or the *restriction of P to W* and denote it by $P|W$.

Given a principal fibre bundle $P(M, G)$, the action of G on P induces a homomorphism σ of the Lie algebra $\mathfrak{g}$ of G into the Lie algebra $\mathfrak{X}(P)$ of vector fields on P by Proposition 4.1. For each $A \in \mathfrak{g}$, $A^* = \sigma(A)$ is called the *fundamental vector field corresponding* to A. Since the action of G sends each fibre into itself, A_u^* is tangent to the fibre at each $u \in P$. As G acts freely on P, A^* never vanishes on P (if $A \neq 0$) by Proposition 4.1. The dimension of each fibre being equal to that of $\mathfrak{g}$, the mapping $A \rightarrow (A^*)_u$ of $\mathfrak{g}$ into $T_u(P)$ is a linear isomorphism of $\mathfrak{g}$ onto the tangent space at u of the fibre through u. We prove

PROPOSITION 5.1. *Let A^* be the fundamental vector field corresponding to $A \in \mathfrak{g}$. For each $a \in G$, $(R_a)_*A^*$ is the fundamental vector field corresponding to* $(\mathrm{ad}\,(a^{-1}))A \in \mathfrak{g}$.

Proof. Since A^* is induced by the 1-parameter group of transformations R_{a_t} where $a_t = \exp tA$, the vector field $(R_a)_*A^*$ is induced by the 1-parameter group of transformations $R_a R_{a_t} R_{a^{-1}} = R_{a^{-1}a_t a}$ by Proposition 1.7. Our assertion follows from the fact that $a^{-1}a_t a$ is the 1-parameter group generated by $(\mathrm{ad}\,(a^{-1}))A \in \mathfrak{g}$. QED.

The concept of fundamental vector fields will prove to be useful in the theory of connections.

In order to relate our intrinsic definition of a principal fibre bundle to the definition and the construction by means of an open covering, we need the concept of transition functions. By (3) for a principal fibre bundle $P(M, G)$, it is possible to choose an open covering $\{U_\alpha\}$ of M, each $\pi^{-1}(U_\alpha)$ provided with a diffeomorphism $u \rightarrow (\pi(u), \varphi_\alpha(u))$ of $\pi^{-1}(U_\alpha)$ onto $U_\alpha \times G$ such that $\varphi_\alpha(ua) = (\varphi_\alpha(u))a$. If $u \in \pi^{-1}(U_\alpha \cap U_\beta)$, then $\varphi_\beta(ua)(\varphi_\alpha(ua))^{-1} = \varphi_\beta(u)(\varphi_\alpha(u))^{-1}$, which shows that $\varphi_\beta(u)(\varphi_\alpha(u))^{-1}$ depends only on $\pi(u)$ not on u. We can define a mapping $\psi_{\beta\alpha}: U_\alpha \cap U_\beta \rightarrow G$ by $\psi_{\beta\alpha}(\pi(u)) = \varphi_\beta(u)(\varphi_\alpha(u))^{-1}$. The family of mappings $\psi_{\beta\alpha}$ are called *transition functions* of the bundle $P(M, G)$ corresponding to the open covering $\{U_\alpha\}$ of M. It is easy to verify that

$$(*) \qquad \psi_{\gamma\alpha}(x) = \psi_{\gamma\beta}(x) \cdot \psi_{\beta\alpha}(x) \qquad \text{for } x \in U_\alpha \cap U_\beta \cap U_\gamma.$$

Conversely, we have

PROPOSITION 5.2. *Let M be a manifold, $\{U_\alpha\}$ an open covering of M and G a Lie group. Given a mapping $\psi_{\beta\alpha}\colon U_\alpha \cap U_\beta \to G$ for every non-empty $U_\alpha \cap U_\beta$, in such a way that the relations (*) are satisfied, we can construct a (differentiable) principal fibre bundle $P(M, G)$ with transition functions $\psi_{\beta\alpha}$.*

Proof. We first observe that the relations (*) imply $\psi_{\alpha\alpha}(x) = e$ for every $x \in U_\alpha$ and $\psi_{\alpha\beta}(x)\psi_{\beta\alpha}(x) = e$ for every $x \in U_\alpha \cap U_\beta$. Let $X_\alpha = U_\alpha \times G$ for each index α and let $X = \bigcup_\alpha X_\alpha$ be the topological sum of X_α; each element of X is a triple (α, x, a) where α is some index, $x \in U_\alpha$ and $a \in G$. Since each X_α is a differentiable manifold and X is a disjoint union of X_α, X is a differentiable manifold in a natural way. We introduce an equivalence relation ρ in X as follows. We say that $(\alpha, x, a) \in \{\alpha\} \times X_\alpha$ is equivalent to $(\beta, y, b) \in \{\beta\} \times X_\beta$ if and only if $x = y \in U_\alpha \cap U_\beta$ and $b = \psi_{\beta\alpha}(x)a$. We remark that (α, x, a) and (α, y, b) are equivalent if and only if $x = y$ and $a = b$. Let P be the quotient space of X by this equivalence relation ρ. We first show that G acts freely on P on the right and that $P/G = M$. By definition, each $c \in G$ maps the ρ-equivalence class of (α, x, a) into the ρ-equivalence class of (α, x, ac). It is easy to see that this definition is independent of the choice of representative (α, x, a) and that G acts freely on P on the right. The projection $\pi\colon P \to M$ maps, by definition, the ρ-equivalence class of (α, x, a) into x; the definition of π is independent of the choice of representative (α, x, a). For $u,v \in P$, $\pi(u) = \pi(v)$ if and only if $v = uc$ for some $c \in G$. In fact, let (α, x, a) and (β, y, b) be representatives for u and v respectively. If $v = uc$ for some $c \in G$, then $y = x$ and hence $\pi(v) = \pi(u)$. Conversely, if $\pi(u) = x = y = \pi(v) \in U_\alpha \cap U_\beta$, then $v = uc$ where $c = a^{-1}\psi_{\beta\alpha}(x)^{-1}b \in G$. In order to make P into a differentiable manifold, we first note that, by the natural mapping $X \to P = X/\rho$, each $X_\alpha = U_\alpha \times G$ is mapped 1:1 onto $\pi^{-1}(U_\alpha)$. We introduce a differentiable structure in P by requiring that $\pi^{-1}(U_\alpha)$ is an open submanifold of P and that the mapping $X \to P$ induces a diffeomorphism of $X_\alpha = U_\alpha \times G$ onto $\pi^{-1}(U_\alpha)$. This is possible since every point of P is contained in $\pi^{-1}(U_\alpha)$ for some α and the identification of (α, x, a) with $(\beta, x, \psi_{\beta\alpha}(x)a)$ is made by means of differentiable mappings. It is easy to check that the action of G on P is differentiable and $P(M, G, \pi)$ is a differentiable principal fibre bundle. Finally, the transition functions of P corresponding to the covering $\{U_\alpha\}$ are precisely the given $\psi_{\beta\alpha}$ if we define $\psi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times G$ by $\psi_\alpha(u) = (x, a)$, where $u \in \pi^{-1}(U)$ is the ρ-equivalence class of (α, x, a). QED.

A *homomorphism* f of a principal fibre bundle $P'(M', G')$ into another principal fibre bundle $P(M, G)$ consists of a mapping $f'\colon P' \to P$ and a homomorphism $f''\colon G' \to G$ such that $f'(u'a') = f'(u')f''(a)$ for all $u' \in P'$ and $a' \in G'$. For the sake of simplicity, we shall denote f' and f'' by the same letter f. Every homomorphism $f\colon P' \to P$ maps each fibre of P' into a fibre of P and hence induces a mapping of M' into M, which will be also denoted by f. A homomorphism $f\colon P'(M', G') \to P(M, G)$ is called an *imbedding* or *injection* if $f\colon P' \to P$ is an imbedding and if $f\colon G' \to G$ is a monomorphism. If $f\colon P' \to P$ is an imbedding, then the induced mapping $f\colon M' \to M$ is also an imbedding. By identifying P' with $f(P')$, G' with $f(G')$ and M' with $f(M')$, we say that $P'(M', G')$ is a *subbundle* of $P(M, G)$. If, moreover, $M' = M$ and the induced mapping $f\colon M' \to M$ is the identity transformation of M, $f\colon P'(M', G') \to P(M, G)$ is called a *reduction* of the structure group G of $P(M, G)$ to G'. The subbundle $P'(M, G')$ is called a *reduced bundle*. Given $P(M, G)$ and a Lie subgroup G' of G, we say that the structure group G is reducible to G' if there is a reduced bundle $P'(M, G')$. Note that we do not require in general that G' is a closed subgroup of G. This generality is needed in the theory of connections.

PROPOSITION 5.3. *The structure group G of a principal fibre bundle $P(M, G)$ is reducible to a Lie subgroup G' if and only if there is an open covering $\{U_\alpha\}$ of M with a set of transition functions $\psi_{\beta\alpha}$ which take their values in G'.*

Proof. Suppose first that the structure group G is reducible to G' and let $P'(M, G')$ be a reduced bundle. Consider P' as a submanifold of P. Let $\{U_\alpha\}$ be an open covering of M such that each $\pi'^{-1}(U_\alpha)$ (π': the projection of P' onto M) is provided with an isomorphism $u \to (\pi'(u), \varphi'_\alpha(u))$ of $\pi'^{-1}(U_\alpha)$ onto $U_\alpha \times G'$. The corresponding transition functions take their values in G'. Now, for the same covering $\{U_\alpha\}$, we define an isomorphism of $\pi^{-1}(U_\alpha)$ (π: the projection of P onto M) onto $U_\alpha \times G$ by extending φ'_α as follows. Every $v \in \pi^{-1}(U_\alpha)$ may be represented in the form $v = ua$ for some $u \in \pi'^{-1}(U_\alpha)$ and $a \in G$ and we set $\varphi_\alpha(v) = \varphi'_\alpha(u)a$.

It is easy to see that $\varphi_\alpha(v)$ is independent of the choice of representation $v = ua$. We see then that $v \to (\pi(v), \varphi_\alpha(v))$ is an isomorphism of $\pi^{-1}(U_\alpha)$ onto $U_\alpha \times G$. The corresponding transition functions $\psi_{\beta\alpha}(x) = \varphi_\beta(v)(\varphi_\alpha(v))^{-1} = \varphi'_\beta(u)(\varphi'_\alpha(u))^{-1}$ take their values in G'.

Conversely, assume that there is a covering $\{U_\alpha\}$ of M with a set of transition functions $\psi_{\beta\alpha}$ all taking values in a Lie subgroup G' of G. For $U_\alpha \cap U_\beta \neq \phi$, $\psi_{\beta\alpha}$ is a differentiable mapping of $U_\alpha \cap U_\beta$ into a Lie group G such that $\psi_{\beta\alpha}(U_\alpha \cap U_\beta) \subset G'$. The crucial point is that $\psi_{\beta\alpha}$ is a differentiable mapping of $U_\alpha \cap U_\beta$ into G' with respect to the differentiable structure of G'. This follows from Proposition 1.3; note that a Lie subgroup satisfies the second axiom of countability by definition, cf. §4. By Proposition 5.2, we can construct a principal fibre bundle $P'(M, G')$ from $\{U_\alpha\}$ and $\{\psi_{\beta\alpha}\}$. Finally, we imbed P' into P as follows. Let $f_\alpha\colon \pi'^{-1}(U_\alpha) \to \pi^{-1}(U_\alpha)$ be the composite of the following three mappings:

$$\pi'^{-1}(U_\alpha) \to U_\alpha \times G' \to U_\alpha \times G \to \pi^{-1}(U_\alpha).$$

It is easy to see that $f_\alpha = f_\beta$ on $\pi'^{-1}(U_\alpha \cap U_\beta)$ and that the mapping $f\colon P' \to P$ thus defined by $\{f_\alpha\}$ is an injection. QED.

Let $P(M, G)$ be a principal fibre bundle and F a manifold on which G acts on the left: $(a, \xi) \in G \times F \to a\xi \in F$. We shall construct a fibre bundle $E(M, F, G, P)$ associated with P with standard fibre F. On the product manifold $P \times F$, we let G act on the right as follows: an element $a \in G$ maps $(u, \xi) \in P \times F$ into $(ua, a^{-1}\xi) \in P \times F$. The quotient space of $P \times F$ by this group action is denoted by $E = P \times_G F$. A differentiable structure will be introduced in E later and at this moment E is only a set. The mapping $P \times F \to M$ which maps (u, ξ) into $\pi(u)$ induces a mapping π_E, called the projection, of E onto M. For each $x \in M$, the set $\pi_E^{-1}(x)$ is called the fibre of E over x. Every point x of M has a neighborhood U such that $\pi^{-1}(U)$ is isomorphic to $U \times G$. Identifying $\pi^{-1}(U)$ with $U \times G$, we see that the action of G on $\pi^{-1}(U) \times F$ on the right is given by

$$(x, a, \xi) \to (x, ab, b^{-1}\xi) \quad \text{for } (x, a, \xi) \in U \times G \times F \quad \text{and} \quad b \in G.$$

It follows that the isomorphism $\pi^{-1}(U) \approx U \times G$ induces an isomorphism $\pi_E^{-1}(U) \approx U \times F$. We can therefore introduce a differentiable structure in E by the requirement that $\pi_E^{-1}(U)$ is an open submanifold of E which is diffeomorphic with $U \times F$ under the isomorphism $\pi_E^{-1}(U) \approx U \times F$. The projection π_E is then a differentiable mapping of E onto M. We call E or more precisely $E(M, F, G, P)$ the *fibre bundle over the base M, with (standard) fibre F and (structure) group G, which is associated with the principal fibre bundle P.*

PROPOSITION 5.4. *Let $P(M, G)$ be a principal fibre bundle and F a manifold on which G acts on the left. Let $E(M, F, G, P)$ be the fibre bundle associated with P. For each $u \in P$ and each $\xi \in F$, denote by $u\xi$ the image of $(u, \xi) \in P \times F$ by the natural projection $P \times F \to E$. Then each $u \in P$ is a mapping of F onto $F_x = \pi_E^{-1}(x)$ where $x = \pi(u)$ and*

$$(ua)\xi = u(a\xi) \qquad \text{for } u \in P,\ a \in G,\ \xi \in F.$$

The proof is trivial and is left to the reader.

By an isomorphism of a fibre $F_x = \pi_E^{-1}(x)$, $x \in M$, onto another fibre F_y, $y \in M$, we mean a diffeomorphism which can be represented in the form $v \circ u^{-1}$, where $u \in \pi^{-1}(x)$ and $v \in \pi^{-1}(y)$ are considered as mappings of F onto F_x and F_y respectively. In particular, an automorphism of the fibre F_x is a mapping of the form $v \circ u^{-1}$ with $u, v \in \pi^{-1}(x)$. In this case, $v = ua$ for some $a \in G$ so that any automorphism of F_x can be expressed in the form $u \circ a \circ u^{-1}$ where u is an arbitrarily fixed point of $\pi^{-1}(x)$. The group of automorphisms of F_x is hence isomorphic with the structure group G.

Example 5.1. $G(G/H, H)$: Let G be a Lie group and H a closed subgroup of G. We let H act on G on the right as follows. Every $a \in H$ maps $u \in G$ into ua. We then obtain a differentiable principal fibre bundle $G(G/H, H)$ over the base manifold G/H with structure group H; the local triviality follows from the existence of a local cross section. It is proved in Chevalley [1; p. 110] that if π is the projection of G onto G/H and e is the identity of G, then there is a mapping σ of a neighborhood of $\pi(e)$ in G/H into G such that $\pi \circ \sigma$ is the identity transformation of the neighborhood. See also Steenrod [1; pp. 28–33].

Example 5.2. *Bundle of linear frames*: Let M be a manifold of dimension n. A *linear frame* u at a point $x \in M$ is an ordered basis $X_1, \ldots, X_n$ of the tangent space $T_x(M)$. Let $L(M)$ be the set of all linear frames u at all points of M and let π be the mapping of $L(M)$ onto M which maps a linear frame u at x into x. The general linear group $GL(n; \mathbf{R})$ acts on $L(M)$ on the right as follows. If $a = (a^i_j) \in GL(n; \mathbf{R})$ and $u = (X_1, \ldots, X_n)$ is a linear frame at x, then ua is, by definition, the linear frame $(Y_1, \ldots, Y_n)$ at x defined by $Y_i = \Sigma_j a^j_i X_j$. It is clear that $GL(n; \mathbf{R})$ acts freely on $L(M)$ and $\pi(u) = \pi(v)$ if and only if $v = ua$ for some $a \in GL(n; \mathbf{R})$. Now in order to introduce a differentiable structure in $L(M)$, let $(x^1, \ldots, x^n)$ be a local coordinate system in a coordinate neighborhood U in M. Every frame u at $x \in U$ can be expressed uniquely in the form $u = (X_1, \ldots, X_n)$ with $X_i = \Sigma_k X^k_i(\partial/\partial x^k)$, where (X^k_i) is a non-singular matrix. This shows that $\pi^{-1}(U)$ is in 1:1 correspondence with $U \times GL(n; \mathbf{R})$. We can make $L(M)$ into a differentiable manifold by taking (x^j) and (X^k_i) as a local coordinate system in $\pi^{-1}(U)$. It is now easy to verify that $L(M)(M, GL(n; \mathbf{R}))$ is a principal fibre bundle. We call $L(M)$ the *bundle of linear frames over* M. In view of Proposition 5.4, a linear frame u at $x \in M$ can be defined as a non-singular linear mapping of $\mathbf{R}^n$ onto $T_x(M)$. The two definitions are related to each other as follows. Let $e_1, \ldots, e_n$ be the natural basis for $\mathbf{R}^n$: $e_1 = (1, 0, \ldots, 0), \ldots, e_n = (0, \ldots, 0, 1)$. A linear frame $u = (X_1, \ldots, X_n)$ at x can be given as a linear mapping u: $\mathbf{R}^n \to T_x(M)$ such that $ue_i = X_i$ for $i = 1, \ldots, n$. The action of $GL(n; \mathbf{R})$ on $L(M)$ can be accordingly interpreted as follows. Consider $a = (a^i_j) \in GL(n; \mathbf{R})$ as a linear transformation of $\mathbf{R}^n$ which maps e_j into $\Sigma_i a^i_j e_i$. Then ua: $\mathbf{R}^n \to T_x(M)$ is the composite of the following two mappings:

$$\mathbf{R}^n \xrightarrow{a} \mathbf{R}^n \xrightarrow{u} T_x(M).$$

Example 5.3. *Tangent bundle*: Let $GL(n; \mathbf{R})$ act on $\mathbf{R}^n$ as above. The *tangent bundle* $T(M)$ *over* M is the bundle associated with $L(M)$ with standard fibre $\mathbf{R}^n$. It can be easily shown that the fibre of $T(M)$ over $x \in M$ may be considered as $T_x(M)$.

Example 5.4. *Tensor bundles*: Let $\mathbf{T}^r_s$ be the tensor space of type (r, s) over the vector space $\mathbf{R}^n$ as defined in §2. The group $GL(n: \mathbf{R})$ can be regarded as a group of linear transformations of the space $\mathbf{T}^r_s$ by Proposition 2.12. With this standard fibre $\mathbf{T}^r_s$, we obtain the *tensor bundle* $T^r_s(M)$ *of type* (r, s) *over* M which is associated with $L(M)$. It is easy to see that the fibre of $T^r_s(M)$ over $x \in M$ may be considered as the tensor space over $T_x(M)$ of type (r, s).

Returning to the general case, let $P(M, G)$ be a principal fibre bundle and H a closed subgroup of G. In a natural way, G acts on the quotient space G/H on the left. Let $E(M, G/H, G, P)$ be the associated bundle with standard fibre G/H. On the other hand, being a subgroup of G, H acts on P on the right. Let P/H be the quotient space of P by this action of H. Then we have

PROPOSITION 5.5. *The bundle $E = P \times_G (G/H)$ associated with P with standard fibre G/H can be identified with P/H as follows. An element of E represented by $(u, a\xi_0) \in P \times G/H$ is mapped into the element of P/H represented by $ua \in P$, where $a \in G$ and ξ_0 is the origin of G/H, i.e., the coset H.*

Consequently, $P(E, H)$ is a principal fibre bundle over the base $E = P/H$ with structure group H. The projection $P \to E$ maps $u \in P$ into $u\xi_0 \in E$, where u is considered as a mapping of the standard fibre G/H into a fibre of E.

Proof. The proof is straightforward, except the local triviality of the bundle $P(E, H)$. This follows from local triviality of $E(M, G/H, G, P)$ and $G(G/H, H)$ as follows. Let U be an open set of M such that $\pi_E^{-1}(U) \approx U \times G/H$ and let V be an open set of G/H such that $p^{-1}(V) \approx V \times H$, where p: $G \to G/H$ is the projection. Let W be the open set of $\pi_E^{-1}(U) \subset E$ which corresponds to $U \times V$ under the identification $\pi_E^{-1}(U) \approx U \times G/H$. If μ: $P \to E = P/H$ is the projection, then $\mu^{-1}(W) \approx W \times H$. QED.

A *cross section* of a bundle $E(M, F, G, P)$ is a mapping σ: $M \to E$ such that $\pi_E \circ \sigma$ is the identity transformation of M. For $P(M, G)$ itself, a cross section σ: $M \to P$ exists if and only if P is the trivial bundle $M \times G$ (cf. Steenrod [1; p. 36]). More generally, we have

PROPOSITION 5.6. *The structure group G of $P(M, G)$ is reducible to a closed subgroup H if and only if the associated bundle $E(M, G/H, G, P)$ admits a cross section σ: $M \to E = P/H$.*

Proof. Suppose G is reducible to a closed subgroup H and let $Q(M, H)$ be a reduced bundle with injection f: $Q \to P$. Let μ: $P \to E = P/H$ be the projection. If u and v are in the same fibre of Q, then $v = ua$ for some $a \in H$ and hence $\mu(f(v)) = \mu(f(u)a) = \mu(f(u))$. This means that $\mu \circ f$ is constant on each fibre of Q and induces a mapping σ: $M \to E$, $\sigma(x) = \mu(f(u))$

where $x = \pi(f(u))$. It is clear that σ is a section of E. Conversely, given a cross section $\sigma: M \to E$, let Q be the set of points $u \in P$ such that $\mu(u) = \sigma(\pi(u))$. In other words, Q is the inverse image of $\sigma(M)$ by the projection $\mu: P \to E = P/H$. For every $x \in M$, there is $u \in Q$ such that $\pi(u) = x$ because $\mu^{-1}(\sigma(x))$ is non-empty. Given u and v in the same fibre of P, if $u \in Q$ then $v \in Q$ when and only when $v = ua$ for some $a \in H$. This follows from the fact that $\mu(u) = \mu(v)$ if and only if $v = ua$ for some $a \in H$. It is now easy to verify that Q is a closed submanifold of P and that Q is a principal fibre bundle $Q(M, H)$ imbedded in $P(M, G)$. QED.

Remark. The correspondence between the sections $\sigma: M \to E = P/H$ and the submanifolds Q is 1:1.

We shall now consider the question of extending a cross section defined on a subset of the base manifold. A mapping f of a subset A of a manifold M into another manifold is called *differentiable on* A if for each point $x \in A$, there is a differentiable mapping f_x of an open neighborhood U_x of x in M into M' such that $f_x = f$ on $U_x \cap A$. If f is the restriction of a differentiable mapping of an open set W containing A into M', then f is clearly differentiable on A. Given a fibre bundle $E(M, F, G, P)$ and a subset A of M, by a cross section on A we mean a differentiable mapping σ of A into E such that $\pi_E \circ \sigma$ is the identity transformation of A.

THEOREM 5.7. *Let $E(M, F, G, P)$ be a fibre bundle such that the base manifold M is paracompact and the fibre F is diffeomorphic with a Euclidean space $\mathbf{R}^m$. Let A be a closed subset (possibly empty) of M. Then every cross section $\sigma: A \to E$ defined on A can be extended to a cross section defined on M. In the particular case where A is empty, there exists a cross section of E defined on M.*

Proof. By the very definition of a paracompact space, every open covering of M has a locally finite open refinement. Since M is normal, every locally finite open covering $\{U_i\}$ of M has an open refinement $\{V_i\}$ such that $\bar{V}_i \subset U_i$ for all i (see Appendix 3).

LEMMA 1. *A differentiable function defined on a closed set of $\mathbf{R}^n$ can be extended to a differentiable function on $\mathbf{R}^n$* (cf. Appendix 3).

LEMMA 2. *Every point of M has a neighborhood U such that every section of E defined on a closed subset contained in U can be extended to U.*

Proof. Given a point of M, it suffices to take a coordinate neighborhood U such that $\pi_E^{-1}(U)$ is trivial: $\pi_E^{-1}(U) \approx U \times F$. Since F is diffeomorphic with $\mathbf{R}^m$, a section on U can be identified with a set of m functions $f_1, \dots, f_m$ defined on U. By Lemma 1, these functions can be extended to U.

Using Lemma 2, we shall prove Theorem 5.7. Let $\{U_i\}_{i \in I}$ be a locally finite open covering of M such that each U_i has the property stated in Lemma 2. Let $\{V_i\}$ be an open refinement of $\{U_i\}$ such that $\bar{V}_i \subset U_i$ for all $i \in I$. For each subset J of the index set I, set $S_J = \bigcup_{i \in J} \bar{V}_i$. Let T be the set of pairs (τ, J) where $J \subset I$ and τ is a section of E defined on S_J such that $\tau = \sigma$ on $A \cap S_J$. The set T is non-empty; take U_i which meets A and extend the restriction of σ to $A \cap \bar{V}_i$ to a section on $\bar{V}_i$, which is possible by the property possessed by U_i. Introduce an order in T as follows: $(\tau', J') < (\tau'', J'')$ if $J' \subset J''$ and $\tau' = \tau''$ on $S_{J'}$. Let (τ, J) be a maximal element (by using Zorn's Lemma). Assume $J \neq I$ and let $i \in I - J$. On the closed set $(A \cup S_J) \cap \bar{V}_i$ contained in U_i, we have a well defined section σ_i: $\sigma_i = \sigma$ on $A \cap \bar{V}_i$ and $\sigma_i = \tau$ on $S_J \cap \bar{V}_i$. Extend σ_i to a section τ_i on $\bar{V}_i$, which is possible by the property possessed by U_i. Let $J' = J \cup \{i\}$ and τ' be the section on $S_{J'}$ defined by $\tau' = \tau$ on S_J and $\tau' = \tau_i$ on $\bar{V}_i$. Then $(\tau, J) < (\tau', J')$, which contradicts the maximality of (τ, J). Hence, $I = J$ and τ is the desired section. QED.

The proof given here was taken from Godement [1, p. 151].

Example 5.5. Let $L(M)$ be the bundle of linear frames over an n-dimensional manifold M. The homogeneous space $GL(n; \mathbf{R})/O(n)$ is known to be diffeomorphic with a Euclidean space of dimension $\frac{1}{2}n(n+1)$ by an argument similar to Chevalley [1, p. 16]. The fibre bundle $E = L(M)/O(n)$ with fibre $GL(n; \mathbf{R})/O(n)$, associated with $L(M)$, admits a cross section if M is paracompact (by Theorem 5.7). By Proposition 5.6, we see that the structure group of $L(M)$ can be reduced to the orthogonal group $O(n)$, provided that M is paracompact.

Example 5.6. More generally, let $P(M, G)$ be a principal fibre bundle over a paracompact manifold M with group G which is a connected Lie group. It is known that G is diffeomorphic with a direct product of any of its maximal compact subgroups H and a Euclidean space (cf. Iwasawa [1]). By the same reasoning as above, the structure group G can be reduced to H.

Example 5.7. Let $L(M)$ be the bundle of linear frames over a manifold M of dimension n. Let $(\ ,\ )$ be the natural inner product in $\mathbf{R}^n$ for which $e_1 = (1, 0, \ldots, 0), \ldots, e_n = (0, \ldots, 0, 1)$ are orthonormal and which is invariant by $O(n)$ by the very definition of $O(n)$. We shall show that each reduction of the structure group $GL(n; \mathbf{R})$ to $O(n)$ gives rise to a Riemannian metric g on M. Let $Q(M, O(n))$ be a reduced subbundle of $L(M)$. When we regard each $u \in L(M)$ as a linear isomorphism of $\mathbf{R}^n$ onto $T_x(M)$ where $x = \pi(u)$, each $u \in Q$ defines an inner product g in $T_x(M)$ by

$$g(X, Y) = (u^{-1}X, u^{-1}Y) \qquad \text{for } X, Y \in T_x(M).$$

The invariance of $(\ ,\ )$ by $O(n)$ implies that $g(X, Y)$ is independent of the choice of $u \in Q$. Conversely, if M is given a Riemannian metric g, let Q be the subset of $L(M)$ consisting of linear frames $u = (X_1, \ldots, X_n)$ which are orthonormal with respect to g. If we regard $u \in L(M)$ as a linear isomorphism of $\mathbf{R}^n$ onto $T_x(M)$, then u belongs to Q if and only if $(\xi, \xi') = g(u\xi, u\xi')$ for all $\xi, \xi' \in \mathbf{R}^n$. It is easy to verify that Q forms a reduced subbundle of $L(M)$ over M with structure group $O(n)$. The bundle Q will be called the *bundle of orthonormal frames over* M and will be denoted by $O(M)$. An element of $O(M)$ is an *orthonormal frame*. The result here combined with Example 5.5 implies that *every paracompact manifold* M *admits a Riemannian metric*. We shall see later that every Riemannian manifold is a metric space and hence paracompact.

To introduce the notion of induced bundle, we prove

PROPOSITION 5.8. *Given a principal fibre bundle $P(M, G)$ and a mapping f of a manifold N into M, there is a unique (of course, unique up to an isomorphism) principal fibre bundle $Q(N, G)$ with a homomorphism f: $Q \to P$ which induces f: $N \to M$ and which corresponds to the identity automorphism of G.*

The bundle $Q(N, G)$ is called the *bundle induced by f from $P(M, G)$* or simply the *induced bundle*; it is sometimes denoted by $f^{-1}P$.

Proof. In the direct product $N \times P$, consider the subset Q consisting of $(y, u) \in N \times P$ such that $f(y) = \pi(u)$. The group G acts on Q by $(y, u) \to (y, u)a = (y, ua)$ for $(y, u) \in Q$ and $a \in G$. It is easy to see that G acts freely on Q and that Q is a principal fibre bundle over N with group G and with projection π_Q given by $\pi_Q(y, u) = y$. Let Q' be another principal fibre bundle over N with group G and f': $Q' \to P$ a homomorphism which induces f: $N \to M$ and which corresponds to the identity automorphism of G. Then it is easy to show that the mapping of Q' onto Q defined by $u' \to (\pi_{Q'}(u'), f'(u'))$, $u' \in Q'$, is an isomorphism of the bundle Q' onto Q which induces the identity transformation of N and which corresponds to the identity automorphism of G. QED.

We recall here some results on covering spaces which will be used later. Given a connected, locally arcwise connected topological space M, a connected space E is called a *covering space* over M with projection p: $E \to M$ if every point x of M has a connected open neighborhood U such that each connected component of $p^{-1}(U)$ is open in E and is mapped homeomorphically onto U by p. Two covering spaces p: $E \to M$ and p': $E' \to M$ are *isomorphic* if there exists a homeomorphism f: $E \to E'$ such that $p' \circ f = p$. A covering space p: $E \to M$ is a *universal covering space* if E is simply connected. If M is a manifold, every covering space has a (unique) structure of manifold such that p is differentiable. From now on we shall only consider covering manifolds.

PROPOSITION 5.9. (1) *Given a connected manifold M, there is a unique (unique up to an isomorphism) universal covering manifold, which will be denoted by $\tilde{M}$.*

(2) *The universal covering manifold $\tilde{M}$ is a principal fibre bundle over M with group $\pi_1(M)$ and projection p: $\tilde{M} \to M$, where $\pi_1(M)$ is the first homotopy group of M.*

(3) *The isomorphism classes of the covering spaces over M are in a 1:1 correspondence with the conjugate classes of the subgroups of $\pi_1(M)$. The correspondence is given as follows. To each subgroup H of $\pi_1(M)$, we associate $E = \tilde{M}/H$. Then the covering manifold E corresponding to H is a fibre bundle over M with fibre $\pi_1(M)/H$ associated with the principal fibre bundle $\tilde{M}(M, \pi_1(M))$. If H is a normal subgroup of $\pi_1(M)$, $E = \tilde{M}/H$ is a principal fibre bundle with group $\pi_1(M)/H$ and is called a regular covering manifold of M.*

For the proof, see Steenrod [1, pp. 67–71] or Hu [1, pp. 89–97].

The action of $\pi_1(M)/H$ on a regular covering manifold $E = \tilde{M}/H$ is properly discontinuous. Conversely, if E is a connected manifold and G is a properly discontinuous group of transformations acting freely on E, then E is a regular covering manifold of

$M = E/G$ as follows immediately from the condition (3) in the definition of properly discontinuous action in §4.

Example 5.8. Consider $\mathbf{R}^n$ as an n-dimensional vector space and let $\xi_1, \ldots, \xi_n$ be any basis of $\mathbf{R}^n$. Let G be the subgroup of $\mathbf{R}^n$ generated by $\xi_1, \ldots, \xi_n$: $G = \{\Sigma\, m_i\, \xi_i;\ m_i \text{ integers}\}$. The action of G on $\mathbf{R}^n$ is properly discontinuous and $\mathbf{R}^n$ is the universal covering manifold of $\mathbf{R}^n/G$. The quotient manifold $\mathbf{R}^n/G$ is called an n-dimensional *torus*.

Example 5.9. Let S^n be the unit sphere in $\mathbf{R}^{n+1}$ with center at the origin: $S^n = \{(x^1, \ldots, x^{n+1}) \in \mathbf{R}^{n+1};\ \Sigma_i (x^i)^2 = 1\}$. Let G be the group consisting of the identity transformation of S^n and the transformation of S^n which maps $(x^1, \ldots, x^{n+1})$ into $(-x^1, \ldots, -x^{n+1})$. Then $S^n, n \geqq 2$, is the universal covering manifold of S^n/G. The quotient manifold S^n/G is called the n-dimensional *real projective space*.

CHAPTER II

Theory of Connections

1. *Connections in a principal fibre bundle*

Let $P(M, G)$ be a principal fibre bundle over a manifold M with group G. For each $u \in P$, let $T_u(P)$ be the tangent space of P at u and G_u the subspace of $T_u(P)$ consisting of vectors tangent to the fibre through u. A *connection* Γ in P is an assignment of a subspace Q_u of $T_u(P)$ to each $u \in P$ such that

(a) $T_u(P) = G_u + Q_u$ (direct sum);

(b) $Q_{ua} = (R_a)_* Q_u$ for every $u \in P$ and $a \in G$, where R_a is the transformation of P induced by $a \in G$, $R_a u = ua$;

(c) Q_u depends differentiably on u.

Condition (b) means that the distribution $u \to Q_u$ is invariant by G. We call G_u the *vertical subspace* and Q_u the *horizontal subspace* of $T_u(P)$. A vector $X \in T_u(P)$ is called *vertical* (resp. *horizontal*) if it lies in G_u (resp. Q_u). By (a), every vector $X \in T_u(P)$ can be uniquely written as

$$X = Y + Z \qquad \text{where } Y \in G_u \quad \text{and} \quad Z \in Q_u.$$

We call Y (resp. Z) the *vertical* (resp. *horizontal*) *component* of X and denote it by vX (resp. hX). Condition (c) means, by definition, that if X is a differentiable vector field on P so are vX and hX. (It can be easily verified that this is equivalent to saying that the distribution $u \to Q_u$ is differentiable.)

Given a connection Γ in P, we define a 1-form ω on P with values in the Lie algebra $\mathfrak{g}$ of G as follows. In §5 of Chapter I, we showed that every $A \in \mathfrak{g}$ induces a vector field A^* on P, called the fundamental vector field corresponding to A, and that $A \to (A^*)_u$ is a linear isomorphism of $\mathfrak{g}$ onto G_u for each $u \in P$. For each $X \in T_u(P)$, we define $\omega(X)$ to be the unique $A \in \mathfrak{g}$ such that

$(A^*)_u$ is equal to the vertical component of X. It is clear that $\omega(X) = 0$ if and only if X is horizontal. The form ω is called the *connection form* of the given connection Γ.

PROPOSITION 1.1. *The connection form ω of a connection satisfies the following conditions:*

(a′) $\omega(A^*) = A$ *for every* $A \in \mathfrak{g}$;

(b′) $(R_a)^*\omega = \mathrm{ad}\,(a^{-1})\omega$, *that is,* $\omega((R_a)_*X) = \mathrm{ad}\,(a^{-1}) \cdot \omega(X)$ *for every $a \in G$ and every vector field X on P, where* ad *denotes the adjoint representation of G in* $\mathfrak{g}$.

Conversely, given a $\mathfrak{g}$-valued 1-form ω on P satisfying conditions (a′) *and* (b′), *there is a unique connection Γ in P whose connection form is ω.*

Proof. Let ω be the connection form of a connection. The condition (a′) follows immediately from the definition of ω. Since every vector field of P can be decomposed into a horizontal vector field and a vertical vector field, it is sufficient to verify (b′) in the following two special cases: (1) X is horizontal and (2) X is vertical. If X is horizontal, so is $(R_a)_*X$ for every $a \in G$ by the condition (b) for a connection. Thus, both $\omega((R_a)_*X)$ and $\mathrm{ad}\,(a^{-1}) \cdot \omega(X)$ vanish. In the case when X is vertical, we may further assume that X is a fundamental vector field A^*. Then $(R_a)_*X$ is the fundamental vector field corresponding to $\mathrm{ad}\,(a^{-1})A$ by Proposition 5.1 of Chapter I. Thus we have

$$(R_a^*\omega)_u(X) = \omega_{ua}((R_a)_*X) = \mathrm{ad}\,(a^{-1})A = \mathrm{ad}\,(a^{-1})(\omega_u(X)).$$

Conversely, given a form ω satisfying (a′) and (b′), we define

$$Q_u = \{X \in T_u(P);\ \omega(X) = 0\}.$$

The verification that $u \to Q_u$ defines a connection whose connection form is ω is easy and is left to the reader. QED.

The projection $\pi\colon P \to M$ induces a linear mapping $\pi\colon T_u(P) \to T_x(M)$ for each $u \in P$, where $x = \pi(u)$. When a connection is given, π maps the horizontal subspace Q_u isomorphically onto $T_x(M)$.

The *horizontal lift* (or simply, the *lift*) of a vector field X on M is a unique vector field X^* on P which is horizontal and which projects onto X, that is, $\pi(X_u^*) = X_{\pi(u)}$ for every $u \in P$.

PROPOSITION 1.2. *Given a connection in P and a vector field X on M, there is a unique horizontal lift X^* of X. The lift X^* is invariant by R_a for every $a \in G$. Conversely, every horizontal vector field X^* on P invariant by G is the lift of a vector field X on M.*

Proof. The existence and uniqueness of X^* is clear from the fact that π gives a linear isomorphism of Q_u onto $T_{\pi(u)}(M)$. To prove that X^* is differentiable if X is differentiable, we take a neighborhood U of any given point x of M such that $\pi^{-1}(U) \approx U \times G$. Using this isomorphism, we first obtain a differentiable vector field Y on $\pi^{-1}(U)$ such that $\pi Y = X$. Then X^* is the horizontal component of Y and hence is differentiable. The invariance of X^* by G is clear from the invariance of the horizontal subspaces by G. Finally, let X^* be a horizontal vector field on P invariant by G. For every $x \in M$, take a point $u \in P$ such that $\pi(u) = x$ and define $X_x = \pi(X_u^*)$. The vector X_x is independent of the choice of u such that $\pi(u) = x$, since if $u' = ua$, then $\pi(X_{u'}^*) = \pi(R_a \cdot X_u^*) = \pi(X_u^*)$. It is obvious that X^* is then the lift of the vector field X. QED.

PROPOSITION 1.3. *Let X^* and Y^* be the horizontal lifts of X and Y respectively. Then*

(1) *$X^* + Y^*$ is the horizontal lift of $X + Y$;*

(2) *For every function f on M, $f^* \cdot X^*$ is the horizontal lift of fX where f^* is the function on P defined by $f^* = f \circ \pi$;*

(3) *The horizontal component of $[X^*, Y^*]$ is the horizontal lift of $[X, Y]$.*

Proof. The first two assertions are trivial. As for the third, we have

$$\pi(h[X^*, Y^*]) = \pi([X^*, Y^*]) = [X, Y].$$

QED.

Let $x^1, \ldots, x^n$ be a local coordinate system in a coordinate neighborhood U in M. Let X_i^* be the horizontal lift in $\pi^{-1}(U)$ of the vector field $X_i = \partial/\partial x^i$ in U for each i. Then $X_1^*, \ldots, X_n^*$ form a local basis for the distribution $u \to Q_u$ in $\pi^{-1}(U)$.

We shall now express a connection form ω on P by a family of forms each defined in an open subset of the base manifold M. Let $\{U_\alpha\}$ be an open covering of M with a family of isomorphisms $\psi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times G$ and the corresponding family of transition functions $\psi_{\alpha\beta}\colon U_\alpha \cap U_\beta \to G$. For each α, let $\sigma_\alpha\colon U_\alpha \to P$ be the cross section on U_α defined by $\sigma_\alpha(x) = \psi_\alpha^{-1}(x, e)$, $x \in U_\alpha$, where e is the identity of G. Let θ be the (left invariant $\mathfrak{g}$-valued) canonical 1-form on G defined in §4 of Chapter I (p. 41).

For each non-empty $U_\alpha \cap U_\beta$, define a $\mathfrak{g}$-valued 1-form $\theta_{\alpha\beta}$ on $U_\alpha \cap U_\beta$ by

$$\theta_{\alpha\beta} = \psi_{\alpha\beta}^*\theta.$$

For each α, define a $\mathfrak{g}$-valued 1-form ω_α on U_α by

$$\omega_\alpha = \sigma_\alpha^*\omega.$$

PROPOSITION 1.4. *The forms $\theta_{\alpha\beta}$ and ω_α are subject to the conditions:*

$$\omega_\beta = \mathrm{ad}\,(\psi_{\alpha\beta}^{-1})\omega_\alpha + \theta_{\alpha\beta} \qquad \textit{on } U_\alpha \cap U_\beta.$$

Conversely, for every family of $\mathfrak{g}$-valued 1-forms $\{\omega_\alpha\}$ each defined on U_α and satisfying the preceding conditions, there is a unique connection form ω on P which gives rise to $\{\omega_\alpha\}$ in the described manner.

Proof. If $U_\alpha \cap U_\beta$ is non-empty, $\sigma_\beta(x) = \sigma_\alpha(x)\psi_{\alpha\beta}(x)$ for all $x \in U_\alpha \cap U_\beta$. Denote the differentials of σ_α, σ_β, and $\psi_{\alpha\beta}$ by the same letters. Then for every vector $X \in T_x(U_\alpha \cap U_\beta)$, the vector $\sigma_\beta(X) \in T_u(P)$, where $u = \sigma_\beta(x)$, is the image of $(\sigma_\alpha(X), \psi_{\alpha\beta}(X)) \in T_{u'}(P) + T_a(G)$, where $u' = \sigma_\alpha(x)$ and $a = \psi_{\alpha\beta}(x)$, under the mapping $P \times G \to P$. By Proposition 1.4 (Leibniz's formula) of Chapter I, we have

$$\sigma_\beta(X) = \sigma_\alpha(X)\psi_{\alpha\beta}(x) + \sigma_\alpha(x)\psi_{\alpha\beta}(X),$$

where $\sigma_\alpha(X)\psi_{\alpha\beta}(x)$ means $R_a(\sigma_\alpha(X))$ and $\sigma_\alpha(x)\psi_{\alpha\beta}(X)$ is the image of $\psi_{\alpha\beta}(X)$ by the differential of $\sigma_\alpha(x)$, $\sigma_\alpha(x)$ being considered as a mapping of G into P which maps $b \in G$ into $\sigma_\alpha(x)b$. Taking the values of ω on both sides of the equality, we obtain

$$\omega_\beta(X) = \mathrm{ad}\,(\psi_{\alpha\beta}(x)^{-1})\omega_\alpha(X) + \theta_{\alpha\beta}(X).$$

Indeed, if $A \in \mathfrak{g}$ is the left invariant vector field on G which is equal to $\psi_{\alpha\beta}(X)$ at $a = \psi_{\alpha\beta}(x)$ so that $\theta(\psi_{\alpha\beta}(X)) = A$, then $\sigma_\alpha(x)\psi_{\alpha\beta}(X)$ is the value of the fundamental vector field A^* at $u = \sigma_\alpha(x)\psi_{\alpha\beta}(x)$ and hence $\omega(\sigma_\alpha(x)\psi_{\alpha\beta}(X)) = A$.

The converse can be verified by following back the process of obtaining $\{\omega_\alpha\}$ from ω. QED.

2. *Existence and extension of connections*

Let $P(M, G)$ be a principal fibre bundle and A a subset of M. We say that a connection is defined over A if, at every point $u \in P$ with $\pi(u) \in A$, a subspace Q_u of $T_u(P)$ is given in such a way that conditions (a) and (b) for connection (see §1) are satisfied and Q_u depends differentiably on u in the following sense. For every point $x \in A$, there exist an open neighborhood U and a connection in $P\,|\,U = \pi^{-1}(U)$ such that the horizontal subspace at every $u \in \pi^{-1}(A)$ is the given space Q_u.

THEOREM 2.1. *Let $P(M, G)$ be a principal fibre bundle and A a closed subset of M (A may be empty). If M is paracompact, every connection defined over A can be extended to a connection in P. In particular, P admits a connection if M is paracompact.*

Proof. The proof is a replica of that of Theorem 5.7 in Chapter I.

LEMMA 1. *A differentiable function defined on a closed subset of $\mathbf{R}^n$ can be always extended to a differentiable function on $\mathbf{R}^n$* (cf. Appendix 3).

LEMMA 2. *Every point of M has a neighborhood U such that every connection defined on a closed subset contained in U can be extended to a connection defined over U.*

Proof. Given a point of M, it suffices to take a coordinate neighborhood U such that $\pi^{-1}(U)$ is trivial: $\pi^{-1}(U) \approx U \times G$. On the trivial bundle $U \times G$, a connection form ω is completely determined by its behavior at the points of $U \times \{e\}$ (e: the identity of G) because of the property $R_a^*(\omega) = \mathrm{ad}\,(a^{-1})\omega$. Furthermore, if σ: $U \to U \times G$ is the natural cross section, that is, $\sigma(x) = (x, e)$ for $x \in U$, then ω is completely determined by the $\mathfrak{g}$-valued 1-form $\sigma^*\omega$ on U. Indeed, every vector $X \in T_{\sigma(x)}(U \times G)$ can be written uniquely in the form

$$X = Y + Z,$$

where Y is tangent to $U \times \{e\}$ and Z is vertical so that $Y = \sigma_*(\pi_* X)$. Hence we have

$$\omega(X) = \omega(\sigma_*(\pi_* X)) + \omega(Z) = (\sigma^*\omega)(\pi_* X) + A,$$

where A is a unique element of $\mathfrak{g}$ such that the corresponding fundamental vector field A^* is equal to Z at $\sigma(x)$. Since A depends only on Z, not on the connection, ω is completely determined by $\sigma^*\omega$. The equation above shows that, conversely, every $\mathfrak{g}$-valued 1-form on U determines uniquely a connection form on $U \times G$. Thus Lemma 2 is reduced to the extension problem for $\mathfrak{g}$-valued 1-forms on U. If $\{A_j\}$ is a basis for $\mathfrak{g}$, then $\omega = \Sigma\, \omega^j A_j$ where each ω^j is a usual 1-form. Thus it is sufficient to consider the extension problem of usual 1-forms on U. Let $x^1, \ldots, x^n$ be a local coordinate system in U. Then every 1-form on U is of the form $\Sigma f_i\, dx^i$ where each f_i is a function on U. Thus our problem is reduced to the extension problem of functions on U. Lemma 2 now follows from Lemma 1.

By means of Lemma 2, Theorem 2.1 can be proved exactly in the same way as Theorem 5.7 of Chapter I. Let $\{U_i\}_{i\in I}$ be a locally finite open covering of M such that each U_i has the property stated in Lemma 2. Let $\{V_i\}$ be an open refinement of $\{U_i\}$ such that $\bar{V}_i \subset U_i$. For each subset J of I, set $S_J = \bigcup_{i\in J} \bar{V}_i$. Let T be the set of pairs (τ, J) where $J \subset I$ and τ is a connection defined over S_J which coincides with the given connection over $A \cap S_J$. Introduce an order in T as follows: $(\tau', J') < (\tau'', J'')$ if $J' \subset J''$ and $\tau' = \tau''$ on $S_{J'}$. Let (τ, J) be a maximal element of T. Then $J = I$ as in the proof of Theorem 5.7 of Chapter I and τ is a desired connection. QED.

Remark. It is possible to prove Theorem 2.1 using Lemma 2 and a partition of unity $\{f_i\}$ subordinate to $\{V_i\}$ (cf. Appendix 3). Let ω_i be a connection form on $\pi^{-1}(U_i)$ which extends the given connection over $A \cap \bar{V}_i$. Then $\omega = \Sigma_i\, g_i\omega_i$ is a desired connection form on P, where each g_i is the function on P defined by $g_i = f_i \circ \pi$.

3. *Parallelism*

Given a connection Γ in a principal fibre bundle $P(M, G)$, we shall define the concept of parallel displacement of fibres along any given curve τ in the base manifold M.

Let $\tau = x_t$, $a \leqq t \leqq b$, be a piecewise differentiable curve of class C^1 in M. A *horizontal lift* or simply a *lift* of τ is a horizontal curve $\tau^* = u_t$, $a \leqq t \leqq b$, in P such that $\pi(u_t) = x_t$ for $a \leqq t \leqq b$. Here a horizontal curve in P means a piecewise differentiable curve of class C^1 whose tangent vectors are all horizontal.

The notion of lift of a curve corresponds to the notion of lift of a vector field. Indeed, if X^* is the lift of a vector field X on M, then the integral curve of X^* through a point $u_0 \in P$ is a lift of the integral curve of X through the point $x_0 = \pi(u_0) \in M$. We now prove

PROPOSITION 3.1. *Let $\tau = x_t$, $0 \leqq t \leqq 1$, be a curve of class C^1 in M. For an arbitrary point u_0 of P with $\pi(u_0) = x_0$, there exists a unique lift $\tau^* = u_t$ of τ which starts from u_0.*

Proof. By local triviality of the bundle, there is a curve v_t of class C^1 in P such that $v_0 = u_0$ and $\pi(v_t) = x_t$ for $0 \leqq t \leqq 1$. A lift of τ, if it exists, must be of the form $u_t = v_t a_t$, where a_t is a curve in the structure group G such that $a_0 = e$. We shall now look for a curve a_t in G which makes $u_t = v_t a_t$ a horizontal curve. Just as in the proof of Proposition 1.4, we apply Leibniz's formula (Proposition 1.4 of Chapter I) to the mapping $P \times G \to P$ which maps (v, a) into va and obtain

$$\dot{u}_t = \dot{v}_t a_t + v_t \dot{a}_t,$$

where each dotted italic letter denotes the tangent vector at that point (e.g., $\dot{u}_t$ is the vector tangent to the curve $\tau^* = u_t$ at the point u_t). Let ω be the connection form of Γ. Then, as in the proof of Proposition 1.4, we have

$$\omega(\dot{u}_t) = \mathrm{ad}(a_t^{-1})\omega(\dot{v}_t) + a_t^{-1}\dot{a}_t,$$

where $a_t^{-1}\dot{a}_t$ is now a curve in the Lie algebra $\mathfrak{g} = T_e(G)$ of G. The curve u_t is horizontal if and only if $\dot{a}_t a_t^{-1} = -\omega(\dot{v}_t)$ for every t. The construction of u_t is thus reduced to the following

LEMMA. *Let G be a Lie group and $\mathfrak{g}$ its Lie algebra identified with $T_e(G)$. Let Y_t, $0 \leqq t \leqq 1$, be a continuous curve in $T_e(G)$. Then there exists in G a unique curve a_t of class C^1 such that $a_0 = e$ and $\dot{a}_t a_t^{-1} = Y_t$ for $0 \leqq t \leqq 1$.*

Remark. In the case where $Y_t = A$ for all t, the curve a_t is nothing but the 1-parameter subgroup of G generated by A. Our differential equation $\dot{a}_t a_t^{-1} = Y_t$ is hence a generalization of the differential equation for 1-parameter subgroups.

Proof of Lemma. We may assume that Y_t is defined and continuous for all t, $-\infty < t < \infty$. We define a vector field X on

$G \times \mathbf{R}$ as follows. The value of X at $(a, t) \in G \times \mathbf{R}$ is, by definition, equal to $(Y_t a, (d/dz)_t) \in T_a(G) \times T_t(\mathbf{R})$, where z is the natural coordinate system in $\mathbf{R}$. It is clear that the integral curve of X starting from $(e, 0)$ is of the form (a_t, t) and a_t is the desired curve in G. The only thing we have to verify is that a_t is defined for all t, $0 \leqq t \leqq 1$. Let $\varphi_t = \exp tX$ be a local 1-parameter group of local transformations of $G \times \mathbf{R}$ generated by X. For each $(e, s) \in G \times \mathbf{R}$, there is a positive number δ_s such that $\varphi_t(e, r)$ is defined for $|r - s| < \delta_s$ and $|t| < \delta_s$ (Proposition 1.5 of Chapter I). Since the subset $\{e\} \times [0, 1]$ of $G \times \mathbf{R}$ is compact, we may choose $\delta > 0$ such that, for each $r \in [0, 1]$, $\varphi_t(e, r)$ is defined for $|t| < \delta$ (cf. Proof of Proposition 1.6 of Chapter I). Choose $s_0, s_1, \ldots, s_k$ such that $0 = s_0 < s_1 < \cdots < s_k = 1$ and $s_i - s_{i-1} < \delta$ for every i. Then $\varphi_t(e, 0) = (a_t, t)$ is defined for $0 \leqq t \leqq s_1$; $\varphi_u(e, s_1) = (b_u, u + s_1)$ is defined for $0 \leqq u \leqq s_2 - s_1$, where $\dot{b}_u b_u^{-1} = Y_{u+s_1}$, and we define $a_t = b_{t-s_1} a_{s_1}$ for $s_1 \leqq t \leqq s_2$; $\ldots$; $\varphi_u(e, s_{k-1}) = (c_u, s_{k-1} + u)$ is defined for $0 \leqq u \leqq s_k - s_{k-1}$, where $\dot{c}_u c_u^{-1} = Y_{u+s_{k-1}}$, and we define $a_t = c_{t-s_{k-1}} a_{s_{k-1}}$, thus completing the construction of a_t, $0 \leqq t \leqq 1$. QED.

Now using Proposition 3.1, we define the parallel displacement of fibres as follows. Let $\tau = x_t$, $0 \leqq t \leqq 1$, be a differentiable curve of class C^1 on M. Let u_0 be an arbitrary point of P with $\pi(u_0) = x_0$. The unique lift τ^* of τ through u_0 has the end point u_1 such that $\pi(u_1) = x_1$. By varying u_0 in the fibre $\pi^{-1}(x_0)$, we obtain a mapping of the fibre $\pi^{-1}(x_0)$ onto the fibre $\pi^{-1}(x_1)$ which maps u_0 into u_1. We denote this mapping by the same letter τ and call it the *parallel displacement along the curve* τ. The fact that $\tau\colon \pi^{-1}(x_0) \to \pi^{-1}(x_1)$ is actually an isomorphism comes from the following

PROPOSITION 3.2. *The parallel displacement along any curve τ commutes with the action of G on P: $\tau \circ R_a = R_a \circ \tau$ for every $a \in G$.*

Proof. This follows from the fact that every horizontal curve is mapped into a horizontal curve by R_a. QED.

The parallel displacement along any piecewise differentiable curve of class C^1 can be defined in an obvious manner. It should be remarked that the parallel displacement along a curve τ is independent of a specific parametrization x_t used in the following sense. Consider two parametrized curves x_t, $a \leqq t \leqq b$, and y_s, $c \leqq s \leqq d$, in M. The parallel displacement along x_t and the one along y_s coincide if there is a homeomorphism φ of the interval $[a, b]$ onto $[c, d]$ such that (1) $\varphi(a) = c$ and $\varphi(b) = d$, (2) both φ and φ^{-1} are differentiable of class C^1 except at a finite number of parameter values, and (3) $y_{\varphi(t)} = x_t$ for all t, $a \leqq t \leqq b$.

If τ is the curve x_t, $a \leqq t \leqq b$, we denote by τ^{-1} the curve y_t, $a \leqq t \leqq b$, defined by $y_t = x_{a+b-t}$. The following proposition is evident.

PROPOSITION 3.3. (a) *If τ is a piecewise differentiable curve of class C^1 in M, then the parallel displacement along τ^{-1} is the inverse of the parallel displacement along τ.*

(b) *If τ is a curve from x to y in M and μ is a curve from y to z in M, the parallel displacement along the composite curve $\mu \cdot \tau$ is the composite of the parallel displacements τ and μ.*

4. *Holonomy groups*

Using the notion of parallel displacement, we now define the holonomy group of a given connection Γ in a principal fibre bundle $P(M, G)$. For the sake of simplicity we shall mean by a curve a piecewise differentiable curve of class C^k, $1 \leqq k \leqq \infty$ (k will be fixed throughout §4).

For each point x of M we denote by $C(x)$ the loop space at x, that is, the set of all closed curves starting and ending at x. If τ and μ are elements of $C(x)$, the composite curve $\mu \cdot \tau$ (τ followed by μ) is also an element of $C(x)$. As we proved in §3, for each $\tau \in C(x)$, the parallel displacement along τ is an isomorphism of the fibre $\pi^{-1}(x)$ onto itself. The set of all such isomorphisms of $\pi^{-1}(x)$ onto itself forms a group by virtue of Proposition 3.3. This group is called the *holonomy group of* Γ *with reference point* x. Let $C^0(x)$ be the subset of $C(x)$ consisting of loops which are homotopic to zero. The subgroup of the holonomy group consisting of the parallel displacements arising from all $\tau \in C^0(x)$ is called the *restricted holonomy group of* Γ *with reference point* x. The holonomy group and the restricted holonomy group of Γ with reference point x will be denoted by $\Phi(x)$ and $\Phi^0(x)$ respectively.

It is convenient to realize these groups as subgroups of the structure group G in the following way. Let u be an arbitrarily fixed point of the fibre $\pi^{-1}(x)$. Each $\tau \in C(x)$ determines an element, say, a, of G such that $\tau(u) = ua$. If a loop $\mu \in C(x)$ determines $b \in G$, then the composite $\mu \cdot \tau$ determines ba because $(\mu \cdot \tau)(u) = \mu(ua) = (\mu(u))a = uba$ by virtue of Proposition 3.2. The set of elements $a \in G$ determined by all $\tau \in C(x)$ forms a subgroup of G by Proposition 3.3. This subgroup, denoted by $\Phi(u)$, is called the *holonomy group of* Γ *with reference point* $u \in P$. The *restricted holonomy group* $\Phi^0(u)$ *of* Γ *with reference point* u can be defined accordingly. Note that $\Phi(x)$ is a group of isomorphisms of the fibre $\pi^{-1}(x)$ onto itself and $\Phi(u)$ is a subgroup of G. It is clear that there is a unique isomorphism of $\Phi(x)$ onto $\Phi(u)$ which makes the following diagram commutative:

$$\begin{tikzcd} & C(x) \arrow[dl] \arrow[dr] & \\ \Phi(x) \arrow[rr] & & \Phi(u). \end{tikzcd}$$

Another way of defining $\Phi(u)$ is the following: When two points u and v of P can be joined by a horizontal curve, we write $u \sim v$. This is clearly an equivalence relation. Then $\Phi(u)$ is equal to the set of $a \in G$ such that $u \sim ua$. Using the fact that $u \sim v$ implies $ua \sim va$ for any $u, v \in P$ and $a \in G$, it is easy to verify once more that this subset of G forms a subgroup of G.

Proposition 4.1. (a) *If $v = ua$, $a \in G$, then $\Phi(v) = \mathrm{ad}\,(a^{-1})(\Phi(u))$, that is, the holonomy groups $\Phi(v)$ and $\Phi(u)$ are conjugate in G. Similarly, $\Phi^0(v) = \mathrm{ad}\,(a^{-1})(\Phi^0(u))$.*

(b) *If two points u and v of P can be joined by a horizontal curve, then $\Phi(u) = \Phi(v)$ and $\Phi^0(u) = \Phi^0(v)$.*

Proof. (a) Let $b \in \Phi(u)$ so that $u \sim ub$. Then $ua \sim (ub)a$ so that $v \sim (va^{-1})ba = va^{-1}ba$. Thus $\mathrm{ad}\,(a^{-1})(b) \in \Phi(v)$. It follows easily that $\Phi(v) = \mathrm{ad}\,(a^{-1})(\Phi(u))$. The proof for $\Phi^0(v) = \mathrm{ad}\,(a^{-1})(\Phi^0(u))$ is similar.

(b) The relation $u \sim v$ implies $ub \sim vb$ for every $b \in G$. Since the relation $\sim$ is transitive, $u \sim ub$ if and only if $v \sim vb$, that is, $b \in \Phi(u)$ if and only if $b \in \Phi(v)$. To prove $\Phi^0(u) = \Phi^0(v)$, let μ^* be a horizontal curve in P from u to v. If $b \in \Phi^0(u)$, then there is a horizontal curve τ^* in P from u to ub such that the curve $\pi(\tau^*)$ in M is a loop at $\pi(u)$ homotopic to zero. Then the composite $(R_b\mu^*) \cdot \tau^* \cdot \mu^{*-1}$ is a horizontal curve in P from v to vb and its projection into M is a loop at $\pi(v)$ homotopic to zero. Thus $b \in \Phi^0(v)$. Similarly, if $b \in \Phi^0(v)$, then $b \in \Phi^0(u)$. QED.

If M is connected, then for every pair of points u and v of P, there is an element $a \in G$ such that $v \sim ua$. It follows from Proposition 4.1 that if M is connected, the holonomy groups $\Phi(u)$, $u \in P$, are all conjugate to each other in G and hence isomorphic with each other.

The rest of this section is devoted to the proof of the fact that the holonomy group is a Lie group.

Theorem 4.2. *Let $P(M, G)$ be a principal fibre bundle whose base manifold M is connected and paracompact. Let $\Phi(u)$ and $\Phi^0(u)$, $u \in P$, be the holonomy group and the restricted holonomy group of a connection Γ with reference point u. Then*

(a) *$\Phi^0(u)$ is a connected Lie subgroup of G;*

(b) *$\Phi^0(u)$ is a normal subgroup of $\Phi(u)$ and $\Phi(u)/\Phi^0(u)$ is countable.*

By virtue of this theorem, $\Phi(u)$ is a Lie subgroup of G whose identity component is $\Phi^0(u)$.

Proof. We shall show that every element of $\Phi^0(u)$ can be joined to the identity element by a piecewise differentiable curve of class C^k in G which lies in $\Phi^0(u)$. By the theorem in Appendix 4, it follows then that $\Phi^0(u)$ is a connected Lie subgroup of G.

Let $a \in \Phi^0(u)$ be obtained by the parallel displacement along a piecewise differentiable loop τ of class C^k which is homotopic to 0. By the factorization lemma (Appendix 7), τ is (equivalent to) a product of small lassos of the form $\tau_1^{-1} \cdot \mu \cdot \tau_1$, where τ_1 is a piecewise differentiable curve of class C^k from $x = \pi(u)$ to a point, say, y, and μ is a differentiable loop at y which lies in a coordinate neighborhood of y. It is sufficient to show that the element of $\Phi^0(u)$ defined by each lasso $\tau_1^{-1} \cdot \mu \cdot \tau_1$ can be joined to the identity element. This element is obviously equal to the element of $\Phi^0(v)$ defined by the loop μ, where v is the point obtained by the parallel displacement of u along τ_1. It is therefore sufficient to show that the element $b \in \Phi^0(v)$ defined by the differentiable loop μ can be joined to the identity element in $\Phi^0(v)$ by a differentiable curve of G which lies in $\Phi^0(v)$.

Let $x^1, \ldots, x^n$ be a local coordinate system with origin at y

and let μ be defined by $x^i = x^i(t)$, $i = 1, \ldots, n$. Set $f^i(t, s) = s + (1 - s)x^i(t)$ for $i = 1, \ldots, n$ and $0 \leqq t, s \leqq 1$. Then $f(t, s) = (f^1(t, s), \ldots, f^n(t, s))$ is a differentiable mapping of class C^k of $I \times I$ into M (where $I = [0, 1]$) such that $f(t, 0)$ is the curve μ and $f(t, 1)$ is the trivial curve y. For each fixed s, let $b(s)$ be the element of $\Phi^0(v)$ obtained from the loop $f(t, s)$, $0 \leqq t \leqq 1$, so that $b(0) = b$ and $b(1) =$ identity. The fact that $b(s)$ is of class C^k in s (as a mapping of I into G) follows from the following

Lemma. *Let f: $I \times I \rightarrow M$ be a differentiable mapping of class C^k and $u_0(s)$, $0 \leqq s \leqq 1$, a differentiable curve of class C^k in P such that $\pi(u_0(s)) = f(0, s)$. For each fixed s, let $u_1(s)$ be the point of P obtained by the parallel displacement of $u_0(s)$ along the curve $f(t, s)$, where $0 \leqq t \leqq 1$ and s is fixed. Then the curve $u_1(s)$, $0 \leqq s \leqq 1$, is differentiable of class C^k.*

Proof of Lemma. Let F: $I \times I \rightarrow P$ be a differentiable mapping of class C^k such that $\pi(F(t, s)) = f(t, s)$ for all $(t, s) \in I \times I$ and that $F(0, s) = u_0(s)$. The existence of such an F follows from local triviality of the bundle P. Set $v_t(s) = F(t, s)$. In the proof of Proposition 3.1, we saw that, for each fixed s, there is a curve $a_t(s)$, $0 \leqq t \leqq 1$, in G such that $a_0(s) = e$ and that the curve $v_t(s)a_t(s)$, $0 \leqq t \leqq 1$, is horizontal. Set $u_t(s) = v_t(s)a_t(s)$. To prove that $u_1(s)$, $0 \leqq s \leqq 1$, is a differentiable curve of class C^k, it is sufficient to show that $a_1(s)$, $0 \leqq s \leqq 1$, is a differentiable curve of class C^k in G. Let ω be the connection form of Γ. Set $Y_t(s) = -\omega(\dot{v}_t(s))$, where $\dot{v}_t(s)$ is the vector tangent to the curve described by $v_t(s)$, $0 \leqq t \leqq 1$, when s is fixed. Then as in the proof of Proposition 3.1, $a_t(s)$ is a solution of the equation $\dot{a}_t(s)a_t(s)^{-1} = Y_t(s)$. As in the proof of the lemma for Proposition 3.1, we define, for each fixed s, a vector field $X(s)$ on $G \times \mathbf{R}$ so that $(a_t(s), t)$ is the integral curve of the vector field $X(s)$ through the point $(e, 0) \in G \times \mathbf{R}$. The differentiability of $a_t(s)$ in s follows from the fact that each solution of an ordinary linear differential equation with parameter s is differentiable in s as many times as the equation is (cf. Appendix 1). This completes the proof of the lemma and hence the proof of (a) of Theorem 4.2.

We now prove (b). If τ and μ are two loops at x and if μ is homotopic to zero, the composite curve $\tau \cdot \mu \cdot \tau^{-1}$ is homotopic to zero. This implies that $\Phi^0(u)$ is a normal subgroup of $\Phi(u)$.

Let $\pi_1(M)$ be the first homotopy group of M with reference point x. We define a homomorphism f: $\pi_1(M) \rightarrow \Phi(u)/\Phi^0(u)$ as follows. For each element α of $\pi_1(M)$, let τ be a continuous loop at x which represents α. We may cover τ by a finite number of coordinate neighborhoods, modify τ within each neighborhood and obtain a piecewise differentiable loop τ_1 of class C^k at x which is homotopic to τ. If τ_1 and τ_2 are two such loops, then $\tau_1 \cdot \tau_2^{-1}$ is homotopic to zero and defines an element of $\Phi^0(u)$. Thus, τ_1 and τ_2 define the same element of $\Phi(u)/\Phi^0(u)$, which is denoted by $f(\alpha)$. Clearly, f is a homomorphism of $\pi_1(M)$ onto $\Phi(u)/\Phi^0(u)$. Since M is connected and paracompact, it satisfies the second axiom of countability (Appendix 3). It follows easily that $\pi_1(M)$ is countable. Hence, $\Phi(u)/\Phi^0(u)$ is also countable. QED.

Remark. In §3, we defined the parallel displacement along any piecewise differentiable curve of class C^1. In this section, we defined the holonomy group $\Phi(u)$ using piecewise differentiable curves of class C^k. If we denote by $\Phi_k(u)$ the holonomy group thus obtained from piecewise differentiable curves of class C^k, then we have obviously $\Phi_1(u) \supset \Phi_2(u) \supset \cdots \supset \Phi_\infty(u)$. We shall prove later in §7 that these holonomy groups coincide.

5. *Curvature form and structure equation*

Let $P(M, G)$ be a principal fibre bundle and ρ a representation of G on a finite dimensional vector space V; $\rho(a)$ is a linear transformation of V for each $a \in G$ and $\rho(ab) = \rho(a)\rho(b)$ for $a, b \in G$. A *pseudotensorial form of degree r on P of type (ρ, V)* is a V-valued r-form φ on P such that

$$R_a^* \varphi = \rho(a^{-1}) \cdot \varphi \qquad \text{for } a \in G.$$

Such a form φ is called a *tensorial form* if it is horizontal in the sense that $\varphi(X_1, \ldots, X_r) = 0$ whenever at least one of the tangent vectors X_i of P is vertical, i.e., tangent to a fibre.

Example 5.1. If ρ_0 is the trivial representation of G on V, that is, $\rho_0(a)$ is the identity transformation of V for each $a \in G$, then a tensorial form of degree r of type (ρ_0, V) is nothing but a form φ on P which can be expressed as $\varphi = \pi^* \varphi_M$ where φ_M is a V-valued r-form on the base M.

Example 5.2. Let ρ be a representation of G on V and E the bundle associated with P with standard fibre V on which G acts through ρ. A tensorial form φ of degree r of type (ρ, V) can be regarded as an assignment to each $x \in M$ a multilinear skew-symmetric mapping $\tilde{\varphi}_x$ of $T_x(M) \times \cdots \times T_x(M)$ (r times) into the vector space $\pi_E^{-1}(x)$ which is the fibre of E over x. Namely, we define

$$\tilde{\varphi}_x(X_1, \ldots, X_r) = u(\varphi(X_1^*, \ldots, X_r^*)), \qquad X_i \in T_x(M),$$

where u is any point of P with $\pi(u) = x$ and X_i^* is any vector at u such that $\pi(X_i^*) = X_i$ for each i. $\varphi(X_1^*, \ldots, X_r^*)$ is then an element of the standard fibre V and u is a linear mapping of V onto $\pi_E^{-1}(x)$ so that $u(\varphi(X_1^*, \cdots, X_r^*))$ is an element of $\pi_E^{-1}(x)$. It can be easily verified that this element is independent of the choice of u and X_i^*. Conversely, given a skew-symmetric multilinear mapping $\tilde{\varphi}_x$: $T_x(M) \times \cdots \times T_x(M) \to \pi_E^{-1}(x)$ for each $x \in M$, a tensorial form φ of degree r of type (ρ, V) on P can be defined by

$$\varphi(X_1^*, \ldots, X_r^*) = u^{-1}(\tilde{\varphi}_x(\pi(X_1^*), \ldots, \pi(X_r^*))), \qquad X_i^* \in T_u(P),$$

where $x = \pi(u)$. In particular, a tensorial 0-form of type (ρ, V), that is, a function f: $P \to V$ such that $f(ua) = \rho(a^{-1})f(u)$, can be identified with a cross section $M \to E$.

A few special cases of Example 5.2 will be used in Chapter III.

Let Γ be a connection in $P(M, G)$. Let G_u and Q_u be the vertical and the horizontal subspaces of $T_u(P)$, respectively. Let h: $T_u(P) \to Q_u$ be the projection.

PROPOSITION 5.1. *If φ is a pseudotensorial r-form on P of type (ρ, V), then*

(a) *The form φh defined by $(\varphi h)(X_1, \ldots, X_r) = \varphi(hX_1, \ldots, hX_r)$, $X_i \in T_u(P)$, is a tensorial form of type (ρ, V);*

(b) *$d\varphi$ is a pseudotensorial $(r+1)$-form of type (ρ, V);*

(c) *The $(r+1)$-form $D\varphi$ defined by $D\varphi = (d\varphi)h$ is a tensorial form of type (ρ, V).*

Proof. From $R_a \circ h = h \circ R_a$, $a \in G$, it follows that φh is a pseudotensorial form of type (ρ, V), It is evident that

$$(\varphi h)(X_1, \ldots, X_r) = 0,$$

if one of X_i's is vertical. (b) follows from $R_a^* \circ d = d \circ R_a^*$, $a \in G$. (c) follows from (a) and (b). QED.

The form $D\varphi = (d\varphi)h$ is called the *exterior covariant derivative of* φ and D is called *exterior covariant differentiation.*

If ρ is the adjoint representation of G in the Lie algebra $\mathfrak{g}$, a (pseudo) tensorial form of type $(\rho, \mathfrak{g})$ is said to be *of type* ad G. The connection form ω is a pseudotensorial 1-form of type ad G. By Proposition 5.1, $D\omega$ is a tensorial 2-form of type ad G and is called the *curvature form* of ω.

THEOREM 5.2 (*Structure equation*). *Let ω be a connection form and Ω its curvature form. Then*

$$d\omega(X, Y) = -\tfrac{1}{2}[\omega(X), \omega(Y)] + \Omega(X, Y) \qquad \textit{for } X, Y \in T_u(P), \quad u \in P.$$

Proof. Every vector of P is a sum of a vertical vector and a horizontal vector. Since both sides of the above equality are bilinear and skew-symmetric in X and Y, it is sufficient to verify the equality in the following three special cases.

(1) X and Y are horizontal. In this case, $\omega(X) = \omega(Y) = 0$ and the equality reduces to the definition of Ω.

(2) X and Y are vertical. Let $X = A^*$ and $Y = B^*$ at u, where $A, B \in \mathfrak{g}$. Here A^* and B^* are the fundamental vector fields corresponding to A and B respectively. By Proposition 3.11 of Chapter I, we have

$$\begin{aligned} 2d\omega(A^*, B^*) &= A^*(\omega(B^*)) - B^*(\omega(A^*)) - \omega([A^*, B^*]) \\ &= -[A, B] = -[\omega(A^*), \omega(B^*)], \end{aligned}$$

since $\omega(A^*) = A$, $\omega(B^*) = B$ and $[A^*, B^*] = [A, B]^*$. On the other hand, $\Omega(A^*, B^*) = 0$.

(3) X is horizontal and Y is vertical. We extend X to a horizontal vector field on P, which will be also denoted by X. Let $Y = A^*$ at u, where $A \in \mathfrak{g}$. Since the right hand side of the equality vanishes, it is sufficient to show that $d\omega(X, A^*) = 0$. By Proposition 3.11 of Chapter I, we have

$$\begin{aligned} 2d\omega(X, A^*) &= X(\omega(A^*)) - A^*(\omega(X)) - \omega([X, A^*]) \\ &= -\omega([X, A^*]). \end{aligned}$$

Now it is sufficient to prove the following

LEMMA. *If A^* is the fundamental vector field corresponding to an element $A \in \mathfrak{g}$ and X is a horizontal vector field, then $[X, A^*]$ is horizontal.*

Proof of Lemma. The fundamental vector field A^* is induced by R_{a_t}, where a_t is the 1-parameter subgroup of G generated by $A \in \mathfrak{g}$. By Proposition 1.9 of Chapter I, we have

$$[X, A^*] = \lim_{t \to 0} \frac{1}{t} [R_{a_t}(X) - X].$$

If X is horizontal, so is $R_{a_t}(X)$. Thus $[X, A^*]$ is horizontal. QED.

COROLLARY 5.3. *If both X and Y are horizontal vector fields on P, then*

$$\omega([X, Y]) = -2\Omega(X, Y).$$

Proof. Apply Proposition 1.9 of Chapter I to the left hand side of the structure equation just proved. QED.

The structure equation (often called "the structure equation of E. Cartan") is sometimes written, for the sake of simplicity, as follows:

$$d\omega = -\tfrac{1}{2}[\omega, \omega] + \Omega.$$

Let $e_1, \ldots, e_r$ be a basis for the Lie algebra $\mathfrak{g}$ and c^i_{jk}, $i, j, k = 1, \ldots, r$, the structure constants of $\mathfrak{g}$ with respect to $e_1, \ldots, e_r$, that is,

$$[e_j, e_k] = \Sigma_i \, c^i_{jk} e_i, \qquad j, k = 1, \ldots, r.$$

Let $\omega = \Sigma_i \, \omega^i e_i$ and $\Omega = \Sigma_i \, \Omega^i e_i$. Then the structure equation can be expressed as follows:

$$d\omega^i = -\tfrac{1}{2}\Sigma_{j,k} \, c^i_{jk} \omega^j \wedge \omega^k + \Omega^i, \qquad i = 1, \ldots, r.$$

THEOREM 5.4 (*Bianchi's identity*). $D\Omega = 0$.

Proof. By the definition of D, it suffices to prove that $d\Omega(X, Y, Z) = 0$ whenever X, Y, and Z are all horizontal vectors. We apply the exterior differentiation d to the structure equation. Then

$$0 = dd\omega^i = -\tfrac{1}{2}\Sigma \, c^i_{jk} \, d\omega^j \wedge \omega^k + \tfrac{1}{2}\Sigma \, c^i_{jk}\omega^j \wedge d\omega^k + d\Omega^i.$$

Since $\omega^i(X) = 0$ whenever X is horizontal, we have

$$d\Omega^i(X, Y, Z) = 0$$

whenever X, Y, and Z are all horizontal. QED.

PROPOSITION 5.5. *Let ω be a connection form and φ a tensorial 1-form of type* ad G. *Then*

$$D\varphi(X, Y) = d\varphi(X, Y) + \tfrac{1}{2}[\varphi(X), \omega(Y)] + \tfrac{1}{2}[\omega(X), \varphi(Y)]$$
$$\textit{for } X, Y \in T_u(P), \quad u \in P.$$

Proof. As in the proof of Theorem 5.2, it suffices to consider the three special cases. The only non-trivial case is the case where X is vertical and Y is horizontal. Let $X = A^*$ at u, where $A \in \mathfrak{g}$. We extend Y to a horizontal vector field on P, denoted also by Y, which is invariant by R_a, $a \in G$. (We first extend the vector πY to a vector field on M and then lift it to a horizontal vector field on P.) Then $[A^*, Y] = 0$. As A^* is vertical, $D\varphi(A^*, Y) = 0$. We shall show that the right hand side of the equality vanishes. By Proposition 3.11 of Chapter I, we have

$$d\varphi(A^*, Y) = \tfrac{1}{2}(A^*(\varphi(Y)) - Y(\varphi(A^*)) - \varphi([A^*, Y]) = \tfrac{1}{2}A^*(\varphi(Y)),$$

so that it suffices to show $A^*(\varphi(Y)) + [\omega(A^*), \varphi(Y)] = 0$ or $A^*(\varphi(Y)) = -[A, \varphi(Y)]$. If a_t denotes the 1-parameter subgroup of G generated by A, then

$$A^*_u(\varphi(Y)) = \lim_{t \to 0} \frac{1}{t} [\varphi_{ua_t}(Y) - \varphi_u(Y)] = \lim_{t \to 0} \frac{1}{t} [(R^*_{a_t}\varphi)_u(Y) - \varphi_u(Y)]$$
$$= \lim_{t \to 0} \frac{1}{t} [\mathrm{ad}\,(a_t^{-1})(\varphi_u(Y)) - \varphi_u(Y)] = -[A, \varphi_u(Y)],$$

since Y is invariant by R_{a_t}. QED.

6. *Mappings of connections*

In §5 of Chapter I, we considered certain mappings of one principal fibre bundle into another such as a homomorphism, an injection, and a bundle map. We now study the effects of these mappings on connections.

PROPOSITION 6.1. *Let $f: P'(M', G') \to P(M, G)$ be a homomorphism with the corresponding homomorphism $f: G' \to G$ such that the induced mapping $f: M' \to M$ is a diffeomorphism of M' onto M. Let Γ' be a connection in P', ω' the connection form and Ω' the curvature form of Γ'. Then*

(a) *There is a unique connection Γ in P such that the horizontal subspaces of Γ' are mapped into horizontal subspaces of Γ by f.*

(b) *If* ω *and* Ω *are the connection form and the curvature form of* Γ *respectively, then* $f^*\omega = f\cdot\omega'$ *and* $f^*\Omega = f\cdot\Omega'$, *where* $f\cdot\omega'$ *or* $f\cdot\Omega'$ *means the* $\mathfrak{g}'$*-valued form on* P' *defined by* $(f\cdot\omega')(X') = f(\omega'(X'))$ *or* $(f\cdot\Omega')(X', Y') = f(\Omega'(X', Y'))$, *where* f *on the right hand side is the homomorphism* $\mathfrak{g}' \to \mathfrak{g}$ *induced by* $f\colon G' \to G$.

(c) *If* $u' \in P'$ *and* $u = f(u') \in P$, *then* $f\colon G' \to G$ *maps* $\Phi(u')$ *onto* $\Phi(u)$ *and* $\Phi^0(u')$ *onto* $\Phi^0(u)$, *where* $\Phi(u)$ *and* $\Phi^0(u)$ (*resp.* $\Phi(u')$ *and* $\Phi^0(u')$) *are the holonomy group and the restricted holonomy group of* Γ (*resp.* Γ') *with reference point* u (*resp.* u').

Proof. (a) Given a point $u \in P$, choose $u' \in P'$ and $a \in G$ such that $u = f(u')a$. We define the horizontal subspace Q_u of $T_u(P)$ by $Q_u = R_a \circ f(Q_{u'})$, where $Q_{u'}$ is the horizontal subspace of $T_{u'}(P')$ with respect to Γ'. We shall show that Q_u is independent of the choice of u' and a. If $u = f(v')b$, where $v' \in P'$ and $b \in G$, then $v' = u'c'$ for some $c' \in G'$. If we set $c = f(c')$, then $u = f(v')b = f(u'c')b = f(u')cb$ and hence $a = cb$. We have $R_b \circ f(Q_{v'}) = R_b \circ f(Q_{u'c'}) = R_b \circ f \circ R_{c'}(Q_{u'}) = R_b \circ R_c \circ f(Q_{u'}) = R_a \circ f(Q_{u'})$, which proves our assertion. We shall show that the distribution $u \to Q_u$ is a connection in P. If $u = f(u')a$, then $ub = f(u')ab$ and $Q_{ub} = R_{ab} \circ f(Q_{u'}) = R_b \circ R_a \circ f(Q_{u'}) = R_b(Q_u)$, thus proving the invariance of the distribution by G. We shall now prove $T_u(P) = Q_u + G_u$, where G_u is the tangent space to the fibre at u. By local triviality of P, it is sufficient to prove that the projection $\pi\colon P \to M$ induces a linear isomorphism $\pi\colon Q_u \to T_x(M)$, where $x = \pi(u)$. We may assume that $u = f(u')$ since the distribution $u \to Q_u$ is invariant by G. In the commutative diagram

$$\begin{array}{ccc} Q_{u'} & \xrightarrow{f} & Q_u \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ T_{x'}(M') & \xrightarrow{f} & T_x(M), \end{array}$$

the mappings $\pi'\colon Q_{u'} \to T_{x'}(M')$ and $f\colon T_{x'}(M') \to T_x(M)$ are linear isomorphisms and hence the remaining two mappings must be also linear isomorphisms. The uniqueness of Γ is evident from its construction.

(b) The equality $f^*\omega = f\cdot\omega'$ can be rewritten as follows:

$$\omega(fX') = f(\omega'(X')) \qquad \text{for } X' \in T_{u'}(P'), \quad u' \in P'.$$

It is sufficient to verify the above equality in the two special cases: (1) X' is horizontal, and (2) X' is vertical. Since $f\colon P' \to P$ maps every horizontal vector into a horizontal vector, both sides of the equality vanish if X' is horizontal. If X' is vertical, $X' = A'^*$ at u', where $A' \in \mathfrak{g}'$. Set $A = f(A') \in \mathfrak{g}$. Since $f(u'a') = f(u')f(a')$ for every $a' \in G'$, we have $f(X') = A^*$ at $f(u')$. Thus

$$\omega(fX') = \omega(A^*) = A = f(A') = f(\omega'(A'^*)) = f(\omega'(X')).$$

From $f^*\omega = f\cdot\omega'$, we obtain $d(f^*\omega) = d(f\cdot\omega')$ and $f^*d\omega = f\cdot d\omega'$. By the structure equation (Theorem 5.2):

$$-\tfrac{1}{2}f^*([\omega, \omega]) + f^*\Omega = -\tfrac{1}{2}f([\omega', \omega']) + f\cdot\Omega',$$

we have

$$-\tfrac{1}{2}[f^*\omega, f^*\omega] + f^*\Omega = -\tfrac{1}{2}[f\cdot\omega', f\cdot\omega'] + f\cdot\Omega'.$$

This implies that $f^*\Omega = f\cdot\Omega'$.

(c) Let τ be a loop at $x = \pi(u)$. Set $\tau' = f^{-1}(\tau)$ so that τ' is a loop at $x' = \pi'(u')$. Let τ'^* be the horizontal lift of τ' starting from u'. Then $f(\tau'^*)$ is the horizontal lift of τ starting from u. The statement (c) is now evident. QED.

In the situation as in Proposition 6.1, we say that f maps the connection Γ' into the connection Γ. In particular, in the case where $P'(M', G')$ is a reduced subbundle of $P(M, G)$ with injection f so that $M' = M$ and $f\colon M' \to M$ is the identity transformation, we say that the connection Γ in P is *reducible* to the connection Γ' in P'. An automorphism f of the bundle $P(M, G)$ is called an *automorphism of a connection* Γ in P if it maps Γ into Γ, and in this case, Γ is said to be *invariant* by f.

PROPOSITION 6.2. *Let* $f\colon P'(M', G') \to P(M, G)$ *be a homomorphism such that the corresponding homomorphism* $f\colon G' \to G$ *maps* G' *isomorphically onto* G. *Let* Γ *be a connection in* P, ω *the connection form and* Ω *the curvature form of* Γ. *Then*

(a) *There is a unique connection* Γ' *in* P' *such that the horizontal subspaces of* Γ' *are mapped into horizontal subspaces of* Γ *by* f.

(b) *If* ω' *and* Ω' *are the connection form and the curvature form of* Γ' *respectively, then* $f^*\omega = f\cdot\omega'$ *and* $f^*\Omega = f\cdot\Omega'$.

(c) *If* $u' \in P'$ *and* $u = f(u') \in P$, *then the isomorphism* $f\colon G' \to G$ *maps* $\Phi(u')$ *into* $\Phi(u)$ *and* $\Phi^0(u')$ *into* $\Phi^0(u)$.

Proof. We define Γ' by defining its connection form ω'. Set $\omega' = f^{-1}\cdot f^*\omega$, where f^{-1}: $\mathfrak{g} \to \mathfrak{g}'$ is the inverse of the isomorphism f: $\mathfrak{g}' \to \mathfrak{g}$ induced from f: $G' \to G$. Let $X' \in T_{u'}(P')$ and $a' \in G'$ and set $X = fX'$ and $a = f(a')$. Then we have

$$\begin{aligned}\omega'(R_{a'}X') &= f^{-1}(\omega(f(R_{a'}X'))) = f^{-1}(\omega(R_aX))\\ &= f^{-1}(\mathrm{ad}\,(a^{-1})(\omega(X))) = \mathrm{ad}\,(a'^{-1})(f^{-1}(\omega(X)))\\ &= \mathrm{ad}\,(a'^{-1})(\omega(X')).\end{aligned}$$

Let $A' \in \mathfrak{g}'$ and set $A = f(A')$. Let A^* and A'^* denote the fundamental vector fields corresponding to A and A' respectively. Then we have

$$\omega'(A'^*) = f^{-1}(\omega(A^*)) = f^{-1}(A) = A'.$$

This proves that the form ω' defines a connection (Proposition 1.1). The verification of other statements is similar to the proof of Proposition 6.1 and is left to the reader. QED.

In the situation as in Proposition 6.2, we say that Γ' is *induced* by f from Γ. If f is a bundle map, that is, $G' = G$ and f: $G' \to G$ is the identity automorphism, then $\omega' = f^*\omega$. In particular, given a bundle $P(M, G)$ and a mapping f: $M' \to M$, every connection in P induces a connection in the induced bundle $f^{-1}P$.

For any principal fibre bundles $P(M, G)$ and $Q(M, H)$, $P \times Q$ is a principal fibre bundle over $M \times M$ with group $G \times H$. Let $P + Q$ be the restriction of $P \times Q$ to the diagonal ΔM of $M \times M$. Since ΔM and M are diffeomorphic with each other in a natural way, we consider $P + Q$ as a principal fibre bundle over M with group $G \times H$. The restriction of the projection $P \times Q \to P$ to $P + Q$, denoted by f_P, is a homomorphism with the corresponding natural homomorphism f_G: $G \times H \to G$. Similarly, for f_Q: $P + Q \to Q$ and f_H: $G \times H \to H$.

Proposition 6.3. *Let Γ_P and Γ_Q be connections in $P(M, G)$ and $Q(M, H)$ respectively. Then*

(a) *There is a unique connection Γ in $P + Q$ such that the homomorphisms f_P: $P + Q \to P$ and f_Q: $P + Q \to Q$ maps Γ into Γ_P and Γ_Q respectively.*

(b) *If ω, ω_P and ω_Q are the connection forms and Ω, Ω_P, and Ω_Q are the curvature forms of Γ, Γ_P, and Γ_Q respectively, then*

$$\omega = f_P^*\omega_P + f_Q^*\omega_Q, \qquad \Omega = f_P^*\Omega_P + f_Q^*\Omega_Q.$$

(c) *Let $u \in P$, $v \in Q$, and $(u, v) \in P + Q$. Then the holonomy group $\Phi(u, v)$ of Γ (resp. the restricted holonomy group $\Phi^0(u, v)$ of Γ) is a subgroup of $\Phi(u) \times \Phi(v)$ (resp. $\Phi^0(u) \times \Phi^0(v)$). The homomorphism f_G: $G \times H \to G$ (resp. f_H: $G \times H \to H$) maps $\Phi(u, v)$ onto $\Phi(u)$ (resp. onto $\Phi(v)$) and $\Phi^0(u, v)$ onto $\Phi^0(u)$ (resp. onto $\Phi^0(v)$), where $\Phi(u)$ and $\Phi^0(u)$ (resp. $\Phi(v)$ and $\Phi^0(v)$) are the holonomy group and the restricted holonomy group of Γ_P (resp. Γ_Q).*

The proof is similar to those of Propositions 6.1 and 6.2 and is left to the reader.

Proposition 6.4. *Let $Q(M, H)$ be a subbundle of $P(M, G)$, where H is a Lie subgroup of G. Assume that the Lie algebra $\mathfrak{g}$ of G admits a subspace $\mathfrak{m}$ such that $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ (direct sum) and $\mathrm{ad}\,(H)(\mathfrak{m}) = \mathfrak{m}$, where $\mathfrak{h}$ is the Lie algebra of H. For every connection form ω in P, the $\mathfrak{h}$-component ω' of ω restricted to Q is a connection form in Q.*

Proof. Let $A \in \mathfrak{h}$ and A^* the fundamental vector field corresponding to A. Then $\omega'(A^*)$ is the $\mathfrak{h}$-component of $\omega(A^*) = A$. Hence, $\omega'(A^*) = A$. Let φ be the $\mathfrak{m}$-component of ω restricted to Q. Let $X \in T_v(Q)$ and $a \in H$. Then

$$\omega(R_aX) = \omega'(R_aX) + \varphi(R_aX),$$
$$\mathrm{ad}\,(a^{-1})(\omega(X)) = \mathrm{ad}\,(a^{-1})(\omega'(X)) + \mathrm{ad}\,(a^{-1})(\varphi(X)).$$

The left-hand sides of the preceding two equalities coincide. Comparing the $\mathfrak{h}$-components of the right hand sides, we obtain $\omega'(R_aX) = \mathrm{ad}\,(a^{-1})(\omega'(X))$. Observe that we used the fact that $\mathrm{ad}\,(a^{-1})(\varphi(X))$ is in $\mathfrak{m}$. QED.

Remark. The connection defined by ω in P is reducible to a connection in the subbundle Q if and only if the restriction of ω to Q is $\mathfrak{h}$-valued. Under the assumption in Proposition 6.4, this means $\omega' = \omega$ on Q.

7. *Reduction theorem*

Unless otherwise stated, a curve will mean a piecewise differentiable curve of class C^∞. The holonomy group $\Phi_\infty(u_0)$ will be denoted by $\Phi(u_0)$.

We first establish

Theorem 7.1 (*Reduction theorem*). *Let $P(M, G)$ be a principal fibre bundle with a connection Γ, where M is connected and paracompact. Let u_0 be an arbitrary point of P. Denote by $P(u_0)$ the set of points in P*

which can be joined to u_0 by a horizontal curve. Then

(1) *$P(u_0)$ is a reduced bundle with structure group $\Phi(u_0)$.*

(2) *The connection Γ is reducible to a connection in $P(u_0)$.*

Proof. (1) We first prove

LEMMA 1. *Let Q be a subset of $P(M, G)$ and H a Lie subgroup of G. Assume: (1) the projection $\pi: P \to M$ maps Q onto M; (2) Q is stable by H, i.e., $R_a(Q) = Q$ for each $a \in H$; (3) if $u,v \in Q$ and $\pi(u) = \pi(v)$, then there is an element $a \in H$ such that $v = ua$; and (4) every point x of M has a neighborhood U and a cross section $\sigma: U \to P$ such that $\sigma(U) \subset Q$. Then $Q(M, H)$ is a reduced subbundle of $P(M, G)$.*

Proof of Lemma 1. For each $u \in \pi^{-1}(U)$, let $x = \pi(u)$ and $a \in G$ the element determined by $u = \sigma(x)a$. Define an isomorphism $\psi: \pi^{-1}(U) \to U \times G$ by setting $\psi(u) = (x, a)$. It is easy to see that ψ maps $Q \cap \pi^{-1}(U)$ 1:1 onto $U \times H$. Introduce a differentiable structure in Q in such a way that $\psi: Q \cap \pi^{-1}(U) \to U \times H$ becomes a diffeomorphism; using Proposition 1.3 of Chapter I as in the proof of Proposition 5.3 of Chapter I, we see that Q becomes a differentiable manifold. It is now evident that Q is a principal fibre bundle over M with group H and that Q is a subbundle of P.

Going back to the proof of the first assertion of Theorem 7.1, we see that, M being paracompact, the holonomy group $\Phi(u_0)$ is a Lie subgroup of G (Theorem 4.2) and that the subset $P(u_0)$ and the group $\Phi(u_0)$ satisfy conditions (1), (2), and (3) of Lemma 1 (cf. the second definition of $\Phi(u_0)$ given before Proposition 4.1 and also Proposition 4.1(b)). To verify condition (4) of Lemma 1, let $x^1, \ldots, x^n$ be a local coordinate system around x such that x is the origin $(0, \ldots, 0)$ with respect to this coordinate system. Let U be a cubical neighborhood of x defined by $|x^i| < \delta$. Given any point $y \in U$, let τ_y be the segment from x to y with respect to the coordinate system $x^1, \ldots, x^n$. Fix a point $u \in Q$ such that $\pi(u) = x$. Let $\sigma(y)$ be the point of P obtained by the parallel displacement of u along τ_y. Then $\sigma: U \to P$ is a cross section such that $\sigma(U) \subset Q$. Now (1) of Theorem 7.1 follows from Lemma 1.

(2) This is an immediate consequence of the following

LEMMA 2. *Let $Q(M, H)$ be a subbundle of $P(M, G)$ and Γ a connection in P. If, for every $u \in Q$, the horizontal subspace of $T_u(P)$ is tangent to Q, then Γ is reducible to a connection in Q.*

Proof of Lemma 2. We define a connection Γ' in Q as follows. The horizontal subspace of $T_u(Q)$, $u \in Q$, with respect to Γ' is by definition the horizontal subspace of $T_u(P)$ with respect to Γ. It is obvious that Γ is reducible to Γ'. QED.

We shall call $P(u)$ the *holonomy bundle* through u. It is evident that $P(u) = P(v)$ if and only if u and v can be joined by a horizontal curve. Since the relation $\sim$ introduced in §4 ($u \sim v$ if u and v can be joined by a horizontal curve) is an equivalence relation, we have, for every pair of points u and v of P, either $P(u) = P(v)$ or $P(u) \cap P(v) =$ empty. In other words, P is decomposed into the disjoint union of the holonomy bundles. Since every $a \in G$ maps each horizontal curve into a horizontal curve, $R_a(P(u)) = P(ua)$ and $R_a: P(u) \to P(ua)$ is an isomorphism with the corresponding isomorphism ad (a^{-1}): $\Phi(u) \to \Phi(ua)$ of the structure groups. It is easy to see that, given any u and v, there is an element $a \in G$ such that $P(v) = P(ua)$. Thus the holonomy bundles $P(u)$, $u \in P$, are all isomorphic with each other.

Using Theorem 7.1, we prove that the holonomy groups $\Phi_k(u)$, $1 \leqq k \leqq \infty$, coincide as was pointed out in Remark of § 4. This result is due to Nomizu and Ozeki [2].

THEOREM 7.2. *All the holonomy groups $\Phi_k(u)$, $1 \leqq k \leqq \infty$, coincide.*

Proof. It is sufficient to show that $\Phi_1(u) = \Phi_\infty(u)$. We denote $\Phi_\infty(u)$ by $\Phi(u)$ and the holonomy bundle through u by $P(u)$. We know by Theorem 7.1 that $P(u)$ is a subbundle of P with $\Phi(u)$ as its structure group. Define a distribution S on P by setting

$$S_u = T_u(P(u)) \qquad \text{for } u \in P.$$

Since the holonomy bundles have the same dimension, say k, S is a k-dimensional distribution. We first prove

LEMMA 1. (1) *S is differentiable and involutive.*

(2) *For each $u \in P$, $P(u)$ is the maximal integral manifold of S through u.*

Proof of Lemma 1. (1) We set

$$S_u = S'_u + S''_u, \qquad u \in P,$$

where S'_u is horizontal and S''_u is vertical. The distribution S' is differentiable by the very definition of a connection. To prove the differentiability of S, it suffices to show that of S''. For each $u \in P$, let U be a neighborhood of $x = \pi(u)$ with a cross section $\sigma: U \to P(u)$ such that $\sigma(x) = u$. (Such a cross section was constructed in the proof of Theorem 7.1.) Let $A_1, \ldots, A_r$ be a basis of the Lie algebra $\mathfrak{g}(u)$ of $\Phi(u)$. We shall define vector fields $\tilde{A}_1, \ldots, \tilde{A}_r$ on $\pi^{-1}(U)$ which form a basis of S'' at every point of $\pi^{-1}(U)$. Let $v \in \pi^{-1}(U)$. Then there is a unique $a \in G$ such that $v = \sigma(\pi(v))a$. Since $\mathrm{ad}\,(a^{-1}): \Phi(u) \to \Phi(v)$ is an isomorphism, $\mathrm{ad}\,(a^{-1})(A_i)$, $i = 1, \ldots, r$, are elements of $\mathfrak{g}(v)$ and form a basis for $\mathfrak{g}(v)$. We set

$$(\tilde{A}_i)_v = (\mathrm{ad}\,(a^{-1})(A_i))^*_v, \qquad i = 1, \ldots, r,$$

where $(\mathrm{ad}\,(a^{-1})(A_i))^*$ is the fundamental vector field on P corresponding to $\mathrm{ad}\,(a^{-1})(A_i) \in \mathfrak{g}(v) \subset \mathfrak{g}$, $i = 1, \ldots, r$. It is easy to see that $\tilde{A}_1, \ldots, \tilde{A}_r$ are differentiable and form a basis of S'' on $\pi^{-1}(U)$.

For each point u, $P(u)$ is an integral manifold of S, since for every $v \in P(u)$, we have $T_v(P(u)) = T_v(P(v)) = S_v$. This implies that S is involutive.

(2) Let $W(u)$ be the maximal integral manifold of S through u (cf. Proposition 1.2 of Chapter I). Then $P(u)$ is an open submanifold of $W(u)$. We prove that $P(u) = W(u)$. Let v be an arbitrary point of $W(u)$ and let $u(t)$, $0 \leq t \leq 1$, be a curve in $W(u)$ such that $u(0) = u$ and $u(1) = v$. Let t_1 be the supremum of t_0 such that $0 \leq t \leq t_0$ implies $u(t) \in P(u)$. Since $P(u)$ is open in $W(u)$, t_1 is positive. We show that $u(t_1)$ lies in $P(u)$; since $P(u)$ is open in $W(u)$, this will imply that $t_1 = 1$, proving that $u(1) = v$ lies in $P(u)$. The point $u(t_1)$ is in $P(u(t_1))$ and $P(u(t_1))$ is open in $W(u(t_1))$. There exists $\varepsilon > 0$ such that $t_1 - \varepsilon < t < t_1 + \varepsilon$ implies $u(t) \in P(u(t_1))$. Let t be any value such that $t_1 - \varepsilon < t < t_1$. By definition of t_1, we have $u(t) \in P(u)$. On the other hand, $u(t) \in P(u(t_1))$. This implies that $P(u) = P(u(t_1))$ so that $u(t_1) \in P(u)$ as we wanted to show. We have thereby proved that $P(u)$ is actually the maximal integral manifold of S through u.

Lemma 2. *Let S be an involutive, C^∞-distribution on a C^∞-manifold. Suppose x_t, $0 \leq t \leq 1$, is a piecewise C^1-curve whose tangent vectors $\dot{x}_t$ belong to S. Then the entire curve x_t lies in the maximal integral manifold W of S through the point x_0.*

Proof of Lemma 2. We may assume that x_t is a C^1-curve. Take a local coordinate system $x^1, \ldots, x^n$ around the point x_0 such that

$\partial/\partial x^1, \ldots, \partial/\partial x^k$, $k = \dim S$, form a local basis for S (cf. Chevalley [1, p. 92]). For small values of t, say, $0 \leq t \leq \varepsilon$, x_t can be expressed by $x^i = x^i(t)$, $1 \leq i \leq n$, and its tangent vectors are given by $\Sigma_i\, (dx^i/dt)(\partial/\partial x^i)$. By assumption, we have $dx^i/dt = 0$ for $k+1 \leq i \leq n$. Thus, $x^i(t) = x^i(0)$ for $k+1 \leq i \leq n$ so that x_t, $0 \leq t \leq \varepsilon$, lies in the slice through x_0 and hence in W. The standard continuation argument concludes the proof of Lemma 2.

We are now in position to complete the proof of Theorem 7.2. Let a be any element of $\Phi_1(u)$. This means that u and ua can be joined by a piecewise C^1-horizontal curve u_t, $0 \leq t \leq 1$, in P. The tangent vector $\dot{u}_t$ at each point obviously lies in S_{u_t}. By Lemma 2, the entire curve u_t lies in the maximal integral manifold $W(u)$ of S through u. By Lemma 1, the entire curve u_t lies in $P(u)$. In particular, ua is a point of $P(u)$. Since $P(u)$ is a subbundle with structure group $\Phi(u)$, a belongs to $\Phi(u)$. QED.

Corollary 7.3. *The restricted holonomy groups $\Phi^0_k(u)$, $1 \leq k \leq \infty$, coincide.*

Proof. $\Phi^0_k(u)$ is the connected component of the identity of $\Phi_k(u)$ for every k (cf. Theorem 4.2 and its proof). Now, Corollary 7.3 follows from Theorem 7.2. QED.

Remark. In the case where $P(M, G)$ is a real analytic principal bundle with an analytic connection, we can still define the holonomy group $\Phi_\omega(u)$ by using only piecewise analytic horizontal curves. The argument used in proving Theorem 7.2 and Corollary 7.3 shows that $\Phi_\omega(u) = \Phi_1(u)$ and $\Phi^0_\omega(u) = \Phi^0_1(u)$.

Given a connection Γ in a principal fibre bundle $P(M, G)$, we shall define the notion of parallel displacement in the associated fibre bundle $E(M, F, G, P)$ with standard fibre F. For each $w \in E$, the *horizontal subspace* Q_w and the *vertical subspace* F_w of $T_w(E)$ are defined as follows. The vertical subspace F_w is by definition the tangent space to the fibre of E at w. To define Q_w, we recall that we have the natural projection $P \times F \to E = P \times_G F$. Choose a point $(u, \xi) \in P \times F$ which is mapped into w. We fix this $\xi \in F$ and consider the mapping $P \to E$ which maps $v \in P$ into $v\xi \in E$. Then the horizontal subspace Q_w is, by definition, the image of the horizontal subspace $Q_u \subset T_u(P)$ by this mapping $P \to E$. We see easily that Q_w is independent of the choice of

$(u, \xi) \in P \times F$. We leave to the reader the proof that $T_w(E) = F_w + Q_w$ (direct sum). A curve in E is *horizontal* if its tangent vector is horizontal at each point. Given a curve τ in M, a *(horizontal) lift* τ^* of τ is a horizontal curve in E such that $\pi_E(\tau^*) = \tau$. Given a curve $\tau = x_t$, $0 \leq t \leq 1$, and a point w_0 such that $\pi_E(w_0) = x_0$, there is a unique lift $\tau^* = w_t$ starting from w_0. To prove the existence of τ^*, we choose a point (u_0, ξ) in $P \times F$ such that $u_0\xi = w_0$. Let u_t be the lift of $\tau = x_t$ starting from u_0. Then $w_t = u_t\xi$ is a lift of τ starting from w_0. The uniqueness of τ^* reduces to the uniqueness of a solution of a system of ordinary linear differential equations satisfying a given initial condition just as in the case of a lift in a principal fibre bundle. A cross section σ of E defined on an open subset U of M is called *parallel* if the image of $T_x(M)$ by σ is horizontal for each $x \in U$, that is, for any curve $\tau = x_t$, $0 \leq t \leq 1$, the parallel displacement of $\sigma(x_0)$ along τ gives $\sigma(x_1)$.

Proposition 7.4. *Let $P(M, G)$ be a principal fibre bundle and $E(M, G/H, G, P)$ the associated bundle with standard fibre G/H, where H is a closed subgroup of G. Let σ: $M \to E$ be a cross section and $Q(M, H)$ the reduced subbundle of $P(M, G)$ corresponding to σ (cf.* Proposition 5.6 *of* Chapter I). *Then a connection Γ in P is reducible to a connection Γ' in Q if and only if σ is parallel with respect to Γ.*

Proof. If we identify E with P/H (cf. Proposition 5.5 of Chapter I), then $\sigma(M)$ coincides with the image of Q by the natural projection μ: $P \to E = P/H$; in other words, if $u \in Q$ and $x = \pi(u)$, then $\sigma(x) = \mu(u)$ (cf. Proposition 5.6 of Chapter I). Suppose Γ is reducible to a connection Γ' in Q. We note that if ξ is the origin (i.e., the coset H) of G/H, then $u\xi = \mu(u)$ for every $u \in P$ and hence if u_t, $0 \leq t \leq 1$, is horizontal in P, so is $\mu(u_t)$ in E. Given a curve x_t, $0 \leq t \leq 1$, in M, choose $u_0 \in Q$ with $\pi(u_0) = x_0$ so that $\sigma(x_0) = \mu(u_0)$. Let u_t be the lift to P of x_t starting from u_0 (with respect to Γ), so that $\mu(u_t)$ is the lift of x_t to E starting from $\sigma(x_0)$. Since Γ is reducible to Γ', we have $u_t \in Q$ and hence $\mu(u_t) = \sigma(x_t)$ for all t. Conversely, assume that σ is parallel (with respect to Γ). Given any curve x_t, $0 \leq t \leq 1$, in M and any point u_0 of Q with $\pi(u_0) = x_0$, let u_t be the lift of x_t to P starting from u_0. Since σ is parallel, $\mu(u_t) = \sigma(x_t)$ and hence $u_t \in Q$ for all t. This shows that every horizontal vector at $u_0 \in Q$ (with respect to Γ) is

tangent to Q. By Proposition 7.2, Γ is reducible to a connection in Q. QED.

8. *Holonomy theorem*

We first prove the following result of Ambrose and Singer [1] by applying Theorem 7.1.

Theorem 8.1. *Let $P(M, G)$ be a principal fibre bundle, where M is connected and paracompact. Let Γ be a connection in P, Ω the curvature form, $\Phi(u)$ the holonomy group with reference point $u \in P$ and $P(u)$ the holonomy bundle through u of Γ. Then the Lie algebra of $\Phi(u)$ is equal to the subspace of $\mathfrak{g}$, Lie algebra of G, spanned by all elements of the form $\Omega_v(X, Y)$, where $v \in P(u)$ and X and Y are arbitrary horizontal vectors at v.*

Proof. By virtue of Theorem 7.1, we may assume that $P(u) = P$, i.e., $\Phi(u) = G$. Let $\mathfrak{g}'$ be the subspace of $\mathfrak{g}$ spanned by all elements of the form $\Omega_v(X, Y)$, where $v \in P(u) = P$ and X and Y are arbitrary horizontal vectors at v. The subspace $\mathfrak{g}'$ is actually an ideal of $\mathfrak{g}$, because Ω is a tensorial form of type ad G (cf. §5) and hence $\mathfrak{g}'$ is invariant by ad G. We shall prove that $\mathfrak{g}' = \mathfrak{g}$. At each point $v \in P$, let S_v be the subspace of $T_v(P)$ spanned by the horizontal subspace Q_v and by the subspace $\mathfrak{g}'_v = \{A^*_v; A \in \mathfrak{g}'\}$, where A^* is the fundamental vector field on P corresponding to A. The distribution S has dimension $n + r$, where $n = \dim M$ and $r = \dim \mathfrak{g}'$. We shall prove that S is differentiable and involutive. Let v be an arbitrary point of P and U a coordinate neighborhood of $y = \pi(v) \in M$ such that $\pi^{-1}(U)$ is isomorphic with $U \times G$. Let $X_1, \ldots, X_n$ be differentiable vector fields on U which are linearly independent everywhere on U and $X^*_1, \ldots, X^*_n$ the horizontal lifts of $X_1, \ldots, X_n$. Let $A_1, \ldots, A_r$ be a basis for $\mathfrak{g}'$ and $A^*_1, \ldots, A^*_r$ the corresponding fundamental vector fields. It is clear that $X^*_1, \ldots, X^*_n, A^*_1, \ldots, A^*_r$ form a local basis for S. To prove that S is involutive, it suffices to verify that the bracket of any two of these vector fields belongs to S. This is clear for $[A^*_i, A^*_j]$, since $[A_i, A_j] \in \mathfrak{g}'$ and $[A_i, A_j]^* = [A^*_i, A^*_j]$. By the lemma for Theorem 5.2, $[A^*_i, X^*_j]$ is horizontal; actually, $[A^*_i, X^*_j] = 0$ as X^*_j is invariant by R_a for each $a \in G$. Finally, set $A = \omega([X^*_i, X^*_j]) \in \mathfrak{g}$, where ω is the connection form of Γ. By Corollary 5.3, $A = \omega([X^*_i, X^*_j]) = -2\omega(X^*_i, X^*_j) \in \mathfrak{g}'$. Since the vertical component of $[X^*_i, X^*_j]$ at $v \in P$ is equal to $A^*_v \in S_v$,

$[X_i^*, X_j^*]$ belongs to S. This proves our assertion that S is involutive.

Let P_0 be the maximal integral manifold of S through u. By Lemma 2 in the proof of Theorem 7.2, we have $P_0 = P$. Therefore,

$$\dim \mathfrak{g} = \dim P - n = \dim P_0 - n = \dim \mathfrak{g}'.$$

This implies $\mathfrak{g} = \mathfrak{g}'$. QED.

Next we prove

THEOREM 8.2. *Let $P(M, G)$ be a principal fibre bundle, where P is connected and M is paracompact. If $\dim M \geqq 2$, there exists a connection in P such that all the holonomy bundles $P(u)$, $u \in P$, coincide with P.*

Proof. Let u_0 be an arbitrary point of P and $x^1, \ldots, x^n$ a local coordinate system with origin $x_0 = \pi(u_0)$. Let U and V be neighborhoods of x_0 defined by $|x^i| < \alpha$ and $|x^i| < \beta$ respectively, where $0 < \beta < \alpha$. Taking α sufficiently small, we may assume that $P \mid U = \pi^{-1}(U)$ is isomorphic with the trivial bundle $U \times G$. We shall construct a connection Γ' in $P \mid U$ such that the holonomy group of the bundle $P \mid V$ coincides with the identity component of G. We shall then extend Γ' to a connection Γ in P in such a way that Γ coincides with Γ' on $P \mid \bar{V}$ (cf. Theorem 2.1).

Let $A_1, \ldots, A_r$ be a basis for the Lie algebra $\mathfrak{g}$ of G. Choose real numbers $\alpha_1, \ldots, \alpha_r$ such that $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_r < \beta$ and let $f_i(t)$, $i = 1, \ldots, r$, be differentiable functions in $-\alpha - \varepsilon < t < \alpha + \varepsilon$ such that $f_i(0) = 0$ for every i and $f_i(\alpha_j) = \delta_{ij}$ (Kronecker's symbol). On $\pi^{-1}(U) = U \times G$, we can define a connection form ω by requiring that

$$\omega_{(x,e)}(\partial/\partial x^1) = \sum_{j=1}^{r} f_j(x^2)A_j$$

and that

$$\omega_{(x,e)}(\partial/\partial x^i) = 0 \qquad \text{for } i = 2, 3, \ldots, n.$$

(Note that, by virtue of the property $R_a^*\omega = \text{ad}\,(a^{-1})(\omega)$, the preceding conditions determine the values of ω at every point (x, a) of $U \times G$.)

Fixing t, $0 < t < \beta$, and α_k, $1 \leqq k \leqq r$, for the moment, consider the rectangle on the x^1x^2-plane in V formed by the line segments τ_1 from $(0, 0)$ to $(0, \alpha_k)$, τ_2 from $(0, \alpha_k)$ to (t, α_k), τ_3 from (t, α_k) to $(t, 0)$ and τ_4 from $(t, 0)$ to $(0, 0)$. (Here and in the following argument, the x^3 to x^n-coordinates of all the points remain 0 and are hence omitted.) In $\pi^{-1}(V) = V \times G$, we determine the horizontal lift of $\tau = \tau_4 \cdot \tau_3 \cdot \tau_2 \cdot \tau_1$ starting from the point $(0, 0; e)$. The lift τ_1^* of τ_1 starting from $(0, 0; e)$ is clearly $(0, s; e)$, $0 \leqq s \leqq \alpha_k$, since its tangent vectors $\partial/\partial x^2$ are horizontal. The lift τ_2^* of τ_2 starting from the end point $(0, \alpha_k; e)$ of τ_1^* is of the form $(s, \alpha_k; c_s)$, $0 \leqq s \leqq t$, where c_s is a suitable curve with $c_0 = e$ in G. Its tangent vector is of the form $(\partial/\partial x^1)_{(s,\alpha_k)} + \dot{c}_s$. By a similar computation to that for Proposition 3.1, we have

$$\begin{aligned}\omega((\partial/\partial x^1)_{(s,\alpha_k)} + \dot{c}_s) &= \text{ad}\,(c_s^{-1})\omega((\partial/\partial x^1))_{(s,\alpha_k;\,e)} + c_s^{-1}\cdot \dot{c}_s \\ &= \text{ad}\,(c_s^{-1})\left(\sum_{j=1}^{r} f_j(\alpha_k)A_j\right) + c_s^{-1}\cdot\dot{c}_s = \text{ad}\,(c_s^{-1})A_k + c_s^{-1}\cdot\dot{c}_s.\end{aligned}$$

Therefore we have $c_s \cdot c_s^{-1} = -A_k$, that is, $c_s = \exp(-sA_k)$. The end point of τ_2^* is hence $(t, \alpha_k; \exp(-tA_k))$. The lift τ_3^* of τ_3 starting from $(t, \alpha_k; \exp(-tA_k))$ is $(t, \alpha_k - s; \exp(-tA_k))$, $0 \leqq s \leqq \alpha_k$. Finally, the lift τ_4^* of τ_4 starting from the end point $(t, 0; \exp(-tA_k))$ of τ_3^* is $(t - s, 0; \exp(-tA_k))$, $0 \leqq s \leqq t$, since $\partial/\partial x^1$ is horizontal at the points with $x^2 = 0$. This shows that the end point of the lift τ^* of τ is $(0, 0; \exp(-tA_k))$, proving that $\exp(-tA_k)$ is an element of the holonomy group of $\pi^{-1}(V)$ with reference point $(0, 0; e)$. Since this is the case for every t, we see that A_k is in the Lie algebra of the holonomy group. The result being valid for any A_k, we see that the holonomy group of the connection in $\pi^{-1}(V)$ coincides with the identity component of G.

Let Γ be a connection in P which coincides with Γ' on $\pi^{-1}(\bar{V})$. Since the holonomy group $\Phi(u_0)$ of Γ obviously contains the identity component of G, the holonomy bundle $P(u_0)$ of Γ has the same dimension as P and hence is open in P. Since P is a disjoint union of holonomy bundles each of which is open, the connectedness of P implies that $P = P(u_0)$. QED.

COROLLARY 8.3. *Any connected Lie group G can be realized as the holonomy group of a certain connection in a trivial bundle $P = M \times G$, where M is an arbitrary differentiable manifold with $\dim M \geqq 2$.*

Theorem 8.2 was proved for linear connections by Hano and Ozeki [1] and then in the general case by Nomizu [5], both by making use of Theorem 8.1. The above proof which is more direct is due to E. Ruh (unpublished).

9. *Flat connections*

Let $P = M \times G$ be a trivial principal fibre bundle. For each $a \in G$, the set $M \times \{a\}$ is a submanifold of P. In particular, $M \times \{e\}$ is a subbundle of P, where e is the identity of G. The *canonical flat connection* in P is defined by taking the tangent space to $M \times \{a\}$ at $u = (x, a) \in M \times G$ as the horizontal subspace at u. In other words, a connection in P is the canonical flat connection if and only if it is reducible to a unique connection in $M \times \{e\}$. Let θ be the canonical 1-form on G (cf. §4 of Chapter I). Let $f\colon M \times G \to G$ be the natural projection and set

$$\omega = f^*\theta.$$

It is easy to verify that ω is the connection form of the canonical flat connection in P. The Maurer-Cartan equation of θ implies that the canonical flat connection has zero curvature:

$$\begin{aligned} d\omega = d(f^*\theta) &= f^*(d\theta) = f^*(-\tfrac{1}{2}[\theta, \theta]) \\ &= -\tfrac{1}{2}[f^*\theta, f^*\theta] = -\tfrac{1}{2}[\omega, \omega]. \end{aligned}$$

A connection in any principal fibre bundle $P(M, G)$ is called *flat* if every point x of M has a neighborhood U such that the induced connection in $P \mid U = \pi^{-1}(U)$ is isomorphic with the canonical flat connection in $U \times G$. More precisely, there is an isomorphism $\psi\colon \pi^{-1}(U) \to U \times G$ which maps the horizontal subspace at each $u \in \pi^{-1}(U)$ upon the horizontal subspace at $\psi(u)$ of the canonical flat connection in $U \times G$.

THEOREM 9.1. *A connection in $P(M, G)$ is flat if and only if the curvature form vanishes identically.*

Proof. The necessity is obvious. Assume that the curvature form vanishes identically. For each point x of M, let U be a simply connected open neighborhood of x and consider the induced connection in $P \mid U = \pi^{-1}(U)$. By Theorems 4.2 and 8.1, the holonomy group of the induced connection in $P \mid U$ consists of the identity only. Applying the Reduction Theorem (Theorem 7.1), we see that the induced connection in $P \mid U$ is isomorphic with the canonical flat connection in $U \times G$. QED.

COROLLARY 9.2. *Let Γ be a connection in $P(M, G)$ such that the curvature vanishes identically. If M is paracompact and simply connected, then P is isomorphic with the trivial bundle $M \times G$ and Γ is isomorphic with the canonical flat connection in $M \times G$.*

We shall study the case where M is not necessarily simply connected. Let Γ be a flat connection in $P(M, G)$, where M is connected and paracompact. Let $u_0 \in P$ and $M^* = P(u_0)$, the holonomy bundle through u_0; M^* is a principal fibre bundle over M whose structure group is the holonomy group $\Phi(u_0)$. Since $\Phi(u_0)$ is discrete by Theorems 4.2 and 8.1 and since M^* is connected, M^* is a covering space of M. Set $x_0 = \pi(u_0)$, $x_0 \in M$. Every closed curve of M starting from x_0 defines, by means of the parallel displacement along it, an element of $\Phi(u_0)$. Since the restricted holonomy group is trivial by Theorems 4.2 and 8.1, any two closed curves at x_0 representing the same element of the first homotopy group $\pi_1(M, x_0)$ give rise to the same element of $\Phi(u_0)$. Thus we obtain a homomorphism of $\pi_1(M, x_0)$ onto $\Phi(u_0)$. Let N be a normal subgroup of $\Phi(u_0)$ and set $M' = M^*/N$. Then M' is a principal fibre bundle over M with structure group $\Phi(u_0)/N$. In particular, M' is a covering space of M. Let $P'(M', G)$ be the principal fibre bundle induced from $P(M, G)$ by the covering projection $M' \to M$. Let $f\colon P' \to P$ be the natural homomorphism (cf. Proposition 5.8 of Chapter I).

PROPOSITION 9.3. *There exists a unique connection Γ' in $P'(M', G)$ which is mapped into Γ by the homomorphism $f\colon P' \to P$. The connection Γ' is flat. If u_0' is a point of P' such that $f(u_0') = u_0$, then the holonomy group $\Phi(u_0')$ of Γ' with reference point u_0' is isomorphically mapped onto N by f.*

Proof. The first statement is contained in Proposition 6.2. By the same proposition, the curvature form of Γ' vanishes identically and Γ' is flat. We recall that P' is the subset of $M' \times P$ defined as follows (cf. Proposition 5.8 of Chapter I):

$$P' = \{(x', u) \in M' \times P;\ \mu(x') = \pi(u)\},$$

where $\mu\colon M' \to M$ is the covering projection. The projection $\pi'\colon P' \to M'$ is given by $\pi'(x', u) = x'$ and the homomorphism $f\colon P' \to P$ is given by $f(x', u) = u$ so that the corresponding homomorphism $f\colon G \to G$ of the structure groups is the identity automorphism. To prove that f maps $\Phi(u_0')$ isomorphically onto

N, it is therefore sufficient to prove $\Phi(u'_0) = N$. Write

$$u'_0 = (x'_0, u_0) \in P' \subset M' \times P.$$

Since $\mu(x'_0) = \pi(u_0)$, there exists an element $a \in \Phi(u_0)$ such that

$$x'_0 = \nu(u_0 a),$$

where $\nu: M^* = P(u_0) \to M' = P(u_0)/N$ is the covering projection. Let $\tau = u'_t$, $0 \leq t \leq 1$, be a horizontal curve in P' such that $\pi'(u'_0) = \pi'(u'_1)$. For each t, we set

$$u'_t = (x'_t, u_t) \in P' \subset M' \times P.$$

Then the curve u_t, $0 \leq t \leq 1$, is horizontal in P and hence is contained in $M^* = P(u_0)$. Since $\mu(x'_t) = \pi(u_t) = \mu \circ \nu(u_t)$ and $x'_0 = \nu(u_0 a)$, we have $x'_t = \nu(u_t a)$ for $0 \leq t \leq 1$. We have

$$\nu(u_1 a) = x'_1 = \pi'(u'_1) = \pi'(u'_0) = x'_0 = \nu(u_0 a)$$

and, consequently,

$$\nu(u_1) = \nu(u_0),$$

which means that $u_1 = u_0 b$ for some $b \in N$. This shows that $\Phi(u'_0) \subset N$. Conversely, let b be any element of N. Let u_t, $0 \leq t \leq 1$, be a horizontal curve in P such that $u_1 = u_0 b$. Define a horizontal curve u'_t, $0 \leq t \leq 1$, in P' by

$$u'_t = (x'_t, u_t),$$

where $x'_t = \nu(u_t a)$. Then $u'_1 = u'_0 b$, showing that $b \in \Phi(u'_0)$. QED.

10. *Local and infinitesimal holonomy groups*

Let Γ be a connection in a principal fibre bundle $P(M, G)$, where M is connected and paracompact. For every connected open subset U of M, let Γ_U be the connection in $P \mid U = \pi^{-1}(U)$ induced from Γ. For each $u \in \pi^{-1}(U)$, we denote by $\Phi^0(u, U)$ and $P(u, U)$ the restricted holonomy group with reference point u and the holonomy bundle through u of the connection Γ_U, respectively. $P(u, U)$ consists of points v of $\pi^{-1}(U)$ which can be joined to u by a horizontal curve in $\pi^{-1}(U)$.

The *local holonomy group* $\Phi^*(u)$ with reference point u of Γ is defined to be the intersection $\bigcap \Phi^0(u, U)$, where U runs through

all connected open neighborhoods of the point $x = \pi(u)$. If $\{U_k\}$ is a sequence of connected open neighborhoods of x such that $U_k \supset \bar{U}_{k+1}$ and $\bigcap_{k=1}^{\infty} U_k = \{x\}$, then we have obviously $\Phi^0(u, U_1) \supset \Phi^0(u, U_2) \supset \cdots \supset \Phi^0(u, U_k) \supset \cdots$. Since, for every open neighborhood U of x, there exists an integer k such that $U_k \subset U$, we have $\Phi^*(u) = \bigcap_{k=1}^{\infty} \Phi^0(u, U_k)$. Since each group $\Phi^0(u, U_k)$ is a connected Lie subgroup of G (Theorem 4.2), it follows that $\dim \Phi^0(u, U_k)$ is constant for sufficiently large k and hence that $\Phi^*(u) = \Phi^0(u, U_k)$ for such k. The following proposition is now obvious.

Proposition 10.1. *The local holonomy groups have the following properties:*

(1) *$\Phi^*(u)$ is a connected Lie subgroup of G which is contained in the restricted holonomy group $\Phi^0(u)$;*

(2) *Every point $x = \pi(u)$ has a connected open neighborhood U such that $\Phi^*(u) = \Phi^0(u, V)$ for any connected open neighborhood V of x contained in U;*

(3) *If U is such a neighborhood of $x = \pi(u)$, then $\Phi^*(u) \supseteqq \Phi^*(v)$ for every $v \in P(u, U)$;*

(4) *For every $a \in G$, we have $\Phi^*(ua) = \mathrm{ad}\,(a^{-1})(\Phi^*(u))$;*

(5) *For every integer m, the set $\{\pi(u) \in M; \dim \Phi^*(u) \leqq m\}$ is open.*

As to (5), we remark that $\dim \Phi^*(u)$ is constant on each fibre of P by (4) and thus can be considered as an integer valued function on M. Then (5) means that this integer valued function is upper semicontinuous.

Theorem 10.2. *Let $\mathfrak{g}(u)$ and $\mathfrak{g}^*(u)$ be the Lie algebras of $\Phi^0(u)$ and $\Phi^*(u)$ respectively. Then $\Phi^0(u)$ is generated by all $\Phi^*(v)$, $v \in P(u)$, and $\mathfrak{g}(u)$ is spanned by all $\mathfrak{g}^*(v)$, $v \in P(u)$.*

Proof. If $v \in P(u)$, then $\Phi^0(u) = \Phi^0(v) \supset \Phi^*(v)$ and $\mathfrak{g}(u) = \mathfrak{g}(v) \supset \mathfrak{g}^*(v)$. By Theorem 8.1, $\mathfrak{g}(u)$ is spanned by all elements of the form $\Omega_v(X^*, Y^*)$ where $v \in P(u)$ and X^* and Y^* are horizontal vectors at v. Since $\Omega_v(X^*, Y^*)$ is contained in the Lie algebra of $\Phi^0(v, V)$ for every connected open neighborhood V of $\pi(v)$, it is contained in $\mathfrak{g}^*(v)$. Consequently, $\mathfrak{g}(u)$ is spanned by all $\mathfrak{g}^*(v)$