CORRIGENDUM: THE SYMPLECTIC SUM FORMULA FOR
GROMOV-WITTEN INVARIANTS

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Abstract. We correct an error and an oversight in [IP]. The sign of the curvature in (8.7) is
wrong, requiring a new proof of Proposition 8.1. Several lemmas address only the basic case of
maps with intersection multiplicity $s = 1$; the general case follows by applying the pointwise
estimates in [IP] with a modified Sobolev norm.

1. Sobolev Norms

In [IP], portions of Sections 6-8 are valid only for maps with intersection multiplicity $s = 1$.
To cover maps with multiplicity vector $s = (s_1, \ldots, s_\ell)$, we modify the Sobolev norms in [IP]
(6.9) by setting

$$\|\zeta\|_{m,p,s}^p = \int_{C_\nu} \left( |\nabla^m \zeta|^p + |\zeta|^p \right) \rho^{-\delta p/2} + \sum_k \int_{C_\nu \cap B_k} \left( |\rho^{1-s_k} \nabla^m \zeta_N|^p + |\rho^{1-s_k} \zeta_N|^p \right) \rho^{-\delta p/2}. \tag{1.1}$$

With this revision, the norms $\|\langle \xi, h \rangle\|_m$ and $\|\eta\|_m$ are again defined by formulas [IP, (6.10)]
and [IP, (6.11)]. For $s = 1$, the above norm is uniformly equivalent to the norm of [IP, (6.9)].
For general $s$, it now has a stronger weighting factor of $\rho^{1-s_k}$ on the normal components near
each node with multiplicity $s_k \geq 2$. Accordingly, one must verify that Lemmas 6.9 and 7.1,
Proposition 7.3 and Lemma 9.2 continue to hold for this new norm. This is easily done using
the pointwise estimates already appearing in the proofs, as follows.

Modifications to Section 6. Lemma 6.8 is unaffected by the change in norms. The statement
of Lemma 6.9 remains valid for the new norms with a slight modification to the exponent in
its conclusion:

$$\|\mathcal{J}_F F - \nu_F\|_0 \leq c |\lambda|^{1/3}$$

with $c$ uniform on each compact set $K_{\delta_0}$. (Throughout [IP], the $\delta$ indexing the sets $K_\delta$ of [IP]
(3.11)] is unrelated to the exponent $\delta$ in (1.1).) The proof of Lemma 6.9 is modified as follows.

Proof of Lemma 6.9. Set $\Phi_F = \mathcal{J}_F F - \nu_F$ and follow the proof in [IP] until (6.14). Outside
$\rho \leq \rho_0$, the new norm is uniformly equivalent to old one, so [IP, (6.13)] continues to hold for
$\Phi = \Phi_F$. Again we focus on the half $A_+$ of one $A_k$ where $|w_k| \leq |z_k|$ where, after omitting
subscripts, $F$ is given by [IP, (6.14)].:

$$F = \left((1 - \beta) h^v, a z^s (1 + (1 - \beta) h^x), b w^s (1 + (1 - \beta) h^x)^{-1} \right) \tag{1.2}$$

where $\beta, h^v$ and $h^x$ are functions of the coordinate $z$ on $C_1 \subset C_0$ and $w = \mu/z$. 

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Introduce the map $\tilde{f} : C_0 \to X$ given by (1.2) with the last entry replaced by zero. Noting that $\Phi_f = 0$ (because $f$ is $(J, \nu)$-holomorphic), we can write

$$(1.3) \quad \Phi_F = (\Phi_F - \Phi_f) + (\Phi_f - \Phi_{\tilde{f}}).$$

To complete the proof, we will bound the two expressions on the righthand side using the following facts, which hold for small $\lambda$:

(i) [IP] Lemma 6.8d implies that $|h^v| + |dh^v| + |h^x| + |dh^x| \leq c\rho \leq \frac{1}{2}$ on $A_+$.

(ii) $|dz| = |z|$ and $|dw| = |w|$ in the cylindrical metric [IP] (4.5), and $0 \leq \beta \leq 1$ and $|d\beta| \leq 2$ by [IP] (5.11).

(iii) On $A_+$, $\rho^2 = |z|^2 + |w|^2 \leq 2|z|^2$, and hence $|z|^{-2} \leq 2\rho^{-2}$ and $\sqrt{|\mu|} \leq |z| \leq \rho \leq \rho_0$.

(iv) By [IP] Lemma 6.8c, $a_k$ and $b_k$ are bounded above and below by positive constants, and hence $|\mu_k^x| \sim |\lambda|$ by [IP] (6.3).

Writing (1.2) as $(F^v, F^x, F^y)$, Facts (i)-(iv) imply the pointwise bounds

$$(1.4) \quad |F^v| + |dF^v| \leq c\rho, \quad |F^x| + |dF^x| \leq c\rho^s, \quad |F^y| + |dF^y| \leq c|w|^s \leq C|\lambda|\rho^{-s}$$

for $s = s_k$, with constants $c, C$ uniform on $K_{\rho_0}$. It follows that $|dF| \leq 3c\rho$ and, since $J$ and $\nu$ are smooth,

$$(1.5) \quad |J_F - J_{\tilde{f}}| + |\nu_F - \nu_{\tilde{f}}| \leq c|F - \tilde{f}| = c|F^y| \leq c|\lambda|\rho^{-s}.$$

(Here, and below, we are updating the constant $c$ as we proceed.)

Now, using the definition of $\Phi_F$, we have

$$(1.6) \quad 2(\Phi_F - \Phi_{\tilde{f}}) = d(F - \tilde{f}) + (J_F - J_{\tilde{f}})dFj + J_{\tilde{f}}(dF - d\tilde{f})j - 2(\nu_F - \nu_{\tilde{f}}),$$

with $F - \tilde{f} = F^y$. Estimating each term, one sees that the above bounds imply that

$$(1.7) \quad |\Phi_F - \Phi_{\tilde{f}}| \leq c|\lambda|\rho^{-s} \quad \text{so} \quad \rho^{1-s}|(\Phi_F - \Phi_{\tilde{f}})^N| \leq c|\lambda|\rho^{1-2s}$$

on $A_+$. Applying the second integral in [IP] (5.10), noting that $s \geq 1$ and that $\rho^2 \geq |\mu|$ on $A_+$ yields

$$(1.8) \quad \|\Phi_F - \Phi_{\tilde{f}}\|_{0,A_+} \leq c|\lambda||\mu|^{\frac{1}{2}(1-2s-\delta/2)} \leq c|\lambda|^{\frac{1}{2\delta}}$$

where the last inequality uses (iv) above and the fact that $0 < \delta < \frac{1}{6}$. By symmetry, a similar estimate holds on the other half of $A_k$. Hence (1.8) holds on the entire set $A \subset C_0$ where $\rho \leq \rho_0$ with a revised constant $c$ and the exponent replaced by $\frac{1}{2|\delta|}$ (since $|s| \geq s_k$ for all $k$).

It remains to estimate the last term in (1.3). On $A_+$, the difference between $f$ and $\tilde{f}$, namely

$$f - \tilde{f} = \beta(h^v, az^x(1 + h^x), 0),$$

is supported in the region $\rho \leq 2|\mu|^{1/4}$. Again expanding as in (1.6) and using (i)-(iv), one obtains

$$|\Phi_{\tilde{f}} - \Phi_{\tilde{f}}| \leq |f - \tilde{f}| + |df - d\tilde{f}| + |J_{\tilde{f}} - J_{\tilde{f}}| + |\nu_{\tilde{f}} - \nu_{\tilde{f}}| \leq c\rho$$

Similarly, using (i)-(iv), the normal component $(\tilde{f} - f)^N = -\beta az^x(1 + h^x)$ satisfies:

$$|(\tilde{f} - f)^N| + |d(\tilde{f} - f)^N| \leq c\rho^s.$$
Next observe that the images of \( f \) and \( \tilde{f} \) both lie in \( Z_0 = X \cup Y \) and, as in [IP] (6.6), \( J \) preserves the normal subbundle \( N_0 \) to \( V \) in \( Z \) along \( Z_0 \). Also noting that [1.4] implies that \(|(d\tilde{f})^N| \leq c\rho^s\), one sees that

\[
\left|((J_f - J\tilde{f}) \circ d\tilde{f})^N + (J_f(d\tilde{f} - df))^N\right| \leq \left|(J_f - J\tilde{f}) \circ (d\tilde{f})^N\right| + \left|J_f(d\tilde{f} - df)^N\right| \leq c\rho^s.
\]

Because \( \nu^N \) vanishes along \( V \), there is also a bound \(|\nu_f^N| \leq C|f^N| \leq c\rho^s\); the same is true for \( \tilde{f} \), and therefore \(|\nu_f^N - \nu_{\tilde{f}}^N| \leq c\rho^s\). Expanding \( \Phi_{\tilde{f}} - \Phi_f \) as in [1.6] and using above estimates yields

\[
(1.9) \quad |\Phi_{\tilde{f}} - \Phi_f| + \rho^{1-s}|(\Phi_{\tilde{f}} - \Phi_f)^N| \leq c\rho
\]

with the lefthand side supported in the region \( \rho \leq 2|\mu|^{1/4} \) in \( A_+ \) and, by symmetry, in \( A_k \) for each \( k \). Applying the first integral in [IP] (5.10) bounds the integrals in the norm (1.1) on the union \( A \) of the \( A_k \). Again using the facts that \(|\lambda| \sim |\mu_k|^{\delta/4}, 0 < \delta < \frac{1}{4}\) and \( s_k \leq |s| \), one obtains the bound

\[
(1.10) \quad \|\Phi_{\tilde{f}} - \Phi_f\|_{0,A} \leq c|\mu|^{1/4(1-\delta/2)} \leq c|\lambda|^{1/|\lambda|}.
\]

The lemma now follows from [IP] (6.13) and the bounds (1.8) and (1.10) on the norms of the two terms in (1.3). \( \square \)

**Modifications to Section 7.** Delete the paragraph that starts after [IP] (7.4) and ends with [IP] (7.6); we no longer need \( D_F^* \).

- The conclusion of Lemma 7.2 should read

\[
(1.11) \quad |(\nabla \zeta^N)| \leq c\rho^s|\zeta^V|, \quad |L_F^N\zeta^V| \leq c\rho^s|\zeta^V|
\]

as the proof shows (keep all powers of \( \rho \) in the last line of the two paragraphs of the proof).

- The statement of Proposition 7.3 remains the same after deleting the statement about \( D_F^* \). The proof is unchanged until two lines before [IP] (7.10)], at which point we have established the estimate

\[
(1.12) \quad |D_{F,C}(\zeta^N, h)| \leq c|\nabla \zeta^N| + c\rho(|\zeta^N| + |\zeta^N| + |h|)
\]

(this is also [IP] (9.7)). As a special case, we have

\[
|D_{F,C}(\zeta^N, 0, 0)| \leq c(|\nabla \zeta^N| + \rho|\zeta^N|).
\]

On the other hand, the normal component is

\[
(D_{F,C}(\zeta^V, \zeta^N, h))^N = \left(L_F(\zeta^V + \sum \beta_k \bar{\zeta}^N)\right)^N + (J_FdFh)^N.
\]

Using (1.11), the first term on the right is bounded by \( c\rho^s(|\zeta^V| + |\bar{\zeta}|) \). The second is dominated by \(|(J_F(dF^V + dF^N))^N| |h|\) with \(|dF^N| \leq c\rho^s\) by (1.4). Furthermore, because \( V \) is \( J \)-holomorphic, \( (Jv)^N = 0 \) along \( V \) for all vectors \( v \) in the \( V \) direction. Hence \(|(J_FdF^V)^N| \leq c|F|^N |dF^V| \leq c\rho^s\), again using (1.4). Altogether, this gives the following pointwise bound:

\[
(1.13) \quad |D_{F,C}(\zeta^N, \zeta^N, h)| \leq c\left(|\nabla \zeta^N| + \rho|\zeta^N|\right) + c\rho^s\left(|\nabla \zeta^V| + |\zeta^V| + |\bar{\zeta}| + |h|\right).
\]

Multiplying both sides of this inequality by \( \rho^{1-s-\delta/2} \), raising to the power \( p \) and integrating shows that [IP] (7.10) holds in the new norms. The proof is completed as before.
Modifications to Section 8. See Section 2 below.

Modifications to Section 9. Section 9 uses Proposition 8.1, but not its proof: the existence of a right inverse is needed, but nothing about its construction. Switching to the new norms \([1.1]\) does not affect Proposition 9.1, and requires only small modifications to the proofs of Lemma 9.2 and Proposition 9.3.

- Lemma 9.2: The statement of Lemma 9.2 remains the same. For the proof, again let \(B_k\) be the region around the \(k\)th node where \(\rho \leq |\mu|^{1/4}\) and let \(A_k\) be the larger region where \(\rho \leq \rho_0\). The new norms \([1.1]\) differ from the old norms only in the weighting of the normal components near the nodes. Thus to show that Lemma 9.2 holds in the new norms we need only bound the \(L^p\) integral of the weighted normal components

\[|\rho^{1-s_k-\delta/2}D_F(\xi, h)^N|,\]

first on \(A_k \setminus B_k\), then on \(B_k\). Fix \(k\) and write \(\mu = \mu_k\) and \(s = s_k\).

Follow the existing proof until two lines below \([IP\ (9.5)]\), at which point we have identified sections of \(F^*TZ\lambda\) over \(C_\mu \setminus B\) with sections of \(f^*TZ_0\) over \(C_0 \setminus B\), and established the estimate:

\[(1.14) \quad |D_F(\xi, h) - D_f(\xi, h)| \leq (|L_F - L_f|)(\xi)| + (|J_F - J_f| + |dF - df|)|h|\]

with \(D_f(\xi, h) = 0\). Formula \([IP\ (1.11)]\) shows that \(L_f\) is a first order differential operator of the form

\[L_f(\xi) = A_f(\nabla \xi) + B_f(\xi)\]

whose coefficients \(A_f\) and \(B_f\) are continuous functions of \(f\) and \(df\). Hence

\[(1.15) \quad |L_F - L_f| \leq c (|F - f| + |dF - df|) (|\nabla \xi| + |\xi|)\]

for some constant \(c\). But in the region \(A_k \setminus B_k\) we have \(F - f = F - \tilde{f} = F^y\). Then \([1.4]\) shows that the \(C^1\) distance between \(F_i\) and \(f_i\) is dominated by \(|\rho^{-s}\), so \([1.14]\) implies the bound

\[|D_F(\xi, h)| \leq c|\lambda|\rho^{-s} (|\nabla \xi| + |\xi| + |h|)\]

on \(A_k \setminus B_k\). After expanding \(\xi\) as in \([IP\ (6.8)]\) and noting that \(|\nabla (\beta_k \xi_k)| \leq c\tilde{\xi}\) by the estimate preceding \([IP\ (7.9)]\), the above bound simplifies to

\[(1.16) \quad |D_F(\zeta, \tilde{\zeta}, h)| \leq c|\lambda|\rho^{-s} (|\nabla \zeta| + |\zeta| + |\tilde{\zeta}| + |h|),\]

a mild strengthening of the displayed equation above \([IP\ (9.6)]\). But in the region \(A_k \setminus B_k\), we have \(|\mu|^{1/4} \leq \rho \leq \rho_0\) and \(|\lambda| \sim |\mu|^{s}\) so \([1.16]\) implies

\[(1.17) \quad |D_F(\zeta, \tilde{\zeta}, h)| + \rho^{1-s} |D_F(\zeta, \tilde{\zeta}, h)^N| \leq c|\lambda|^{1/4}\rho^{1+s} (|\nabla \zeta| + |\zeta| + |\tilde{\zeta}| + |h|).\]

Taking the norms defined by \([IP\ (6.10)]\) and \([1.1]\), shows that \([IP\ (9.6)]\) continues to hold in the new norms.

Now focus on one \(B_k\). Proceed as in \([IP\], using the new estimate \([1.13]\) to strengthen \([IP\ (9.7)]\). For \(\xi = \xi^V\), \([1.13]\) gives

\[(1.18) \quad |D_F(\xi^V, \tilde{\zeta}, h)^N| \leq cp^s (|\nabla \xi^V| + |\zeta^V| + |\tilde{\zeta}| + |h|)\]

on each \(B_k\). For \(\xi = \xi^N\), we again have \(\zeta = 0\) and \(\xi = \zeta\), so \([1.13]\) and the last displayed equation on page 988 give

\[(1.19) \quad |D_F(\xi^N, 0)^N| \leq c (|\nabla \xi^N| + |\rho \xi^N|) \leq cp^s(|\tilde{a}| + |\tilde{b}|).\]

Combining \([1.18]\) and \([1.19]\) with the argument on top of page 989 shows that the conclusion of the first displayed equation on top of page 989 continues to hold in the new norms. The
proof is completed as before.

- **Proposition 9.3**: Replace the last 4 lines on page 989 of [IP] by the following: write $F_n - f_n = (\zeta_n, \bar{\xi}_n)$ in the notation of (6.7) and (6.8). Then $\bar{\xi}_n \to 0$ because $f_n \to f_0$ in $C^0$. By Lemma 5.4, the norm $\|f_n\|_1$ on $A_k(\rho_0)$ is bounded by $c\rho_0^{1/6}$. Inserting the bounds (1.4) into (1.1) and integrating using [IP] (5.10) gives the similar inequality $\|F_n\|_1 \leq c\rho_0^{1/6}$ on $A_k(\rho_0)$. Therefore $\|\zeta_n\|_1 \leq \|F_n\|_1 + \|f_n\|_1 + |\bar{\xi}_n| \leq 3c\rho_0^{1/6}$ on $A_k(\rho_0)$ for all large $n$. Combining ... Continue at the top of page 990, and change 2|s| ↦→ 5|s| on page 990, line 14.

## 2. Revised Section 8

An incorrect formula [IP] (7.5) for the adjoint and a sign error in the curvature formula [IP] (8.7) invalidate the proof of Proposition 8.1. The following replacement for Section 8 retains everything up to and including the statement of Proposition 8.1, and then gives a new proof of Proposition 8.1. Instead of establishing eigenvalue estimates, this new proof transfers the partial right inverse $P$ from the nodal curve $C_0$ to its smoothing $C_{\mu}$. The proof is then easier, the adjoint $D_F^*$ never appears, and again the required estimates follow from pointwise bounds already in [IP].

Retain the beginning of Section 8 of [IP] up to Proposition 8.1.

To simplify notation, note that for $F \in \text{Approx}^{\delta_0}(Z_\lambda)$, [IP] Proposition 7.3] shows that the linearizations $D_F$ of [IP] (7.4) are uniformly bounded operators

$$D_F : \mathcal{E}_F \to \mathcal{F}_F$$

between the spaces

$$\mathcal{E}_F = L_{1;0}^{1}(F^*T_{Z_\lambda}) \oplus T_{\bar{\nu}}V^{\ell} \oplus T_{C_{\mu}}M_{g,n} \quad \text{and} \quad \mathcal{F}_F = L_{s}(A_{1}^{01}(F^*T_{Z_\lambda})),$$

while the linearization $D_f = D_{f,C_0}$ of [IP] (7.3)] at each $f \in M_{s}^{V}(X) \times \text{ev} M_{s}^{V}(Y)$ is a map between the corresponding spaces $\mathcal{E}_f$ and $\mathcal{F}_f$.

The aim of this section is to prove the following analytic result.

**Proposition 2.1.** For each generic $(J, \nu) \in J(Z)$, there are positive constants $\lambda_0$ and $E$ such that, for all non-zero $\lambda \leq \lambda_0$, the linearization $D_F$ at an approximate map $(F, C_{\mu}) = F_{f,C_0,\mu} \in \text{Approx}^{\delta_0}(Z_\lambda)^*$ has a right inverse

$$P_F : \mathcal{E}_F \to \mathcal{F}_F$$

that satisfies $D_F P_F = \text{id}$ and

$$E^{-1} \|\eta\|_0 \leq \|P_F \eta\|_1 \leq E \|\eta\|_0. \quad (2.21)$$

Proposition 8.1 is proved by constructing an approximation to $P_F$ in the following sense.

**Definition 2.2.** An approximate right inverse to $D_F$ is a linear map

$$A_F : \mathcal{F}_F \to \mathcal{E}_F$$

such that, for all $\eta \in \mathcal{F}_F$,

$$\|D_F A_F \eta - \eta\|_0 \leq \frac{1}{2} \|\eta\|_0 \quad \text{and} \quad \|A_F \eta\|_1 \leq C \|\eta\|_0. \quad (2.22)$$
Such an approximate right inverse defines an actual right inverse by the formula

\[ P_F = A_F \sum_{k \geq 0} (I - D_F A_F)^k. \]

The bounds (2.22) ensure that this series converges and defines a bounded operator \( P_F \), which satisfies \( D_F P_F = I \). Because both \( P_F \) and \( D_F \) are bounded (cf. [IP, (Lemma 7.3)]), we have \( \|P_F \eta\|_1 \leq c\|\eta\|_0 \) and \( \|\eta\|_0 = \|D_F P_F \eta\|_0 \leq c\|P_F \eta\|_1 \), which gives (2.21).

Thus Proposition 8.1 follows from the existence of an approximate right inverse \( A_F \) as in Definition 2.2, where the constant \( C \) in (2.22) is uniform in \( \lambda \) for small \( \lambda \). The remainder of this section is devoted to constructing such an \( A_F \).

We start by observing that, under the hypotheses of Proposition 2.1, we may assume that \( f \) is regular (cf. [IP, (Lemma 3.4)]). Thus \( D_f : E_f \to F_f \) is a bounded surjective map, so has a bounded right inverse \( P_f : F_f \to E_f \). We will use a splicing construction to transfer \( P_f \) from an operator on \( C_0 \) to one on the domain \( C_\mu \) of \( F \), and show that the resulting operator \( A_F \) satisfies (2.22). The construction is summarized by the following (noncommutative) diagram:

\[
\begin{array}{ccc}
\mathcal{F}_F & \xrightarrow{A_F} & \mathcal{E}_F \\
\gamma_F \downarrow & & \pi_F \downarrow \\
\mathcal{F}_f & \xrightarrow{P_f} & \mathcal{E}_f \\
\end{array}
\quad
\begin{array}{ccc}
\mathcal{D}_F & \xrightarrow{\Gamma_F} & \mathcal{F}_F \\
\pi_F \downarrow & & \Gamma_F \downarrow \\
\mathcal{D}_f & \xrightarrow{\Gamma_f} & \mathcal{F}_f \\
\end{array}
\]

Each of the maps \( \gamma_F, \pi_F \) and \( \Gamma_F \) will be defined by regarding the two halves of \( C_\mu \) as graphs over \( C_0 \), and similarly regarding \( Z_\lambda \) as graphs over \( Z_0 \). The desired approximate right inverse is then defined by

\[ A_F = \Gamma_F \circ P_f \circ \pi_F. \]

Our notation for splicing is as in Lemma 9.2 of [IP]. For each \( \mu \neq 0 \), let

\[ C_1(\mu) = C_1 \cap \{ |z| \geq |\mu|^3/4 \} \quad \text{and} \quad C_2(\mu) = C_2 \cap \{ |w| \geq |\mu|^3/4 \}. \]

and let \( C_\mu^+ \) and \( C_\mu^- \) be the corresponding parts of \( C_\mu \) (see the figure). We identify \( C_1(\mu) \) with \( C_\mu^+ \) by the projection \( (z, w) \mapsto z \). With this identification, \( z \) is a coordinate on both \( C_1(\mu) \) and \( C_\mu^+ \) and, similarly, \( w \) is a coordinate on both \( C_2(\mu) \) and \( C_\mu^- \).

There is a corresponding picture in the target: the projections \( (v, x, y) \mapsto (v, x) \) and \( (v, x, y) \mapsto (v, y) \) give identifications \( Z_\lambda = X \) in the region \( Z_\lambda^+ \) where \( |x| \geq |\lambda|^{3/4} \), and \( Z_\lambda = Y \) in the region \( Z_\lambda^- \) where \( |y| \geq |\lambda|^{3/4} \). This trivializes \( f^*TZ_0 \) and \( F^*TZ_\lambda \) inside the coordinate chart \((v, x, y)\).
These identifications, together with [IP] Definition 6.2, induce isomorphisms
\[(2.24) \quad \Omega^{0,q}(C_1(\mu), f^*TX) \rightarrow \Omega^{0,q}(C_\mu^+, F^*TZ_\lambda) \quad \text{by } \xi_1 \mapsto \hat{\xi}_1\]
for \(q = 0, 1\) defined by \(\hat{\xi}_1(z) = \xi(z)\) in \(B_k\) under the above identifications of \(C_1(\mu)\) with \(C_\mu^+\) and of \(X\) with \(Z_\lambda^+\), extended by setting \(\hat{\xi}_1 = \xi\) outside the union of the balls \(B_k\) of radius \(2|\mu_k|^{1/4}\) (where \(C_0\) is identified with \(C_\mu\) and \(F = f\)). Permuting \(z \leftrightarrow w\) and \(x \leftrightarrow y\) gives similar isomorphisms
\[\Omega^{0,q}(C_2(\mu), f^*TX) \rightarrow \Omega^{0,q}(C_\mu^+, F^*TZ_\lambda) \quad \text{by } \xi_1 \mapsto \hat{\xi}_2.\]

**Lemma 2.3.** For each region \(\Omega^+_M\) defined by \(M^{-1}|w| \leq |z| \leq 1\), there are constants \(c_M, \lambda_M > 0\) such that the map \((2.24)\) satisfies the pointwise estimates
\[(2.25) \quad c_M^{-1} |\xi| \leq |\hat{\xi}| \leq c_M |\xi|, \quad |\nabla \hat{\xi}| \leq c_M (|\nabla \xi| + |\xi|)\]
whenever \(|\lambda| \leq \lambda_M\) small. Furthermore, if \(\xi = \xi^V\) then \((\hat{\xi})^N = 0\).

**Proof.** For each non-zero small \(\mu\), equations [IP] (4.4), (4.5)] show that the cylindrical metric on \(C_\mu \cap \Omega^+_M\) is independent of \(\mu\) (the ratio \(g_\mu/g_0\) of the metrics is \(r^2 \rho^{-2} (1 + |\frac{\xi}{z}|^2) = 1\)). On the target, the corresponding formula shows that the (smooth) metrics \(g_X\) on \(X\) and \(g_\lambda\) on \(Z_\lambda\) have the form
\[(2.26) \quad g_X = (g_V + dx \tilde{\xi}) + O(R) \quad g_\lambda = \left[g_V + \left(1 + \frac{|\lambda|^2}{|x|^2}\right) dx \tilde{\xi} + O(R),\right]
where \(g_V\) is the metric on \(V\), and \(R^2 = |x|^2 + |y|^2\). Using the formula for the Christoffel symbols, one sees that the difference of the corresponding Levi-Civita connections is a 1-form \(\alpha dx\) on \(X\) with \(|\alpha| \leq c(1 + |\lambda|^2 |x|^{-5})\).

As in [1.2]–[1.4], the coordinates of each approximate map \(F\) satisfy \(|x| \sim |z|^\delta\) and \(|y| \sim |w|^\delta\) and \(xy = \lambda\). Thus the image of \(\Omega^+_M\) lies in the region \(Z_\lambda\) where \(|y| \leq c_1(M) |x|\) for a constant \(c_1(M)\) independent of \(\lambda\). In this region, the metrics \((2.26)\) are uniformly equivalent with a similar constant \(c_2(M)\), giving the first part of \((2.25)\). Furthermore, the covariant derivatives are related by
\[\nabla \hat{\xi} = \nabla \xi + \alpha_F \xi \quad \text{with } |\alpha_F \xi| \leq c_3(M) (1 + |\lambda|^2 |x|^{-5}) \cdot (|df^N| + |dF^N|) |\xi|,\]
Noting that \(xy = \lambda\) and \(|x| \sim |z|^\delta \sim \rho^\delta\), the term \(|\lambda|^2 |x|^{-5}\) is dominated by \(|\frac{\mu}{z}|^2 \rho^{-5} \leq c_1^2(M) \rho^{-\delta}\). We also have \(|dF^N| \leq c\rho^\delta\) by \((1.4)\) and the bound \(|dF^N| \leq c\rho^\delta\) obtained similarly by taking \(\beta = 0\) in \((1.2)\). The last part of \((2.25)\) follows. \(\square\)

To define cutoff functions, consider the central annular region \(\Omega_M\) of \(C_\mu\) defined by
\[(2.27) \quad M^{-1} \leq \left|\frac{w}{z}\right| \leq M.\]
In cylindrical coordinates (defined by \(z = \sqrt{|\mu|} e^{i \theta + i \varphi}\)), this is a region of length \(\log(1 + M^2)\). Thus we can choose a smooth cutoff function \(\varphi_M(z, w)\) that vanishes for \(|w| > M|z|\), is equal to 1 for \(M|w| \leq |z|\), and satisfies \(0 \leq \varphi_M \leq 1\) and
\[(2.28) \quad |d\varphi_M| \leq \frac{1}{|\log M|}.\]
To maintain symmetry, we can also assume (after appropriately symmetrizing) that
\[ \varphi_M(z, w) + \varphi_M(w, z) = 1. \]

With this setup, the maps \( \gamma_F, \pi_F \) and \( \Gamma_F \) in Diagram (2.23) are defined as follows.

- **The map** \( \pi_F : F_F \to F_f \). The map (2.24) with \( q = 1 \) has an inverse
  \[ \tau^+ : \Omega^{0,1}(C^+ \mu, F^*T\lambda) \to \Omega^{0,1}(C_\mu, f^*TX). \]

Then each \( F^*T\lambda \)-valued (0,1)-form \( \eta \) on \( C_\mu \) restricts to a form \( \eta^+ \) on \( C^+ \mu \), and we define \( \pi_F(\eta) \) on \( C_1 \) by
  \[ (\pi_F(\eta))(z) = \begin{cases} \tau^+\eta^+(z) & \text{for } |z| > |\mu|^{1/2} \\ 0 & \text{for } |z| \leq |\mu|^{1/2}. \end{cases} \]

The restriction of \( \pi_F(\eta) \) to \( C_2 \) is defined symmetrically.

- **The map** \( \gamma_F : F_f \to F_F \). This map takes a \( f^*T\lambda \)-valued (0,1)-form \( \eta \) on \( C_0 \) to a \( F^*T\lambda \)-valued (0,1)-form \( \eta \) on \( C_\mu \). It is given by
  \[ \gamma_F(\eta) = \varphi_M \hat{\eta}_1 + (1 - \varphi_M) \hat{\eta}_2 \]
  where \( \hat{\eta}_1 \) and \( \hat{\eta}_2 \) are defined in terms of the restrictions \( \eta|_{C_1} = \xi_1 d\bar{z} \) and \( \eta|_{C_2} = \xi_2 d\bar{w} \) by \( \hat{\eta}_1 = \hat{\xi}_1 d\bar{z} \) and \( \hat{\eta}_2 = \hat{\xi}_2 d\bar{w} \) inside each coordinate chart \((z, w)\), and \( \gamma_F = id \) outside.

- **The map** \( \Gamma_F : \mathcal{E}_f \to \mathcal{E}_F \). This map takes a section of \( f^*T\lambda \) on \( C_0 \) to a section of \( F^*T\lambda \) on \( C_\mu \), and a variation \( h \) in the complex structure of \( C_0 \) to a variation in the complex structure of \( C_\mu \). It is given by
  \[ \Gamma_F(\xi, h_0) = (\varphi_M \hat{\xi}_1 + (1 - \varphi_M) \hat{\xi}_2, h_\mu) \]
  where \( \hat{\xi}_1 \) and \( \hat{\xi}_2 \) are obtained from the restrictions \( \xi|_{C_1} = \xi_1 \) and \( \xi|_{C_2} = \xi_2 \) inside these neighborhoods, and \( h_\mu = (h_0, 0) \) in the notation of [IP, (4.9)]. Again, \( \Gamma_F \) extends outside as \( \Gamma_F = id \). Thus, in the notation of [IP] (7.3), (7.4), \( \Gamma_F \) is a map
  \[ \Gamma_F : L_1^s(f^*T\lambda) \oplus T_{C_1}\tilde{\mathcal{M}} \oplus T_{C_2}\tilde{\mathcal{M}} \to L_1^s(F^*T\lambda) \oplus T_{C_\mu}\tilde{\mathcal{M}}_{g,\lambda}. \]

**Corollary 2.4.** The maps \( \pi_F, \gamma_F \) and \( \Gamma_F \) satisfy
  \[ \pi_F \gamma_F = id, \]
  and for small \( \lambda \)
  \[ \|\pi_F \eta\|_0 \leq 2\|\eta\|_0 \quad \|\gamma_F \eta\|_0 \leq c_M \|\eta\|_0 \quad \|\Gamma_F(\xi, h)\|_1 \leq c_M \|\xi, h\|_1, \]
where \( c_M \) depends only on the constant \( M \) in (2.27).

**Proof.** The equation \( \pi_F \gamma_F = id \) follows directly from the definitions of \( \pi_F \) and \( \gamma_F \). As in first paragraph of the proof of Lemma 2.3, the projection \( C_\mu \to C_1 \) is an isometry in the region where \( \rho \leq 1 \), and \( g_\lambda \) is greater than \( g_X \) on its image. It follows that the operator norm of \( \pi_F \) is at most 2 for small \( \lambda \).

Similarly, (2.29), the fact that \( 0 \leq \varphi_M \leq 1 \), and Lemma 2.3 show that
  \[ \|\gamma_F(\eta)\|_0 \leq \|\hat{\eta}_1\|_0 + \|\hat{\eta}_2\|_0 \leq c_M \|\eta\|. \]


Using (2.30) in exactly the same way, we also have
\[ \| \Gamma_F(\xi, h) \|_0 \leq \| \hat{\xi}_1 \|_0 + \| \hat{\xi}_2 \|_0 + \| h \| \leq C_M \| (\xi, h) \|_0. \]
Differentiating (2.30) and again applying Lemma 2.3 yields the last inequality in (2.31).

The next lemma shows that the difference \( D_F \Gamma_F - \gamma_F D_f \) can be made small. The statement again involves the constant \( M \) in the bounds (2.27) and (2.28) associated with the cutoff functions \( \varphi_M \).

**Lemma 2.5.** Fix the compact subset \( K_{\delta_0} \) of \( M_\delta^V(X) \times ev M_\delta^V(Y) \) of \( \delta_0 \)-flat maps. For any \( \varepsilon > 0 \), there exists a slope \( M = M_\varepsilon > 1 \) and a \( \lambda_M > 0 \) such that each approximate map \( F \) constructed from \( f \in K_{\delta_0} \) with \( |\lambda| \leq \lambda_M \) satisfies
\[ \| (D_F \Gamma_F - \gamma_F D_f)(\xi, h) \|_0 \leq \varepsilon \| (\xi, h) \|_1 \]
for all \( (\xi, h) \in \mathcal{E}_f \).

**Proof.** We use the set-up of Lemma 9.2 above, except that we do not make the assumption that \( D_f(\xi_0, h_0) = 0 \) in (1.14). Outside the region \( B = \bigcup B_k \), \( \Gamma_{F,M} \) and \( \gamma_f \) are both the identity for \( |\mu| \leq M^{-2} \). Thus the discussion from (1.14) to (1.17) implies the bound
\[ \| (D_F \Gamma_F - \gamma_F D_f)(\xi, h) \|_{0,C_0,B} \leq C |\lambda| \frac{1}{4} \| (\xi, h) \|_1. \]
Next restrict attention to the region \( B_k \) around one node of \( C_0 \), where \( \rho \leq 2|\mu|^{1/4} \), and consider a deformation \( (\xi, h) \) on \( C_1 \) (an identical analysis applies on \( C_2 \)). Then \( \xi \in \Gamma(C_1, f^*TX) \) lifts by (2.24) to \( \hat{\xi} \in \Gamma(C_1^+, f^*TZ_\lambda) \); this is the identification implicitly used in [9, (9.5)] and in (1.14). With this notation, (1.14) can be written as
\[ \| D_f(\hat{\xi}, h) - D_f(\xi, h) \| \leq \| L_F \hat{\xi} - L_f \xi \| + c|J_F - J_f||\xi| + |dF - df||h|. \]
For the following estimates, we restrict attention to the annular subregion \( A_M^+ \subset B_k \) where
\[ \frac{w}{z} \leq M \quad \text{and thus} \quad |z| \leq \rho \leq |z| \sqrt{1 + M^2}. \]
In this subregion, the \( C^1 \) norm of \( F - f \) is \( O(\rho) \), as shown in the proof of Lemma 6.9. Using (1.14) and (1.15) we obtain
\[ \| D_f(\hat{\xi}, h) - D_f(\xi, h) \| \leq c\rho (|\nabla \xi| + |\hat{\xi}| + |\nabla \xi| + |\xi| + |h|). \]
Lemma 2.3 then shows that we may remove the hats on the righthand side, giving
\[ \| D_f(\hat{\xi}, h) - D_f(\xi, h) \| \leq c\rho (|\nabla \xi| + |\xi| + |h|). \]
Here the left-hand side is regarded as a function of \( z \) and \( w = \mu/z \) on \( C_1 \cap A_M^+ \), while \( \xi \) and \( h \) are functions of \( z \) on the corresponding region of \( C_1 \), and \( \rho^2 = |z|^2 + |w|^2 \).

As in previous lemmas, we need separate bounds for the normal components. First, (2.33) for \( \xi = \xi^N \) and \( h = 0 \) gives
\[ \| D_F(\hat{\xi}^N, 0) - D_f(\hat{\xi}^N, 0) \| \leq c\rho (|\nabla \xi^N| + |\xi^N|). \]
On the other hand, if \( \xi = \xi^V \) is tangent to \( V \), Lemma 2.3 shows that its lift \( \hat{\xi} \) is also tangent to \( V \). Writing \( D_F(\xi, h) = L_F \xi + J_f dFh \), we can apply (1.11) and the argument made before (1.13) to obtain
\[ \| D_F(\hat{\xi}^V, h)^N \| \leq c\rho^s \left( |\nabla \xi^V| + |\hat{\xi}^V| + |h| \right). \]
By the same argument, a similar inequality holds with $D_F$ replaced by $D_f$ and $\hat{\xi}^V$ by $\xi^V$; together these give
\[
|D_F(\hat{\xi}^V, h)^N - (D_f(\hat{\xi}^V, h))^N| \leq c\rho^s (|\nabla \xi^V| + |\xi^V| + |h|)
\]
after again using (2.25) to remove hats on the right. Combining this with (2.34) gives
\[(2.35) \quad |D_F(\hat{\xi}, h)^N - (D_f(\xi, h))^N| \leq c\rho(|\nabla \xi^N| + |\xi^N|) + c\rho^s (|\nabla \xi^V| + |\xi^V| + |h|) .
\]

Now set $\Psi(\xi, h) = D_F(\hat{\xi}, h) - D_f(\xi, h)$. With this notation, we can combine (2.33) and (2.35), and then decompose $\xi$ into $(\zeta, \bar{\xi})$ as in (6.9), noting that $|\nabla \xi| + |\xi| \leq |\nabla \zeta| + |\zeta| + |\bar{\xi}|$ as before (1.16). The result is
\[
|\Psi(\xi, h)| + \rho^{1-s}(|\Psi(\xi, h)|)^N \leq c\rho(\rho(|\nabla\zeta| + |\zeta|) + \rho^{1-s}(|\nabla\bar{\xi}| + |\bar{\xi}|)) 
\]
Multiplying by $\rho^{-\delta/2}$ and computing the integral (1.1) over $A_M^+ \cap C_\mu$, where $\rho \sim |\zeta|$ by (2.32) and $|\lambda| \sim |\mu|$, one sees that
\[(2.36) \quad \|D_F(\hat{\xi}_1, h_1) - D_f(\hat{\xi}_2, h_2)\|_{0, A_M^+ \cap C_\mu} \leq c_M |\mu|^{1/4(1-\delta/2)} \|\xi_1, h_1\|_1 \leq c_M |\lambda|^{\frac{1}{4\mu}} \|\xi_1, h_1\|_1
\]
for all pairs $(\xi_1, h_1)$ on $C_1$. A similar estimate holds for pairs $(\xi_2, h_2)$ on $C_2$.

To complete the proof, recall that both $\Gamma_F$ and $\gamma_F$ are obtained by splicing in the region $\Omega_M$ of (2.27). Using (2.36), the formula $D_F(\xi, h) = L_F \xi + JdFh$ and the Leibnitz rule, we obtain
\[
|D_F \Gamma_F(\xi, h) - \varphi_M D_F(\hat{\xi}_1, h_\mu) - (1 - \varphi_M) D_F(\hat{\xi}_2, h_\mu)| \leq |d \varphi_M| \cdot |\xi| \leq |\log M|^{-1} |\xi|.
\]
Combining this with (2.29) and (2.36) gives the following uniform estimate:
\[
\|(D_F \Gamma_F, M - \gamma_F D_f)(\xi, h)\|_0 \leq \left( c_M \lambda^{\frac{1}{4\mu}} + c |\log M|^{-1} \right) \|(\xi, h)\|_1.
\]
The lemma follows by first choosing $M$ so that $c |\log M|^{-1} \leq \varepsilon/2$, then choosing $\lambda_M$ so that $c_M \lambda_M^{\frac{1}{4\mu}} \leq \varepsilon/2$.

We are now able to define the approximate inverse of $D_F$. Recall that the linearization $D_f$ of a regular map is onto. Fix the compact set $\mathcal{K} = K_{\delta_0}$ of $\delta_0$-flat regular maps. Then we can choose a family $P_f$ of partial right inverses of $D_f$ which are uniformly bounded
\[
\|P_f \eta\|_0 \leq K \|\eta\|_1.
\]
on $\mathcal{K}$. Recall that the operator norm of $\pi_F$ is at most 2.

**Lemma 2.6.** In the above context, there exist positive constants $C, M$ and $\lambda_0$ such that for any approximate map $f$ (obtained from $f \in \mathcal{K}$ for $|\lambda| \leq \lambda_0$), the operators
\[
A_F = \Gamma_F, M \circ P_f \circ \pi_F : \mathcal{E} \to \mathcal{F} \quad \text{obtained from splicing region (2.27)}
\]
for any $\eta \in \mathcal{F}$.

\[(2.37) \quad \|D_F A_F \eta - \eta\|_0 \leq \frac{1}{2}\|\eta\|_0 \quad \text{and} \quad \|A_F \eta\|_1 \leq C \|\eta\|_0
\]
Proof. Write $D_F A_F - I$ as
\[
D_F \circ \Gamma_F \circ P_f \circ \pi_F - id = (D_F \Gamma_F - \gamma_F D_f) \circ P_f \circ \pi_F + \gamma_F D_f \circ P_f \circ \pi_F - id \\
= (D_F \Gamma_F - \gamma_F D_f) \circ P_f \circ \pi_F.
\]
We know that $\|\pi_F\| \leq 2$ and $\|P_f\| \leq K$ are uniformly bounded on the compact $K$. Hence there is a bound on the operator norm:
\[
\|D_F A_F - I\| \leq \|D_F \Gamma_F - \gamma_F D_f\| \cdot \|P_f\| \cdot \|\pi_F\| \leq 2K\|D_F \Gamma_F - \gamma_F D_f\|.
\]
Now take $\varepsilon = \frac{1}{4K} > 0$ in Lemma 2.5 to obtain the first inequality of (2.22). This choice of $\varepsilon$ fixes the slope $M = M_\varepsilon$ in Lemma 2.5. With this choice, $\Gamma_{F,M}$ are bounded, with a bound that depends on $M$, and hence
\[
\|A_F\| = \|\Gamma_{F,M} \circ P_f \circ \pi_F\| \leq \|\Gamma_{F,M}\| \cdot \|P_f\| \cdot \|\pi_F\| \leq 2K\|\Gamma_{F,M}\|.
\]

3. Typographical errors

The following typographical errors in [IP] have no consequences, but may cause confusion.

- Line after (5.1): insert “after passing to a subsequence”.
- In (5.18), delete +1/3.
- In (1.11), the second + should be a −.
- Page 946, line 4: $J(\nabla_{\nu(w)}J)\xi \to (\nabla_{J\nu(w)}J)\xi$. One can also note that the tensor $\hat{\nabla}J$ is zero by (C.7.5) of [MS] for compatible structures $(\omega,J,g)$.
- Page 1003, line 8: (10.6) $\mapsto$ (10.11).

REFERENCES


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