Convergence of the heat flow for closed geodesics

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Abstract

On a closed Riemannian manifold (M, g), the geodesic heat flow deforms loops through an energy-decreasing homotopy $u(t)$. It is known that there is a sequence $t_n \to \infty$ so that $u(t_n)$ converges in C^0 to a closed geodesic, but it is also known that convergence fails in general. We show that for a generic set of metrics on M, the heat flow converges for all initial loops. We also show that convergence, when it holds, is in C^∞. The proofs are based on a Morse theory approach using the Palais-Smale condition.

Geometric heat flows have been extensively studied, yet one of the simplest examples is not yet fully understood. This note clarifies several issues concerning the convergence as $t \to \infty$ for the heat flow associated with maps from S^1 to a closed Riemannian manifold (M, g).

It is well-known that every map $u_0 : S^1 \to M$ into a closed Riemannian manifold (M, g) is homotopic to a closed geodesic. Intuitively, this can be proven by deforming u_0 along the flow of the downward gradient vector field of the energy function

$$E(u) \;=\; \frac{1}{2}\int_{S^1} |du|^2 \; d\theta \tag{0.1}$$

on the space of maps $u : S^1 \to M$. There are two standard ways of directly realizing this intuition. In the "Morse theory" approach, one works on the infinite-dimensional manifold $\mathcal{L}_M$ of finite-energy loops in M and uses the gradient of E defined by the Sobolev $W^{1,2}$ metric; this approach provides techniques for showing that the flow paths converge (cf. Section 3). Alternatively, one can use the gradient of E defined by the L^2 metric. The resulting downward gradient flow is a solution of the "geodesic heat flow equation"

$$\dot{u} \;=\; \nabla^* du \qquad\qquad u(\theta, 0) = u_0(\theta). \tag{0.2}$$

This equation, which is the 1-dimensional harmonic map heat flow equation, is typically studied using parabolic techniques, without reference to the manifold $\mathcal{L}_M$. Our main purpose here is to show that Morse theory methods can be productively applied to the heat flow.

It is known that the geodesic heat flow exists and is smooth for all $t > 0$. Thus central issue is convergence as $t \to \infty$. Specifically, do all solutions of (0.2) converge to closed geodesics as $t \to \infty$, and in which norms does one have convergence?

The subtleties of this problem have been repeatedly underestimated. In 1965 Eells and Sampson [ES2] asserted that the geodesic heat flow converges to a closed geodesic at $t \to \infty$ (see also [Sa] and ([J]). Ottarsson [O] rigorously proved the long-time existence of the heat flow with smooth initial data. Very recently, L. Lin and L. Wang [LW], building on arguments of Struwe [St], extended Ottarsson's theorem to $W^{1,2}$ initial data, proving that the heat flow exists for all time and is unique. Ottarsson also showed that there is a sequence $t_n \to \infty$ so that $u(t_n)$ converges in C^0 to a closed geodesic. On the other hand, the flow itself does not necessarily converge: an elegant construction of Topping yields the following fact:

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Theorem (Topping). *There is a closed Riemannian manifold (M, g) and a smooth loop u_0 in M such that the geodesic heat flow (0.2) does not converge in C^0 as $t \to \infty$.*

This is proved by taking $M = S^1 \times S^1$ and replacing S^2 by S^1 in Section 5 of [T].

In Sections 1-3 we obtain some partial convergence results by viewing the heat flow as a path in the loop space and applying Morse theory methods. The key points are the finiteness of the "action integral" proved in Section 2, and the simple but surprising observation (Lemma 3.1) that the Palais-Smale compactness condition applies to the heat flow. The Palais-Smale condition does not imply convergence; rather it implies that the flow is asymptotic to a critical set K of the energy function. The flow can "orbit" around K, as occurs in Topping's example. The results of Sections 3 can be summarized as follows.

Theorem A. *Let (M, g) be a closed Riemannian manifold and let $u(t)$ be a geodesic heat flow. Then as $t \to \infty$,*

(a) $\dot{u}(t) \to 0$ in C^∞.

(b) $u(t)$ is asymptotic in C^∞ to a compact component K of the critical set of E.

In fact, we can ensure convergence by making a mild assumption about the Riemannian metric. Morse Theory points to an appropriate condition: the energy function $E : \mathcal{L}_M \to \mathbb{R}$ should be a Morse or Morse-Bott function. This is true for a generic set of metrics, the so-called bumpy metrics. In Section 4 we prove that on manifolds with bumpy metrics all heat flows paths converge in the $W^{1,2}$ topology of $\mathcal{L}_M$. The bootstrap arguments given in Section 5 then show that this convergence is actually in C^∞. This gives our main result:

Theorem B. *Let M be a closed Riemannian manifold. There is a Baire set $\mathcal{B}$ in the space of all metrics on M such that if $g \in \mathcal{B}$, then every heat flow path $u(t)$ on (M, g) converges in C^∞ to a smooth closed geodesic γ homotopic to $u(0)$.*

Finally, we return to the case of a general Riemannian metric, but now restrict attention to those flow lines $u(t)$ that are asymptotic, as in Theorem A to a *stable* critical set K as $t \to \infty$. With our definitions, there are two cases of stability. In the first, K is a single orbit of the circle action on $\mathcal{L}_M$ on which the energy function E is non-zero that the Hessian of E is non-degenerate on the normal bundle to K. The second is the simple case when K is the manifold of geodesics with energy $E = 0$, i.e. maps to a single point. In Section 6 we use this stability condition to prove that $u(t)$ converges at an exponential rate. The argument for trivial geodesics, given in Section 7, is different, but the conclusion is the same. Thus:

Theorem C. *Let (M, g) be a closed Riemannian manifold and let $u(t)$ be a geodesic heat flow that is asymptotic to a stable critical set K as $t \to \infty$. Then $u(t)$ converges in C^∞ at an exponential rate to a smooth closed geodesic γ that is homotopic to $u(0)$.*

Theorem B is related to L. Simon's theorems [**?**] on the convergence of gradient flows of real analytic functionals, which apply to the energy function (0.1) on a analytic Riemannian manifold. This implies the convergence of Theorem B for a dense set of metrics because Whitney showed that analytic structures are C^k dense in space of metrics on a compact Riemannian manifold.

Simon's proof is based on a gradient inequality of Lojasiewicz, which requires analyticity. But Whitney showed that a compact Riemannian manifold is C^k arbitrarily close to an analytic Riemannian manifold.

These theorems highlight two themes that may also pertain to other heat flow problems. First, convergence issues can be approached by "Morse theory" methods rather than relying entirely on parabolic estimates. Second, one should expect convergence as $t \to \infty$ only in generic situations, not in general.

1 Gradient flows for geodesics

Solutions of the heat flow equation (0.2) can be viewed as paths in the Hilbert manifold $\mathcal{L}_M$ of finite-energy loops that decrease the energy function. This is the context for the well-known Morse theory developed by Palais and others in the 1960s. In this section we review the analytic setup; detailed proofs can be found in [P]. We then make some initial observations about the heat flow on $\mathcal{L}_M$.

Fix an isometric embedding $M \hookrightarrow \mathbb{R}^r$ and let $\mathcal{L}_M$ be the space of all $L^{1,2}$ maps $u : S^1 \to M$. Then $\mathcal{L}_M$ is a smooth, closed Hilbert submanifold of the Hilbert space $L^{1,2}(S^1, \mathbb{R}^r)$, the tangent space $T_u\mathcal{L}_M$ is the space of $W^{1,2}$ sections of the pullback bundle u^*TM, and $E : \mathcal{L}_M \to \mathbb{R}$ is a smooth function. The first variation of E is

$$(dE)_u(X) \;=\; -\int_{S^1} \langle \nabla_T T, X\rangle \tag{1.1}$$

where $T = \partial_\theta u$, and the Hessian of E at a geodesic u is given by the bilinear form

$$B_u(X, X) \;=\; \int_{S^1} |\nabla_T X|^2 + \langle X, R_u(T, X)T\rangle. \tag{1.2}$$

The restriction of the inner product on $W^{1,2}(S^1, \mathbb{R}^r)$ defines a Riemannian metric on $\mathcal{L}_M$; the corresponding norm is given at each $u \in \mathcal{L}_M$ by

$$\begin{aligned} |||X|||^2 \;&=\; \int_{S^1} |\nabla X|^2 + |X|^2 \, d\theta \\ &=\; \langle X, (I + \Delta)X\rangle_{L^2} \qquad X \in T_u\mathcal{L}_M \end{aligned} \tag{1.3}$$

where ∇ is the Levi-Civita connection on M and Δ is the Laplacian $\nabla^*\nabla = -\nabla_T\nabla_T$ on sections of u^*TM. With this metric, $\mathcal{L}_M$ is a complete Riemannian Hilbert manifold. Using (1.1) and noting that $I + \Delta$ extends to an invertible bounded linear map $T\mathcal{L}_M \to T^*\mathcal{L}_M$, the gradient of E for this metric is

$$\boldsymbol{\nabla} E \;=\; (I + \Delta)^{-1}\nabla_T T. \tag{1.4}$$

This is a smooth vector field on $\mathcal{L}_M$ and satisfies the following *Palais-Smale Condition.*

> **P-S** Every sequence $\{u_k\}$ in $\mathcal{L}_M$ with $E(u_k) < C$ and $|||\boldsymbol{\nabla} E(u_k)||| \to 0$ has a convergence subsequence.

Most of Morse theory carries over to this infinite-dimensional context. For our purposes, the most important conclusions are the following statements about the long-time convergence of the downward gradient flow paths.

Proposition 1.1. *For a closed Riemannian manifold (M, g), the energy function $E : \mathcal{L}_M \to \mathbb{R}$ is smooth and satisfies the P-S condition. Consequently, each downward gradient flow $u(t)$ exists for all $t \geqslant 0$, and*

a) $u(t)$ is asymptotic to one component of the critical set of E as $t \to \infty$, and there is a sequence $t_n \to \infty$ such that $u(t_n)$ converges to a closed geodesic.

b) If E is a Morse or Morse-Bott function, then $u(t)$ converges to a closed geodesic as $t \to \infty$.

Proposition 1.1 can be deduced from results in [P] using arguments like those in Sections 5 and 6 below. As explained in Section 6, the Morse-Bott condition on E is actually a condition on the Riemannian metric g on M.

Notice the distinction, which is seldom emphasized in the literature, between statements (a) and (b) above: the asymptotic statement holds in general, while the convergence statement requires a condition on the Riemannian metric. The next several sections show that the same distinction holds for geodesic heat flows.

2 The heat flow and the action integral

On the space $\mathcal{L}_M$ of $W^{1,2}$ loops, we can also consider the gradient of the energy function E with respect to the (weak) L^2 Riemannian metric. By (1.1), this gradient is given at a loop u by

$$(\nabla E)_u = -\nabla_T T$$

(we use boldface $\boldsymbol{\nabla} E$ for the $W^{1,2}$ gradient (1.4) and ∇E for the L^2 gradient). The Morse theory of the previous section does not apply because this L^2 gradient ∇E is not a continuous vector field on $\mathcal{L}_M$. However, the associated downward gradient flow lines are solutions of the heat equation

$$\dot{u} \;=\; \nabla_T T \qquad\qquad u(\theta, 0) = u_0(\theta) \tag{2.1}$$

and parabolic theory gives the following strong existence theorem.

Proposition 2.1. *For each $u_0 \in \mathcal{L}_M$, the heat flow (2.1) exists and is unique for all $t \geqslant 0$, is smooth for all $t > 0$, and the map $[0, \infty) \to \mathcal{L}_M$ by $t \mapsto u(t)$ is uniformly continuous in t with a constant depending only on $E(u_0)$.*

Proof. Existence and uniqueness were proven for smooth initial data in [O] and for $W^{1,2}$ initial data in [LW]. The uniform continuity of the flow is proved as Lemma 2.4 in [LW]; the following alternative argument is more in keeping with the approach we will take in subsequent sections.

Consider the length of the path $u(t)$ for $s < t < T$, measured in the $W^{1,2}$ Riemannian metric on $\mathcal{L}_M$. By Holder's inequality,

$$\text{Length}(s, T) \;=\; \int_s^T \|\!|\dot{u}(\tau)|\!\|\, d\tau \;\leqslant\; \sqrt{T-s}\,\sqrt{A(s,T)} \tag{2.2}$$

where $A(s,T)$ is the *action integral*, defined as the time integral of the square of the $W^{1,2}$ norm of $\dot{u}$:

$$A(s,T) \;=\; \int_s^T \|\!|\dot{u}(t)|\!\|^2\, dt. \tag{2.3}$$

(In physics language, this is the action of a free particle moving in the Riemannian manifold $\mathcal{L}_M$.) By Lemma 2.3 below, the action is finite and bounded by a constant depending only on $E(u_0)$. Then (2.2) shows that the path $t \mapsto u(t)$ is uniformly continuous. □

The rest of this section is devoted to proving that the action (2.3) is finite. To that end, we review some basic facts about the heat flow. By Proposition 2.1 the flow defines a smooth map $u(\theta, t) : S^1 \times (0, \infty) \to M$ and vector fields $T = u_*(\frac{\partial}{\partial\theta})$ and $\dot{u} = u_*(\frac{\partial}{\partial t})$. To a considerable extent, the behavior of the flow is controlled by the evolution of two functions: the energy density $e = \frac{1}{2}|du|$ and $|\dot{u}|^2$. These satisfy the following equations.

Lemma 2.2. *Suppose that $u(t)$ is a solution of the geodesic heat flow (2.1). Then for $t > 0$*

a) $(\partial_t + \Delta)e = -|\dot{u}|^2$, and

b) $\partial_t |\dot{u}|^2 \;=\; -2\langle \dot{u}, \nabla^*\nabla\dot{u} + R(T, \dot{u})T\rangle$

where Δ and $\nabla^*\nabla$ are the Laplacians on functions and sections of u^*TM respectively.

Proof. Noting that $[T, \dot{u}] = u_*[\frac{\partial}{\partial\theta}, \frac{\partial}{\partial t}] = 0$, we have $\nabla_T \dot{u} = \nabla_{\dot{u}} T$. Then $e = \frac{1}{2}|T|^2$ satisfies

$$\begin{aligned}(\partial_t + \Delta)e &= \langle T, \nabla_{\dot{u}} T\rangle + d^*\langle T, \nabla_T T\rangle \\ &= \langle T, \nabla_T \dot{u}\rangle - \big(\langle T, \nabla_T\nabla_T T\rangle + |\nabla_T T|^2\big)\end{aligned}$$

Using (2.1), this reduces to a). Next, noting that $\partial_t|\dot{u}|^2 = 2\langle \dot{u}, \nabla_{\dot{u}}\dot{u}\rangle$ where, again using (2.1),

$$\nabla_{\dot{u}}\dot{u} \;=\; \nabla_{\dot{u}}\nabla_T T \;=\; \nabla_T\nabla_{\dot{u}} T + R(\dot{u}, T)T$$

with $\nabla_T\nabla_{\dot{u}} T = \nabla_T\nabla_T\dot{u} = -\nabla^*\nabla\dot{u}$. This gives b). □

Integrating the equation of Lemma 2.2a) immediately shows that the function $E(t) = E(u(t))$ satisfies

$$E'(t) \;=\; -\int_{S^1} |\dot u|^2 \;=\; -\|\dot u\|^2 \tag{2.4}$$

Thus $E(t)$ is non-increasing and

$$\int_s^t \|\dot u\|^2 \;=\; E(s) - E(t). \tag{2.5}$$

Differentiating again and using Lemma 2.2b) then gives

$$E''(t) \;=\; 2\int_{S^1} |\nabla \dot u|^2 + \langle \dot u, R(T,\dot u)T\rangle \tag{2.6}$$

These formulas, in turn, can be used to bound the action.

Lemma 2.3. *There is a constant C depending only on $E_0 = E(u(0))$ such that, whenever $u(t)$ is a solution of the geodesic heat flow (2.1) and $0 \leqslant s < s+\sigma < t$, the action satisfies*

$$A(s+\sigma,t) \;\leqslant\; \left(C + \frac{1}{\sigma}\right)\Big(E(s) - E(t)\Big). \tag{2.7}$$

In particular, $A(1,\infty) \leqslant (C+2)E_0$ is finite.

Proof. Start by writing (2.6) as

$$\|\nabla \dot u\|^2 \;=\; \tfrac{1}{2}E'' + \langle \dot u, R_u(\dot u, T)T\rangle_{L^2}.$$

Since the curvature is bounded and $\|T\|^2 = 2E(t)$ is decreasing with initial value $2E(s) \leqslant E(0)$, interpolation using the compact embedding $W^{1,2} \subset C^0$ gives an inequality

$$\begin{aligned} |\langle \dot u, R_u(\dot u,T)T\rangle_{L^2}| \;&\leqslant\; c\|T\|^2\,\|\dot u\|_\infty^2 \\ &\leqslant\; cE(0)\left(\delta \|\!|\dot u|\!\|_{1,2}^2 + C(\delta)\|\dot u\|^2\right) \end{aligned}$$

for any $\delta > 0$. We can now bound $\|\!|\dot u|\!\|^2 \;=\; \|\nabla \dot u\|^2 \;+\|\dot u\|^2$ by combining the last two displayed equations, taking $\delta = (2cE_0)^{-1}$ and rearranging, and noting that $E' = -\|\dot u\|^2$ by (2.4). The result is

$$\|\!|\dot u|\!\|^2 \;\leqslant\; \tfrac{1}{2}\left(E'' - CE'\right) \tag{2.8}$$

where the constant C depends only on the geometry of M and on $E(u_0)$.

Now for fixed $t > s$, the function $A(s,t)$ is a non-increasing function of s, hence

$$A(s+\sigma,t) \;\leqslant\; \frac{1}{\sigma}\int_s^{s+\sigma} A(\rho,t)\;d\rho \;=\; \frac{1}{\sigma}\int_s^{s+\sigma}\int_\rho^t \|\!|\dot u(\tau)|\!\|^2\;d\tau\,d\rho.$$

Applying (2.8) and evaluating the integrals gives

$$\begin{aligned} A(s,t) \;&\leqslant\; \frac{1}{\sigma}\int_s^{s+\sigma} \big[E'(t) - E'(\rho)\big] - C\left[E(t) - E(\rho)\right]\;d\rho \\ &\leqslant\; E'(t) + \frac{1}{\sigma}\Big[E(s) - E(s+\sigma)\Big] - CE(t) + \frac{C}{\sigma}\int_s^{s+\sigma} E(\sigma)\;d\sigma. \end{aligned}$$

But $E'(t) \leqslant 0$ and $E(\sigma)$ is decreasing, so this inequality implies (2.7). □

Alternatively, one can use Lemma 2.2a) and the parabolic maximum principle to obtain the pointwise bound $|du| \leqslant c$. Then (2.8) follows easily, but with a constant that does not depend continuously on the initial map $u_0 \in \mathcal{L}_M$.

3 Asymptotics

The next step is to examine the behavior of the geodesic heat flow paths as $t \to \infty$. This is usually done using parabolic estimates. In this section we take a different approach, showing how information about the long-time behavior of the flow follows from the Palais-Smale condition. The key observation is the following reformulation of the Palais-Smale lemma.

Lemma 3.1(L^2 Palais-Smale). *Every sequence $\{u_k\}$ in $\mathcal{L}_M$ with $E(u_k) < C$ such that the L^2 norm of the L^2 gradient satisfies $\|\nabla E(u_k)\| \to 0$ has a subsequence that converges in the $W^{1,2}$ topology of $\mathcal{L}_M$.*

Proof. The L^2 gradient $\nabla E = \nabla_T T$ and the $W^{1,2}$ gradient (1.4) are related by $\boldsymbol{\nabla} E = (I + \Delta)^{-1} \nabla E$. Hence

$$\begin{aligned} |||\boldsymbol{\nabla} E|||^2 \;=\; \langle (I+\Delta)\boldsymbol{\nabla} E, \boldsymbol{\nabla} E\rangle_{L^2} \;&=\; \langle \nabla E, (I+\Delta)^{-1}\nabla E\rangle_{L^2} \\ &\leqslant\; \|\nabla E\|^2. \end{aligned} \tag{3.1}$$

Thus the hypothesis $\|\nabla E(u_k)\| \to 0$ implies that $|||\boldsymbol{\nabla} E(u_k)||| \to 0$. The lemma then follows from the standard P-S condition. □

Now for any initial map $u_0 \in \mathcal{L}_M$, the heat flow $u(t)$ exists for all time by Proposition 2.1, and the energy is non-increasing along the flow. Because the integral (2.5) is finite, there is a sequence $t_k \to \infty$ such that maps $u_k = u(t_k)$ satisfy $\|\dot{u}_k\| = \|\nabla E(u_k)\| \to 0$. Lemma 5.1 then implies that the flow is adherent to a critical point u_∞ of the energy.

One must next ask whether u_∞ is actually the limit of the flow as $t \to \infty$. Observe that Lemma 5.1 implies that, for each energy level $\alpha \geqslant 0$, the set $K(\alpha)$ of all critical points in $\mathcal{L}_M(\alpha) = \{u \in \mathcal{L}_M \,|\, E(u) \leqslant \alpha\}$ is compact, and hence is the disjoint union of finitely many compact connected components $K_i(\alpha)$.

Proposition 3.2. *For each $u_0 \in \mathcal{L}_M(\alpha)$, the heat flow (2.1) is asymptotic to one component K_i of the critical set of E as $t \to \infty$. In particular, there is a sequence $t_n \to \infty$ so that $u(t_n)$ converges to some $u_\infty \in K_i$.*

Proof. For each $\varepsilon > 0$ let $\mathcal{N}_\varepsilon$ denote the ε-neighborhood of $K(\alpha)$ in the $W^{1,2}$ topology of $\mathcal{L}_M$. Then there is a $\delta > 0$ such that $|||\boldsymbol{\nabla} E||| \geqslant \delta$ on $\mathcal{L}_M(\alpha)\backslash\mathcal{N}_\varepsilon$: otherwise, there would be a sequence $\{u_k\}$ in the closed set $\mathcal{L}_M(\alpha)\backslash\mathcal{N}_\varepsilon$ with $|||(\boldsymbol{\nabla} E)_{u_k}||| \to 0$; the P-S condition would then provide a subsequence converging to a critical point in $\mathcal{L}_M(\alpha)\backslash\mathcal{N}_\varepsilon$, which does not exist.

For each $T > 0$, let $R_T = \{t \geqslant T \,|\, u(t) \notin \mathcal{N}_\varepsilon\}$. Then for each $t \in R_T$ we have $|||(\boldsymbol{\nabla} E)_{u(t)}||| \geqslant \delta$, so $\|\nabla E\| \geqslant \delta$ by (3.1) and hence $|||\dot{u}||| \geqslant \|\dot{u}\| = \|\nabla E\| \geqslant \delta$. The total length of the part of the path $u(t)$, $t \geqslant T$, that lies outside $\mathcal{N}_\varepsilon$ is therefore bounded by

$$\int_{R_T} |||\dot{u}|||\; dt \;\leqslant\; \frac{1}{\delta}\int_{R_T} |||\dot{u}|||^2\; dt \;\leqslant\; \frac{1}{\delta}\int_T^\infty |||\dot{u}|||^2\; dt.$$

The last integral is finite by Lemma 2.3. Thus there is a $T = T(\varepsilon)$ such that $u(t)$ lies in one component of $\mathcal{N}_{2\varepsilon}$ for all $t \geqslant T$. The lemma follows because, for small ε, $\mathcal{N}_{2\varepsilon}$ is the disjoint union of neighborhoods $\mathcal{N}_{\varepsilon,i}$ of the components K_i of the compact set $K(\alpha)$. □

Topping's theorem shows that Proposition 3.2 is the optimal statement that holds for general Riemannian metrics on M. In his example the flow, regarded as a path in $\mathcal{L}_M$, spirals around as it converges to a circle $C \subset \mathcal{L}_M$ consisting of minimal closed geodesics. The results of the next section show that this spiraling behavior does not occur for generic metrics.

4 Convergence for generic metrics

Morse theory for the energy fuction $E : \mathcal{L}_M \to \mathbb{R}$ is complicated by the fact that E is invariant under the S^1 action defined by rotations of the domain of loops $u : S^1 \to p \in M$. As a result, all critical points are degenerate for all Riemannian metrics on M. These critical points are geodesics in (M, g), and the degeneracy occurs for two reasons:

- At a non-trivial geodesic, the tangent to the S^1 orbit is a non-zero null vector for the Hessian of E.
- The trivial loops $u : S^1 \to p \in M$ form a critical submanifold $M_0 = E^{-1}(0) \subset \mathcal{L}_M$ diffeomorphic to M; these are the absolute minima of E and are fixed by the S^1-action.

Both cases fit into the context of Morse-Bott functions. Recall that a smooth function $f : M \to \mathbb{R}$ on a manifold, a submanifold $K \subset M$ is a called a *non-degenerate critical submanifold* if each point $p \in K$ is a critical point of f and T_pK is precisely the kernel of the Hessian, that is,

$$(Hf)_p(X, Y) = 0 \ \forall Y \in T_pM \ \Leftrightarrow \ X \in T_pK.$$

A function f is called a *Morse-Bott* if its critical set is a union of non-degenerate critical submanifolds. For the energy function, the Hessian of E is non-degenerate on the normal bundle to M_0 in $\mathcal{L}_M$ (the curvature term in (1.2) vanishes). But E is not Morse-Bott for general metrics on M.

A *bumpy metric* is a Riemannian metric without degenerate periodic geodesics; this implies that the energy function E on $\mathcal{L}_M$ is a Morse-Bott function whose critical set is a disjoint union of M_0 and embedded circles. The *Bumpy Metric Theorem* asserts that on a closed manifold M, the set of bumpy metrics are generic (i.e. are a Baire set) in the space of metrics with the C^k-topology, $2 \leqslant k \leqslant \infty$. The proof was outlined by R. Abraham [Ab] and a completed by D. Anosov [An].

Our proof of the convergence of the geodesic heat flow on closed manifolds with bumpy metrics is based on a well-known equivariant Morse-Bott Lemma for the energy function on the space $\mathcal{L}_M$ (see [K] or the more general version in [GM]).

Let (M, g) be a closed Riemannian manifold with a bumpy metric. Suppose that $K_\alpha = E^{-1}(\alpha)$ is a non-degenerate critical set in $\mathcal{L}_M$. Then K_α is a submanifold isometric to a circle, and the normal bundle N to K_α is equivariantly trivial. For each $\gamma \in K_\alpha$ let $D_\gamma(\varepsilon)$ denote the ε-disk in the fiber of N at γ.

Morse-Bott Lemma. *Let K_α be a non-degenerate critical submanifold. Then there is a $D_\gamma(\varepsilon)$ as above, a neighborhood U of K_α in $\mathcal{L}_M$, and an equivariant diffeomorphism $\phi : S^1 \times D_\gamma(\varepsilon) \to U$ so that $\phi^{-1}(K_\alpha) = S^1 \times \{0\}$ and*

$$(\phi^* E)(x, \xi) \ = \ \alpha + \|P\xi\|_{1,2}^2 - \|Q\xi\|_{1,2}^2 \tag{4.1}$$

where $P : N \to N$ and $Q = I - P$ are smooth equivariant orthonormal bundle projections.

Lemma 4.1. *Each non-degenerate critical set K_α has a neighborhood in which the L^2 norm of the L^2 gradient satisfies $E \leqslant \alpha + k\|\nabla E\|^2$ for some $k > 0$.*

Proof. By the Morse-Bott Lemma, we can identify a neighborhood U of K_α with $S^1 \times D_\gamma(\varepsilon)$. Then U has two $W^{1,2}$ metrics: the product metric g_0 (whose norm appears in (4.1)) and the ambient metric g of $\mathcal{L}_M$. Fixing $x \in K_\alpha$ and sections ξ, η of the fiber N_x, we can differentiate (4.1) to obtain

$$(dE)_{(x,\xi)}\eta \ = \ 2\langle P_x\xi, P_x\eta\rangle - 2\langle Q_x\xi, Q_x\eta\rangle \ = \ 2\langle P_x\xi - Q_x\xi, \eta\rangle$$

after noting that $P^*P = P$. Consequently, the g_0-norm of the 1-form dE satisfies

$$\tfrac{1}{4}\,\|dE_{(x,\xi)}\|_{-1,2}^2 \ = \ \|P_x\xi - Q_x\xi\|_{1,2}^2 \ = \ \|P_x\xi\|_{1,2}^2 - \|Q_x\xi\|_{1,2}^2 \ = \ E(x, \xi) - \alpha.$$

Furthermore, still using g_0-norms, $\|dE_{(x,\xi)}\|_{-1,2}$ is equal to $|\!|\!|\nabla E|\!|\!|$. But E and the metrics are smooth and K_α is compact, so is a neighborhood of K_α on which the g-gradient of E satisfies $k|\!|\!|\nabla E|\!|\!|^2 \geqslant E - \alpha$. The lemma then follows from inequality (3.1). □

Proposition 4.2. *Let (M, g) be a closed Riemannian manifold with a bumpy metric. Then for each $u_0 \in \mathcal{L}_M$, the heat flow (0.2) converges in $W^{1,2}$ as $t \to \infty$ to a smooth geodesic $\gamma \in \mathcal{L}_M$ that is homotopic to $u(0)$.*

Proof. By Lemma 3.2 the flow $u(t)$ is asymptotic to a component $K_i(\alpha)$ of a critical set $E^{-1}(\alpha)$ for some α. Thus for all sufficiently large t, $u(t)$ lies in a neighborhood of $K_i(\alpha)$ in which Lemma 4.1 applies. Noting that $|||\dot{u}||| \geqslant \|\dot{u}\|$ and applying (2.8)

$$|||\dot{u}||| \;\leqslant\; \frac{|||\dot{u}|||^2}{\|\dot{u}\|} \;\leqslant\; \frac{1}{2}\left(\frac{E''}{\|\dot{u}\|} - \frac{CE'}{\|\dot{u}\|}\right). \tag{4.2}$$

Rewriting the denominator of the E'' term using (2.4), and bounding the denominator $\|\dot{u}\| = \|\nabla E\|$ of the E' term using the inequality of Lemma 4.1, this becomes

$$|||\dot{u}||| \;\leqslant\; \frac{-|E'|'}{2\sqrt{|E'|}} - \frac{CE'}{\sqrt{(E-\alpha)}}.$$

Integrating and again noting that $|E'| = \|\dot{u}\|^2 \geqslant 0$, we see that the length of the path $u(t)$ in the $W^{1,2}$ Riemannian metric satisfies

$$\text{Length}(t, s) \;=\; \int_t^s |||\dot{u}(\tau)|||\, d\tau \;\leqslant\; \|\dot{u}(t)\| + 2C\left(\sqrt{(E(t)-\alpha)} - \sqrt{(E(s)-\alpha)}\right). \tag{4.3}$$

Thus $\text{Length}(t, \infty)$ is finite for each t. But if the path $u(t)$ were adherent to two different points in $\mathcal{L}_M$, this length would be infinite.

We conclude that $u(0)$ converges in $\mathcal{L}_M$ to a critical point u_∞ of E. This is an $W^{1,2}$ weak solution to the geodesic equation $\nabla_T T = 0$, so by standard elliptic theory is a smooth closed geodesic. Finally, observe that each $u(t)$ is homotopic to $u(0)$ because the heat flow is continuous for $t \geqslant 0$, and $u(t)$ is homotopic to γ because convergence in $W^{1,2}$ implies convergence in C^0. □

5 C^∞ convergence

The convergence statements in both Proposition 3.2 and Proposition 4.2 refer to the $W^{1,2}$ topology of $\mathcal{L}_M$. We can now use a bootstrap argument to show that, in both cases, the convergence is actually C^∞ convergence. The proof is largely a matter of extending of Lemma 2.2 to higher derivatives and applying the parabolic maximum principle. Accordingly, we begin by deriving some pointwise estimates on the derivatives of $\dot{u}$. We then show that certain combinations of these derivatives — the functions f_k of Lemma 5.4 and their integrals F_k — are monotone under the geodesic heat flow. With these functions, it is straightforward to show that $W^{1,2}$ convergence implies C^∞ convergence. The result is the following theorem.

Theorem 5.1. *Let $u(t)$ be a geodesic heat flow on a closed Riemannian manifold (M, g). Then as $t \to \infty$,*

(a) $\dot{u}(t) \to 0$ in C^∞ and $u(t)$ is asymptotic in C^∞ to a component of the critical set of E.

(b) If g is a bumpy metric then $u(t)$ converges in C^∞ to a closed geodesic u_∞ homotopic to $u(0)$.

All of the lemmas of this section hold for arbitrary Riemannian metrics. The assumption about bumpy metrics appears only in the last few lines of the section when we invoke Proposition 4.2.

Lemma 5.2. *If $\nabla_T T = \dot{u}$, then for each $k \geqslant 0$ we have*

$$\nabla_T^{k+2}\dot{u} = \nabla_{\dot{u}}\nabla_T^k\dot{u} + \mathcal{R}_k$$

where $\mathcal{R}_k$ is a tensor that is a specific linear combination of terms of the form

$$\text{either} \quad R(T,\nabla_T^k\dot{u})T \quad \text{or} \quad (\nabla^\ell R)(X_1, X_2, \dots X_{\ell+2})X_{\ell+3} \tag{5.1}$$

for $\ell \leqslant k$ where each X_i is one of the vectors $T, \dot{u}, \nabla_T\dot{u}, \dots, \nabla_T^{k-1}\dot{u}$.

Proof. This holds for $k = 0$ because $\nabla_T^2\dot{u} - \nabla_{\dot{u}}\dot{u}$ is, as in Lemma 2.2, equal to $\nabla_T\nabla_{\dot{u}}T - \nabla_{\dot{u}}\nabla_T T = R(T,\dot{u})T$. Assuming inductively that (5.1) holds for k, we have

$$\begin{aligned}
\nabla_T(\nabla_T^{k+2}\dot{u}) &= \nabla_T\left(\nabla_{\dot{u}}\nabla_T^k\dot{u} + \mathcal{R}_k\right) \\
&= \nabla_{\dot{u}}\nabla_T\nabla_T^k\dot{u} + R(T,\dot{u})\nabla_T^k\dot{u} + \nabla_T\mathcal{R}_k.
\end{aligned}$$

Write this as $\nabla_T^{k+3}\dot{u} = \nabla_{\dot{u}}\nabla_T^{k+1}\dot{u} + \mathcal{R}_{k+1}$. One then sees, after repeatedly applying the product rule and using the equation $\dot{u} = \nabla_T T$ when appropriate, that $\mathcal{R}_{k+1}$ is a sum of terms of the form (5.1) where each X_i is one of the vectors $T, \dot{u}, \nabla_T\dot{u}, \dots, \nabla_T^k\dot{u}$ and the term $R(T,\nabla_T^{k+1}\dot{u})T$ occur only when the product rule is applied to a term $\nabla_T^{k+1}\big(R(T,\dot{u})T\big)$. □

Now let $H = \frac{1}{2}(\partial_t + \Delta)$ be (half) the heat operator and let $e = \frac{1}{2}|T|^2$ be the energy density. For $k \geqslant 0$, define functions g_k on $S^1 \times [0,\infty)$ by

$$g_k \;=\; \tfrac{1}{2}\,|\nabla_T^k\dot{u}|^2$$

and define functions $e_0(t)$ and $h_k(t)$ by

$$e_0(t) \;=\; \sup_{S^1\times\{t\}} e \qquad \text{and} \qquad h_k(t) \;=\; \sup_{S^1\times\{t\}}\Big(e + g_0 + \cdots + g_k\Big).$$

Lemma 5.3. *There are constants c_0 depending only on e_0 such that*

$$He = -\tfrac{1}{2}|\dot{u}|^2 = -g_0 \qquad \text{and} \qquad Hg_0 \;\leqslant\; -g_1 + c_0\,g_0. \tag{5.2}$$

For each $k \geqslant 1$ constants $c_{k\ell}$ depending only on e_0 and h_{k-1} such that

$$Hg_k \;\leqslant\; -g_{k+1} + \sum_{\ell\leqslant k} c_{k\ell}\,g_\ell. \tag{5.3}$$

Proof. The equation for He holds by Lemma 2.2. To find Hg_k for $k \geqslant 0$, differentiate and use Lemma 5.2:

$$\begin{aligned}
2Hg_k \;=\; H\left(|\nabla_T^k\dot{u}|^2\right) &= \langle\nabla_{\dot{u}}\nabla_T^k\dot{u}, \nabla_T^k\dot{u}\rangle - T\cdot\langle\nabla_T^{k+1}\dot{u}, \nabla_T^k\dot{u}\rangle \\
&= \langle\nabla_{\dot{u}}\nabla_T^k\dot{u} - \nabla_T^{k+2}\dot{u}, \nabla_T^k\dot{u}\rangle - |\nabla_T^{k+1}\dot{u}|^2 \\
&= -|\nabla_T^{k+1}\dot{u}|^2 - \langle\nabla_T^k\dot{u}, \mathcal{R}_k\rangle.
\end{aligned} \tag{5.4}$$

For $k = 0$ we have $\mathcal{R}_0 = R(T,\dot{u})T$, so the last term in (5.4) is bounded by $|\langle\dot{u},\mathcal{R}_0\rangle| \leqslant 2c_0 g_0$ where $c_0 = 2e\|R\|_\infty$. For $k \geqslant 1$, $\mathcal{R}_k$ is a sum of terms of the form (5.1). For terms of the first type we have

$$|R(T,\nabla_T^k\dot{u})T|^2 \leqslant\; \|R\|_\infty^2\,|T|^4\,|\nabla_T^k\dot{u}|^2 \;=\; 2c_0^2\,g_k.$$

For the remaining terms in (5.1), note that each vector X_i is either equal to T or satisfies $|X_i| \leqslant \sqrt{2g_\ell}$ for some $\ell < k$. Furthermore, the tensor $\nabla^\ell R$ is bounded and is skew in two of its entries, so not all of the X_i are equal to T. Thus, altogether,

$$|\mathcal{R}_k|^2 \;\leqslant\; 2c_0^2\,g_k \;+\; 2\sum_{\ell\leqslant k-1} c_{k\ell}\,g_\ell.$$

for some constants $c_{k\ell}$ depending only on e_0 and h_{k-1}. Then (5.3) follows from (5.4) after noting that the last term in (5.4) is bounded by $2g_k + |\mathcal{R}_k|^2$ and renaming the constants. □

Lemma 5.4. *If $u(0) \in C^\infty$ then for each $k \geqslant 0$ there are constants $a_{k\ell}$, b_k, and C_k depending only on $e_0(0)$ and $h_k(0)$ such that the functions*

$$f_k \;=\; g_k \;+\; \sum_{\ell=0}^{k-1} a_{k\ell} g_\ell \;+\; b_k e \tag{5.5}$$

satisfy $\sup |f_k| \leqslant C_k$ *and*

$$H f_k \leqslant -g_{k+1}. \tag{5.6}$$

Proof. By Lemma 5.3 the energy density e satisfies $He \leqslant 0$. The parabolic maximum principle shows that $e \leqslant \sup_{S^1 \times \{0\}} |e| = e_0(0)$. Hence e and c_0 are uniformly bounded in spacetime. Now let

$$f_0 \;=\; g_0 \;+\; c_0\, e.$$

Using (5.2), we have

$$H f_0 \;=\; H g_0 \;+\; c_0\, He \;\leqslant\; -g_1 \;\leqslant\; 0. \tag{5.7}$$

Hence the maximum principle implies that f_0 is pointwise bounded by the constant

$$C_0 \;=\; \sup_{S^1} f_0(\cdot, 0) \;\leqslant\; h_0(0) + c_0\, e_0(0).$$

Now suppose inductively that Lemma 5.4 holds for $\ell = 0, 1, \ldots, k-1$ and let

$$f_k \;=\; g_k \;+\; \sum_{1 \leqslant \ell \leqslant k} c_{k\ell} f_{\ell-1} \;+\; c_{k0}\, e. \tag{5.8}$$

Using (5.2)-(5.3) and (5.6), we have

$$H f_k \;\leqslant\; -g_{k+1}.$$

Moreover, substituting for the $f_{\ell-1}$ ($1 \leqslant \ell \leqslant k$), one can find the coefficients $a_{k\ell}$ and b_k in the form (5.5) such that $a_{k\ell}$ and b_k depend on $e_0(0)$ and $h_{k-1}(0)$. The maximum principle now implies that $\sup |f_k| \leqslant C_k$ where C_k depends on $e_0(0)$ and $h_k(0)$. □

Notice that (5.8) can be rearranged to express g_k in terms of the f_ℓ and e:

$$g_k \;=\; f_k \;-\; \sum_{1 \leqslant \ell \leqslant k} c_{k\ell} f_{\ell-1} \;-\; c_{k0}\, e. \tag{5.9}$$

Lemma 5.5. *As $t \to \infty$, we have $\dot u(t) \to 0$ in C^k for all k.*

Proof. Let $E_\infty = \lim_{t\to\infty} E(t)$. Then the function $G_0(t) = \frac{1}{2}\|\dot u(t)\|^2 = -\frac{1}{2}E'(t)$ satisfies

$$\int_t^\infty 2G_0(s)\; ds \;=\; E(t) - E_\infty \;\to 0.$$

In particular, G_0 is integrable on $(0, \infty)$. We proceed inductively for $k \geqslant 0$ using the functions

$$\begin{cases} G_k(t) \;=\; \displaystyle\int_{S^1} g_k(\theta, t)\; d\theta \\[2ex] F_k(t) \;=\; \displaystyle\int_{S^1} f_k(\theta, t)\; d\theta \;-\; b_k E_\infty \end{cases}$$

where b_k is the constant in (5.5). Integrating (5.6) over the domain shows that each F_k is non-increasing:

$$F_k'(t) \;=\; \int_{S^1} \tfrac{\partial}{\partial t} f_k \;=\; \int_{S^1} H f_k \;\leqslant\; -\int_{S^1} g_{k+1} \;=\; -G_{k+1}(t) \;\leqslant\; 0. \tag{5.10}$$

Integrating (5.10) gives

$$\int_t^T G_{k+1}(s)\ ds \ \leqslant\ -\int_t^T F_k'(s)\ ds \ =\ F_k(t) - F_k(T). \tag{5.11}$$

Our induction step will show that if G_ℓ is integrable on $(0,\infty)$ for all $0 \leqslant \ell \leqslant k$, then

(i) $G_\ell(t) \to 0$ and $F_\ell(t) \to 0$ as $t \to \infty$ for all $\ell \leqslant k$, and

(ii) G_ℓ is integrable for all $\ell \leqslant k+1$.

Under this hypothesis, we can choose a sequence $\{t_n\} \to \infty$ so that $G_\ell(t_n) \to 0$ for all $\ell \leqslant k$. Integrating formula (5.5) over S^1 and using the definition of F_k then shows that

$$F_\ell \ =\ G_\ell \ +\ \sum_{m=0}^{k-1} a_{km}\, G_m \ +\ b_\ell(E - E_\infty) \tag{5.12}$$

satisfies $F_\ell(t_n) \to 0$ for all $\ell \leqslant k$. But each F_ℓ is smooth and non-increasing, so we conclude that $F_\ell(t) \to 0$ for $\ell \leqslant k$. Then (5.11) implies that G_ℓ is integrable for all $\ell \leqslant k+1$. Finally, integrating (5.9) over S^1 shows that we also have $G_\ell(t) \to 0$ for all $0 \leqslant \ell \leqslant k+1$. This completes the induction.

We now have $\|\nabla_T^k \dot{u}_t\|^2 = 2G_k(t) \to 0$ as $t \to \infty$ for all k. Consequently, $\dot{u} \to 0$ in the Sobolev $W^{k,2}$ norm for all k and therefore, applying the Sobolev embedding $W^{k,2} \subset C^{k-1}$, in C^k for all k. □

To complete our analysis we need the following estimate in the Hölder spaces $C^{k,\alpha}$.

Lemma 5.6. *For each $k \geqslant 0$ there is a constant c_k such that whenever $\|u(t) - \gamma\|_{0,\alpha} < 1$ for some geodesic γ, we have*

$$\|u(t) - \gamma\|_{k+2,\alpha} \ \leqslant\ c_k \left(\|\dot{u}(t)\|_{k,\alpha} \ +\ \|u(t) - \gamma\|_{0,\alpha} \right). \tag{5.13}$$

Proof. Using the isometric embedding $M \subset \mathbb{R}^r$, we can regard $u(t)$ as an $\mathbb{R}^r$-valued function and write the heat equation as $\dot{u} = \Delta u + A(du, du)$ where A is the second fundamental form. Then the Schauder estimate for the Laplacian $\Delta = -\partial_\theta^2$ on the circle give a bound

$$\|u(t) - \gamma\|_{k+2,\alpha} \quad \leqslant \quad c_1 \left(\|\Delta u(t) - \Delta\gamma\|_{k,\alpha} \ +\ \|u(t) - \gamma\|_{0,\alpha} \right).$$

Substituting the heat equation and $\delta\gamma = 0$, this becomes

$$\|u(t) - \gamma\|_{k+2,\alpha} \ \leqslant\ c_1 \left(\|\dot{u}(t)\|_{k,\alpha} \ +\ \|\Phi(du(t), d\gamma)\|_{k,\alpha} \ +\ \|u(t) - \gamma\|_{0,\alpha} \right) \tag{5.14}$$

where $\Phi(du, dv) \ =\ A_u(du, du) - A_v(dv, dv)$. By expanding

$$\Phi(du, dv) \ =\ \big(A_u - A_v\big)(du, du) + A_v(du - dv, du) + A_v(dv, du - dv)$$

and noting that $\|du\|_{k,\alpha} \leqslant \|d\gamma\|_{k,\alpha} + \|du - d\gamma\|_{k,\alpha}$, we see that

$$\begin{aligned} \|\Phi(du, d\gamma)\|_{k,\alpha} \quad &\leqslant \quad c_2\, \|u - \gamma\|_{k,\alpha} \cdot \|du\|_{k,\alpha}^2 \ +\ c_3\, \|du - d\gamma\|_{k,\alpha} \left(\|du\|_{k,\alpha} + \|d\gamma\|_{k,\alpha} \right) \\ &\leqslant \quad c_4\, \|u - \gamma\|_{k+1,\alpha} \cdot \big(1 + \|u - \gamma\|_{k+1,\alpha}^2\big) \end{aligned}$$

where the constants depend on $\|\gamma\|_{k+1,\alpha}$ and on the second fundamental form. Furthermore, there is an interpolation inequality

$$\|u - \gamma\|_{k+1,\alpha} \ \leqslant\ \varepsilon\, \|u - \gamma\|_{k+2,\alpha} \ +\ c_5(\varepsilon)\, \|u - \gamma\|_{0,\alpha}$$

for each $\varepsilon > 0$. Substituting the last two displayed equations into (5.14) and taking $\varepsilon = (4c_1c_4)^{-1}$ shows that whenever $\|u(t) - \gamma\|_{k+1,\alpha} < c_6$ we have

$$\|u(t) - \gamma\|_{k+2,\alpha} \;\leqslant\; c_7 \Big(\|\dot{u}(t)\|_{k,\alpha} \;+\; \|u(t) - \gamma\|_{0,\alpha} \Big).$$

The lemma follows by induction. □

Using the Sobolev embedding $W^{1,2} \subset C^{0,\alpha}$ and the inclusions $C^{k+1} \subset C^{k,\alpha} \subset C^k$, we can recast Lemma 5.6 in terms of the $W^{1,2}$ distance in $\mathcal{L}_M$ as follows.

Corollary 5.7. *For $k \geqslant 0$ there are constants c'_k such that if dist$(u(t),\gamma) < 1$ for some geodesic γ then*

$$\|u(t) - \gamma\|_{C^{k+1}} \;\leqslant\; c'_k \Big(\|\dot{u}(t)\|_{C^k} \;+\; \mathit{dist}\,(u(t),\gamma) \Big). \qquad \square \tag{5.15}$$

Theorem 5.1 follows readily. Given $\varepsilon > 0$ we know from Lemma 5.5 that there is an T_1 so that $\|\dot{u}\|_{C^k} < \varepsilon/2c'_k$ for all $t > T_1$. For a general metric, we also know from Proposition 3.2 that there is a T_2 such that, for each $t > T_2$, there is a geodesic $\gamma_t \in K_i$ with $\mathrm{dist}\,(u(t),\gamma_t) < \varepsilon/2c'_k$. Then for any $t \geqslant T_k = \max\{T_1, T_2\}$

$$\mathrm{dist}_{\,C^{k+1}}(u(t),\gamma_t) \;\leqslant\; \|u(t) - \gamma_t\|_{k+1} \;\leqslant\; c'_k \left(\frac{\varepsilon}{2c'_k} + \frac{\varepsilon}{2c'_k} \right) \;=\; \varepsilon.$$

Thus $u(t)$ is asymptotic to K_i in C^k. For a bumpy metric, we can take $\gamma_t = \gamma$ for all t, where to γ is the geodesic that is the limit in Proposition 4.2.

6 Exponential convergence

This last section proves a statement about the *rate* of convergence in the case when the limit is a non-trivial stable geodesic. Specifically, if a geodesic heat flow approaches a non-trivial stable critical point of E as $t \to \infty$, then the convergence occurs at an exponential rate as $t \to \infty$ in the every C^k norm. (Convergence to a trivial geodesic is addressed in the next section.)

We will say that a critical S^1-orbit $K \subset \mathcal{L}_M$ is *non-trivial and stable* if $E > 0$ on K and the Hessian of the energy function is positive definite on normal bundle to K (using the Riemannian metric on $\mathcal{L}_M$). As in (1.2), the Hessian of E is given by the bilinear form

$$B_u(X,X) \;=\; \int_{S^1} |\nabla_T X|^2 + \langle X, R_u(T,X)T\rangle \tag{6.1}$$

when u is a geodesic. Thus a critical orbit K is stable if there is a $\lambda > 0$ such that, for each $u \in K$, the second variation of the energy E satisfies

$$B_u(X,X) \geqslant 2\lambda \|X\|^2 \tag{6.2}$$

for all $X \in u^*(TN)$ that are $W^{1,2}$ orthogonal to the tangent vector field $T = u^*\left(\frac{\partial}{\partial\theta}\right)$. Because the energy function and the Riemannian metric on $\mathcal{L}_M$ are both S^1-invariant, the constant λ is independent of $u \in K$, and if (6.2) holds for one $u \in K$ then it holds for all $u \in K$. The Morse-Bott Lemma then implies that K is an isolated component of the critical set of E.

Lemma 6.1. *B is a smooth 2-tensor on $\mathcal{L}_M$ that is a bounded symmetric bilinear form on $\mathcal{L}_M$.*

Proof. As in Section 3, the isometric embedding $M \hookrightarrow \mathbb{R}^N$ induces a smooth embedding $\mathcal{L}_M$ into the Hilbert space $W^{1,2}(S^1, \mathbb{R}^N)$. The first term of B is the restriction to $\mathcal{L}_M$ of the (constant) L^2 inner product $\langle dX, dX \rangle$; hence this term depends smoothly on $u \in \mathcal{L}_M$.

For the second term of B, note that the curvature is a smooth tensor on M, so induces a smooth type $(3,1)$ tensor $\mathcal{R}$ on $\mathcal{L}_M$, that is, a smooth bundle map $T\mathcal{L}_M \otimes T\mathcal{L}_M \otimes T\mathcal{L}_M \otimes \to T\mathcal{L}_M$. Let $\hat{T}\mathcal{L}_M$ be the vector bundle over $\mathcal{L}_M$ obtained by completing the tangent bundle $T\mathcal{L}_M$ in the L^2 norm. Noting that the curvature is bounded and there is a Sobolev inequality $\|X\|_\infty \leqslant c|\!|\!|X|\!|\!|$, we see that

$$|\langle R_u(X,Y)X, Y\rangle_{L^2}| \;\leqslant\; \|R\|_\infty \, \|Y\|^2 \, \|X\|_\infty^2 \;\leqslant\; c\, \|Y\|^2 \, |\!|\!|X|\!|\!|^2 \qquad \text{for all } X \in T\mathcal{L}_M, Y \in \hat{T}\mathcal{L}_M.$$

Thus $\mathcal{R}$ extends to a smooth bundle map $\hat{T}\mathcal{L}_M \otimes \hat{T}\mathcal{L}_M \otimes T\mathcal{L}_M \otimes \to T\mathcal{L}_M$. Because $u \mapsto T = \partial_\theta u$ is a smooth section of $\hat{T}\mathcal{L}_M$, we conclude that the curvature term in (6.1) is a smooth 2-tensor on $\mathcal{L}_M$.

Finally, to see that B is bounded, we note that the curvature is bounded, that the L^2 norm of $T = du$ is twice the energy of u, and that there is a Sobolev inequality $\|X\|_\infty \leqslant c|\!|\!|X|\!|\!|$:

$$\begin{aligned} |B_u(X,X)| &\leqslant \|\nabla_T X\|^2 + \|R\|_\infty \, \|du\|^2 \, \|X\|_\infty^2 \\ &\leqslant (1 + c\, E(u)) \, |\!|\!|X|\!|\!|^2. \end{aligned}$$

□

Lemma 6.2. *Fix a non-trivial stable orbit K. Then there is a neighborhood $\mathcal{N}$ of K in $\mathcal{L}_M$ such that at each $u \in \mathcal{N}$ the tangent $\dot{u} = \nabla_T T$ to the heat flow satisfies*

$$B_u(\dot{u}, \dot{u}) \geqslant \lambda |\!|\!|\dot{u}|\!|\!|^2. \tag{6.3}$$

Proof. Over the set of non-trivial maps, the tangent vector field $u \mapsto T = du$ is a non-zero section of the bundle $\hat{T}\mathcal{L}_M$. The span of this section defines a real line bundle $N \subset \hat{T}\mathcal{L}_M$. By the stability hypothesis, this bundle N is exactly the null space of B_u for each $u \in K$. Lemma 6.1 then implies that there is a neighborhood $\mathcal{N}$ of K in which (6.2) holds – with 2λ replaced by λ – for all $X \in \hat{T}_u\mathcal{L}_M$ that are $W^{1,2}$ orthogonal to N along $u \in \mathcal{N}$. To complete the proof, we observe that $\dot{u}$ is such a vector field, as follows.

Because the energy E is invariant under the S^1-action, its L^2 gradient $\nabla_T T$ is L^2 orthogonal to the orbits. In fact, integration by parts shows that $\dot{u}$ is also $W^{1,2}$ orthogonal to the tangent vectors T along the heat flow for $t > 0$:

$$\begin{aligned} \langle \dot{u}, T\rangle_{1,2} \;=\; \int_{S^1} \langle \nabla_T \dot{u}, \nabla_T T\rangle \;+\; \langle \dot{u}, T\rangle \;&=\; \int_{S^1} \langle \nabla_T \dot{u}, \dot{u}\rangle \;+\; \langle \nabla_T T, T\rangle \\ &=\; \frac{1}{2}\int_{S^1} \frac{\partial}{\partial\theta} \left(|\dot{u}|^2 + |T|^2\right) \;=\; 0. \end{aligned}$$

□

Theorem 6.3. *Suppose that a geodesic heat flow $u(t)$ on a closed Riemannian manifold is adherent to a non-trivial stable S^1-orbit K as $t \to \infty$. Then $u(t) \to \gamma$ in C^∞ for some geodesic γ in K, and there are constants C_k, T_k and λ such that*

$$\|u(t) - \gamma\|_{C^k} \;\leqslant\; C_k \, e^{-\lambda t} \tag{6.4}$$

for all $k \geqslant 0$, and $t \geqslant T_k$.

Proof. By Proposition 3.2, there is a T such that $u(t)$ lies in the neighborhood $\mathcal{N}$ of Lemma 6.2 for all $t \geqslant T$ and satisfies $\lim_{t\to\infty} E(t) = E(\gamma_0)$. Using (2.6), (6.2) and (2.4) and the fact that $\|\dot{u}\| \leqslant |\!|\!|\dot{u}|\!|\!|$, we then have

$$E''(t) \;=\; 2B_{u_t}(\dot{u}_t, \dot{u}_t) \;\geqslant\; 2\lambda \|\dot{u}_t\|^2 \;=\; -2\lambda E'(t),$$

Integrating over $[t, \infty)$ with $t \geqslant T$ gives

$$-E'(t) \;=\; \|\dot{u}_t\|^2 \;\leqslant\; c_1 e^{-2\lambda t} \tag{6.5}$$

where $c_1 = \|\dot{u}(T)\|^2$. Integrating again,

$$E(t) - E(\gamma_0) \;\leqslant\; c_2\, e^{-2\lambda t} \tag{6.6}$$

where $c_2 = c_1 \lambda^{-1}$. We can also use (6.5) to rewrite equation (4.2) as

$$2|\!|\!|\dot{u}|\!|\!| \;\leqslant\; -\Phi^{-\frac{1}{2}}\,\Phi' + c_3\, e^{-\lambda t}$$

where $\Phi(t) = -E'(t) = \|\dot{u}\|^2 \geqslant 0$. Integrating as in (4.3) and using (6.5) then shows that the $W^{1,2}$ distance from $u(t)$ to $u(s)$ for $s > t$ satisfies

$$\mathrm{Length}(t,s) \;=\; \int_t^s |\!|\!|\dot{u}(\tau)|\!|\!|\; d\tau \;\leqslant\; c_4\, e^{-\lambda t}.$$

This bound, together with the fact that $\mathcal{L}_M$ is complete with respect to the $W^{1,2}$ metric, implies that $u(t)$ has a limit γ as $t \to \infty$ and that

$$\mathrm{dist}(u(t), \gamma) \;\leqslant\; c_4\, e^{-\lambda t}. \tag{6.7}$$

The remainder of the proof adapts the bootstrap argument from the previous section to show that the functions $G_k(t) = \frac{1}{2}\,\|\nabla_T^k \dot{u}(t)\|^2$ decay exponentially. To that end, we inductively suppose that

$$\begin{cases} (a) \;\; \int_t^\infty G_\ell(s)\, ds \;\leqslant\; Ce^{-2\lambda t} & \text{for all } 0 \leqslant \ell \leqslant k, \text{ and} \\ (b) \;\; G_\ell(t) \;\leqslant\; Ce^{-2\lambda t} & \text{for all } 0 \leqslant \ell \leqslant k-1. \end{cases} \tag{6.8}$$

For $\ell = 0$, (a) holds by integrating (6.5), and (b) is vacuously true. Integrating (5.12) using (6.8a) and (6.6), we then have

$$\int_t^\infty F_\ell(s)\, ds \;\leqslant\; Ce^{-2\lambda t} \quad \forall \ell \leqslant k.$$

Thus $F_\ell(s)$ is smooth, integrable, and non-increasing by (5.10). It follows that $F_\ell(s) \geqslant 0$ and $F_\ell(s) \to 0$ as $s \to \infty$. Hence for each $\ell \leqslant k$ we have

$$F_\ell(t) \;\leqslant\; \int_{t-1}^t F_\ell(s)\, ds \;\leqslant\; \int_{t-1}^\infty F_\ell(s)\, ds \;\leqslant\; Ce^{-2\lambda(t-1)} \;=\; Ce^{-2\lambda t}. \tag{6.9}$$

Putting this bound into (5.11) then gives

$$\int_t^\infty G_{\ell+1}(s)\, ds \;\leqslant\; F_\ell(t) - F_\ell(\infty) \;\leqslant\; F_\ell(t) \;\leqslant\; Ce^{-2\lambda t} \quad \forall t.$$

Also, putting (6.9) into (5.12) and using (6.6) and the induction hypothesis, we see that $G_\ell(t) \leqslant Ce^{-2\lambda t}$ for all $\ell \leqslant k$. This completes the induction.

Finally, Lemma 5.6 and Sobolev embeddings $W^{k+1,2} \hookrightarrow C^{k,\alpha}$ and $W^{1,2} \hookrightarrow C^{0,\alpha}$ give

$$\|u(t) - u_\infty\|_{k+2,\alpha} \;\leqslant\; c_k \left(\|\dot{u}\|_{k+1,2} \;+\; \mathrm{dist}(u_t, \gamma_\infty) \right). \tag{6.10}$$

But, by the definition of G_ℓ and (6.8),

$$\|\dot{u}(t)\|_{k,2}^2 \;=\; 2 \sum_{\ell \leqslant k} G_\ell(t) \;\leqslant\; Ce^{-2\lambda t}.$$

Putting this inequality and (6.7) into (6.10) gives (6.4). $\square$

7 Convergence to trivial geodesics

The analysis in Section 8 did not address the case of heat flow paths whose energy satisfies $E(t) \to 0$. Here, for completeness, we show that any such heat flow converges in C^∞ at an exponential rate to a trivial geodesic, that is, a map to a single point. The theorem requires no genericity assumption about the Riemannian metric.

Theorem 7.1. *There are constants ε_0, c_k and T_k depending only on the geometry of (M,g) such that for every heat flow $u(t)$ with $E(t) \to 0$, there is a trivial geodesic $\gamma_p : S^1 \to p \in M$*

$$\|u(t) - \gamma_p\|_{C^k} \;\leqslant\; c_k\, e^{-t/4\pi^2} \qquad \text{for all } t \geqslant T + T_k \text{ and all } k \geqslant 0. \tag{7.1}$$

where T is the first time $t \geqslant 1$with $E(u(T)) \leqslant \varepsilon_0$.

Proof. By a result of Ottarsson (Theorem 5A of [O]) and the compactness of M, there is a constant $c_0 > 0$ depending only (M,g) such that the inequality

$$E(u) \;\leqslant\; 2\pi^2 \int_{S^1} |\dot{u}|^2 \tag{7.2}$$

holds for every map $u : S^1 \to M$ whose image lies in some geodesic ball of radius c_0. But, by Holder's inequality, the length of the image is bounded by $\sqrt{2\pi E(u)}$. Thus (7.2) holds whenever $E(t) \leqslant \varepsilon_0 = c_0^2/2\pi$.

Now if $u(t)$ is a heat flow with $E(u) \to 0$ then, because $E(t)$ is decreasing, there is a $T \geqslant 1$ such that $E(t) \leqslant \varepsilon_0$ for all $t \geqslant T$. Then (7.2) and (2.4) give the differential inequality $E(t) \leqslant -2\pi^2 E'(t)$ so, after integrating,

$$E(t) \;\leqslant\; \varepsilon_0\, e^{-t/2\pi^2} \qquad \text{for all } t \geqslant T. \tag{7.3}$$

Substituting into (2.7) shows that the action satisfies $A(t+n, t+n+1) \leqslant (C+1)\varepsilon_0\, e^{-(t+n)/2\pi^2}$. The length of the path $u(t)$ in the $W^{1,2}$ Riemannian metric can then be bounded as in (2.2): for each $t \geqslant T$

$$\begin{aligned} \text{Length}(t,\infty) \;=\; \int_t^\infty |||\dot{u}||| &= \sum_{n=0}^{\infty} \text{Length}(t+n, t+n+1) \\ &\leqslant \sum_{n=0}^{\infty} \sqrt{A(t+n, t+n+1)} \\ &\leqslant \sqrt{(C+1)\,\varepsilon_0}\;\; e^{-t/2\pi^2} \end{aligned} \tag{7.4}$$

after summing the geometric series. Thus as $T \to \infty$, $u(t)$ converges in $\mathcal{L}_M$ to a limit $\gamma \in \mathcal{L}_M$. Then (7.3) shows that $E(\gamma) = 0$, so γ maps S^1 to a single point $p \in M$ and, by the Sobolev embedding $W^{1,2} \subset C^0$,

$$\text{dist}(u(t), p) \;\leqslant\; c\, e^{-t/2\pi^2}.$$

From (7.4) we also have that $|||\dot{u}|||$ decays exponentially. The bootstrap argument that begins at (6.8) then leads to (7.1). $\square$

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