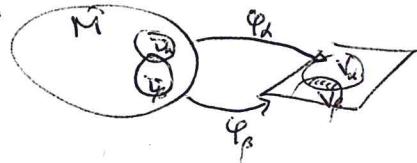


Math 868 — Midterm Exam

1. (30 points) Finish the sentence to make a precise complete definition:

(a) An ~~atlas~~^(ch) on a metric space M is a cover $\{U_\alpha\}$ of M by homeomorphisms $\phi_\alpha : U_\alpha \rightarrow V_\alpha \subset \mathbb{R}^n$ that satisfy the compatibility condition: if $U_\alpha \cap U_\beta \neq \emptyset$, then

$$\varphi_\beta \circ \varphi_\alpha^{-1} : V_\alpha \cap V_\beta \rightarrow V_\alpha \cap V_\beta \text{ is } C^k.$$



- (b) A linear map $D : C^\infty(M) \rightarrow C^\infty(M)$ is a *derivation* if

$$D(fg) = Df \cdot g + f \cdot Dg \quad \forall f, g \in C^\infty M.$$

- (c) A *Lie algebra* over \mathbb{R} is a real vector space V with a bracket operation $[\cdot, \cdot]$ that satisfies:

(i) *Bilinear*: $[ax + by, z] = a[x, z] + b[y, z]$ and $[x, ay + bz] = a[x, y] + b[x, z]$

(ii) *Skew*: $[x, y] = -[y, x]$.

(iii) *Jacobi Identity*: $[x [y, z]] + [y [z, x]] + [z [x, y]] = 0$.

$$\left. \begin{array}{l} \forall x, y, z \in V \\ a, b \in \mathbb{R} \end{array} \right]$$

- (d) Let $f : M \rightarrow N$ is a smooth map between manifolds and $p \in M$. Suppose that

$$(Df)_p : T_p M \rightarrow T_{f(p)} N$$

is injective. Then, by the Local Immersion Theorem, there are local coordinates $\{x^1, \dots, x^n\}$ around p in M and $\{y^1, \dots, y^m\}$ around $f(p)$ in N such that $f(x)$ is the canonical map

$$f(x^1, \dots, x^n) = (x^1, \dots, x^n, 0, \dots, 0)$$

- (e) The *differential* of a function $f \in C^\infty(M)$ is the 1-form df on M defined by

$$df(X) = X \cdot f \quad \text{for all vector fields } X$$

In local coordinates $\{x^i\}$, df is given by the formula

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i$$

- (f) (i) A 1-form ω on a manifold is *exact* if ... $\omega = df$ for some $f \in C^\infty M$

- (ii) A 1-form $\omega = \sum_i w_i(x) dx^i$ on \mathbb{R}^n is *closed* if ... $\frac{\partial \omega_i}{\partial x^j} = \frac{\partial \omega_j}{\partial x^i} \quad \forall i, j$.

2. (20 points) Define $f : \mathbf{R}^3 \rightarrow \mathbf{R}$ by $f(x, y, z) = x^4 + y^4 + y^2z^2 + z^2$. Consider the level set

$$M = \{(x, y, z) \mid f(\vec{x}) = 1\}.$$

Prove that M is a manifold. Name the theorem that you are using.

Proof Note that $M = f^{-1}(1)$. By the Regular Level Set Theorem it suffices to show that $(Df)_p$ is surjective for all points $p = (x, y, z) \in M$. But

$$\begin{aligned} (Df)_p &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = (4x^3, 4y^3 + 2yz^2, 2y^2z + 2z) \\ &= (4x^3, 2y(2y^2 + z^2), 2z(y^2 + 1)) \end{aligned}$$

is surjective unless all components are 0, i.e.

$$\begin{cases} 4x^3 = 0 \Rightarrow x = 0 \\ 2y(2y^2 + z^2) = 0 \Rightarrow y = 0 \\ 2z(y^2 + 1) = 0 \Rightarrow z = 0. \end{cases}$$

Since $(0, 0, 0) \notin M$, we conclude that Df_p is surjective $\forall p \in M$ □

3. (10 points) Let X and Y be the vector fields on \mathbf{R}^2 defined by $X = x^2 \frac{\partial}{\partial x}$ and $Y = \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$.

(a) Compute $[X, Y]$.

$$\begin{aligned} X Y - Y X &= x^2 \cancel{\partial_x} (2 + y \cancel{\partial_y}) - (\cancel{\partial_x} + y \cancel{\partial_y})(x^2 \cancel{\partial_x}) \\ &= x^2 \cancel{\partial_x^2} + x^2 \cancel{\partial_x \partial_y} - [(\cancel{\partial_x} \partial_x + x^2 \cancel{\partial_x^2}) + y \cancel{x^2 \partial_y \partial_x}] \\ &= \boxed{-\cancel{\partial_x} \frac{\partial}{\partial x}} \end{aligned}$$

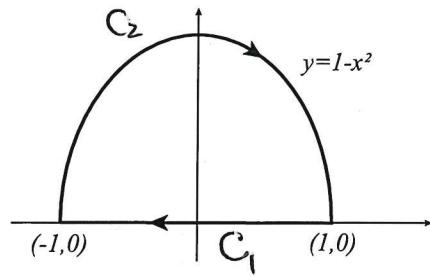
- (b) Can one find coordinates (z, w) in a neighborhood of $(0, 0)$ such that $X = \frac{\partial}{\partial z}$ and $Y = \frac{\partial}{\partial w}$? Explain your answer.

No, $[\frac{\partial}{\partial z}, \frac{\partial}{\partial w}] = 0$ but $[X, Y] \neq 0$ as above.

4. (10 points) For the 1-form $\omega = y dx$, compute the integral

$$\int_C \omega$$

where C is the closed oriented curve in \mathbb{R}^2 in the picture. (C is the union of the segment C_1 along the x -axis, and the graph C_2 of $y = 1 - x^2$).

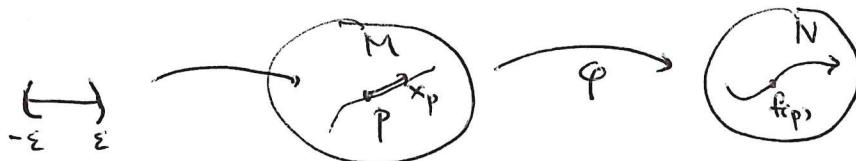


- Along C_1 , $\omega = 0$ (since $y=0$), so $\int_{C_1} \omega = 0$
- Parameterize C_2 by $\sigma: [-1, 1] \rightarrow \mathbb{R}^2$, $\sigma(t) = (t, 1-t^2)$. Then $\sigma^* \omega = (1-t^2) dt$ so.

$$\int_{C_2} \omega = \int_{-1}^1 (1-t^2) dt = \left[t - \frac{t^3}{3} \right]_{-1}^1 = 2 - \frac{2}{3} = \frac{4}{3}.$$

Altogether, $\int_C \omega = \int_{C_1} \omega + \int_{C_2} \omega = 0 + \frac{4}{3} = \boxed{\frac{4}{3}}$

5. (10 points) Let $\phi: M \rightarrow N$ be a smooth map between manifolds. Given $p \in M$ and a vector $X \in T_p M$, define the pushforward vector $\phi_* X$ in terms of paths. *Include a picture.*



Choose a path $\sigma: (-\varepsilon, \varepsilon) \rightarrow M$ with $\sigma(0) = p$ and $\left. \frac{d}{dt} \sigma(t) \right|_{t=0} = X_p$. Then $\tau = \phi \circ \sigma: (-\varepsilon, \varepsilon) \rightarrow N$ is a path in N .

$$(\phi_* X)_{f(p)} = \left. \frac{d}{dt} \tau(t) \right|_{t=0}$$

6. (10 points) Let $\Phi : \mathbf{R}^2 \rightarrow \mathbf{R}^3$ be the map $\Phi(u, v) = (u + u^3, v, v^3)$. Assume that the image $N = \Phi(\mathbf{R}^2)$ is an embedded submanifold \mathbf{R}^3 (which is true).

(a) Write down $D\Phi$ as a matrix.

$$D\Phi = \begin{pmatrix} \frac{\partial \Phi}{\partial u} & \frac{\partial \Phi}{\partial v} \\ \frac{\partial \Phi^2}{\partial u} & \frac{\partial \Phi^2}{\partial v} \\ \frac{\partial \Phi^3}{\partial u} & \frac{\partial \Phi^3}{\partial v} \end{pmatrix} = \begin{pmatrix} 1+3u^2 & 0 \\ 0 & 1 \\ 0 & 3v^2 \end{pmatrix}$$

- (b) Write down two linearly independent vectors A and B in \mathbf{R}^3 that are tangent to N at $p = (2, 1, 1)$.

Note that $p = \Phi(1, 1)$. Take

$$\left\{ \begin{array}{l} A = \Phi_{\ast}\left(\frac{\partial}{\partial x}\right) = (D\Phi)_{(1,1)}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 4 & 0 \\ 0 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} \\ B = \Phi_{\ast}\left(\frac{\partial}{\partial y}\right) = (D\Phi)_{(1,1)}\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 4 & 0 \\ 0 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \end{array} \right.$$

7. (10 points) Let M be a non-compact manifold. Given a vector $v \in T_p M$ at one point $p \in M$, how can one construct a compactly supported vector field X on M with $X(p) = v$?

Just list the key steps.

STEP 1: Choose local coordinates $\{x_i\}$ on a nbd. U of p .

Then $v = \sum_{i=1}^n a^i \frac{\partial}{\partial x^i}$ for some constants a^i .

STEP 2. For these same a^i , $\sum_{i=1}^n a^i \frac{\partial}{\partial x^i}$ is a vector field on U that is equal to v at p .

STEP 3. Choose $\epsilon > 0$ st. the ball $B(p, 2\epsilon)$ lies in U . Let $\beta(x) \in C_c^\infty(U)$ be a smooth bump function with $\begin{cases} \beta \equiv 1 & \text{on } B(p, \epsilon) \\ \beta \equiv 0 & \text{outside } B(p, 2\epsilon) \end{cases}$

Then

$$X = \begin{cases} \sum_{i=1}^n \beta(x) a^i \frac{\partial}{\partial x^i} & x \in U \\ 0 & x \in M - U \end{cases}$$

is a smooth v. field with $X(p) = v$.

