Math 309, Section 2

SOLUTION TO SUPPLEMENTAL PROBLEM 4. First, row reduce:

$$\begin{pmatrix} 3 & 1 \\ 9 & 5 \end{pmatrix} \underset{R_2-3R_1}{\approx} \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix} \underset{\frac{1}{3}R_1}{\approx} \begin{pmatrix} 1 & \frac{1}{3} \\ 0 & 2 \end{pmatrix} \underset{\frac{1}{2}R_2}{\approx} \begin{pmatrix} 1 & \frac{1}{3} \\ 0 & 1 \end{pmatrix} quad \underset{R_1-\frac{1}{3}R_2}{\approx} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \qquad (0.1)$$

For each step, write down the corresponding elementary matrix and its inverse:

$R_2 - 3R_1$ $\quad E_1 = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$ $\quad E_1^{-1} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$ $\qquad \frac{1}{2}R_2$ $\quad E_3 = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ $\quad E_3^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

$\frac{1}{3}R_1$ $\quad E_2 = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix}$ $\quad E_2^{-1} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$ $\qquad R_1 - \frac{1}{3}R_2$ $\quad E_4 = \begin{pmatrix} 1 & -\frac{1}{3} \\ 0 & 1 \end{pmatrix}$ $\quad E_4^{-1} = \begin{pmatrix} 1 & \frac{1}{3} \\ 0 & 1 \end{pmatrix}$

Then, reading (0.1) starting from the A side, gives

$$E_4E_3E_2E_1A = I$$

Multiply both sides by A^{-1} to get $A^{-1} = E_4E_3E_2E_1$. Similar, reading (0.1) backwards (so the row operations are given by the inverse matrices, one sees that

$$E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}I = A.$$

These last two formula express A and A^{-1} as products of elementary matrices.

Definition of a vector space

A *vector space* is a set V (whose elements are called vectors) endowed with

- a rule for addition that associates to each pair $\mathbf{x}, \mathbf{y} \in V$ an element $\mathbf{x} + \mathbf{y} \in V$, and
- a rule for scalar multiplication that associates to each $\mathbf{x} \in V$ and $r \in \mathbb{R}$ an element $r\mathbf{x} \in V$,

such that, for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$ and $\alpha, \beta \in \mathbb{R}$,

A1. $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$

A2. $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{y} + (\mathbf{x} + \mathbf{z})$

A3. $\exists$ a vector $\mathbf{0} \in V$ s.t. $\mathbf{x} + \mathbf{0} = \mathbf{x}$

A4. For each $\mathbf{x} \in V$, $\exists$ an "opposite vector" $-\mathbf{x} \in V$ s.t. $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$

A5. $\alpha(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}$

A6. $(\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$

A7. $\alpha(\beta\mathbf{x}) = (\alpha\beta)\mathbf{x}$

A8. $1 \cdot \mathbf{x} = \mathbf{x}$.

Notes (a) These axioms implicitly assume that the properties of the real numbers, of sets, and of the symbol $=$ (e.g. adding the same thing to both sides preserves equality) are known and will be used freely. Thus in proofs you will have occasion to use the abbreviations

$\mathbb{R}$ Prop. $\qquad$ Set Prop. $\qquad$ $=$ Prop.

for "property of the real numbers" , "property of sets" and "property of equality".

(b) When checking if a given set is a vector space, the two bulleted requirements are the most important.

From the axioms, one can derive numerous simple consequences that are useful in calculations. Each is proved from the axioms *and from previously proved facts*. Once a fact is proved, it gets added to our basket of "known facts" and can be used from then on.