Section 4.2 More on linear transformations and matrix multiplication

These notes give further examples and practice the concepts Section 4.2 in the textbook. Recall our two principles: for a linear transformation $L : V \to W$

- L is determined by what it does to the basis elements.
- After we choose a basis for V and a basis for W, L is described by a matrix A.

The matrix of a linear transformation $L : V \to W$ is built as follows:

1. Fix a basis $C = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ of V and a basis $D = \{\mathbf{w}_1, \dots, \mathbf{w}_m\}$ of W.
2. For each i, write $L(\mathbf{v}_i)$ as a linear combination $\sum a_{ij}\mathbf{w}_j$ of the D-basis vectors, and put the coordinates into a column vector: $[L\mathbf{v}_i]_D = \begin{pmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{pmatrix}$
3. Build the matrix A whose columns are $[L\mathbf{v}_1]_D, [L\mathbf{v}_2]_D$, etc. :

$$A \;=\; [L]_{DC} \;=\; \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}$$

   This is called the *matrix of L with respect to the bases C and D.*

The linear transformation L is then given by multiplication by the matrix A in the following sense:

Theorem 4.2.2 For a vector $\mathbf{x} \in V$ with coordinates $[\mathbf{x}]_C$ in the basis C, the D-coordinates of $L\mathbf{x}$ are given by

$$[L\mathbf{x}]_D \;=\; A[\mathbf{x}]_C$$

where $A = [L]_{DC}$ is the above matrix.

Special Case. For linear transformations $L : \mathbb{R}^n \to \mathbb{R}^m$, we can use the standard bases. The procedure for finding the matrix is then easy:

1. Write $L(1, 0, 0, \dots 0)$, $L(0, 1, 0, \dots 0)$, $\dots$ as a column vectors.
2. Assemble these column vectors into a matrix A.

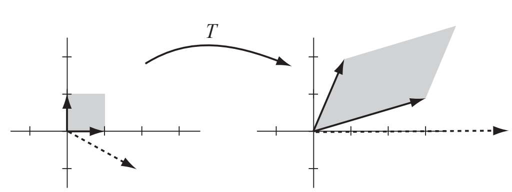
Then the transformation is given by matrix multiplication by A.

Example 1. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that takes $(1, 0)^T$ to $(3, 1)^T$ and $(0, 1)^T$ to $(1, 2)^T$. In column form

$$L\begin{pmatrix} 1 \\ 0 \end{pmatrix} \;=\; \begin{pmatrix} 3 \\ 1 \end{pmatrix} \qquad \text{and} \qquad L\begin{pmatrix} 0 \\ 1 \end{pmatrix} \;=\; \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

so the matrix of L is $A = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$. To find where L takes the vector $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$, we calculate

$$L\begin{pmatrix} 2 \\ -1 \end{pmatrix} \;=\; \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} \;=\; \begin{pmatrix} 5 \\ 0 \end{pmatrix}.$$



Example 2. What is the matrix of the linear transformation $T\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}4x-2y\\-x+7y\end{pmatrix}$?

Solution: taking $x=1$ and $y=0$, then $x=0$ and $y=1$ we obtain:

$$T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}4\\-1\end{pmatrix} \quad\text{and}\quad T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-2\\7\end{pmatrix} \qquad \text{so the matrix is} \qquad A = \begin{pmatrix}4 & -2\\-1 & 7\end{pmatrix}.$$

Examples of six types of linear transformations $L: \mathbb{R}^2 \to \mathbb{R}^2$.

1. Dilation by factor of 3 horizontally, factor of 5 vertically: $D = \begin{pmatrix}3 & 0\\0 & 5\end{pmatrix}$.
2. Rotation counterclockwise by 90°: $R_{90} = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$ and by angle θ: $R_\theta = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$.
3. The reflection across the x-axis: $R_x = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$.
4. A *shear* that fixes the first basis vector $\mathbf{e}_1$ and moves $\mathbf{e}_2$ to $\mathbf{e}_1 + \mathbf{e}_2$: $S = \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}$.
5. The orthogonal projection onto the x-axis: $P = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$.
6. The *embedding* $E: \mathbb{R}^2 \to \mathbb{R}^3$ that takes $\mathbb{R}^2$ into the xy-plane: $E : \begin{pmatrix}1 & 0\\0 & 1\\0 & 0\end{pmatrix}$.

Supplemental Homework Problems

1. Let $L: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that takes $\mathbf{e}_1 = (1,0)$ to $3\mathbf{e}_1 + 4\mathbf{e}_2$ and takes $\mathbf{e}_2 = (0,1)$ to $-2\mathbf{e}_1 + 5\mathbf{e}_2$. What is the matrix for L with respect to the (standard) basis $\{\mathbf{e}_1, \mathbf{e}_2\}$?
2. Let $R: \mathbb{R}^2 \to \mathbb{R}^2$ be counterclockwise rotation about the origin by 60°.
   (a) What is $R(\mathbf{e}_1)$? What is $R(\mathbf{e}_2)$?
   (b) What is the matrix of R with respect to the standard basis?
   (c) What is the image $R(\mathbf{v})$ of the vector $\mathbf{v} = (1,1)$?
3. Let $P: \mathbb{R}^3 \to \mathbb{R}^3$ be the orthogonal projection onto the yz plane, given by $P(x,y,z) = (0,y,z)$.
   (a) What are $P(\mathbf{e}_1)$, $P(\mathbf{e}_2)$ and $P(\mathbf{e}_3)$?
   (b) What is the matrix of L with respect to the standard basis?
   (c) What is the image $P(\mathbf{v})$ of the vector $\mathbf{v} = (1,2,-1)$?
4. (a) Using the formula given in Example 2 above, find the matrix of the rotation $R: \mathbb{R}^2 \to \mathbb{R}^2$ about the origin by 30°.
   (b) This rotation takes $\begin{pmatrix}4\\9\end{pmatrix}$ to what vector?
5. Given points $\mathbf{v}_0$ and $\mathbf{v}_1$ in a vector space, the line segment between them is
   $$S = \Big\{ v_t = (1-t)\mathbf{v}_0 + t\mathbf{v}_1 \;\Big|\; 0 \le t \le 1 \Big\}.$$
   Prove that a linear transformation $L: V \to W$ takes each line segment to the line segment between the image points $L(\mathbf{v}_0)$ and $L(\mathbf{v}_1)$.

6. Let $\square$ denote the unit square in $\mathbb{R}^2$ with corners at $(0,0),(1,0),(0,1)$ and $(1,1)$. Using the fact that you proved in Problem 4, sketch the image of $\square$ under the linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^2$ whose matrix is

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

7. Find the matrix of the orthogonal projection $P : \mathbb{R}^3 \to \mathbb{R}^3$ onto the z-axis.

8. Let $\lambda = \{y = -x\}$ be the "anti-diagonal" line in $\mathbb{R}^2$. Reflection through the line λ (obtained by moving along the line segment perpendicular to λ, going an equal distance to the opposite side of λ) is a linear transformation.

   (a) What is $R(1,0)$? What is $R(0,1)$?

   (b) Find the matrix of this reflection R.

   (c) Where does R take the vector $\begin{pmatrix} -2 \\ 5 \end{pmatrix}$?

9. Find a non-zero 2×2 matrix A such that $A\mathbf{v}$ is perpendicular to $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ for all $\mathbf{v} \in \mathbb{R}^2$. *Hint: Find one perpendicular vector* $\mathbf{w}$ *and define* A *by* $A\mathbf{e}_1 = \mathbf{w}$ *and* $A\mathbf{e}_2 = r\mathbf{w}$ *for you favorite* $r \in \mathbb{R}$.

10. Find the matrices of the following transformations from $\mathbb{R}^3$ to $\mathbb{R}^3$.

    (a) The reflection across the xz-plane.

    (b) The rotation about the z-axis through an angle θ counterclockwise as viewed from the positive z-axis.

    (c) Reflection across the plane $y = z$.

11. Let D be the dilation $\begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$. Show that D takes the unit circle $x^2 + y^2 = 1$ to an ellipse. Sketch the ellipse. *Hint: What is the image point* $\begin{pmatrix} z \\ w \end{pmatrix}$ *of* $\begin{pmatrix} x \\ y \end{pmatrix}$*? What equation do* z *and* w *satisfy?*

12. Let $T : P_3 \to P_3$ be the linear transformation that takes a polynomial $p(x) = ax^3 + bx^2 + cx + d$ to $p(x-2)$. What is the matrix for T with respect to the basis $\{x^3, x^2, x, 1\}$ of P_3? (Be sure to keep the basis elements in this order.)