Compositions and Inverses

Chapter 4 of the textbook is missing some important concepts. These notes fill them in.

Definition. The *composition* of transformations $L : U \to V$ and $M : V \to W$ is the transformation $M \circ L : U \to W$ defined by

$$(M \circ L)(\mathbf{u}) = M(L(\mathbf{u})).$$

Caution – read backwards! $M \circ L$ means "do L first, then do M". To keep things straight, write down this diagram:

$$U \xrightarrow{L} V \xrightarrow{M} W$$

Composing maps in this manner keeps us in the world of linear transformations:

Lemma. If L and M are linear transformations, then so is $M \circ L$.

Proof. If $\mathbf{u}, \mathbf{v} \in U$ and $\alpha, \beta \in \mathbb{R}$ then

$$\begin{aligned}
(M \circ L)(\alpha\mathbf{u} + \beta\mathbf{v}) &= M(L(\alpha\mathbf{u} + \beta\mathbf{v})) && \text{Def. of } M \circ L \\
&= M(\alpha L(\mathbf{u} + \beta L(\mathbf{v})) && L \text{ is a LM} \\
&= (\alpha M(L\mathbf{u})) + \beta M(L(\mathbf{v})) && M \text{ is a LM} \\
&= \alpha(M \circ L)(\mathbf{u}) + \beta(M \circ L)(\mathbf{v})) && \text{Def. of } M \circ L
\end{aligned}$$

□

After we choose bases of the vector spaces U, V and W, then we can write $L : U \to V$ and $M : V \to W$ as matrices. What is the matrix of $M \circ L$?

Theorem. Fix bases of the vector spaces U, V and W. If the matrix of $L : U \to V$ is A and the matrix of $M : V \to W$ is B, then the matrix of $M \circ L$ is the matrix product BA.

Example. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the dilation by a factor of 2 in the x direction and 4 in the y direction, and let $M : \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection across the line $x = y$. In terms of matrices,

$$L = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} \qquad M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad M \circ L = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 4 & 0 \end{pmatrix}$$

Homework – due Monday Oct. 31

1. Sketch the image of the unit square under the linear transformation $M : \mathbb{R}^2 \to \mathbb{R}^2$ whose matrix is $\begin{pmatrix} -2 & -3 \\ 0 & 4 \end{pmatrix}$.
   This image will be a parallelogram.

2. Write down the matrix for the linear transformation $M = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6x - 9y + z \\ 5x + 8y - 2z \\ 4x - 3y + 7z \end{pmatrix}$

3. Construct the 2×2 matrix for the linear transformations $\mathbb{R}^2 \to \mathbb{R}^2$ defined by the following compositions. *In each case, write down the matrix of each transformation, then multiply the matrices in the correct order.*

(a) A dilation by a factor of 4, then a reflection across the x-axis.

(b) A counterclockwise rotation through $\frac{\pi}{2}$, then a dilation by a factor of $\frac{1}{2}$.

(c) A reflection about the line $x = y$, then a rotation though an angle of π.

4. (a) Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the composition: A scaling factor of 6 in the x direction and 2 in the y direction, then a *clockwise* rotation of 45° (= a counterclockwise rotation of −45°).

   (a) Write down the matrix for L.

   (b) Find the images under L of each of the four corners of the unit square.

   (c) Plot the points found in (b) and sketch the image of the unit square under L.

5. (a) Write down the matrix R for a counterclockwise rotation through $\frac{\pi}{4}$ radians.
   (b) Compute R^2, R^4 (square R^2), and show that $R^8 = Id$.
   (c) Give a geometric explanation why $R^8 = Id$.

6. Verify the associative property of matrix multiplication in one example by computing $A(BC)$ and $(AB)C$ for the matrices

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 0 \\ 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 4 \\ -2 & 3 \end{pmatrix} \qquad C = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

7. Consider the matrices

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 5 & 4 \\ -2 & 1 & 3 \end{pmatrix} \qquad C = \begin{pmatrix} 2 & 3 \\ 6 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 2 & -2 \\ 1 & 3 \end{pmatrix}$$

   Calculate, if possible, (a) AB and BA, (b) AC and CA, (c) AD and DA.
   Observe that $AB \neq BA$ since one of these does not exist, $AC \neq CA$, and $AD = DA$, illustrating all possibilities when the order of multiplication is reversed.

8. Let $A = \begin{pmatrix} 2 & 3 \\ -1 & 5 \end{pmatrix}$ and let $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ be the 2×2 identity matrix. Calculate A^2 and use your answer to find the matrices

$$\text{(a) } A^2 + 2A - 5I_2 \qquad \text{and} \qquad \text{(b) } A^2 - 7A + 13I_2.$$

9. The matrix $L = \begin{pmatrix} 0 & 3 & 4 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}$ is called *strictly upper triangular* for the obvious reason.

   (a) Compute L^2 and L^3.

   (b) Formulate and prove a theorem about general strictly upper triangular 3×3 matrices.

Inverses and Isomorphisms

This section considers the important question "When does a linear transformation have an inverse?" Most of the theorems stated here are in the textbook, but are scattered through several sections and are partly done in exercises.

First recall the definition of the kernel and the range of a linear transformation $L : V \to W$:

$$\ker L = \{\mathbf{v} \in V \mid L\mathbf{v} = \mathbf{0}\} \qquad R(L) = \{L\mathbf{v} \mid \mathbf{v} \in V\}.$$

We call the number $\dim \ker L$ the *nullity of* L, and the number $\dim R(L)$ the *rank of* L.

Also recall that the inverse of a linear transformation (if it exists) is automatically linear.

Lemma. *If a linear transformation $L : V \to W$ has an inverse, then L^{-1} is also linear.*

Proof. Given any $\mathbf{w}_1, \mathbf{w}_2 \in W$, set $\mathbf{v}_1 = L^{-1}(\mathbf{w}_1)$ and $\mathbf{v}_2 = L^{-1}(\mathbf{w}_2)$. Then $L\mathbf{v}_1 = \mathbf{w}_1$ and $L\mathbf{v}_2 = \mathbf{w}_2$, so for any $\alpha, \beta \in \mathbb{R}$ we have

$$\begin{aligned}
L^{-1}(\alpha\mathbf{w}_1 + \beta\mathbf{w}_2) &= L^{-1}(aL(\mathbf{v}_1) + \beta L(\mathbf{v}_2)) && \text{substitution} \\
&= L^{-1}(L(\alpha\mathbf{v}_1 + \beta\mathbf{v}_2)) && L \text{ is linear.} \\
&= \alpha\mathbf{v}_1 + \beta\mathbf{v}_2 && L^{-1}L = Id. \\
&= \alpha L^{-1}(\mathbf{w}_1) + \beta L^{-1}(\mathbf{w}_2) && \text{substitution.}
\end{aligned}$$

Thus L^{-1} is a linear transformation. □

Rank-Nullity Theorem (version 2). *For any linear transformation $L : V \to W$*

$$\dim R(L) + \dim\ker L \;=\; \dim V,$$

that is, rank(L)+ nullity(L) $= \dim V$.

Proof. We proceed in four steps:

(1) Choose a basis $\mathbf{v}_1, \ldots, \mathbf{v}_k$ of $\ker L \subset V$. Then k is the nullity.

(2) Expand this to a basis $\mathbf{v}_1, \ldots, \mathbf{v}_k, \ldots, \mathbf{v}_n$ of V. Then $n = \dim V$.

(3) Set $\mathbf{w}_i = L\mathbf{v}_i$. Then $\mathbf{w}_{k+1}, \ldots \mathbf{w}_n$ are a basis for $R(L)$ (see below). This consists of $n-k$ vectors, so $\dim R(L) = n - k$.

(4) Expand to a basis $\mathbf{w}_{k+1}, \ldots \mathbf{w}_n, \ldots \mathbf{w}_m$ of W.

Then

$$\dim R(L) + \dim\ker L = (n-k) + k = n = \dim V.$$

To verify the claim in Step (3), we argue as follows.

(a) For $i = 1, \ldots, k$, $\mathbf{v}_i \in \ker L$, so $L\mathbf{v}_i = 0$. Hence

$$R(L) \;=\; \operatorname{span}(L\mathbf{v}_1, \ldots, L\mathbf{v}_n) \;=\; \operatorname{span}(L\mathbf{v}_{k+1}, \ldots, L\mathbf{v}_n) \;=\; \operatorname{span}(\mathbf{w}_{k+1}, \ldots, \mathbf{w}_n)$$

so $\{\mathbf{w}_{k+1}, \ldots, \mathbf{w}_n\}$ spans the range.

(b) To see that $\{\mathbf{w}_{k+1}, \ldots, \mathbf{w}_n\}$ are linearly independent, suppose that

$$0 = \sum_{k+1}^{n} \alpha_i \mathbf{w}_i = \sum \alpha_i L\mathbf{v}_i = L\left(\sum \alpha_i \mathbf{v}_i\right).$$

then $\mathbf{v} = \sum \alpha_i \mathbf{v}_i \in \ker L$. But then $\mathbf{v} \in \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$, so $\mathbf{v} = \sum_i^n \beta_i \mathbf{v}_i$. This shows that

$$0 \;=\; \mathbf{v} - \mathbf{v} \;=\; \sum \sum_i^n \beta_i \mathbf{v}_i + \sum_{k+1}^{n} \alpha_i \mathbf{v}_i$$

But then all α_i and β_i are 0 because $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ are LI. Thus $\{\mathbf{w}_{k+1}, \ldots, \mathbf{w}_n\}$ are linearly independent.
□

In the bases $\mathbf{v}_1, \dots, \mathbf{v}_n$ and $\mathbf{w}_{k+1}, \dots, \mathbf{w}_m$ defined in this proof, we have $L\mathbf{v}_i = 0$ for $\imath = 1, \dots, k$ and $L\mathbf{v}_i = \mathbf{w}_i$ for $i = k+1, \dots n$. Hence the matrix for L in these bases has the block form

$$A_L \;=\; \left(\begin{array}{c|c} 0 & \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} \\ \hline 0 & 0 \end{array}\right) \;=\; \left(\begin{array}{c|c} 0 & I_{n-k} \\ \hline 0 & 0 \end{array}\right) \tag{21.1}$$

Isomorphisms

Definition (a) A linear map $L : V \to W$ is called an *isomorphism* if it is invertible.

(b) Two vector spaces V and W are *isomorphic*, written $V \cong W$, if there exists an isomorphism $L : V \to W$.

Caution about (b): If there is one isomorphism, then there will be many different isomorphisms L.

Isomorphism Theorem. Let V and W be finite-dimensional vector spaces. Then

(a) V and W are isomorphic if and only if $\dim V = \dim W$.

(b) For a linear transformation $L : V \to W$ between vector spaces *of the same dimension*

$$L \text{ is an isomorphism} \iff N(L) = 0 \iff R(L) = W.$$

Proof. (a) If V and W are isomorphic, then there is an invertible transformation $L : V \to W$. Then

- If $L\mathbf{v} = 0$ then $\mathbf{v} = L^{-1}L\mathbf{v} = L^{-1}(0) = 0$, so $\ker L = 0$.
- For any $\mathbf{w} \in W$, we can write $\mathbf{w} = LL^{-1}\mathbf{w} = L\mathbf{v}$ for $v = L^{-1}\mathbf{w} \in V$. This shows that $\mathbf{w} \in R(L)$, so $R(L) = W$.

Then by the Rank-Nullity Theorem, $\dim V = \dim R(L) + \dim \ker L = \dim W + 0$.

Conversely, if $\dim V = \dim W$ then we can choose bases $\{\mathbf{v}_1, \dots \mathbf{v}_n\}$ of V and $\{\mathbf{w}_1, \dots \mathbf{w}_n\}$ of W with the same number of elements. Lhen

$$L(a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n) \;=\; a_1\mathbf{w}_1 + \cdots + a_n\mathbf{w}_n.$$

defines a linear transformation that has an inverse, namely $L^{-1}(a_1\mathbf{w}_1 + \cdots + a_n\mathbf{w}_n) \;=\; a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n$. Hence $V \cong W$.

(b) For a linear transformation $L : V \to W$ with $\dim V = \dim W = n$, we can find bases as in the proof of the Rank-Nullity Theorem. In these bases, the matrix A_L of L has the block form (21.1):

$$A_L \;=\; \left(\begin{array}{c|c} 0 & I_{n-k} \\ \hline 0 & 0 \end{array}\right)$$

But then L is an isomorphism $\Leftrightarrow \det A_L \neq 0$. But $\det A_L$ is the product of the diagonal entries, so is zero only if the $(n-k) \times (n-k)$ block I_{n-k} is the entire matrix. This means that $k = 0$ and $A_l = I_n$, so $\ker L = 0$ and $R(L) = W$. $\square$

Summary.

After fixing bases $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of V and $\{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ of W, each linear transformation $L : V \to W$ has an associated matrix $A = A_L$. Then L is invertible is and only if the matrix A is invertible. Combining lots of previous facts, we have:

Invertibility Theorem. A linear transformation $L : V \to W$ can be an isomorphism only if $\dim V = \dim W = n$. If so, then $A = A_L$ is an $n \times n$ matrix and the following are are equivalent:

- L is an isomorphism.
- L^{-1} exists and is linear.
- A is invertible.
- The columns of A are linearly independent.
- The columns span $\mathbb{R}^n$.
- The columns are a basis of $\mathbb{R}^n$.
- The equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for all $\mathbf{b} \in \mathbb{R}^n$.
- $\det A \neq 0$.
- $A \approx I_n$ by row operations.

Homework – due Wednesday Nov. 2

1. Let $Q : \mathbb{R}^4 \to M(2,2)$ be the transformation $Q\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.
   (a) Show that Q is linear.
   (b) Show that Q is invertible by defining a linear map $R : M(2,2) \to \mathbb{R}^4$ and showing that $Q \circ R = I$ and $R \circ Q = I$.

2. Prove that the vector spaces P_4 and $M(2,2)$ are isomorphic by defining a transformation $L : P_4 \to M(2,2)$ and showing that it is linear and invertible.

3. Which pairs of the following vector spaces are isomorphic?

$$\mathbb{R}^7 \qquad \mathbb{R}^{12} \qquad M(3,3) \qquad M(3,4) \qquad M(4,3) \qquad P_6 \qquad P_8 \qquad P_{11}.$$

4. For each of the following linear transformations, determine whether L is invertible and justify your answer.

   (a) $L : \mathbb{R}^2 \to \mathbb{R}^3$ by $L(x, y) = (x - y, y, 3x + y)^T$.
   (b) $L : \mathbb{R}^3 \to \mathbb{R}^3$ by $L(x, y, z) = (3x - 2z, y, 3x + 4y)^T$.
   (c) $L : P_4 \to P_3$ by $L(p(x)) = p'(x)$.
   (d) $L : M(2,2) \to M(2,2)$ by $L\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a + b & a \\ c & c - d \end{pmatrix}$.

5. Prove that if A is an invertible matrix and $AB = \mathbf{0}$, then $B = \mathbf{0}$ (here $\mathbf{0}$ denotes the matrix with all entries equal to 0). *This can be done in one line!*

6. Let A be an $n \times n$ matrix.

   (a) Suppose that $A^2 = \mathbf{0}$. Use determinants to prove that A is not invertible.

   (b) Suppose that $AB = \mathbf{0}$ for some non-zero $n \times n$ matrix B. Could A be invertible? Explain. *Suppose that A^{-1} exists. What can you say?*

7. Let $L : V \to W$ be an isomorphism between finite-dimensional vector spaces, and let S be a vector subspace of V.

   (a) Prove that $L(S)$ is a subspace of W. *See Theorem 4.1.1 of the textbook.*

   (b) Prove that $\dim(L(S)) = \dim S$. *One approach: pick a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ of S and show that $\{L(\mathbf{v}_1), \ldots, L(\mathbf{v}_k)\}$ is a basis of $L(S)$.*