Math 309-003 **Sample Final Exam** Fall, 2018

Name: SOLUTIONS PID: A

1. (20 pts, 4 pts each) State clearly and concisely the following definitions.

   (a) An $n \times n$ matrix A is **nonsingular**.

   $A\bar{x} = \bar{0}$ has the only solution $\bar{x} = \bar{0}$
   [MANY EQUIVALENT DEFINITIONS: A is row-equivalent to I; $\det A \neq 0$]

   (b) A set S is a **subspace** of a vector space V. [NO NEED TO CHECK ALL AXIOMS AS IT IS GIVEN THAT $\vec{V}$ is a vectorspace]

   $\forall \vec{x} \in S$ and $\vec{y} \in S$ we have $\alpha \vec{x} \in S$ for any real α and $\bar{x} + \bar{y} \in S$ [OR UNIFY IT INTO ONE EQUALITY $\alpha \bar{x} + \beta \bar{y} \in S$ FOR ALL REAL α AND β]

   (c) Vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ in a vector space V are **linearly independent**.

   The equation $\alpha_1 \bar{v}_1 + \alpha_2 \bar{v}_2 + \cdots + \alpha_n \bar{v}_n = \vec{0}$ has the only solution $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$.

   (d) A set of vectors $\mathbf{x}$ in $\mathbb{R}^n$ is the **orthogonal complement** of a subspace Y of $\mathbb{R}^n$

   RECALL THE NOTATION $Y^\perp$

   $\vec{x} \in Y^\perp$ is the set of all vectors in $\mathbb{R}^n$ such that $\vec{x} \cdot \vec{y} = 0$ FOR ANY VECTOR $\bar{y} \in Y$.

   (e) An operation $\langle \, , \, \rangle$ is an **inner product** on a vector space V.

   RECALL THREE NECESSARY PROPERTIES OF AN INNER PRODUCT:

   positivity [I] $\langle \vec{x}, \vec{x} \rangle \geq 0$ FOR ALL $\vec{x} \in V$ and $\langle \bar{x}, \bar{x} \rangle = 0$ ONLY FOR $\bar{x} = \bar{0}$

   symmetricity [II] $\langle \bar{x}, \bar{y} \rangle = \langle \bar{y}, \bar{x} \rangle$ FOR ALL $\bar{x}, \bar{y} \in V$

   linearity [III] $\langle \bar{y}, \alpha \bar{x} \rangle = \alpha \langle \bar{y}, \bar{x} \rangle$ FOR ALL $\bar{x}, \bar{y} \in V$ AND $\alpha \in \mathbb{R}$.

2. (15 pts) Let a and b be real numbers and consider the following linear system:

$$\begin{array}{rcrcrcl} x_1 & + & 5x_2 & + & 2x_3 & = & b \\ & & x_2 & + & 3x_3 & = & b^2 \\ & & 2x_2 & + & ax_3 & = & 4 \end{array}$$

(a) Find all values of a and b such that the system has no solutions.

(b) Find all values of a and b such that the system has a unique solution.

(c) Find all values of a and b such that the system has infinitely many solutions.

$$\begin{array}{c} \\ \\ -2 \end{array}\left[\begin{array}{ccc|c} 1 & 5 & 2 & b \\ 0 & 1 & 3 & b^2 \\ 0 & 2 & a & 4 \end{array}\right] \xrightarrow{\text{REF}} \left[\begin{array}{ccc|c} 1 & 5 & 2 & b \\ 0 & 1 & 3 & b^2 \\ 0 & 0 & a-6 & 4-2b^2 \end{array}\right]$$

NOW ANALYSE:

(a) MUST BE $a-6=0$ AND $4-2b^2 \neq 0$,
SO $\boxed{a=6,\ b \notin \{\sqrt{2}, -\sqrt{2}\}}$

(b) b - ANY, $a-6 \neq 0$, SO $\boxed{a \neq 6}$

(c) MUST BE BOTH $a-6=0$ AND $4-2b^2=0$,
SO $\boxed{a=6,\ b=\sqrt{2} \text{ OR } a=6,\ b=-\sqrt{2}}$

3. (15 pts, 5 pts each) Determine whether the set S given below is a subspace of a vector space V.

(a) S is a set of all **skew-symmetric** $n \times n$ matrices A (that is $A^T = -A$) in the set V of all $n \times n$ matrices.

LET $A^T = -A$ AND $B^T = -B$:

$[\alpha A + \beta B]^T = \alpha \cdot A^T + \beta \cdot B^T = \alpha \cdot (-A) + \beta(-B) =$

$= (-1) \cdot [\alpha A + \beta B]$, so it holds

subspace

(b) S is a set of all **orthogonal** $n \times n$ matrices Q (that is $Q^T = Q^{-1}$) in the set V of all $n \times n$ matrices.

NOTE THAT $[A+B]^{-1} \neq A^{-1} + B^{-1}$ IN GENERAL

NOT A SUBSPACE! BUT HERE IT IS EVEN EASIER: RECALL THAT $\det A = \det A^T$, SO $\det[Q \cdot Q^T] = \det Q \cdot \det Q^T = [\det Q]^2$ BUT $\det[QQ^T] = \det I = 1$, SO $\det Q = \pm 1$. FOR INSTANCE, IF $Q^T = Q^{-1}$, $[\alpha Q]^T = \alpha \cdot Q^T = \alpha \cdot Q^{-1}$ AND $[\alpha Q]^T \cdot [\alpha Q] = \alpha^2 Q Q^{-1} = \alpha^2 \cdot I$ IF $|\alpha| \neq 1$, THEN αQ IS NOT ORTHOGONAL

(c) S is a set of all continuous functions $f(x) \in C[0,1] = V$ such that $f(0) \geq 0$.

AGAIN, IF $f(0) > 0$, THEN FOR $(-1) \cdot f(x) \in S$ WE HAVE $(-1) \cdot f(0) = -f(0) < 0$, NOT IN THE SET!

SO NOT A SUBSPACE

[IF WE IMPOSE $f(0) = 0$, THEN A SUBSPACE! INDEED

$\alpha f(0) + \beta g(0) = 0$, SO $\alpha f(x) + \beta g(x) \in S$

IF WE IMPOSE $f(0) \neq 0$, THEN AGAIN NOT A SUBSPACE! BECAUSE $0 \cdot f(x)$ MUST BE IN THE SET S]

4. (15 pts) Let $A = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & -2 & 3 & 0 \\ -3 & 3 & -4 & 1 \\ 3 & -3 & 4 & -1 \end{bmatrix}$. Use row operations to find the basis in the **row-vector** space $R[A^{\mathrm{T}}]$, **column-vector** space $R[A]$, and the basis in $N[A]$. What is the interrelation between $R[A^{\mathrm{T}}]$ and $N[A]$?

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & -2 & 3 & 0 \\ -3 & 3 & -4 & 1 \\ 3 & -3 & 4 & -1 \end{bmatrix} \xrightarrow{\text{RRF}} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & -2 & -4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} \boxed{1} & -1 & 0 & -3 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ V}$$

free

lead

NOW, ANALYSE:

$R[A^T] = \text{Span}\left\{ [1\ -1\ 0\ -3]^T, [0\ 0\ 1\ 2]^T \right\}$

$R[A] = \text{Span}\left\{ \underbrace{\begin{bmatrix} 1 \\ 2 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ -4 \\ 4 \end{bmatrix}} \right\}$

column vectors of original A taken for <u>lead</u> variables

$N[A]$: $[x_1, \alpha, x_3, \beta]^T$ where $x_1 = 1\cdot\alpha + 3\cdot\beta$, $x_2 = -2\beta$,

so ALL vectors of the form $\begin{bmatrix} \alpha+3\beta \\ \alpha \\ -2\beta \\ \beta \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 3 \\ 0 \\ -2 \\ 1 \end{bmatrix} =$

$= \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$

$$\boxed{R[A^T] = [N[A]]^{\perp}}$$

5. (15pts) Let $L : P_2 \to P_3$ be a mapping defined by

$$L(p(x)) = 2p(x) + x^2 p'(x).$$

(a) verify that L is a **linear transformation**

$$L(\alpha p(x) + \beta q(x)) = 2(\alpha p(x) + \beta q(x)) + x^2(\alpha p'(x) + \beta q'(x))$$

$$= \alpha(2p(x) + x^2 p'(x)) + \beta(2q(x) + x^2 q'(x)) =$$

$$= \alpha L(p(x)) + \beta L(q(x))$$ proven

Find the matrix representation of L with respect to the ordered bases $[1 + x, 1 - x]$ of P_2 and $[x^2, x, 1]$ of P_3.

$$L(1+x) = 2(1+x) + x^2 \cdot (1+x)' = 2 \cdot 1 + 2 \cdot x + 1 \cdot x^2 =$$

$$= \begin{array}{cc} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} & \begin{array}{c} x^2 \\ x \\ 1 \end{array} \end{array}$$

$$L(1-x) = 2(1-x) + x^2(1-x)' = 2 + (-2) \cdot x + (-1) \cdot x^2 =$$

$$= \begin{array}{cc} \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix} & \begin{array}{c} x^2 \\ x \\ 1 \end{array} \end{array}, \text{ so } \quad L\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \ L\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix},$$

$$\boxed{L = \begin{bmatrix} 1 & -1 \\ 2 & -2 \\ 2 & 2 \end{bmatrix}}$$

6. (20pt) The linear operator $T : \mathbb{R}^2 \to \mathbb{R}^2$ is given by its action in the standard basis: $T(\mathbf{e}_1) = \frac{1}{3}\mathbf{e}_1 + \frac{1}{2}\mathbf{e}_2$, $T(\mathbf{e}_2) = \frac{2}{3}\mathbf{e}_1 + \frac{1}{2}\mathbf{e}_2$. Find the matrix representation of this operator in the standard basis and in the basis of orthonormal vectors $\mathbf{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$.

$$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/2 \end{bmatrix} \quad T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/2 \end{bmatrix}, \text{ so}$$

$$T = \begin{bmatrix} 1/3 & 2/3 \\ 1/2 & 1/2 \end{bmatrix}$$

$$\begin{tikzcd} \{\bar{e}_1, \bar{e}_2\} \arrow[r, "T"] & \{\bar{e}_1, \bar{e}_2\} \arrow[d, "V^{-1}"] \\ \{\bar{v}_1, \bar{v}_2\} \arrow[u, "V"] \arrow[r, "V^{-1}TV"] & \{\bar{v}_1, \bar{v}_2\} \end{tikzcd}$$

$$V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \quad \det V = -1$$

$$V^{-1} = \frac{1}{-1}\begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$V^{-1}TV = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1/3 & 2/3 \\ 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\frac{1}{\sqrt{2}} =$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1/3 \\ 1 & 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & -1/3 \\ 0 & -1/3 \end{bmatrix} = \begin{bmatrix} 1 & -1/6 \\ 0 & -1/6 \end{bmatrix}$$

CHECK THAT $\det T = \det T' = \det(V^{-1}TV)$.

$$\det T = \frac{1}{6} - \frac{2}{6} = -\frac{1}{6} \qquad \det(V^{-1}TV) = -1/6$$

7. Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 2 & 2 & 1 \end{bmatrix}$.

(a) (6pts) Find all eigenvalues of A

$a^2 - b^2 = (a-b)(a+b)$

$$\det \begin{bmatrix} 2-\lambda & 1 & 0 \\ 1 & 2-\lambda & 0 \\ 2 & 2 & 1-\lambda \end{bmatrix} = (1-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (1-\lambda)\left[(2-\lambda)^2 - 1^2\right] = (1-\lambda)(1-\lambda)(3-\lambda)$$

$\lambda_1 = 1 \qquad \lambda_2 = 3$

(b) (8pts) for each of the eigenvalue find the corresponding eigenvectors.

solve homogeneous system $[A - \lambda_i I]$ ~~$\bar{x} N[A] = \vec{0}$~~ $\cdot \bar{x} = \bar{0}$.

$\lambda_1 = 1$: $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 0 \end{bmatrix} \xrightarrow{RRF} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\alpha \; \beta$ $\quad N[A-1\cdot I] = \begin{bmatrix} -\alpha \\ \alpha \\ \beta \end{bmatrix} = \text{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\bar{X}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \bar{X}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\lambda_2 = 3$ $\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 2 & 2 & -2 \end{bmatrix} \xrightarrow{-2} \begin{bmatrix} 1 & -1 & 0 \\ 2 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 4 & -2 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix} \to$

$\to \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix} \quad \bar{X}_3 = \begin{bmatrix} \alpha/2 \\ \alpha/2 \\ \alpha \end{bmatrix} = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}$

(c) (6pts) Find a nonsingular matrix X and a diagonal matrix D such that $A = XDX^{-1}$.

$$\boxed{X = \left[\bar{X}_1 \,|\, \bar{X}_2 \,|\, \bar{X}_3\right] = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}} \qquad \boxed{D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}}$$

$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 2 & 2 & 1 \end{bmatrix} = A = XDX^{-1}$ DO NOT EVALUATE

8. (15pts) Let $L : V \to W$ be a linear transformation from a vector space V to a vector space W and let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be vectors in V. Prove that if the vectors $\mathbf{w}_1 = L(\mathbf{v}_1)$, $\mathbf{w}_2 = L(\mathbf{v}_2)$, $\ldots$, $\mathbf{w}_n = L(\mathbf{v}_n)$ are linearly independent, then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ must be linearly independent.

Let $\bar{w}_1 \ldots \bar{w}_n$ be linearly independent. For any linear combination $\alpha_1 \bar{w}_1 + \cdots + \alpha_n \bar{w}_n$ we have

$$\alpha_1 \bar{w}_1 + \cdots + \alpha_n \bar{w}_n = \alpha_1 L(V_1) + \alpha_2 L(\bar{V}_2) + \cdots + \alpha_n L(\bar{V}_n) =$$

$= L\left[\alpha_1 \bar{V}_1 + \alpha_2 \bar{V}_2 + \cdots + \alpha_n \bar{V}_n\right]$. If $\bar{V}_1 \ldots \bar{V}_n$ are linearly dependent, we have $\alpha_1 \ldots \alpha_n$ not all equal to zero such that $\alpha_1 \bar{V}_1 + \cdots + \alpha_n \bar{V}_n = \bar{0}$. But then

$\vec{0} = L(\vec{0}) = L(\alpha_1 \bar{V}_1 + \cdots + \alpha_n \bar{V}_n) = \alpha_1 \bar{W}_1 + \cdots + \alpha_n \bar{W}_n$ with not all $\alpha_j = 0$, so $\bar{W}_1 \ldots \bar{W}_n$ ARE linearly dependent, which contadict the condition. So no such combination exists and $\bar{V}_1 \ldots \bar{V}_n$ are linearly independent.

9. (15pts) Consider the vector space $\mathbb{R}^n$ with the inner product $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^{\mathrm{T}}\mathbf{y}$ and an $n \times n$ matrix A.

(a) Prove that for any $\mathbf{x}$ in $\mathbb{R}^n$, $\langle \mathbf{x}, A^{\mathrm{T}}A\mathbf{x} \rangle = \|A\mathbf{x}\|^2$.

(b) Prove that if λ is an eigenvalue of the matrix $A^{\mathrm{T}}A$, then $\lambda \geq 0$.

[a] $\langle \bar{x}, A^T A \bar{x} \rangle = \bar{x}^T \cdot A^T A \bar{x} = (\bar{x}^T A^T) \cdot (A\bar{x}) = (A\bar{x})^T \cdot A\bar{x} =$

$= \|\bar{y}\|^2$ where $\bar{y} = A\bar{x}$, i.e. $\|\bar{y}\|^2 = \|A\bar{x}\|^2$

[b] Let λ be an e-value of $A^T A$

[NOTE: $A^T A$ is symmetric for any A, even not necessarily square matrix: $[A^T A]^T = [A]^T \cdot [A^T]^T = A^T \cdot A$.

ANY $n \times n$ symmetric matrix has n real eigenvalues and n eigenvectors]

Then $\exists$ nonzero $\vec{x}$ such that $A^T A \vec{x}_\lambda = \lambda \vec{x}_\lambda$.

Calculate now $\vec{x}_\lambda^T \cdot A^T A \vec{x}_\lambda = \|A\bar{x}_\lambda\|^2 \geq 0$ BUT SINCE

$A^T A \bar{x}_\lambda = \lambda \bar{x}_\lambda$, we have $\bar{x}_\lambda^T \cdot A^T A \bar{x}_\lambda = \bar{x}_\lambda^T \lambda \bar{x}_\lambda = \lambda \|\bar{x}_\lambda\|^2$

By condition, $\bar{x}_\lambda \neq \bar{0}$, so $\|\bar{x}_\lambda\|^2 > 0$, so

$$\lambda = \frac{\|A\bar{x}_\lambda\|^2}{\|\bar{x}_\lambda\|^2} \geq 0.$$

[Note that $A\bar{x}_\lambda$ can be zero, so λ can be zero as well].

10. (a) (6pts) Let λ be a real number. Prove by induction that for any $n \geq 1$,

$$A^n = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}$$

P_1: $A^1 = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$

P_n: true

P_{n+1}: $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}^{n+1} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}^n \cdot \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \underset{\text{BY } P_n}{=} \begin{bmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{bmatrix} \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$

$= \begin{bmatrix} \lambda^{n+1} & \lambda^n + n\lambda^n \\ 0 & \lambda^{n+1} \end{bmatrix} = \begin{bmatrix} \lambda^{n+1} & (n+1)\lambda^{(n+1)-1} \\ 0 & \lambda^{n+1} \end{bmatrix}$, so

it is true for $n+1$, so it is true for all natural n.

(b) (6pts)

What is e^A? Use that $e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n$.

$A^0 = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sum_{n=1}^{\infty} \frac{1}{n!} \begin{bmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{bmatrix} =$

$= \begin{bmatrix} 1 + \sum_{n=1}^{\infty} \frac{1}{n!}\lambda^n & \sum_{n=1}^{\infty} \overbrace{\boxed{\frac{1}{n!} \cdot n}}^{1/(n-1)!} \lambda^{\boxed{n-1}} \\ 0 & 1 + \sum_{n=1}^{\infty} \frac{1}{n!}\lambda^n \end{bmatrix} = \begin{bmatrix} \sum_{n=0}^{\infty} \frac{1}{n!}\lambda^n & \sum_{m=0}^{\infty} \frac{1}{m!}\lambda^m \\ 0 & \sum_{n=0}^{\infty} \frac{1}{n!}\lambda^n \end{bmatrix}$

Let $m = n-1$

$= \begin{bmatrix} e^\lambda & e^\lambda \\ 0 & e^\lambda \end{bmatrix}$

11. (a) (6pts) Find the projection of the function x on the function $\sin(kx)$ in the space $C[-\pi,\pi]$ with the inner product $\langle p,q\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} p(x)q(x)dx$

$$\vec{P} = \frac{\langle \vec{u},\vec{v}\rangle}{\langle \vec{v},\vec{v}\rangle}\cdot\vec{v} \qquad u = x \quad v = \sin(kx)$$

$$\boxed{\cos(\pi k) = (-1)^k}$$

$$\langle \vec{u},\vec{v}\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cdot\sin(kx)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\,(-d\cos(kx))\cdot\frac{1}{k} \overset{\text{IBP}}{=}$$

$$= \frac{1}{\pi}\cdot\frac{1}{k}\left[-\cos(kx)\cdot x\right]\Big|_{-\pi}^{\pi} + \underbrace{\frac{1}{\pi k}\int_{-\pi}^{\pi}\cos(kx)\,dx}_{\text{vanishes}} = \frac{1}{\pi k}\cdot 2\pi(-1)^{k+1}$$

$$= \frac{2(-1)^{k+1}}{k}; \quad \langle \vec{v},\vec{v}\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi}\sin^2(kx)\,dx = 1.$$

$$\boxed{\vec{P} = \frac{2(-1)^{k+1}}{k}\sin(kx)}$$

(b) (6pts) Find the dimension of the null space of A^T if A is a 7×9 matrix whose rank is 4.

$$N[A^T] = R[A]^\perp \qquad \dim R[A] = \dim R[A^T] = \operatorname{rank} A$$

$$A = m\times n \text{ matrix}; \; m = 7,\; n = 9 \qquad R[A]\oplus R[A]^\perp = \mathbb{R}^m$$

$$\underbrace{\dim R[A]}_{=\operatorname{rank} A = 4} + \underbrace{\dim R[A]^\perp}_{=\dim N[A^T]} = \underbrace{m}_{=7}, \text{ so } \boxed{\dim N[A^T] = 7-4 = 3}$$

(c) Let A be a defective 3×3 matrix with two distinct eigenvalues λ_1 and λ_2. Find $\dim N[A-\lambda_1 I]$ and $\dim N[A-\lambda_2 I]$

Given is that Span$\{\bar{x}_i\}$ where $\bar{x}_i$ are eigenvectors of A has dimension strictly lesser than $n=3$ [otherwise A is diagonalizable and thus not defective]. For every eigenvalue we have at least one eigenvector, since we have two distinct e-values λ_1 and λ_2 we have the corresponding two e-vectors $\bar{x}_1, \bar{x}_2$ which are linearly independent. So Span$\{\bar{x}_i\}$ has dimension at least two, which means exactly two, so $\dim N[A-\lambda_1 I] = \dim N(A-\lambda_2 I) = 1$.

Scratch Paper

Do not write in the area below. (For recording YOUR SCORES only.)

1. ______	OUT OF 20	7. ______	OUT OF 10
2. ______	OUT OF 15	8. ______	OUT OF 15
3. ______	OUT OF 15	9. ______	OUT OF 15
4. ______	OUT OF 15	10. ______	OUT OF 12
5. ______	OUT OF 15	11. (bonus) ______	OUT OF 15
6. ______	OUT OF 10	**TOTAL:** ______	OUT OF 150