Math 309 §2 Name SOLUTIONS

Fall 2018

Exam 3

Directions: Do all problems (100 points total). You must *show all steps and explain your reasoning* to receive full credit. No books, notes, or electronic devices are allowed.

1. (*18 points*) Let V and W be vector spaces. Complete the definitions *as briefly as possible:*

(a) Two $n \times n$ matrices A and B are *similar* if there is an invertible $n \times n$ matrix S such that:

$$B = S^{-1}AS$$

(b) An $n \times n$ matrix Q is *orthogonal* if

$$Q^T Q = I_n$$

(c) The *orthogonal complement* of a subspace S of $\mathbb{R}^n$ is defined by

$$S^\perp = \{ x \in \mathbb{R}^n \mid x \cdot y = 0 \ \forall y \in S \}$$

(d) In a vector space V with inner product $\langle \, , \rangle$,

- The *norm* of a vector $\mathbf{x}$ is defined by the formula $\|\mathbf{x}\| = \sqrt{\langle x, x \rangle}$
- $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_k\}$ is an *orthonormal set* if ...

$$\langle e_i, e_j \rangle = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j. \end{cases}$$

(e) To compute the correlation coefficient between to data vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, one first the deviation vectors $\mathbf{x}_D$ and $\mathbf{y}_D$ (by subtracting the means $\overline{\mathbf{x}}, \overline{\mathbf{y}}$). Then, in terms of $\mathbf{x}_D$ and $\mathbf{y}_D$,

- The formula for the *correlation coefficient* is $r = \dfrac{x_D \cdot y_D}{\|x_D\| \cdot \|y_D\|}$
- Geometrically, r is ... $\cos\theta$ where θ = angle between x_D and y_D.

2. (*14 points*) Circle (T) for TRUE, circle (F) for FALSE.

(a) If A and B are similar matrices, then $\det A = \det B$. T F

(b) If A and B are similar matrices, then A^2 is similar to B^2. T F

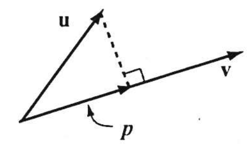
(c) For an $m \times n$ matrix A, the null space $N(A)$ and the range $R(A)$ are orthogonal subspaces of $\mathbb{R}^n$. T F

(d) If $\langle \, , \rangle$ is an inner product on a vector space V, then $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$ for all $\mathbf{x}, \mathbf{y} \in V$. T F

(e) If $\langle\ ,\ \rangle$ is an inner product on a vector space V, then $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ only if $\mathbf{x} = \mathbf{0}$. T F

(f) Each vector space V has a unique inner product. T F

(g) Every finite-dimensional inner product space has an orthonormal basis. T F



3. *(12 points)* Let θ denote the angle between the vectors $\mathbf{u} = (1, 1, 2, 2)^T$ and $\mathbf{v} = (-2, 1, 2, 0)^T$ in $\mathbb{R}^4$. Let $\mathbf{p}$ be the orthogonal projection of $\mathbf{u}$ onto $\mathbf{v}$. Find:

$\mathbf{u} \cdot \mathbf{v} = 1 \cdot (-2) + 1 \cdot 1 + 2 \cdot 2 + 2 \cdot 0 = 3$

$\cos\theta = \dfrac{u \cdot v}{\|u\| \cdot \|v\|} = \dfrac{3}{\sqrt{10}\sqrt{9}} = \dfrac{1}{\sqrt{10}}$

$\mathbf{p} = \dfrac{u \cdot v}{\|v\|^2} v = \dfrac{3}{9}\vec{v} = \dfrac{1}{3}\begin{pmatrix} -2 \\ 1 \\ 2 \\ 0 \end{pmatrix}$

4. *(10 points)* Consider the vector space $C[0, 1]$ with inner product defined by $\langle f, g \rangle = \int_0^1 f(x)g(x)\,dx$.

(a) Show that 1 and $2x - 1$ are orthogonal:

$$\langle 1, 2x-1 \rangle = \int_0^1 1 \cdot (2x-1)\,dx = x^2 - x\Big|_0^1 = 1 - 1 = 0$$

(b) Calculate $\|2x - 1\|$:

$$\|2x-1\|^2 = \langle 2x-1, 2x-1 \rangle = \int_0^1 (2x-1)^2\,dx$$
$$= \int_0^1 4x^2 - 4x + 1\,dx$$
$$= \tfrac{4}{3}x^3 - 2x^2 + x\Big|_0^1$$
$$= \left(\tfrac{4}{3} - 2 + 1\right) - 0$$
$$= \tfrac{1}{3}$$

Answer: $\|2x - 1\| = \boxed{\dfrac{1}{\sqrt{3}}}$

don't forget to take square root!

5. (*10 points*) Let V be a vector space with inner product $\langle\, ,\, \rangle$. Prove that

$$\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2 = 2\left(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2\right) \qquad \forall \mathbf{x}, \mathbf{y} \in V.$$

Use two-column format, giving a reason for each step in terms of properties of the inner product.

$$\|x+y\|^2 + \|x-y\|^2 = \langle x+y, x+y\rangle + \langle x-y, x-y\rangle \qquad \text{Def. of norm}$$

$$= [\langle x,x\rangle + \cancel{\langle x,y\rangle} + \cancel{\langle y,x\rangle} + \langle y,y\rangle] + [\langle x,x\rangle - \cancel{\langle x,y\rangle} - \cancel{\langle y,x\rangle} + \langle y,y\rangle] \qquad \text{inner product is bilinear}$$

$$= 2[\langle x,x\rangle + \langle y,y\rangle]$$

$$= 2(\|x\|^2 + \|y\|^2) \qquad \text{Def. of norm.}$$

6. (*20 points*) Use the Gram-Schmidt process to find an orthonormal basis of of $\mathbb{R}^3$ starting from the vectors

$$\mathbf{x}_1 = \begin{pmatrix}2\\1\\2\end{pmatrix} \qquad \mathbf{x}_2 = \begin{pmatrix}1\\1\\0\end{pmatrix} \qquad \mathbf{x}_3 = \begin{pmatrix}-1\\4\\2\end{pmatrix}$$

Number your steps and show your work.

Step 1 $\quad e_1 = \dfrac{x_1}{\|x_1\|} = \dfrac{1}{3}\begin{pmatrix}2\\1\\2\end{pmatrix}$

Step 2 $\quad y_2 = x_2 - \langle x_2, e_1\rangle e_1 = \begin{pmatrix}1\\1\\0\end{pmatrix} - \underbrace{\left[\begin{pmatrix}1\\1\\0\end{pmatrix}\cdot\frac{1}{3}\begin{pmatrix}2\\1\\2\end{pmatrix}\right]}_{=1}\cdot\frac{1}{3}\begin{bmatrix}2\\1\\2\end{bmatrix}$

$$= \frac{1}{3}\begin{pmatrix}3\\3\\0\end{pmatrix} - \frac{1}{3}\begin{pmatrix}2\\1\\2\end{pmatrix} = \frac{1}{3}\begin{pmatrix}1\\2\\-2\end{pmatrix}$$

$e_2 = \frac{1}{3}\begin{pmatrix}1\\2\\-2\end{pmatrix}$ already normalized!

Step 3 $\quad y_3 = x_3 - \langle x_3, e_1\rangle e_1 - \langle x_3, e_2\rangle e_2.$

$$= \begin{pmatrix}-1\\4\\2\end{pmatrix} - \underbrace{\left[\begin{pmatrix}-1\\4\\2\end{pmatrix}\cdot\frac{1}{3}\begin{pmatrix}2\\1\\2\end{pmatrix}\right]}_{\frac{6}{3}=2}\cdot\frac{1}{3}\begin{bmatrix}2\\1\\2\end{bmatrix} - \underbrace{\left[\begin{pmatrix}-1\\4\\2\end{pmatrix}\cdot\frac{1}{3}\begin{pmatrix}1\\2\\-2\end{pmatrix}\right]}_{\frac{3}{3}=1}\cdot\frac{1}{3}\begin{pmatrix}1\\2\\-2\end{pmatrix}$$

easier to make common factor of $\frac{1}{3}$

$$= \frac{1}{3}\begin{pmatrix}-3\\12\\6\end{pmatrix} - \frac{2}{3}\begin{pmatrix}2\\1\\2\end{pmatrix} - \frac{1}{3}\begin{pmatrix}1\\2\\-2\end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix}-8\\8\\4\end{pmatrix} \longleftarrow \text{normalizing this is the same as normalizing } \begin{pmatrix}-2\\2\\1\end{pmatrix}$$

e_3 = normalization of $\begin{pmatrix}-2\\2\\1\end{pmatrix} = \frac{1}{3}\begin{pmatrix}-2\\2\\1\end{pmatrix}$

ON basis:
$\frac{1}{3}\begin{pmatrix}2\\1\\2\end{pmatrix}, \frac{1}{3}\begin{pmatrix}1\\2\\-2\end{pmatrix}, \frac{1}{3}\begin{pmatrix}-2\\2\\1\end{pmatrix}$

7. (*16 points*) Use the Least Squares Procedure to find the line $y = ax + b$ in the plane that best fits the data points $\{(-2,-2),\ (-1,0),\ (1,1),\ (2,3)\}$.

(a) What is the overdetermined linear system to be solved?

Passing thru $(-2,-2) \Rightarrow -2a+b=-2$
thru $(-1,0) \Rightarrow -a+b=0$
thru $(1,1) \Rightarrow a+b=1$
thru $(2,3) \Rightarrow 2a+b=3$

L System.

(b) This system has the form $A\mathbf{x} = \mathbf{b}$ for what matrix A and vector $\mathbf{b}$?

$$A = \begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \qquad b = \begin{pmatrix} -2 \\ 0 \\ 1 \\ 3 \end{pmatrix}$$

(c) Find the Least Square solution.

$$A^TA = \begin{pmatrix} -2 & -1 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 10 & 0 \\ 0 & 4 \end{pmatrix} \Rightarrow (A^TA)^{-1} = \frac{1}{40}\begin{pmatrix} 4 & 0 \\ 0 & 10 \end{pmatrix}$$

$$A^Tb = \begin{pmatrix} -2 & -1 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} -2 \\ 0 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 11 \\ 2 \end{pmatrix}$$

Least Square Sol'n : $\vec{x} = (A^TA)^{-1}A^Tb = \frac{1}{20}\begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 11 \\ 2 \end{pmatrix} = \frac{1}{20}\begin{pmatrix} 22 \\ 10 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1.1 \\ \frac{1}{2} \end{pmatrix}$$

(d) What is the equation of the line of best fit?

$$\boxed{y = 1.1\,x + \tfrac{1}{2}}$$