Math 309 §2 Name SOLUTIONS

Fall 2018

Exam 2

Directions: Do all problems (100 points total). You must *show all steps and explain your reasoning* to receive full credit. No books, notes, or electronic devices are allowed.

1. (*20 points*) Let V and W be vector spaces. Complete the definitions *as briefly as possible:*

(a) A collection of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ are *linearly independent* if ...

$\sum_{i=1}^{n} \alpha_i v_i = 0$ implies $\alpha_i = 0 \ \forall i$.

(b) We say that the *dimension* of V is n if ...

$\exists$ a basis with n elements.

(c) A mapping $L : V \to W$ between vector spaces is a *linear transformation* if ...

$L(\alpha v_1 + \beta v_2) = \alpha L(v_1) + \beta L(v_2)$ $\forall \alpha, \beta \in \mathbb{R}$ $\forall v_1, v_2 \in V$.

(d) The *kernel* of a linear transformation $L : V \to W$ is defined by

$$\ker L = \{ v \in V \mid Lv = 0 \}.$$

(e) For an $n \times n$ matrice A and B, one has:

- $\det(AB) =$ $\det A \cdot \det B$
- A is non-singular $\Leftrightarrow$ $\det A \neq 0$.

2. (*14 points*) Circle (T) for TRUE, circle (F) for FALSE.

(a) For any $a, b, c \in \mathbb{R}$, $L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x+1 \\ x+y \end{pmatrix}$ defines a linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^2$. T F

Since $L(0) \neq 0$

(b) If $L : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation and $L\mathbf{x} = L\mathbf{y}$, then the vectors $\mathbf{x}$ and $\mathbf{y}$ must be equal. T F

e.g. the map $L\vec{x} = 0 \ \forall x$ is linear

(c) Every set of 3 vectors in $\mathbb{R}^5$ can be expanded to a basis. T F

Tricky! $\Rightarrow$ Any linearly independent set v_1, v_2, v_3 can be extended to a basis. But $\{v, 2v, 3v\}$ cannot be.

(d) Any set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ of four linearly independent vectors in $\mathbb{R}^4$ is a basis of $\mathbb{R}^4$. T F

YES, by the "Two-for-One" Lemma.

(e) The range of a linear map $T : \mathbb{R}^5 \to \mathbb{R}^9$ has dimension at least 5. T F

Again consider $T(x) = 0$ $\forall x$.

(f) For every $n \times n$ matrix A, $\det(3A) = 3 \det A$. T F

e.g. $\det\begin{pmatrix} 3a & 3b \\ 3c & 3d \end{pmatrix} = 3 \det\begin{pmatrix} a & b \\ 3c & 3d \end{pmatrix} = 9 \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

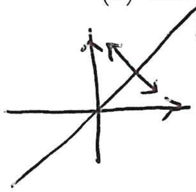
(g) If the row echelon form of an $n \times n$ matrix A has a pivot in every column then $\det(A) \neq 0$. T F

3. (*2+4+4+4 points*) Quick answers:

(a) If a 4×7 matrix $A = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$ has rank 2, then its nullity is $\underline{5}$.

rank + nullity = # of columns = 7

(b) Let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be reflection through the line $y = x$. By drawing a picture, one sees that



$R\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $R\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Hence the matrix of R is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

(c) Do the polynomials $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\} = \{x^3 + x,\ x^2 - 2x,\ x\}$ span $\mathbf{P}_4$? Why or why not?

No, $\dim P_4 = 4$, so we need 4 vectors to span P_4.

(d) Are the functions $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\} = \{x,\ x^2,\ e^x\}$ linearly independent in $C[0,1]$? Why or why not?

YES The Wronskian $W = \begin{vmatrix} x & x^2 & e^x \\ 1 & 2x & e^x \\ 0 & 2 & e^x \end{vmatrix} = x \begin{vmatrix} 2x & e^x \\ 2 & e^x \end{vmatrix} - 1 \cdot \begin{vmatrix} x^2 & e^x \\ 2 & e^x \end{vmatrix}$

$= (2x^2 - 2x)e^x - (x^2 - 2)e^x = (x^2 - 2x + 2)e^x$ is $\neq 0$.

4. (*16 points*) Let $L : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear transformation given by the matrix $A = \begin{pmatrix} 1 & 3 & 9 \\ 1 & 0 & 3 \\ 0 & 1 & 2 \\ -2 & 3 & 0 \end{pmatrix}$.

(a) Put in RRE form: $\begin{pmatrix} 1 & 3 & 9 \\ 1 & 0 & 3 \\ 0 & 1 & 2 \\ -2 & 3 & 0 \end{pmatrix} \approx \begin{pmatrix} 1 & 3 & 9 \\ 0 & -3 & -6 \\ 0 & 1 & 2 \\ 0 & 9 & 18 \end{pmatrix} \approx \begin{pmatrix} 1 & 3 & 9 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix} \approx \begin{pmatrix} 1 & 3 & 9 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \approx \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(b) Write down a basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ for the column space of A (*Caution: the column space is NOT preserved by row operations*):

Pivots in 1^{st} and 2^{nd} $\Rightarrow$ take 1^{st} and 2^{nd} columns of original matrix A

$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}$ $v_2 = \begin{pmatrix} 3 \\ 0 \\ 1 \\ 3 \end{pmatrix}$.

(c) What is the rank(A)? 2 [Column space = span of v_1, v_2 above has dimension 2]

(d) Using the Rank-Nullity Theorem, what is nullity(A)? *Show your computation.*

Rank L + Nullity(L) = # columns = 3

$\Rightarrow$ nullity(A) = 3 − 2 = 1

(e) Write down a basis for the null space $N(A)$: Solve $A\vec{x} = 0$. Using the RRE matrix,

$$\begin{array}{c} \begin{matrix} x & y & \alpha & \end{matrix} \\ \left(\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \end{array} \Rightarrow \begin{cases} x + 3\alpha = 0 \\ y + 2\alpha = 0 \\ z = \alpha \end{cases} \quad \text{So} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \alpha \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix}$$

Hence $v = \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix}$ is a basis of $N(A)$.

5. (*10 points*) Let $A = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 5 & 4 & -1 \\ 0 & -1 & 1 & 6 \\ 1 & 5 & 11 & 0 \end{pmatrix}$. One can find $\det A$ using row operations.

(a) Complete the calculation below. *Fill in the constant (possibly 1) that appears in front at each step.*

$$\begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 5 & 4 & -1 \\ 0 & -1 & 1 & 6 \\ 1 & 5 & 11 & 0 \end{vmatrix} = (-1) \begin{vmatrix} 1 & 5 & 4 & -1 \\ 0 & 1 & 2 & 3 \\ 0 & -1 & 1 & 6 \\ 1 & 5 & 11 & 0 \end{vmatrix} = (-1) \begin{vmatrix} 1 & 5 & 4 & -1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & 9 \\ 0 & 0 & 7 & 1 \end{vmatrix} = (-3) \begin{vmatrix} 1 & 5 & 4 & -1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 7 & 1 \end{vmatrix}$$

$$= (-3) \begin{vmatrix} 1 & 5 & 4 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & -20 \end{vmatrix} = \underbrace{(-3) \cdot 1 \cdot 1 \cdot 1 \cdot (-20)}_{\text{product of diagonal entries}} = 60$$

(b) Are the column vectors of the matrix A linearly independent? YES NO.

Since $\det A \neq 0$.

6. (*12 points*) Complete the following proof:

Lemma 0.1. *If $L : V \to W$ is a linear transformation, then image(L) is a vector subspace of W.*

Proof. For any $\mathbf{x}, \mathbf{y} \in$ image(L) there are vectors $\mathbf{v}, \mathbf{w} \in V$ with $L\mathbf{v} =$ __x__ and $L\mathbf{w} =$ __y__. Then for any $\alpha, \beta \in \mathbb{R}$,

$$\begin{aligned} \alpha\mathbf{x} + \beta\mathbf{y} &= \alpha Lv + \beta Lw && \text{substitution} \\ &= L(\alpha v + \beta w) && \text{because } \underline{L \text{ is a linear transformation}} \end{aligned}$$

Therefore __$\alpha v + \beta w$__ is in __im L__, so image(L) is a vector subspace. □

7. (*14 points*) Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation

$$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x - y \\ x + 4y \end{pmatrix},$$

and let $B = \{\mathbf{v}_1, \mathbf{v}_2\}$ be the basis consisting of $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 7 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$.

(a) What is the matrix of L in the standard basis?

$$A = [L]_{EE} = \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix}$$

(b) What is the matrix $[L]_{EB}$ of L from the B-basis to the standard basis? (This is the easy case).

$$[L]_{EB} = \begin{pmatrix} | & | \\ Lv_1 & Lv_2 \\ | & | \end{pmatrix} = \begin{pmatrix} -3 & -2 \\ 30 & 17 \end{pmatrix} \quad \text{since} \quad \begin{cases} Lv_1 = \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 2 \\ 7 \end{pmatrix} = \begin{pmatrix} -3 \\ 30 \end{pmatrix} \\ Lv_2 = \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -2 \\ 17 \end{pmatrix} \end{cases}$$

(c) What is the transition matrix U_{EB} from the B-basis to the standard basis?

$$U_{EB} = \begin{pmatrix} 2 & 1 \\ 7 & 4 \end{pmatrix}$$

(d) What is U_{BE}?

$$U_{BE} = U_{EB}^{-1} = \begin{pmatrix} 2 & 1 \\ 7 & 4 \end{pmatrix}^{-1} = \frac{1}{1}\underline{\begin{pmatrix} 4 & -1 \\ -7 & 2 \end{pmatrix}}$$

(e) What is the matrix $[L]_{BB}$ of L in the A-basis?

$$[L]_{BB} = \underbrace{U_{BE}}_{\text{from (d)}} \underbrace{[L]_{EB}}_{\text{from (b)}}$$

$$= \begin{pmatrix} 4 & -1 \\ -7 & 2 \end{pmatrix}\begin{pmatrix} -3 & -2 \\ 30 & 17 \end{pmatrix} = \begin{pmatrix} -42 & -25 \\ 81 & 48 \end{pmatrix}$$

Alternatively, use $[L]_{BB} = U_{BE}[L]_{EE}U_{EB}$

and matrices from (d) (a) (c)