Name $\boxed{\text{SOLUTIONS}}$

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Math 309 §2 — Exam 1

Do all 7 problems. You must show your work to receive credit. No books, notes, or electronic devices.

1. (*21 points*) Consider the system $\begin{cases} x + y - w &= 0 \\ 3x + 2y - z &= 1 \\ 4x + 5y + z - 5w &= -1 \end{cases}$

(a) Write the system as an augmented matrix and find the reduced row echelon form. *Label your row operations.*

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & -1 & 0 \\ 3 & 2 & -1 & 0 & 1 \\ 4 & 5 & 1 & -5 & -1 \end{array}\right) \underset{\substack{R_2-3R_1 \\ R_3-4R_1}}{\approx} \left(\begin{array}{cccc|c} 1 & 1 & 0 & -1 & 0 \\ 0 & -1 & -1 & 3 & 1 \\ 0 & 1 & 1 & -1 & -1 \end{array}\right) \underset{\substack{R_1+R_2 \\ -R_2}}{\approx} \left(\begin{array}{cccc|c} 1 & 0 & -1 & 2 & 1 \\ 0 & 1 & 1 & -3 & -1 \\ 0 & 1 & 1 & -1 & -1 \end{array}\right)$$

$$\underset{R_3-R_2}{\approx} \left(\begin{array}{cccc|c} 1 & 0 & -1 & 2 & 1 \\ 0 & 1 & 1 & -3 & -1 \\ 0 & 0 & 0 & 2 & 0 \end{array}\right) \underset{\substack{\frac{1}{2}R_3 \\ R_1-R_3}}{\approx} \left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & -3 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right) \underset{R_2+3R_3}{\approx} \underbrace{\left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)}_{\text{RRE form}}$$

(b) Write down the solution set S in terms of free variables. *Use set notation.*

$$\begin{aligned} x - \alpha &= 1 \\ y + \alpha &= -1 \\ z &= \alpha \\ w &= 0 \end{aligned} \quad \Rightarrow \quad \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 1+\alpha \\ -1-\alpha \\ \alpha \\ 0 \end{pmatrix}$$

Solution Set: $S = \left\{ \begin{pmatrix} 1+\alpha \\ -1-\alpha \\ \alpha \\ 0 \end{pmatrix} \middle| \ \alpha \in \mathbb{R} \right\}$

2. (*18 points*) Write (T) if TRUE or (F) if FALSE.

(a) For a homogeneous linear system $A\mathbf{x} = \mathbf{0}$, the number of free variables in the solution set is equal to the number of non-pivot columns. T

(b) If the augmented matrix of a linear system has only 0s in its bottom row, then the system is inconsistent. F

(c) The product AB of matrices is defined only if A and B are both $n \times n$ square matrices. F

(d) For matrices A and B, $AB = BA$ whenever both sides are defined. F

(e) For matrices A, B and C, $(AB)C = A(BC)$ whenever both sides are defined. T

(f) $(AB)^T = A^T B^T$ for all $n \times n$ matrices A and B. F

(g) If matrices A and B are row-equivalent, then $A = EB$ for some elementary matrix E. T

(h) The xy plane is a vector subspace of $\mathbb{R}^3$. T

(i) If S is a subspace of a vector space V and $\mathbf{x}, \mathbf{y}, \mathbf{z} \in S$, then $2\mathbf{x} + 8\mathbf{y} - 3\mathbf{z} \in S$. T

3. (*12 points*) Complete the definition or statement. *Make your wording precise.*

(a) The **inverse** of an $n \times n$ matrix A is an $n \times n$ matrix B such that ...

$AB = I_n$ and $BA = I_n$

(b) The **null space** of an $n \times n$ matrix A is

$$N(A) = \{ \vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0} \}.$$

(c) **Theorem.** *For an $n \times n$ matrix A, the following are equivalent:*

(a) A is invertible.

(a) The linear system $A\mathbf{x} = \mathbf{0}$ has no solution except $\vec{x} = \vec{0}$

(c) A is row equivalent to I_n

(d) The matrix $\begin{pmatrix} 1 & \alpha \\ -3 & 4 \end{pmatrix}$ is non-singular for every value of α except $\alpha =$ $-4/3$

Reason: Invertible unless $1 \cdot 4 - \alpha(-3) = 0 \Rightarrow 4 + 3\alpha = 0 \Rightarrow \alpha = -4/3$.

4. (*9 points*) Complete the proof by filling in the blanks. *The axioms* **A1** – **A8** *are listed at the end of the exam.*

Cancellation Law: If $\mathbf{v} + \mathbf{x} = \mathbf{w} + \mathbf{x}$ then $\mathbf{v} = \mathbf{w}$.

Proof:

$\mathbf{v}$	$= \mathbf{v} + \mathbf{0}$	A3
	$= v + (x + -x)$	A4
	$= (\mathbf{v} + \mathbf{x}) + -\mathbf{x}$	**A2**
	$= (w + x) + (-x)$	*given*
	$= w + (x + -x)$	A2
	$= w + \vec{0}$	A4
	$= \mathbf{w}$	A3

□

5. (*15 points*) Find the inverse of $B = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 2 & -1 \\ 0 & 0 & 3 \end{pmatrix}$. *Do not label row operations.*

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array}\right) \underset{\substack{R_2+R_1 \\ \frac{1}{3}R_3}}{\approx} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 3 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{3} \end{array}\right) \underset{\frac{1}{3}R_2}{\approx} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{3} \end{array}\right)$$

$$\underset{R_1-R_2-R_3}{\approx} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{3} \end{array}\right)$$

$$\Rightarrow \quad B^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

6. (*10 points*) A 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is called *symmetric* if $b = c$.

Let V be the vector space of all 2×2 matrices, and S the set of all symmetric 2×2 matrices. **Prove that S is a subspace of V.** *Use sentences to make your reasoning clear.*

Proof. For any two elements $A, B \in S$, we can write $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ and $B = \begin{pmatrix} d & e \\ e & f \end{pmatrix}$ for some $a,b,c,d,e,f \in \mathbb{R}$.

Then

- $A + B = \begin{pmatrix} a & b \\ b & c \end{pmatrix} + \begin{pmatrix} d & e \\ e & f \end{pmatrix} = \begin{pmatrix} a+d & b+e \\ b+e & c+f \end{pmatrix}$ is symmetric, so in S
- $\alpha A = \alpha \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} \alpha a & \alpha b \\ \alpha b & \alpha c \end{pmatrix}$ is also symmetric, so in S

Therefore S is a subspace of V $\blacksquare$

7. (*15 points*) Suppose that a matrix X satisfies $XA + B = A$ for some matrices A and B.

(a) Algebraically solve for X in terms of A and B.

$$XA + B = A$$

$$\Rightarrow \quad XA = A - B$$

$$\Rightarrow \quad \underbrace{XA \cdot A^{-1}}_{= \text{Identity.}} = (A-B)A^{-1}$$

$$\Rightarrow \quad X = (A-B)A^{-1}$$

(b) Find X for in the case where $A = \begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 \\ 0 & 7 \end{pmatrix}$.

$$A^{-1} = \frac{1}{2\cdot 5 - 3\cdot 1}\begin{pmatrix} 5 & -3 \\ -1 & 2 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 5 & -3 \\ -1 & 2 \end{pmatrix}$$

$$A - B = \begin{pmatrix} 2-2 & 3-1 \\ 1-0 & 5-7 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 1 & -2 \end{pmatrix}$$

$$X = (A-B)A^{-1} = \begin{pmatrix} 0 & 2 \\ 1 & -2 \end{pmatrix} \cdot \frac{1}{7}\begin{pmatrix} 5 & -3 \\ -1 & 2 \end{pmatrix}$$

$$X = \frac{1}{7}\begin{pmatrix} -2 & 4 \\ 7 & -7 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} -\frac{2}{7} & \frac{4}{7} \\ 1 & -1 \end{pmatrix}.$$

The axioms of a vector space V: for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$ and $\alpha, \beta \in \mathbb{R}$,

A1. $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$

A2. $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{y} + (\mathbf{x} + \mathbf{z})$

A3. $\exists$ a vector $\mathbf{0} \in \mathbf{V}$ s.t. $\mathbf{x} + \mathbf{0} = \mathbf{x}$ for all $\mathbf{x} \in V$.

A4. For each $\mathbf{x} \in V$, there is a vector $-\mathbf{x} \in V$ such that $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$

A5. $\alpha(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}$

A6. $(\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$

A7. $\alpha(\beta\mathbf{x}) = (\alpha\beta)\mathbf{x}$

A8. $1 \cdot \mathbf{x} = \mathbf{x}$.