Example 3.9 Evaluate the following integrals

(a) \( \int_{1}^{8} \frac{dx}{\sqrt{x^2}} \)

Solution:

\[
\int_{1}^{8} \frac{dx}{\sqrt{x^2}} = \int_{1}^{8} x^{-2/3} \, dx
\]
\[
= \left[ 3x^{1/3} \right]_{1}^{8}
\]
\[
= 3\sqrt[3]{8} - 3\sqrt[3]{1}
\]
\[
= 2 - 1
\]
\[
= 1
\]

(b) \( \int_{0}^{\pi/3} \sin(\theta) \, d\theta \)

Solution:

\[
\int_{0}^{\pi/3} \sin(\theta) \, d\theta = \left[ -\cos(\theta) \right]_{0}^{\pi/3}
\]
\[
= -\cos\left(\frac{\pi}{3}\right) + \cos(0)
\]
\[
= -\frac{1}{2} + 1
\]
\[
= \frac{1}{2}
\]

(c) \( \int_{-2}^{2} f(x) \, dx \), where \( f(x) = \begin{cases} 
2 & \text{if } -2 \leq x \leq 0 \\
4 - x^2 & \text{if } 0 < x \leq 2 
\end{cases} \)

Solution:

\[
\int_{-2}^{2} f(x) \, dx = \int_{-2}^{0} 2 \, dx + \int_{0}^{2} (4 - x^2) \, dx
\]
\[
= 2x \bigg|_{-2}^{0} + \left[ 4x - \frac{x^3}{3} \right]_{0}^{2}
\]
\[
= 2(0) - 2(-2) + 4(2) - \frac{8}{3}
\]
\[
= 12 \frac{8}{3}
\]
\[
= \frac{28}{3}
\]
(d) \[ \int_{-1}^{5} |3x - 6| \, dx \]

**Solution:** Since \(|3x - 6| = 0\) when \(x = 2\), we have

\[
\int_{-1}^{5} |3x - 6| \, dx = \int_{-1}^{2} (6 - 3x) \, dx + \int_{2}^{5} (3x - 6) \, dx
\]

\[
= \left[ 6x - \frac{3}{2}x^2 \right]_{-1}^{2} + \left[ \frac{3}{2}x^2 - 6x \right]_{2}^{5}
\]

\[
= (12 - 6 + 6) + \left( \frac{75}{2} - 30 - 6 + 12 \right)
\]

\[
= \frac{78}{2} - 12
\]

\[
= 39 - 12
\]

\[
= 27
\]

**Example 3.10** Identify what is wrong with the evaluation:

\[\int_{-1}^{1} \frac{1}{x^2} \, dx = \left[ -x^{-1} \right]_{-1}^{1}
\]

\[
= (-1)^{-1} + (-1)^{-1}
\]

\[
= -1 - 1
\]

\[
= -2
\]

**Solution:** The **Fundamental Theorem of Calculus** only applies if the function is continuous on the interval \([a, b]\), which in this case is \([-1, 1]\). But the function \(\frac{1}{x^2}\) is not continuous at \(x = 0\), so the theorem does not apply.

**Example 3.11** Find the derivatives of the following functions:

(a) \(F(x) = \int_{1}^{x} \sin^4(t) \, dt\)

**Solution:** If we let \(f(x) = \int_{1}^{x} \sin^4(t) \, dt\), then the **Fundamental Theorem** says that

\[
f'(x) = \sin^4(x)
\]

Since \(F(x) = f\left(\frac{1}{x}\right)\), the chain rule says that

\[
F'(x) = f'\left(\frac{1}{x}\right) \cdot \left( -\frac{1}{x^2} \right)
\]

\[
= -\frac{1}{x^2} \sin^4\left(\frac{1}{x}\right)
\]

(b) \(G(x) = \int_{\sin(x)}^{1} \cos^2(\theta) \, d\theta\)
Solution: If we let $g(x) = \int_x^1 \cos^2(\theta) d\theta = -\int_1^x \cos^2(\theta) d\theta$, then the **Fundamental Theorem** says that

$$g'(x) = -\cos^2(x)$$

Since $G(x) = g(sin(x))$, the chain rule says that

$$G'(x) = g'(sin(x)) \cdot \cos(x)$$

$$= -\cos^2(sin(x)) \cdot \cos(x)$$

(c) $H(x) = \int_{\tan(x)}^{x^2} \frac{dt}{\sqrt{2 + t^4}}$

Solution: If we let $h(x) = \int_0^x \frac{dt}{\sqrt{2 + t^4}}$, then the **Fundamental Theorem** says that

$$h'(x) = \frac{1}{\sqrt{2 + x^4}}$$

Since $H(x) = h(x^2) - h(\tan(x))$, the chain rule says that

$$H'(x) = h'(x^2) \cdot 2x - h'(\tan(x)) \cdot \sec^2(x)$$

$$= \frac{2x}{\sqrt{2 + x^4}} - \frac{\sec^2(x)}{\sqrt{2 + \tan^4(x)}}$$