Example 9.8 Use Newton’s method to approximate all roots of the equation \( \frac{1}{x} = 1 + x^3 \). Sketch these graphs and use the Intermediate Value Theorem to select the best integer values for starting points. Stop at \( x_2 \).

Solution. If we graph the functions \( x^3 + 1 \) and \( \frac{1}{x} \), we see:

![Graphs of f(x) = x^3 + 1 and g(x) = 1/x]

We can see that there are two intersection points — one is between \(-2\) and \(-1\), and the other is between \(0\) and \(1\). Define the function

\[
f(x) = x^3 + 1 - \frac{1}{x}
\]

The intersection points we are looking for are the zeros of \( f(x) \). The derivative is given by

\[
f'(x) = 3x^2 + \frac{1}{x^2}
\]

First let’s approximate the left intersection point (which is a zero of \( f(x) \)), using \( x_1 = -1 \) for our initial guess:

\[
x_2 = -1 - \frac{f(-1)}{f'(-1)} = -1 - \frac{1}{4} = -\frac{5}{4}
\]

Now let’s approximate the right intersection point, using \( x_1 = 1 \) as our initial guess:

\[
x_2 = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{1}{4} = \frac{3}{4}
\]

Example 9.9 Use Newton’s method to approximate \( \sqrt[3]{29} \). Calculate up to \( x_3 \). Do not simplify.

Solution. We will use the function \( f(x) = x^3 - 29 \), with initial guess \( x_1 = 3 \) (since \( 3^3 = 27 \)):

\[
x_2 = 3 - \frac{f(3)}{f'(3)} = 3 - \frac{-2}{27} = 3 + \frac{2}{27} = \frac{83}{27}
\]

\[
x_3 = \frac{83}{27} - \frac{\left(\frac{83}{27}\right)^3 - 29}{3 \left(\frac{83}{27}\right)^2}
\]
Example 9.10 Find the most general antiderivative of the functions below:

(a) \( f(x) = (3 + 2x)^2 \)

**Solution.** If we multiply it out, we get

\[
 f(x) = (3 + 2x)^2 = 9 + 12x + 4x^2
\]

The antiderivative is then

\[
9x + 6x^2 + \frac{4}{3}x^3 + C
\]

(b) \( f(x) = \frac{3 - x + 5x^3}{x^3} \)

**Solution.** We can re-write \( f \) as:

\[
f(x) = \frac{3}{x^3} - \frac{1}{x^2} + 5 = 3x^{-3} - x^{-2} + 5
\]

The antiderivative is then

\[
-\frac{3}{2}x^{-2} + x^{-1} + 5x + C = -\frac{3}{2x^2} + \frac{1}{x} + 5x + C
\]