Example 7.8 Find the point on the curve \( y = \sqrt{x} \) that is closest to the point \((3, 0)\).

Solution. All points \((x, y)\) on the graph are of the form \((x, \sqrt{x})\). If \(D\) is the distance from \((x, y)\) to the point \((3, 0)\), then the Pythagorean distance formula gives us that

\[
D^2 = (x - 3)^2 + y^2 = (x - 3)^2 + (\sqrt{x})^2 = x^2 - 5x + 9
\]

We want to minimize \(D\), since we are looking for the closest point. So we need to find critical points of \(D\). Differentiating both sides (using implicit differentiation), we get

\[
2D \cdot D' = 2x - 5
\]

Solving this for \(D'\), we get

\[
D' = \frac{2x - 5}{2D} = \frac{2x - 5}{2\sqrt{x^2 - 5x + 9}}
\]

This is zero when \(x = \frac{5}{2}\), so this is a critical point (in fact, the only one). Since \(D' < 0\) when \(x < \frac{5}{2}\), and \(D' > 0\) when \(x > \frac{5}{2}\), this is indeed a local (and global) minimum. So the closest point on the graph to the point \((3, 0)\) is the point \((\frac{5}{2}, \sqrt{\frac{5}{2}})\).

Example 7.9 A right circular cylinder is inscribed in a right circular cone of radius 2 and height 3. Find the largest possible volume of such a cylinder.

Solution. Let \(r\) be the radius of the base of the cylinder and \(h\) the height of the cylinder. The space inside the cone on top of the cylinder is another, smaller, cone, which is similar (proportional) to the original cone. The base of this small cone is the same as the top of the cylinder, so the small cone has radius \(r\). Since “height-to-radius” ratio of the big cone is \(\frac{3}{2}\), it must be the same for the small cone. This means that the height of the small cone is \(\frac{3}{2}r\). Finally, we can see that the height of the cylinder plus the height of this small cone must add up to the height of the big cone, so we get

\[
h = 3 - \frac{3r}{2}
\]

Therefore the volume of the cylinder is

\[
V = \pi r^2 \cdot h = 3\pi r^2 - \frac{3\pi}{2} r^3
\]

Notice that the radius cannot exceed 2 (the radius of the big cone), so the natural domain for the function \(V(r)\) is \([0, 2]\). We want to maximize this function on this domain. So we differentiate and find critical points:

\[
V' = 6\pi r - \frac{9\pi}{2} r^2 = 3\pi r \left(2 - \frac{3}{2}r\right)
\]

We see that \(V' = 0\) either when \(r = 0\) or when \(2 - \frac{3}{2}r = 0\). The latter happens when \(r = \frac{4}{3}\). We have found the critical point. So we just have to test this and the endpoints:

\[
V(0) = 0
V \left(\frac{4}{3}\right) = 16\pi \frac{9}{9}
V(2) = 0
\]

Therefore the maximum volume of such a cylinder is \(16\pi \frac{9}{9}\), which happens when the radius of the base is \(r = \frac{4}{3}\).

Example 7.10 A right circular cylinder is inscribed in a sphere of radius 2. Find the largest possible surface area of such a cylinder.
Solution. Let $h$ be the height of the inscribed cylinder, and $r$ its radius. If you look at a cross section of the picture, you can form a right triangle where the horizontal leg is a radius of the cylinder, the hypotenuse is a radius of the sphere, and the vertical leg has length $\frac{h}{2}$. From this we get the relation

$$h = 2\sqrt{4 - r^2}$$

The surface area of the cylinder is given by

$$S = 2\pi rh = 4\pi r\sqrt{4 - r^2}$$

We want to maximize this quantity. So let’s differentiate:

$$S' = 4\pi \left(\sqrt{4 - r^2} - \frac{r^2}{\sqrt{4 - r^2}}\right)$$

To find the critical points we solve the equation $S' = 0$, and we get

$$\sqrt{4 - r^2} = \frac{r^2}{\sqrt{4 - r^2}}$$
$$4 - r^2 = r^2$$
$$4 = 2r^2$$
$$r^2 = 2$$
$$\sqrt{2} = r$$

Therefore $r = \sqrt{2}$ is a critical point, and we only have to test this and the endpoints:

$$S(0) = 0$$
$$S(\sqrt{2}) = 8\pi$$
$$S(2) = 0$$

We see that the maximum surface area is $8\pi$, which happens when the radius of the cylinder is $r = \sqrt{2}$.

7.11 What is the minimum vertical distance between the parabolas $y = x^2 + 1$ and $y = x - x^2$?

Solution. The difference, $D$, between them is given by

$$D(x) = (x^2 + 1) - (x - x^2) = 2x^2 - x + 1$$

We wish to minimize this, so let’s differentiate:

$$D'(x) = 4x - 1$$

We have a critical point at $x = \frac{1}{4}$, which is a local minimum. Since $D'$ is negative for all $x < \frac{1}{4}$ and $D'$ is positive for all $x > \frac{1}{4}$, we have that $x = \frac{1}{4}$ is in fact the global minimum.

Example 7.12 Find the area of the largest trapezoid that can be inscribed in a circle of radius 1, and whose base is a diameter of the circle.
**Solution.** The area of a trapezoid is

\[ A = \frac{b_1 + b_2}{2} \cdot h \]

where \( b_1 \) and \( b_2 \) are the lengths of the two bases, and \( h \) is the height. In this case, \( b_2 \) is the diameter of the circle, which is 2. If we draw the appropriate right triangles, then \( h = \sin(\theta) \) and \( b_1 = 2 \cos(\theta) \). So the area formula becomes

\[ A = \frac{2 \cos(\theta) + 2}{2} \cdot h = (\cos(\theta) + 1)\sin(\theta) \]

Notice that only \( \theta \) such that \( 0 \leq \theta \leq \frac{\pi}{2} \) makes sense. Taking the derivative, we get

\[ A' = \cos(\theta) (\cos(\theta) + 1) - \sin^2(\theta) \]
\[ = \cos^2(\theta) + \cos(\theta) - \sin^2(\theta) \]
\[ = \cos^2(\theta) + \cos(\theta) - 1 + (1 - \sin^2(\theta)) \]
\[ = 2 \cos^2(\theta) + \cos(\theta) - 1 \]

Using the quadratic formula, we see that \( \cos(\theta) \) is either \(-1\) or \(\frac{1}{2}\). This happens when \( \theta = \pi \) (which is not in our domain) or when \( \theta = \frac{\pi}{3} \). So \( \theta = \frac{\pi}{3} \) is a critical point for \( A \). So we only have to test this and the endpoints:

\[ A(0) = 0 \]
\[ A\left(\frac{\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) \left(\cos\left(\frac{\pi}{3}\right) + 1\right) = \frac{3\sqrt{3}}{4} \]
\[ A\left(\frac{\pi}{2}\right) = 0 \]

---

**Example 7.13** Show that of all the rectangles with given fixed perimeter, the one with the largest area is a square.

**Solution.** Let \( P \) be the perimeter of a rectangle, with side lengths \( x \) and \( y \). Then we have

\[ P = 2x + 2y \]

We can solve this for \( y \) to get \( y = \frac{P - 2x}{2} = \frac{P}{2} - x \). The area is then given by

\[ A = xy = x \left(\frac{P}{2} - x\right) \]

Notice that only \( x \) such that \( 0 \leq x \leq \frac{P}{2} \) makes sense. We want to maximize the area, so we need to differentiate and find critical points:

\[ A' = \left(\frac{P}{2} - x\right) - x = \frac{P}{2} - 2x \]

We see that \( A' = 0 \) when \( x = \frac{P}{4} \). Since \( A = 0 \) when \( x = 0 \) or \( x = \frac{P}{2} \), this must be the maximum. In this case, \( y = \frac{P}{4} \) as well, and so the rectangle is actually a square.

---

**Example 7.14** An open-top box is to be made by cutting small squares (all the same size) from the corners of a 12in-by-12in sheet of cardboard, and bending up the sides. How large should the squares cut from the corners be to make the box have the maximum possible volume (and what is the maximum volume)?
Solution. The base of the box will be a square with side length \( 12 - 2x = 2(6 - x) \), and the height of the box will be \( x \). So the volume is given by

\[
V = 4x(6 - x)^2
\]

We want to maximize this, so let’s differentiate:

\[
V' = 4(6 - x)^2 - 8x(6 - x) = 8(6 - x)(2 - x)
\]

The critical points are then \( x = 6 \) and \( 6 = 2 \). But \( x = 6 \) doesn’t make sense, because \( V(6) = 0 \). So we test the critical point and the endpoints:

\[
V(0) = 0
\]
\[
V(2) = 4(2)(4)^2 = 128
\]
\[
V(6) = 0
\]

So the maximum volume is 128, which happens when you cut 2in-by-2in squares from the corners.