Example 5.2(d) Sketch a graph of \( f(x) = \sin^3(x) = (\sin(x))^3 \).

Solution.

1. The domain of \( f \) is \((-\infty, \infty)\).
2. Since \( f(0) = 0 \), \((0, 0)\) is the \( y \)-intercept (and also an \( x \)-intercept). The \( x \)-intercepts occur when \( f(x) = 0 \), which happens at integer multiples of \( \pi \).
3. The function is odd, since \( f(-x) = -f(x) \), and it is also periodic, with period \( 2\pi \). So we will only focus on the interval \([0, 2\pi]\) from here on out.
4. There are no asymptotes.
5. Taking the derivative, we get
   \[
   f'(x) = 3\sin(x)^2 \cdot \cos(x)
   \]
   Since \( 3\sin(x)^2 \) is always non-negative, \( f'(x) \) will be positive when \( \cos(x) \) is positive, and \( f'(x) \) is negative when \( \cos(x) \) is negative. So \( f'(x) > 0 \) (and \( f(x) \) is increasing) for \( x \) in \((0, \pi/2) \cup (3\pi/2, 2\pi)\), and \( f'(x) < 0 \) (and \( f(x) \) is decreasing) for \( x \) in \((\pi/2, 3\pi/2)\).
6. Using the first derivative test, we see that \( x = \pi/2 \) is a local max, and \( x = 3\pi/2 \) is a local min.
7. First let’s take the second derivative:
   \[
   f''(x) = 3\sin(x) \left( 2\cos(x)^2 - \sin(x)^2 \right)
   \]
   This is zero when either \( 3\sin(x) = 0 \) or when \( 2\cos(x)^2 - \sin(x)^2 = 0 \). In the first case, \( 3\sin(x) = 0 \) when \( x = \pi \) in the interval \((0, 2\pi)\). In the second case, we have to solve:
   \[
   2\cos(x)^2 - \sin(x)^2 = 0
   \]
   \[
   2\cos(x)^2 = \sin(x)^2
   \]
   \[
   2 = \frac{\sin(x)^2}{\cos(x)^2}
   \]
   \[
   2 = \tan(x)^2
   \]
   \[
   \pm \sqrt{2} = \tan(x)
   \]
Let’s call $\alpha = \tan^{-1}(\sqrt{2})$. We get four solutions for $x$, since $\tan(x) = \sqrt{2}$ when $x = \alpha$ and when $x = \pi + \alpha$, and $\tan(x) = -\sqrt{2}$ when $x = \pi - \alpha$ and when $x = 2\pi - \alpha$. Plugging in points, we see that $f''(x)$ changes sign at all of these points, so they are all inflection points. The graph is concave up on

$$\{(0, \alpha) \cup (\pi - \alpha, \pi) \cup (\pi + \alpha, 2\pi - \alpha)\}$$

and concave down on

$$\{(\alpha, \pi - \alpha) \cup (\pi, \pi + \alpha) \cup (2\pi - \alpha, 2\pi)\}$$

Finally, here is the picture (the inflection points are colored blue):
Example 5.3(a) Sketch a graph of \( f(x) = x^5 - 5x \).

Solution.

1. The domain is \((-\infty, \infty)\), since it’s a polynomial.

2. \((0,0)\) is on the graph, since \(f(0) = 0\). This is the \(y\)-intercept (and also an \(x\)-intercept). To find the other \(x\)-intercepts, we solve:

\[
\begin{align*}
x^5 - 5x &= 0 \\
x(x^4 - 5) &= 0 \\
x(x^2 + \sqrt{5})(x^2 - \sqrt{5}) &= 0
\end{align*}
\]

So \( f(x) = 0 \) when either \( x = 0 \), \( x^2 + \sqrt{5} = 0 \), or \( x^2 - \sqrt{5} = 0 \). We know that \( x^2 + \sqrt{5} \) is never zero, and \( x^2 - \sqrt{5} = 0 \) when \( x = \pm \sqrt{5} \). So there are 3 \(x\)-intercepts, which are 0, \( \sqrt{5} \), and \( -\sqrt{5} \).

3. Plug in \(-x\), and compute:

\[
\begin{align*}
f(-x) &= (-x)^5 - 5(-x) \\
&= -x^5 + 5x \\
&= -(x^5 - 5x) \\
&= -f(x)
\end{align*}
\]

So \( f(x) \) is an odd function.

4. Polynomials do not have asymptotes.

5. Taking the derivative, we get

\[
f'(x) = 5x^4 - 5
\]

This is zero when \( x = 1 \) or \( x = -1 \), and so these are the critical points. To see where \( f(x) \) is increasing, we need to solve \( f'(x) > 0 \), which can be written \( 5x^4 - 5 > 0 \). This is equivalent to \( x^4 > 1 \), which happens when \( x \) is in \((-\infty, -1) \cup (1, \infty)\). Similarly, \( f'(x) < 0 \) when \( x^4 < 1 \), which happens when \( x \) is in \((-1, 1)\).

6. Using the first derivative test, we see that \( x = -1 \) is a local maximum, and \( x = 1 \) is a local minimum.
7. The second derivative is given by

\[ f''(x) = 20x^3 \]

This is positive when \( x \) is positive, and negative when \( x \) is negative.

8. The only inflection point is \( x = 0 \).

Here is the graph:
Example 5.3(b) Sketch a graph of $f(x) = \frac{x}{x-1}$.

Solution.

1. The domain is $(-\infty, 1) \cup (1, \infty)$.
2. Since $f(0) = 0$, the point $(0, 0)$ is the $y$-intercept and an $x$-intercept. It is in fact the only $x$-intercept.
3. $f(-x) = \frac{-x}{-x-1} = \frac{x}{x+1}$. This is not necessarily equal to $f(x)$ or $-f(x)$, and so this function neither even nor odd.
4. There is a vertical asymptote at $x = 1$ (since the denominator is zero there). To find horizontal asymptotes, we must take the limit of $f$ as $x \to \infty$ and $x \to -\infty$.

$$
\lim_{x \to \infty} \frac{x}{x-1} = \lim_{x \to \infty} \frac{1}{1 - \frac{1}{x}} = \frac{1}{1 - 0} = 1
$$

$$
\lim_{x \to -\infty} \frac{x}{x-1} = \lim_{x \to -\infty} \frac{1}{1 - \frac{1}{x}} = \frac{1}{1 - 0} = 1
$$

So the line $y = 1$ is the only horizontal asymptote.
5. Before we take the derivative, let’s re-write $f(x)$, by performing polynomial long division, to get

$$
f(x) = \frac{x}{x-1} = 1 + \frac{1}{x-1}
$$

In fact, at this point, we can already completely tell what the graph will look like. Notice that when we write it like this we can see that it is just a transformation of the graph of $\frac{1}{x}$, obtained by shifting right by 1, and shifting up by 1. But anyways, this formula also makes it easier to take the derivative:

$$
f'(x) = \frac{-1}{(x-1)^2}
$$

The derivative is always negative, so the function is always decreasing.
6. There are no local max or min points, since it is always decreasing.
7. Taking the second derivative, we get

\[ f''(x) = \frac{2}{(x - 1)^3} \]

This is positive when \( x > 1 \), and negative when \( x < 1 \).

8. There are no inflection points. Although the concavity changes (\( f'' \) changes from negative to positive) at \( x = 1 \), this point is not in the domain, so it is not an inflection point.

Here is the graph:
Example 5.3(c) Sketch a graph of $1 + \frac{1}{x} + \frac{1}{x^2}$.

Solution.

1. The domain is $(-\infty, 0) \cup (0, \infty)$.

2. There is no $y$-intercept, since the function is not defined at $x = 0$. There are also no $x$-intercepts, since the function is never equal to zero.

3. $f(-x) = 1 - \frac{1}{x} + \frac{1}{x^2}$, which is neither equal to $f(x)$ or to $-f(x)$, so the function is neither even nor odd.

4. There is a vertical asymptote at $x = 0$. Taking the limit as $x \to \infty$, we see
   \[
   \lim_{x \to \infty} 1 + \frac{1}{x} + \frac{1}{x^2} = 1 + 0 + 0 = 1
   \]
   Similarly, the limit as $x \to -\infty$ is also equal to 1. This means that $y = 1$ is a horizontal asymptote. Notice that for large positive values of $x$, $f(x) > 1$, but for large negative values of $x$, $f(x) < 1$. This means that the graph approaches the asymptote from above on the right, but from below on the left.

5. Taking the derivative, we get
   \[
   f'(x) = \frac{-1}{x^2} + \frac{-2}{x^3} = \frac{-(x + 2)}{x^3}
   \]
   We see that the only critical point is at $x = -2$. When $x < -2$, and when $x > 0$, $f'(x)$ is negative, and $f(x)$ is decreasing. When $-2 < x < 0$, then $f'(x) > 0$, and $f(x)$ is increasing.

6. By the first derivative test, $x = -2$ is a local minimum.
7. Taking the second derivative, we get

\[ f''(x) = \frac{2}{x^3} + \frac{6}{x^4} = \frac{2x + 6}{x^4} \]

We see that \( f''(x) = 0 \) when \( x = -3 \), \( f''(x) < 0 \) when \( x < -3 \), and \( f''(x) > 0 \) when \( x > -3 \).

8. There is one inflection point, at \( x = -3 \).

Here is the graph:
Example 5.3(d) Sketch a graph of $f(x) = x + \cos(x)$.

Solution.

1. The domain is $(-\infty, \infty)$.
2. The $y$-intercept is $(0, 1)$, since $f(0) = 1$. The $x$-intercept is not easy to write down. It happens where $x = -\cos(x)$.
3. $f(-x) = -x + \cos(-x) = -x + \cos(x)$, which is neither equal to $f(x)$ nor $-f(x)$. So this is neither even nor odd.
4. There are no asymptotes.
5. The derivative is given by
   \[ f'(x) = 1 - \sin(x) \]
   Since $\sin(x) \leq 1$, we see that $f'(x) \geq 0$ for all $x$. Therefore $f(x)$ is always increasing.
6. There are no local max or min points, since the function is always increasing.
7. The second derivative is

\[ f''(x) = -\cos(x) \]

This is zero when \( x = \frac{\pi}{2} + k\pi \) for all integers \( k \). For example, on the interval \([0, 2\pi]\), \( f''(x) = 0 \) when \( x = \frac{\pi}{2} \) and when \( x = \frac{3\pi}{2} \). It is positive on \((\pi/2, 3\pi/2)\), and changes sign at each of the zeros, and so they are all inflection points.

Here is the graph: