Example 4.9

Suppose that the graph to the right is of is of $f''$.
For which values of $x$ does $f$ have an inflection point?

Solution. Inflection points are where $f''$ changes sign. This only happens on the graph at $x = 1$, where it changes from negative to positive.

Example 4.10 Find where $f(x) = x - 4\sqrt{x}$ is concave up and where it is concave down. Where are the inflection points?

Solution. First let’s note that the domain of the function is $[0, \infty)$. Let’s differentiate the function twice:

$$f'(x) = 1 - \frac{4}{2\sqrt{x}} = 1 - 2\frac{1}{\sqrt{x}} = 1 - 2x^{-1/2}$$

$$f''(x) = -2 \cdot \left(\frac{-1}{2}\right)x^{-3/2} = x^{-3/2} = \frac{1}{(\sqrt{x})^3}$$

On the domain of $f$, we see that $f''$ is always positive, so it is always concave up. Therefore there are no inflection points.

Example 4.11 Use the second derivative test to classify the critical points of $f(x) = \frac{x^2}{x-1}$.

Solution. First we must locate the critical points. So let’s take the derivative:

$$f'(x) = \frac{2x(x-1) - x^2}{(x-1)^2} = \frac{x^2 - 2x}{(x-1)^2} = \frac{x(x-2)}{(x-1)^2}$$

We see that the critical points are $x = 0$ and $x = 2$ (since these values make $f' = 0$). To use the second derivative test, we must of course take the second derivative:

$$f''(x) = \frac{(2x-2)(x-1)^2 - 2x(x-2)(x-1)}{(x-1)^4} = \frac{2(x-1)[(x-1)^2 - x(x-2)]}{(x-1)^4} = \frac{2[x^2 - 2x + 1 - x^2 + 2x]}{(x-1)^3} = \frac{2}{(x-1)^3}$$

This is positive if $x > 1$ and negative if $x < 1$. So, using the second derivative test, $x = 0$ is a local maximum, and $x = 2$ is a local minimum.

Example 4.12 Find the limits or show that they do not exist.

$$(a) \lim_{x \to \infty} \frac{(2x + 1)^2}{(x - 1)^2(x^2 + x)}$$
**Solution.** The numerator is degree 2, and the denominator is degree 4. Since the degree on bottom is larger, the limit is zero. If you want to calculate this explicitly, you can multiply everything out, and then use the trick we did in class:

\[
\lim_{x \to \infty} \frac{(2x + 1)^2}{(x - 1)(x^2 + x)} = \lim_{x \to \infty} \frac{4x^2 + 4x + 1}{x^4 - x^3 - x^2 + x}
\]

\[
= \lim_{x \to \infty} \frac{4x^2 + 4x^3 + x^4}{1 - x^{-1} - x^{-2} + x^{-3}}
\]

\[
= 0 + 0 + 0
\]

\[
= \frac{1}{0 - 0 + 0}
\]

\[
= 0
\]

(b) \( \lim_{x \to -\infty} \frac{x^2}{\sqrt{3x^4 + 1}} \)

**Solution.** Since in the denominator, there is a 4-th degree polynomial under a square root, the denominator actually behaves like a degree 2 polynomial (since \( \frac{4}{2} = 2 \)). So divide top and bottom by \( x^2 \), and then take the limit:

\[
\lim_{x \to -\infty} \frac{x^2}{\sqrt{3x^4 + 1}} = \lim_{x \to -\infty} \frac{1}{\sqrt{\frac{3x^4 + 1}{x^4}}}
\]

\[
= \lim_{x \to -\infty} \frac{1}{\sqrt{3 + x^{-4}}}
\]

\[
= \frac{1}{\sqrt{3}}
\]

(c) \( \lim_{x \to \infty} \sqrt{x^2 + 1} \)

**Solution.** This function increases without bound as \( x \) increases. So the limit is just \( \infty \). To make this more precise, let’s show that for any number \( M \) (as large as you like), we can find an \( x \) big enough so that \( \sqrt{x^2 + 1} = M \). Just solve the equation:

\[
\sqrt{x^2 + 1} = M
\]

\[
x^2 + 1 = M^2
\]

\[
x^2 = M^2 - 1
\]

\[
x = \sqrt{M^2 - 1}
\]

**Example 4.13** Find the horizontal asymptotes of \( f(x) = \frac{4x^3 + 6x^2 - 2}{5x^3 - x + 1} \).

**Solution.**

\[
\lim_{x \to \infty} \frac{4x^3 + 6x^2 - 2}{5x^3 - x + 1} = \lim_{x \to \infty} \frac{4 + 6x^{-1} - 2x^{-3}}{5 - x^{-2} + x^{-3}}
\]

\[
= \frac{4}{5}
\]

You get the same thing if you take the limit as \( x \to -\infty \). So the only horizontal asymptote is the line \( y = 4/5 \).