Example 8.7 A sphere is growing, its volume increasing at a constant rate of 10 in$^3$ per second. Let $r(t)$, $V(t)$, and $S(t)$ be the radius, volume, and surface area of the sphere at time $t$. If $r(1) = 2$, then compute:

(a) $r'(1)$

Solution. Relate the volume and the radius by the equation

$$V = \frac{4\pi}{3}r^3$$

Remember that everything is a function of $t$, so we really have

$$V(t) = \frac{4\pi}{3}r(t)^3$$

We can differentiate both sides (with respect to $t$):

$$V'(t) = \frac{4\pi}{3} \cdot 3r(t)^2 \cdot r'(t)$$

The right-hand side follows from the chain rule. Now solve for $r'(t)$ to get:

$$r'(t) = \frac{10}{4\pi r(t)^2}$$

Now plug in $t = 1$, and use the fact that we know $r(1) = 2$:

$$r'(1) = \frac{10}{4\pi(2)^2} = \frac{10}{16\pi} = \frac{5}{8\pi}$$

So the radius is increasing at a rate $\frac{5}{8\pi}$ inches per second at $t = 1$.

(b) $S'(1)$

Solution. First write down the equation for the surface area of a sphere

$$S = 4\pi r^2$$

Remember that everything is a function of $t$, so this is

$$S(t) = 4\pi r(t)^2$$

Differentiate both sides:

$$S'(t) = 4\pi \cdot 2r(t) \cdot r'(t) = 8\pi r(t) \cdot r'(t)$$

Plugging in $t = 1$, and using part (a), we get

$$S'(1) = 8\pi r(1) \cdot r'(1) = 8\pi \cdot 2 \cdot \frac{5}{8\pi} = 10$$

So the surface area is increasing at a rate of 10 square inches per second at $t = 1$. 
Example 8.8 A circle is growing, its radius (in inches) give by \( r(t) = \sqrt{t} \) for \( t \) in seconds. How fast is the area growing at time \( t = 4 \)?

Solution. Write down the equation for the area of a circle:

\[
A = \pi r^2
\]

Remember everything is a function of \( t \), so we have

\[
A(t) = \pi r(t)^2
\]

Differentiate, to get

\[
A'(t) = 2\pi r(t) \cdot r'(t)
\]

We want to find \( A'(4) \). To do so, we need to know \( r'(t) \), so differentiate:

\[
r'(t) = \frac{1}{2\sqrt{t}}
\]

So we get

\[
A'(4) = 2\pi \cdot \sqrt{4} \cdot \frac{1}{2\sqrt{4}} = \pi
\]

Example 8.9 A right triangle is growing, with its vertical side growing at a constant rate of 1 unit per second, and its horizontal side growing at a constant rate of 2 units per second. At time \( t = 2 \) seconds, how fast is the hypotenuse growing?

Solution. Let’s call the horizontal side \( x \) and the vertical \( y \), and the hypotenuse \( h \). We need to find \( h'(2) \). First write down the Pythagorean Theorem:

\[
x^2 + y^2 = h^2
\]

Now differentiate (with respect to \( t \)), to get

\[
2x \cdot x' + 2y \cdot y' = 2h \cdot h'
\]

Solve for \( h' \) to get

\[
h' = \frac{x \cdot x' + y \cdot y'}{h}
\]

We know that \( x' = 1 \) and \( y' = 2 \). Assuming that \( x(t) = 2t \) and \( y(t) = t \), we get that at \( t = 2 \),

\[
h' = \frac{4 \cdot 2 + 2 \cdot 1}{\sqrt{4^2 + 2^2}} = \frac{10}{\sqrt{20}} = \sqrt{5}
\]