1 Area Between Curves

**Theorem 1.1.** If \( f(x) \) and \( g(x) \) are continuous functions on \([a, b]\) where \( f(x) \geq g(x) \) for all \( x \) in \([a, b]\), then the area of the region in between the graphs of \( y = f(x) \) and \( y = g(x) \) between \( x = a \) and \( x = b \) is given by

\[
\text{Area} = \int_{a}^{b} (f(x) - g(x)) \, dx
\]

**Remark 1.2.** This theorem only applies if \( f(x) \geq g(x) \) on \([a, b]\). In the more general case (where the graphs cross), we can use the following theorem.

**Theorem 1.3.** If \( f(x) \) and \( g(x) \) are continuous functions on \([a, b]\), then the area of the region in between the graphs of \( y = f(x) \) and \( y = g(x) \) between \( x = a \) and \( x = b \) is given by

\[
\text{Area} = \int_{a}^{b} |f(x) - g(x)| \, dx
\]

**Remark 1.4.** If it is more convenient, you can think of \( x \) as a function of \( y \), and integrate with respect to \( y \). For example, to find the area between the graphs of \( x = f(y) \) and \( x = g(y) \) between \( y = a \) and \( y = b \), compute

\[
\text{Area} = \int_{a}^{b} |f(y) - g(y)| \, dy
\]

**Example 1.5** (Instructor). Find the area between the graphs of \( y = -x \) and \( y = \cos(x) \) between \( x = 0 \) and \( x = \frac{\pi}{2} \).

**Example 1.6** (Instructor). Find the area enclosed by the curves \( y = \sec^2(x) \) and \( y = 8 \cos(x) \) on the interval \([-\frac{\pi}{3}, \frac{\pi}{3}] \).

**Example 1.7** (Instructor). Find the area of the region bounded by the graphs of \( y = 5x - x^2 \) and \( y = x \).

**Example 1.8** (Instructor). Find a positive value of \( c \) so that the area between \( y = x^2 + 1 \) and \( y = x - c \) from \( x = 0 \) to \( x = 1 \) is equal to 1.

**Example 1.9** (Instructor). Find the area enclosed by the line \( y = x - 1 \) and the parabola \( y^2 = 2x + 6 \).

**Example 1.10** (Student). (WW1) Find the area of the region bounded by the curves \( y = \frac{1}{2}x^2 + 6 \) and \( y = x \) and the vertical lines \( x = -3 \) and \( x = 5 \).

**Example 1.11** (Student). (WW4) Sketch the region bounded by the curves \( 2x^2 + y = 19 \) and \( x^4 - y = 5 \), then find the area of the region.

**Example 1.12** (Student). (WW6) Find \( c > 0 \) such that the area of the region enclosed by the parabolas \( y = x^2 - c^2 \) and \( y = c^2 - x^2 \) is 18.

**Example 1.13** (Student). (WW7) Consider the area between the graphs \( x + 6y = 8 \) and \( x + 8 = y^2 \). Compute this area in two different ways:

(a) by integrating with respect to \( x \)  
(Hint: split into two integrals)  
(b) by integrating with respect to \( y \)