3 The Fundamental Theorem of Calculus

**Theorem 3.1** (FTC, Part 1). If \( f \) is continuous on \([a, b]\), then the function \( g \) defined by

\[
g(x) = \int_a^x f(t) \, dt, \quad a \leq x \leq b
\]

is continuous on \([a, b]\) and differentiable on \((a, b)\), and \( g'(x) = f(x) \)

**Remark 3.2.** Here is an idea of the proof:

**Theorem 3.3** (FTC, Part 2). If \( f \) is continuous on \([a, b]\), then

\[
\int_a^b f(x) \, dx = F(b) - F(a)
\]

where \( F \) is any antiderivative of \( f \), that is, a function such that \( F' = f \).

**Remark 3.4.** Here is an idea of the proof:

**Remark 3.5.** The two parts of the FTC together state that differentiation and integration are inverse processes.

**Remark 3.6.** From our perspective FTC, Part 2 is the most important because it allows us to calculate definite integrals without having to take limits of Riemann sums!
Example 3.7 (Instructor). Evaluate the following integrals

(a) \( \int_1^9 \sqrt{x} \, dx \)

(b) \( \int_{-1}^5 (u + 1)(u - 2) \, dt \)

(c) \( \int_1^2 \frac{x^4 + 1}{x^2} \, dx \)

(d) \( \int_{-1}^2 |x| \, dx \)

Example 3.8 (Instructor). Find the derivatives of the following functions:

(a) \( f(x) = \int_1^x t^2 \, dt \)

(b) \( g(x) = \int_x^2 \sqrt{t^2 + 3} \, dt \)

(c) \( h(x) = \int_{3x}^{2x} \frac{u^2 - 1}{u^2 + 1} \, du \)

Example 3.9 (Student). Evaluate the following integrals

(a) \( \int_1^8 \frac{1}{\sqrt{x^2}} \, dx \)

(b) \( \int_0^{\pi/3} \sin \theta \, d\theta \)

(c) \( \int_{-2}^2 f(x) \, dx \) where \( f(x) = \begin{cases} 2 & \text{if } -2 \leq x \leq 0 \\ 4 - x^2 & \text{if } 0 < x < 2 \end{cases} \)

(d) \( \int_{-1}^5 |3x - 6| \, dx \)

Example 3.10 (Student). Identify what is wrong with the evaluation:

\( \int_{-1}^1 \frac{1}{x^2} \, dx = [-x^{-1}]_{-1}^{1} = [-(1)^{-1} + (-1)^{-1}] = [-1 - 1] = -2 \)

Example 3.11 (Student). Find the derivatives of the following functions:

(a) \( F(x) = \int_2^{1/x} \sin^4 t \, dt \)

(b) \( G(x) = \int_{\sin x}^1 \cos^2 \theta \, d\theta \)

(c) \( H(x) = \int_{\tan x}^{2} \frac{1}{\sqrt{2 + t^4}} \, dt \)