9 (b) - Antiderivatives

**Definition(s) 9.1.** A differential equation is an equation involving the derivatives of an unknown function.

**Definition(s) 9.2.** An initial value problem is a differential equation for \( y = f(x) \) along with an initial condition, such as \( f(c) = a \) for some constants \( c \) and \( a \). The solution to the initial value problem is a solution to the differential equation that also satisfies the initial condition.

**Remark 9.3.** In an initial value problem, if the unknown function \( f(t) \) represents position as a function of time, then the initial condition \( f(t_0) = a \) means that at time \( t = t_0 \), the object is at position \( a \). If the unknown function represents velocity, then the same initial condition means the velocity is \( a \) at time \( t = t_0 \).

1 Areas and Distances

The goal of this section is to understand how to approximate areas with rectangles. Suppose that we want to approximate the area between the graph of a continuous function \( f(x) \) and the \( x \)-axis between \( x = a \) and \( x = b \) (suppose for now that \( f \) is positive). Let’s sub-divide the interval \([a, b]\) into \( n \) sub-intervals \([x_1, x_2]\) through \([x_{n-1}, x_n]\) of equal width \( \Delta x \). If we pick a point \( x_i^* \) in each interval \([x_i, x_{i+1}]\), then we can estimate the area under the graph by the sum of the areas of the rectangles with width \( \Delta x \) and height \( f(x_i^*) \):

\[
\text{Area} \approx f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x
\]

**Definition(s) 1.1.**

- An **upper sum** is when the \( x_i^* \) are all chosen to be the global maximum on \([x_i, x_{i+1}]\) for each \( i \).
- A **lower sum** is when the \( x_i^* \) are all chosen to be the global minimum on \([x_i, x_{i+1}]\) for each \( i \).
- A **left-hand sum** is when \( x_i^* \) is the left endpoint of \([x_i, x_{i+1}]\) for each \( i \).
- A **right-hand sum** is when \( x_i^* \) is the right endpoint of \([x_i, x_{i+1}]\) for each \( i \).

**Remark 1.2.** To get better approximations of area, use more rectangles.

**Remark 1.3.** If the velocity of an object is constant, then we have

\[
\text{distance} = \text{velocity} \times \text{time}
\]

We can think of this product as being the area of a rectangle of width “time” and height “velocity”. Thought of in this way, the distance traveled by an object from time \( t = a \) to \( t = b \) is given by the area under the graph of the velocity from \( t = a \) to \( t = b \).
Example 1.4 (Instructor). A ball is thrown upward from an initial height of 1 meter, with an initial velocity of 2 meters per second. Given that the acceleration due to gravity on earth is $-9.8$ meters per second per second, come up with the function $h(t)$ giving the height of the ball at time $t$.

Example 1.5 (Instructor). Solve the initial value problem

$$f'(x) = 5x^4 - 3x^2 + 4, \quad f(-1) = 2$$

Example 1.6 (Instructor). Solve the initial value problem

$$f''(x) = 8x^3 + 5, \quad f(1) = 0, \quad f'(1) = 8$$

Example 1.7 (Instructor). Approximate the area under the graph of $f(x) = \sqrt{x}$ from $x = 0$ to $x = 4$ using 4 rectangles of equal width, using a:

(a) left-hand sum

(b) right-hand sum

Example 1.8 (Instructor). The speed of a runner (in ft/s) is given in the table below at different times (in seconds).

<table>
<thead>
<tr>
<th>$t$</th>
<th>0</th>
<th>0.5</th>
<th>1.0</th>
<th>1.5</th>
<th>2.0</th>
<th>2.5</th>
<th>3.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v(t)$</td>
<td>0</td>
<td>6.2</td>
<td>10.8</td>
<td>14.9</td>
<td>18.1</td>
<td>19.4</td>
<td>20.2</td>
</tr>
</tbody>
</table>

(a) Give an upper estimate of the distance traveled by the runner.

(b) Give a lower estimate of the distance traveled by the runner.

Example 1.9 (Student). Solve the initial value problem

$$f'(x) = 1 + 3\sqrt{x}, \quad f(4) = 25$$

Example 1.10 (Student). Solve the initial value problem

$$f''(x) = 2 + \cos(x), \quad f(0) = -1, \quad f(\pi/2) = 0$$

Example 1.11 (Student). Estimate the area under the graph of $f(x) = 28 + 12x - x^2$ from $x = -2$ to $x = 14$ using:

(a) An upper sum

(b) A lower sum

Example 1.12 (Student). Below is the graph of the velocity of a car (in ft/s) as it is coming to a stop.

Estimate the distance the car travels as it comes to a stop using the various types of sums.