9 Linear Approximations and Differentials

Remark 9.1. Linear approximation and tangent line approximation are two names for using the equation of a tangent line to approximate a function.

Definition(s) 9.2. \[ L(x) = f(x) + f'(a)(x - a) \]
is called the linearization of \( f \) at \( a \).

Note: compare this to the equation of a tangent line through \((a, f(a))\): \( y - f(a) = f'(a)(x - a) \).

Remark 9.3. An equivalent notion to linearization is differentials. Consider the definitions:

\[ dx = x - a \quad dy = y - f(a) \]
(Note \( y \) and \( x \) are from the tangent line, not the function \( f \).)

Then we can plug these into our tangent line equation to get

\[ dy = f'(a)dx \]

More generally it can be written as

\[ dy = f'(x)dx \]

Where \( dy \) and \( dx \) are considered variables in their own right.

Definition(s) 9.4. \[ \Delta x = x - a \quad \Delta y = f(x + \Delta x) - f(x) \]

Example 9.5 (Instructor).

(a) Find the linearization \( L(x) \) of the function \( f(x) = \sqrt{x} \) at \( a = 9 \)
(b) Use the linearization to approximate \( \sqrt{10} \)

Example 9.6 (Instructor).

(a) Find the differential \( dy \) of \( y = \cos(\pi x) \)
(b) Evaluate \( dy \) for \( x = 1/3 \) and \( dx = -0.02 \)

Example 9.7 (Instructor).

Use linear approximation to estimate \( \sqrt{1001} \)

Example 9.8 (Student).

(a) Find the linearization \( L(x) \) of the function \( f(x) = \sin x \) at \( a = \pi/4 \)
(b) Use the linearization to approximate \( \sin(11\pi/40) \)

Example 9.9 (Student).

(a) Find the differential \( dy \) of \( y = \sqrt{x^2 + 8} \)
(b) Evaluate \( dy \) for \( x = 1 \) and \( dx = 0.02 \)

Example 9.10 (Student).

Use linear approximation to estimate \( \frac{1}{4.002} \)