1 Derivatives and Rates of Change

Definition(s) 1.1.
The **derivative of a function** $f$ at a number $c$, denoted by $f'(c)$, is the number

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h},$$

if the limit exists. An equivalent formulation is

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

Remark 1.2.
The tangent line to the graph of $y = f(x)$ at the point $(c, f(c))$ is the line through $(c, f(c))$ whose slope is equal to $f'(c)$.

Example 1.3.
If $f(t)$ measures distance of a moving object, and $t$ is time, then the **velocity** (or **instantaneous velocity**) of the moving object, denoted $v(t)$, is the limit of the average velocities (as defined in Section 1.4).

$$v(t) = \lim_{s \to t} \frac{f(s) - f(t)}{s - t}.$$

2 The Derivative as a Function

Definition(s) 2.1.
The **derivative of a function** $f$, denoted by $f'$, is the function $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$. Another common notation is to write $\frac{df}{dx}$ or $\frac{d}{dx} f(x)$ instead of $f'(x)$. The derivative at $x = c$ in this notation is written $\frac{df}{dx} \bigg|_{x=c}$.

Definition(s) 2.2.
A function $f$ is **differentiable at** $c$ if $f'(c)$ exists. It is **differentiable on the interval** $(a, b)$ if it is differentiable at every number in $(a, b)$.

Theorem 2.3.
If $f$ is differentiable at $c$, then $f$ is continuous at $c$.

How Can a Function Fail to be Differentiable (at a point $c$)?

- $f$ is discontinuous at $c$
- $\lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h} \neq \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h}$
- $\lim_{x \to c} |f'(x)| = \infty$

Definition(s) 2.4.
The **second derivative of** $f$ is the derivative of $f'(x)$, denoted by $f''(x)$ or $\frac{d^2f}{dx^2}$. In general, the $n^{th}$ **derivative**, denoted by $f^{(n)}(x)$ or $\frac{d^n f}{dx^n}$, is the derivative of $f^{(n-1)}(x)$.

Example 2.5.
If $f(t)$ measures distance of a moving object, then the **acceleration** of the object, $a(t)$, is the second derivative of $f$, and the first derivative of the velocity, $v(t)$.
Example 2.6 (Instructor).

Compute (using the limit definition) the derivative of the following functions:

(a) \( f(x) = x^2 + 3x + 7 \)

(b) \( f(x) = \frac{1}{x} \)

(c) \( f(x) = \sqrt{x} \)

Example 2.7 (Instructor).

Explain why \( f(x) = |x| \) is not differentiable at \( x = 0 \).

Example 2.8 (Instructor).

Are the following functions differentiable at \( x = 2 \)? Why or why not?

(a) \( f(x) = \begin{cases} 
  x & \text{if } x \leq 2 \\
  3 & \text{if } x > 2 
\end{cases} \)

(b) \( f(x) = \begin{cases} 
  x^2 & \text{if } x < 2 \\
  4x - 4 & \text{if } x \geq 2 
\end{cases} \)

(c) \( f(x) = \begin{cases} 
  x^2 & \text{if } x < 2 \\
  4 & \text{if } x \geq 2 
\end{cases} \)

Example 2.9 (Student).

Compute (using the limit definition) the derivative of the following functions:

(a) \( f(x) = 3x^2 + x - 8 \)

(b) \( f(x) = \frac{5}{x-3} \)

(c) \( f(x) = \frac{1}{x^2} \)

Example 2.10 (Student).

Is the function \( f(x) = \begin{cases} 
  2x + 1 & \text{if } x < 0 \\
  x^2 + 1 & \text{if } x \geq 0 
\end{cases} \) differentiable at \( x = 0 \)? Why or why not?

Example 2.11 (Student).

Assuming a function \( f(x) \) is differentiable at \( x = c \), come up with a general equation for the tangent line to \( f \) at \( c \).

Example 2.12 (Student).

The graph of a function \( f(x) \) is shown on the left. Use it to sketch a graph of \( f'(x) \).