SOLUTIONS OF MATHEMATICS OLYMPIAD 2008
5–6 grades

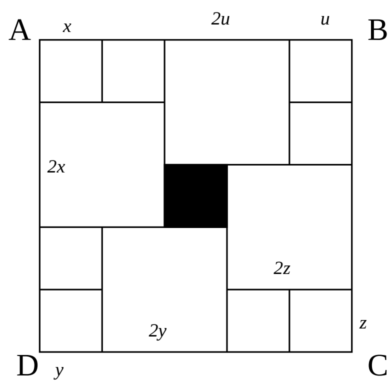
1. Insert "+" signs between some of the digits in the following sequence to obtain correct equality:

$$1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6\ \ 7\ = 100$$

Solution:

$$1\ +\ 23\ +\ 4\ +\ 5\ + 67 = 100$$

2. A square is tiled by smaller squares as shown in the figure. Find the area of the black square in the middle if the perimeter of the big square $ABCD$ is 40 cm.



Solution: If we denote by x, y, z, u the sides of the corresponding small squares (see figure) then they must satisfy

$$3x + 2y = 3y + 2z = 3z + 2u = 3u + 2x$$

Hence, $x = y = z = u$. The side of the big square is $5x$. Therefore, $5x = 40/4$, and $x = 2\,cm$. The area of the black square is $2^2 = 4\,cm^2$.

3. Jack made 3 quarts of fruit drink from orange and apple juice. $\frac{2}{5}$ of his drink is orange juice and the rest is apple juice. Dick prefers more orange juice in the drink. How much orange juice should he add to the drink to obtain a drink composed of $\frac{3}{5}$ of orange juice?

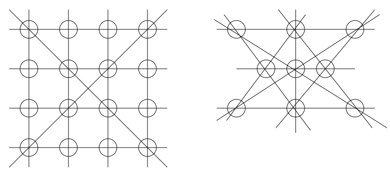
Solution: Let V denote the amount of drink Jake made, A denote the amount of apple juice in the drink, U be the amount of drink after he adds more orange juice. Then $A = (1 - \frac{2}{5})V = \frac{3}{5}V$. Moreover, $A = \frac{2}{5}U$. Hence, $U = \frac{(\frac{3}{5})}{(\frac{2}{5})}V = \frac{3}{2}V$. So, Jack has to add $\frac{1}{2}V$ orange juice to his drink, more exactly, 1.5 quarts of orange juice.

4. A train moving at 55 miles per hour meets and is passed by a train moving in the opposite direction at 35 miles per hour. A passenger in the first train sees that the second train takes 8 seconds to pass him. How long is the second train?

Solution: The speed of the passenger in the first train, in relation to the movement of the second train, is $55 + 35 = 90$ miles per hour, or: $\frac{90}{60\times 60} = \frac{1}{40}$ miles per second. Therefore, the length of the second train is $8 \times \frac{1}{40} = \frac{1}{5}$ mile.

5. It is easy to arrange 16 checkers in 10 rows of 4 checkers each, but harder to arrange 9 checkers in 10 rows of 3 checkers each. Do both.

Solution: See figure.



a) b)

6. Every human that lived on Earth exchanged some number of handshakes with other humans. Show that the number of people that made an odd number of handshakes is even.

Solution: If we add together the numbers of handshakes all people on Earth made we will get twice the number of handshakes (because each handshake is counted twice: one time for each of two participants). Hence this number must be even. The contribution of all people that made an even number of handshakes is also even. Hence the total contribution of the people with odd number of handshakes is even too. But if the number of such people is odd then the total number of their handshakes is odd. This contradiction proves that the number of such people must be even.

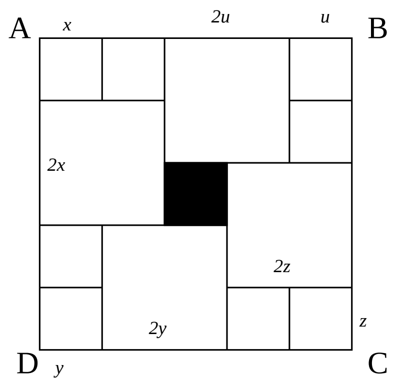
MATHEMATICS OLYMPIAD 2008
7–9 grades

Calculators are not allowed!

1. Jack made 3 quarts of fruit drink from orange and apple juice. His drink contains 45% of orange juice. Dick prefers more orange juice in the drink. How much orange juice should he add to the drink to obtain a drink composed of 60% of orange juice?

Solution: Let V denote the amount of drink Jake made, A denote the amount of apple juice in the drink, U be the amount of drink after he adds more orange juice. Then $A = (1 - 0.45)V = 0.55V$. Moreover, $A = 0.4U$. Hence, $U = \frac{0.55}{0.4}V = \frac{11}{8}V$. So, Jack has to add $\frac{3}{8}V$ orange juice to his drink, more exactly, $\frac{9}{8} = 1.125$ quarts of orange juice.

2. A square is tiled by smaller squares as shown in the figure. Find the area of the black square in the middle if the perimeter of the big square $ABCD$ is 40 cm.



Solution: If we denote by x, y, z, u the sides of the corresponding small squares (see figure) then they must satisfy

$$3x + 2y = 3y + 2z = 3z + 2u = 3u + 2x$$

Hence, $x = y = z = u$. The side of the big square is $5x$. Therefore, $5x = 40/4$, and $x = 2\,cm$. The area of the black square is $2^2 = 4\,cm^2$.

3. For one particular number $a > 0$ the function f satisfies the equality

$$f(x+a) = \frac{1+f(x)}{1-f(x)}$$

for all x. Show that f is periodic function.

Solution:

$$f(x+2a) = \frac{1+f(x+a)}{1-f(x+a)} = \frac{1+\frac{1+f(x)}{1-f(x)}}{1-\frac{1+f(x)}{1-f(x)}} = \frac{\frac{1-f(x)+1+f(x)}{1-f(x)}}{\frac{1-f(x)-1-f(x)}{1-f(x)}} = -\frac{1}{f(x)}$$

Therefore,

$$f(x+4a) = f(x).$$

Thus, f is periodic with period $4a$.

4. If a, b, c, x, y, z are numbers so that

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \qquad \text{and} \qquad \frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 0$$

Show that

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

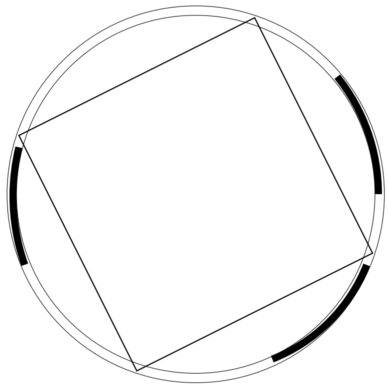
Solution: $\alpha = \frac{x}{a}$, $\beta = \frac{y}{b}$, $\gamma = \frac{z}{c}$. Then, $1 = (\alpha+\beta+\gamma)^2 = \alpha^2+\beta^2+\gamma^2+2(\alpha\beta+\alpha\gamma+\beta\gamma)$. Note that $0 = \frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} = \frac{\alpha\beta+\alpha\gamma+\beta\gamma}{\alpha\beta\gamma}$. Therefore, $\alpha\beta+\alpha\gamma+\beta\gamma = 0$ and $\alpha^2+\beta^2+\gamma^2 = 1$.

5. Is it possible that a four-digit number $AABB$ is a perfect square? (Same letters denote the same digits).

Solution: $88 \times 88 = 7744$.

6.

6. A finite number of arcs of a circle are painted black (see figure). The total length of these arcs is less than $\frac{1}{5}$ of the circumference. Show that it is possible to inscribe a square in the circle so that all vertices of the square are in the unpainted portion of the circle.

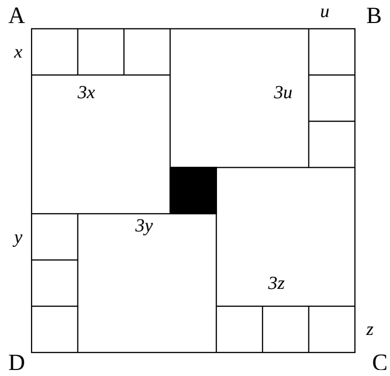


Solutions: Consider all squares at least one of whose vertices is in black area. When this vertex moves along black portion of the circle of the total length L the other three vertices moves along the circle and cover additionally a part of the circle of the length $3L$. Therefore all four vertices will cover the part of the circle of the total length $4L$. Since L is less than $\frac{1}{5}$ part of the circumference, then the vertices of such squares will cover at most $\frac{4}{5}$ of the circumference. Therefore there exists a square all whose vertices are in the unpainted part of the circle.

MATHEMATICS OLYMPIAD 2008
10–12 grades

Calculators are not allowed!

1. A square is tiled by smaller squares as shown in the figure. Find the area of the black square in the middle if the perimeter of the square $ABCD$ is 14 cm.



Solution: If we denote by x, y, z, u the sides of the corresponding small squares (see figure) then they must satisfy

$$3x + 4u = 3u + 4z = 3z + 4y = 3y + 4x$$

Hence, $x = y = z = u$. The side of the big square is $7x$. Therefore, $7x = 14/4$, and $x = \frac{1}{2}\,cm$. The area of the black square is $(\frac{1}{2})^2 = \frac{1}{4}\,cm^2$.

2. If a, b, and c are numbers so that

$$a + b + c = 0 \qquad \text{and } a^2 + b^2 + c^2 = 1$$

Compute $a^4 + b^4 + c^4$.

Solution: Note that $1 = (a^2 + b^2 + c^2)^2 = a^4 + b^4 + c^4 + 2(a^2b^2 + a^2c^2 + b^2c^2)$. Also, $0 = (a+b+c)^2 = a^2+b^2+c^2+2(ab+ac+bc) = 1 + 2(ab + ac + bc)$. Hence $ab + ac + bc = -\frac{1}{2}$. Square both sides. We get $a^2b^2 + a^2c^2 + b^2c^2 + 2(a^2bc + ab^2c + abc^2) = \frac{1}{4}$. Note also that $a^2bc + ab^2c + abc^2 = abc(a+b+c) = 0$. Therefore, $a^2b^2 + a^2c^2 + b^2c^2 = \frac{1}{4}$. Finally, $a^4 + b^4 + c^4 = 1 - 2(a^2b^2 + a^2c^2 + b^2c^2) = \frac{1}{2}$.

3. A given fraction $\frac{a}{b}$ (a, b are positive integers, $a \neq b$) is transformed by the following rule: first, 1 is added to both the numerator and the denominator, and then the numerator and the denominator of the new fraction are each divided by their greatest common divisor (in other words, the new fraction is put in simplest form). Then the same transformation is applied again and again. Show that after some number of steps the denominator and the numerator differ exactly by 1.

Solution: Note that the difference between numerator and denominator does not change when we add 1 to both of them, and the difference decreases d times when numerator and denominator have a common divisor $d > 1$. So to prove the statement it is enough to show that for any a and b there is a positive integer q such that $a + q$ and $b + q$ have a common divisor $d > 1$. Without loss of generality we can assume that $a < b$, and $b = a + k$, where $k > 1$. Let $a = tk - p$, where $0 < p < k$. Then $\frac{a}{b} = \frac{tk-p}{tk-p+k}$. Take $q = p$. We get $\frac{a+p}{b+p} = \frac{tk}{tk+k}$. Both numerator and denominator are divisible by k. Therefore, starting to add the same positive integer from 1 to k to numerator and denominator we necessarily reach such moment when both numerator and denominator have both common divisor that is bigger than 1. Repeating that many times we decrease the difference to 1.

4. A goat uses horns to make the holes in a new 30×60 cm large towel. Each time it makes two new holes. Show that after the goat repeats this 61 times the towel will have at least two holes whose distance apart is less than 6 cm.

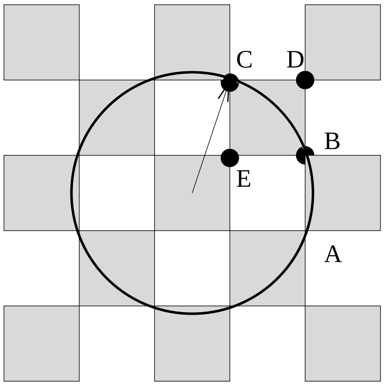
Solution: After 61 times the goat makes 122 holes. Divide the 30×60 towel in 120 rectangles of size 3×5. Two holes must occur in one rectangle. But the distance between two arbitrary points in 3×5 rectangle is less than $\sqrt{3^2 + 5^2} = \sqrt{34} < 6$.

5. You are given 555 weights weighing 1 g, 2 g, 3 g,…, 555 g. Divide these weights into three groups whose total weights are equal.

Solution: Split first the sequence $1 \ldots 555$ into 37 subsequences $1 \ldots 15$, $16 \ldots 30$, …, $541 \ldots 555$, each subsequence contains 15 weights. Divide each subsequence into 3 groups in the following way: the 1st, 5th, 9th, 10th, and 15th element of each subsequence go to the first group; the 2nd, 6th,7th, 12th, and 13th go to the second group; and, finally,the rest goes to the third group. It is easy to check that total weights of all three groups are equal.

6. Draw on the regular 8×8 chessboard a circle of the maximal possible radius that intersects only black squares (and does not cross white squares). Explain why no larger circle can satisfy the condition.

Solution: The solution is shown on the figure.



Let us discuss why the larger radius is not possible. Note first that the circle can cross the boundary of black squares only at corners. Since the circle has a finite radius it must cross at some level two neighboring corners (placed as A and B on the figure). Then it can not cross D because A, B, D lie on a line, hence it must cross either E or C. In the first case its radius is $\sqrt{\frac{1}{2}}$. The second case is shown on the figure and the circle radius is $\frac{\sqrt{10}}{2}$.