1. A boy has as many sisters as brothers. However, his sister has twice as many brothers as sisters. How many boys and girls are there in the family?

**Solution:** Assume that there are $n$ boys and $m$ girls. Since a boy has as many brothers as sisters then $n - 1 = m$. On the other hand, $n = 2(m - 1)$. Combining that together $m + 1 = 2(m - 1)$. Hence, $m = 3$, and $n = 4$. There are 4 boys and 3 girls in the family. □

2. Solve each of the following problems.
(1) Find a pair of numbers with a sum of 11 and a product of 24.
(2) Find a pair of numbers with a sum of 40 and a product of 400.
(3) Find three consecutive numbers with a sum of 333.
(4) Find two consecutive numbers with a product of 182.

**Solution:**
(1) $a = 8, b = 3$.
(2) $a = b = 20$.
(3) $n + (n + 1) + (n + 2) = 333$. Then $3n = 330$. Finally, $n = 110$. Three consecutive integers, 110, 111, 112.
(4) $182 = 2 \cdot 7 \cdot 13 = 13 \cdot 14$. Answer: 13, 14. □
3. 2008 integers are written on a piece of paper. It is known that the sum of any 100 numbers is positive. Show that the sum of all numbers is positive.

**Solution.** Since the sum of any 100 numbers is positive then totally there no more than 99 negative numbers in the set. Let us take the sum of 100 numbers including these 99 negative integers. This sum is positive. The remaining numbers are also positive. Hence the total sum is positive. □

4. Let $p$ and $q$ be prime numbers greater than 3. Prove that $p^2 - q^2$ is divisible by 24.

**Solution.** $p^2 - q^2 = (p - q)(p + q)$. Since both $p, q$ are prime greater than 3 both are odd numbers, and their remainders $mod\ 4$ are 1 or 3. If both remainders coincide then $p - q$ is divisible by 4 while $p + q$ is even. Hence, $(p - q)(p + q)$ is divisible by 8. If remainders are distinct then $p + q$ is divisible by 4 while $p - q$ is even. Again, $(p - q)(p + q)$ is divisible by 8.

Consider now, remainders modulo 3. The possible remainders are 1 or 2. Again, either $p - q$ is divisible by 3 if remainders coincide, or $p + q$ is divisible by 3 if remainders are distinct.

Finally, $(p - q)(p + q)$ is divisible by both 3 and 8, and hence by 24. □
5. Four villages $A,B,C,$ and $D$ are connected by trails as shown on the map.

On each path $A \to B \to C$ and $B \to C \to D$ there are 10 hills, on the path $A \to B \to D$ there are 22 hills, on the path $A \to D \to B$ there are 45 hills.

A group of tourists starts from $A$ and wants to reach $D$. They choose the path with the minimal number of hills. What is the best path for them?

**Solution.** One need to compare 3 different minimal paths: $A \to D$, $A \to B \to D$, and $A \to B \to C \to D$.

Denote the number of hills on segment $AB$ by $(AB)$, on segment $BD$ by $(BD)$, etc.

$(AB) + (BC) + (CD) \leq (AB) + (BC) + (BD) = 20$.

$(AB) + (BD) = 22$. Therefore, $(BD) \leq 22$. Then, $(AD) + (BD) = 45$, and $(AD) = 45 - (BD) \geq 45 - 22 = 23$.

Summarizing, we conclude that the most convenient path is $A \to B \to C \to D$. \qed