Transp Porous Med (2017) 117:337–347
DOI 10.1007/s11242-017-0836-x

Instability of fully developed mixed convection with viscous dissipation in a vertical porous channel

A. Barletta[1] · M. Miklavčič[2]

Received: 27 October 2016 / Accepted: 9 February 2017 / Published online: 23 February 2017

Abstract Flow driven by an externally imposed pressure gradient in a vertical porous channel is analysed. The combined effects of viscous dissipation and thermal buoyancy are taken into account. These effects yield a basic mixed convection regime given by dual flow branches. Duality of flow emerges for a given vertical pressure gradient. In the case of downward pressure gradient, i.e. upward mean flow, dual solutions coincide when the intensity of the downward pressure gradient attains a maximum. Above this maximum no stationary and parallel flow solution exists. A nonlinear stability analysis of the dual solution branches is carried out limited to parallel flow perturbations. This analysis is sufficient to prove that one of the dual solution branches is unstable. The evolution in time of a solution in the unstable branch is also studied by a direct numerical solution of the governing equation.

Keywords Porous medium · Vertical channel · Fully developed flow · Viscous dissipation · Mixed convection · Instability

1 Introduction

Viscous dissipation acts as a heat source in channel flows. If coupled with effects of thermal buoyancy or temperature-dependent viscosity, viscous dissipation may be the cause of flow instability. This phenomenon is known from several decades, ever since the pioneering work by Joseph (1964, 1965). A survey on this topic and on the relative recent literature can be found in Barletta (2015).

✉ A. Barletta
antonio.barletta@unibo.it

M. Miklavčič
milan@math.msu.edu

1 Department of Industrial Engineering, Alma Mater Studiorum Università di Bologna, Viale Risorgimento 2, 40136 Bologna, Italy

2 Department of Mathematics, Michigan State University, East Lansing, MI 48824, USA

Most of the existing literature on dissipation instability in channel flows deals with horizontal channels, either filled with a porous medium or not. On the other hand, vertical channels or inclined channels have received little attention. An exception is the paper by Nield et al. (2011), which, however, investigates primarily small inclinations to the horizontal. The main reason for restrictive assumptions is the general difficulty to express the basic velocity and temperature fields in terms of an analytical solution when the channel is inclined to the horizontal. As a consequence, the stability analysis is even more complicated from a computational viewpoint.

A well-known feature of viscous dissipation flows in vertical channels is the lack of uniqueness in the steady parallel flow solution with a given value of the imposed pressure gradient. The steady parallel flows belong to dual branches whose characteristics have been widely investigated for either clear fluids (Barletta et al. 2005, 2008; Miklavčič and Wang 2011) or fluid saturated porous media (Barletta et al. 2007). Indeed, studies about stability of the dual branches of solutions have been carried out only for clear fluids (Miklavčič 2015; Barletta and Miklavčič 2016). An interesting aspect pointed out in these studies is the role played by the choice of the reference temperature to be used within the Oberbeck-Boussinesq approximation. In fact, this choice influences not only the basic parallel flow (Barletta and Zanchini 1999), but also its stability analysis. In Miklavčič (2015) and in Barletta and Miklavčič (2016), the same type of flow is investigated, but with two different choices of the reference temperature: either the temperature of the channel walls (Miklavčič 2015) or the mean temperature over a channel cross section (Barletta and Miklavčič 2016). The last choice leads to a more complicated mathematical formulation, but it is to be preferred on physical grounds (Barletta and Zanchini 1999).

We mention that a nonlinear analysis of instability induced by viscous dissipation has been recently carried out by Celli et al. (2016) with reference to a horizontal porous channel. We also point out that a linear stability analysis of dissipation-induced dual flows in a horizontal porous channel has been performed by Barletta and Rees (2009).

The aim of the present contribution is to study the nonlinear instability in a vertical porous channel along the lines sketched by Barletta and Miklavčič (2016). Here, the local momentum balance equation is formulated according to Darcy's law instead of the Navier–Stokes equation. The simpler mathematical nature of Darcy's law allows a straightforward manipulation of the coupled local momentum and energy balance equations leading to a single nonlinear governing equation whose solution yields the temperature field. The basic stationary flows display a structure of dual solutions as in the case of a Navier–Stokes fluid, parametrised by the vertical pressure gradient within the channel. The stability theory for semilinear parabolic equations is invoked to test the evolution in time of perturbations superposed to the basic stationary dual flows. A class of parallel flow perturbations is considered. These perturbations are unstable for a branch of dual solutions and stable for the other one. Finally, a direct numerical simulation of the time evolution of a basic flow belonging to the unstable branch is carried out.

2 Governing equations

We consider fully developed buoyant flow in a vertical porous channel. The flow regime is assumed to be two-dimensional in the (y, z)–plane, with $y \in [-L, L]$ and $z \in \mathbb{R}$. We denote by $\mathbf{g} = -g\,\mathbf{e}_z$ the gravitational acceleration, where g is its modulus, and $\mathbf{e}_z$ is the unit vector along the vertical z axis. The flow is partly driven by the buoyancy force and partly by a prescribed constant vertical pressure gradient $P = \partial p/\partial z$. To be precise, we denote by p the local difference between the fluid pressure and the hydrostatic pressure. Under the assumption of fully developed regime, the velocity field is parallel to the z axis, and symbol W is used to denote the z component of velocity. According to Darcy's law and to the Oberbeck–Boussinesq approximation (Barletta and Zanchini 1999; Barletta 2009), the local momentum balance equation and the local energy balance equations can be written as

$$\frac{\mu}{K}\, W = -P + \rho g \beta \left(T - T_m\right), \tag{1a}$$

$$\langle \rho c \rangle \frac{\partial T}{\partial t} = \langle k \rangle \frac{\partial^2 T}{\partial y^2} + \frac{\mu}{K}\, W^2. \tag{1b}$$

Here, μ is the dynamic viscosity of the fluid, K is the permeability, ρ is the fluid density, β is the fluid coefficient of thermal expansion, $\langle \rho c \rangle$ is the average value of the product between the density and the specific heat of the saturated porous medium, $\langle k \rangle$ is the average thermal conductivity of the saturated porous medium. The boundary conditions can be written as

$$y = \pm L : T = T_0. \tag{2}$$

We point out that, unlike the case of Navier–Stokes' flow (Barletta and Miklavčič 2016), Darcy's flow implies in general slip conditions ($W \neq 0$) at the impermeable boundary walls, $y = \pm L$ (Nield and Bejan 2013).
In Eqs. (1) and (2), T stands for temperature and t for time. The fluid properties are considered as constants. The reference temperature T_m that appears in Eq. (1a) is the mean temperature, defined as

$$T_m = \frac{1}{2L} \int_{-L}^{L} T \, \mathrm{d}y. \tag{3}$$

2.1 Dimensional analysis

Equations (1)–(3) can be written in a dimensionless form through the scaling

$$\frac{y}{L} \to y, \quad \frac{t}{L^2 \langle \rho c \rangle / \langle k \rangle} \to t, \quad \frac{W}{\langle k \rangle / (\rho g \beta L^2)} \to W, \quad \frac{T - T_0}{\langle k \rangle \mu / (K \rho^2 g^2 \beta^2 L^2)} \to T,$$
$$\frac{T_m - T_0}{\langle k \rangle \mu / (K \rho^2 g^2 \beta^2 L^2)} \to T_m, \quad \frac{P}{\langle k \rangle \mu / (\rho g \beta K L^2)} \to P. \tag{4}$$

Therefore, Eq. (1) can be expressed as,

$$W = -P + T - T_m, \tag{5a}$$

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial y^2} + W^2, \tag{5b}$$

where the boundary conditions (2) and the additional integral constraint (3) are written as

$$y = \pm 1 : T = 0, \tag{6a}$$

$$\frac{1}{2} \int_{-1}^{1} T \, \mathrm{d}y = T_m. \tag{6b}$$

Equations (1) can be collapsed into a single partial differential equation,

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial y^2} + (T - T_m - P)^2 . \tag{7}$$

In Eqs. (6) and (7), T is a function of y and t, while T_m is a function of t. These equations yield effectively an integro-differential initial value problem whose solution can be sought for prescribed values of the input parameter P and initial temperature distribution.

2.2 Basic dual flows

Stationary solutions of (6) and (7) satisfy

$$\frac{d^2 T}{dy^2} + (T - T_m - P)^2 = 0, \tag{8a}$$

$$y = \pm 1 : T = 0, \tag{8b}$$

$$\frac{1}{2} \int_{-1}^{1} T \, dy = T_m . \tag{8c}$$

By employing the usual ODE techniques, one can show that stationary solutions T have to be even functions of y. Equation (8) can be solved numerically, for any large enough value of P, by adopting a shooting method based on a Runge–Kutta solver. In the "Appendix", it will be shown that the solutions can also be represented in terms of some nonstandard functions.

Figure 1 shows that, for a given value of P, there exists a pair of solutions of Eq. (8): one on the *upper branch* and one on the *lower branch* of either T_m, $T(0)$ or $W(0)$ graph. Equation (5a) implies that these dual solutions have the same mean flow velocity

$$W_m = \frac{1}{2} \int_{-1}^{1} W \, dy = - P . \tag{9}$$

The flow marked by A in Fig. 1 corresponds to the trivial solution of Eq. (8) and describes the system at rest. The dual of the rest state is marked by B in Fig. 1, and it is a rather peculiar flow since it is caused by viscous dissipation alone, without any externally imposed pressure gradient. Some authors (Miklavčič and Wang 2011; Barletta and Miklavčič 2016) call such flows completely passive flows. The branches merge at point C in Fig. 1 where P is at its minimum. No stationary flow is possible when $P < -2.11512$. Some details about these flows are summarised in Table 1.

Figure 2 illustrates the temperature and velocity profiles of flows B and C. Observe that upward flow, $W(y) > 0$, occurs at the central core of the porous layer, while downward flow, $W(y) < 0$, takes place close to the boundary planes $y = \pm 1$, for both flows B and C.

3 Stability analysis

At given P, we can write a time-dependent solution of Eq. (7) as

$$T(y) + u(y, t) \tag{10}$$

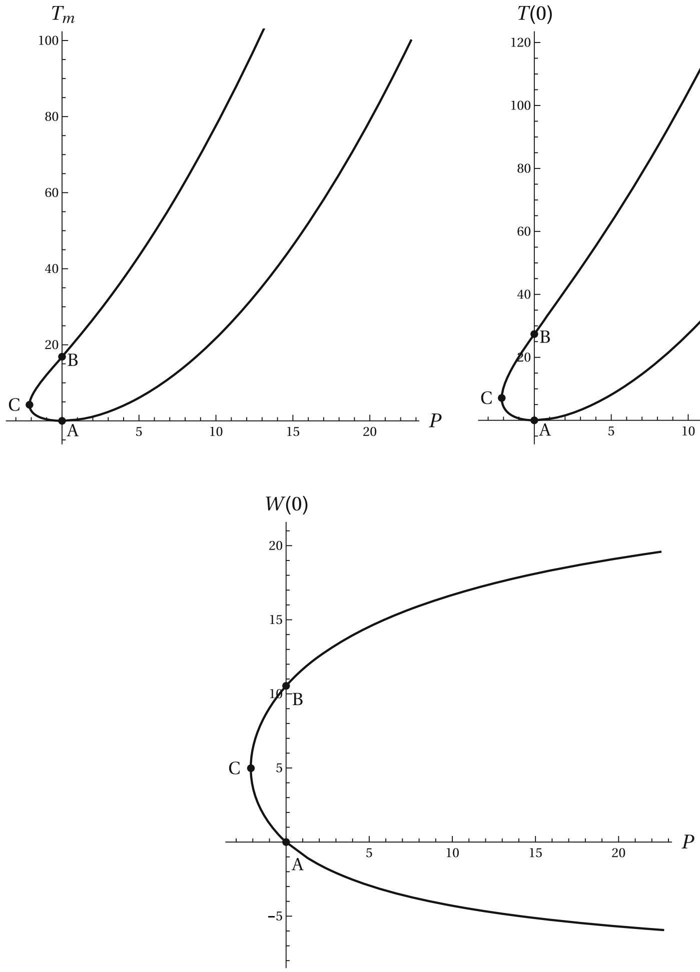

Fig. 1 Dual basic flows: plots of T_m, $T(0)$ and $W(0)$ versus P

Table 1 Some data about the flows marked in Fig. 1

Point	P	T_m	$T(0)$	$W(0)$	λ_1
A	0	0	0	0	$-\pi^2/4$
B	0	16.8633	27.4021	10.5388	2.70885
C	−2.11512	4.23024	7.09844	4.98332	0

where T is a stationary solution of Eqs. (8) and u is the time varying perturbation whose evolution is governed by

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial y^2} + 2(T - T_m - P)\left(u - \frac{1}{2}\int_{-1}^{1} u\,\mathrm{d}y\right) + \left(u - \frac{1}{2}\int_{-1}^{1} u\,\mathrm{d}y\right)^2, \tag{11a}$$

$$u(\pm 1, t) = 0. \tag{11b}$$

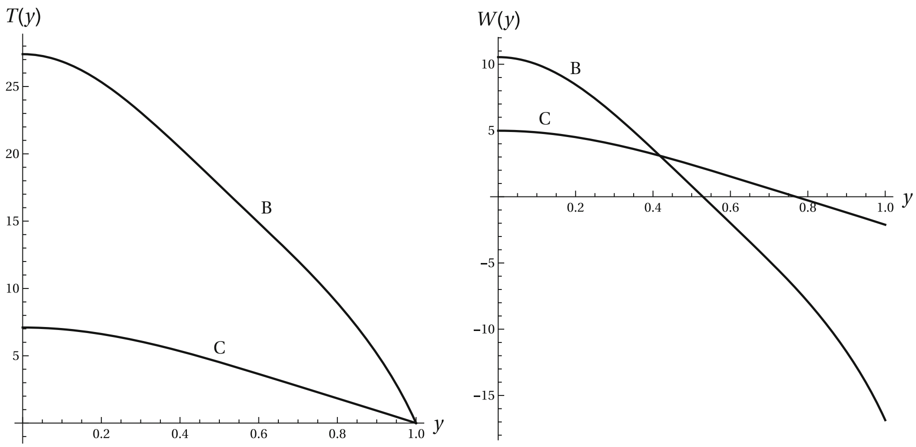

Fig. 2 Dual basic flows: plots of $T(y)$ and $W(y)$ for solutions B and C

Equations (11) define a semilinear parabolic equation in $L^2(-1, 1)$ and growth/decay of u is determined by the corresponding eigenvalues (Henry 1981; Miklavčič 1998). A complex number λ is a corresponding eigenvalue if there exists $v \in L^2(-1, 1)$ such that

$$\lambda v = \frac{d^2 v}{dy^2} + 2(T - T_m - P)\left(v - \frac{1}{2}\int_{-1}^{1} v\, dy\right), \tag{12a}$$

$$v(\pm 1) = 0, \ \ v \not\equiv 0. \tag{12b}$$

It can be shown (Henry 1981; Miklavčič 1998) that all perturbations u, which are initially small enough, will decay with time if all the corresponding eigenvalues have negative real parts. If, on the other hand, there exists an eigenvalue which has a positive real part, then there exists a fixed threshold value such that for arbitrary small ε one can find a solution of Eq. (11) which is initially smaller than ε but grows eventually past the threshold value. In other words, a stationary solution T of Eq. (8) is stable if all the eigenvalues determined by Eq. (12) have negative real parts. A stationary solution T of Eq. (8) is unstable if Eq. (12) have an eigenvalue with a positive real part.

We point out that the usual reasoning in applied stability analysis goes as follows. The perturbation u in Eq. (11) is initially small. Hence, one would expect that the last term in Eq. (11a), which is quadratic in u, does not initially affect the growth of u. This leads in fact to a linearisation of Eq. (11). One may drop the last term in Eq. (11a) and look for solutions in the form $u = e^{\lambda t} v(y)$. Then, λ and v have to satisfy Eq. (12). The sign of the real part of λ determines whether the small perturbations will grow or decay. This kind of reasoning has been proven to be applicable to general semilinear parabolic equations (Henry 1981; Miklavčič 1998). However, the detailed proofs are very technical and will be omitted here.

Equations (12) have infinitely many eigenvalues. We ordered them so that the real parts are decreasing. For the system at rest, at point A in Fig. 1, the eigenvalues are

$$-(n\pi/2)^2, \quad n = 1, 2, \ldots. \tag{13}$$

We discretised Eqs. (12) using central differences and use Richardson extrapolation to calculate the eigenvalues.

Dependence of the first eigenvalue on $T(0)$ is shown in Fig. 3. Note that it changes sign at C. One can show that there is an eigenvalue 0 at point C as follows. Consider T and P in

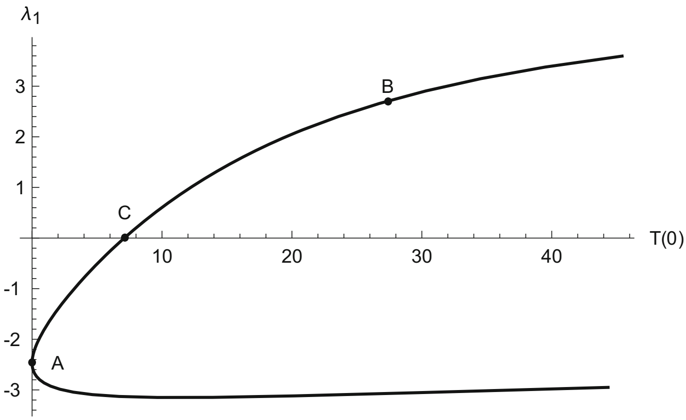

Fig. 3 Dependence of the first eigenvalue on $T(0)$

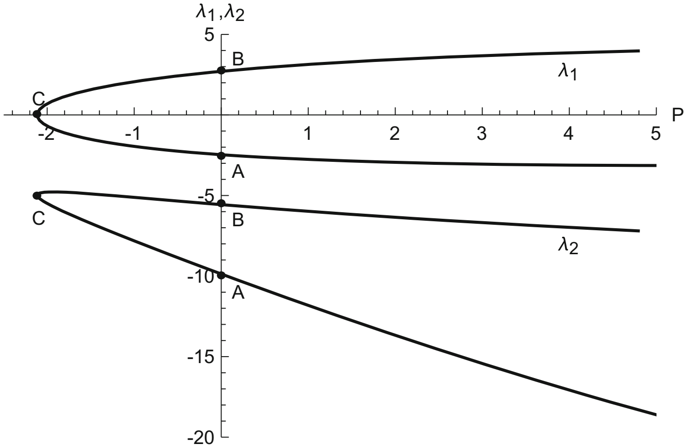

Fig. 4 Dependence of the first two eigenvalues λ_1, λ_2 on P

Eqs. (8) as functions of a parameter $\xi = T(0)$ and take the derivative of Eq. (8a) with respect to ξ to obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}y^2}\frac{\partial T}{\partial \xi} + 2(T - T_m - P)\left(\frac{\partial T}{\partial \xi} - \frac{1}{2}\int_{-1}^{1}\frac{\partial T}{\partial \xi}\,\mathrm{d}y - \frac{\mathrm{d}P}{\mathrm{d}\xi}\right) = 0. \tag{14}$$

P has minimum at C hence $\mathrm{d}P/\mathrm{d}\xi = 0$ and $v = \partial T/\partial \xi$ satisfies Eq. (12) with $\lambda = 0$.

Computed eigenvalues happened to be purely real. The largest two eigenvalues are shown in Fig. 4. This shows that solutions of Eq. (8) that are on the upper branch in Fig. 1 are unstable, and that the lower branch in Fig. 1 consists of stable solutions.

4 Evolution of perturbations

Equation (7) was solved numerically to determine evolution of perturbations of stationary solutions. Small perturbations of solutions on the lower branch in Fig. 1 decay since those

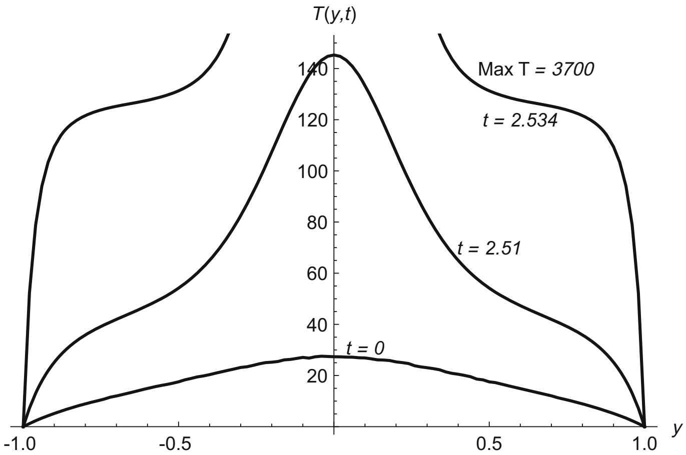

Fig. 5 Blow up of a slightly perturbed completely passive solution when the average of the initial perturbation is 0. The initial size of perturbations is under 1%

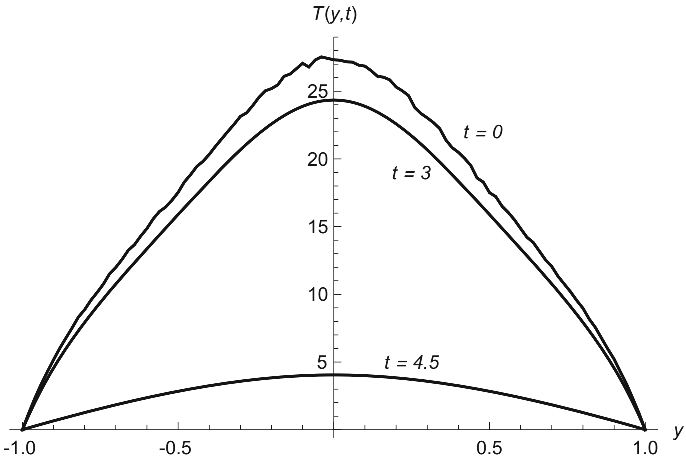

Fig. 6 Perturbations of the completely passive flow with initially slightly negative average lead to transition to the rest state

stationary solutions are stable, which was established in the previous section. For perturbations of stationary solutions on the upper branch, we found two kind of outcomes. When the average of the perturbation is a bit negative, the perturbed solution will approach the stationary solution at the same P on the lower branch. However, when the average of the perturbation is bigger than this threshold, the perturbation grows explosively. We will present the details for perturbations of the completely passive flow, which is marked by B in Fig. 1.

We made random, up to $\pm 1\%$, initial changes in the temperature profile at $n = 99$ equally spaced points on the interval $-1 < y < 1$ and solved the discrete version of Eq. (7) directly with $P = 0$. The solution blew up at about $t = 2.534$ as is shown in Fig. 5.

However, when the average initial perturbation was in the range -1.07 to $+0.93\%$, the solution collapsed to the trivial solution as is illustrated in Fig. 6. The same perturbation but raised into range -1.06 to $+0.94\%$ leads to blow up. To study the transition to the rest state

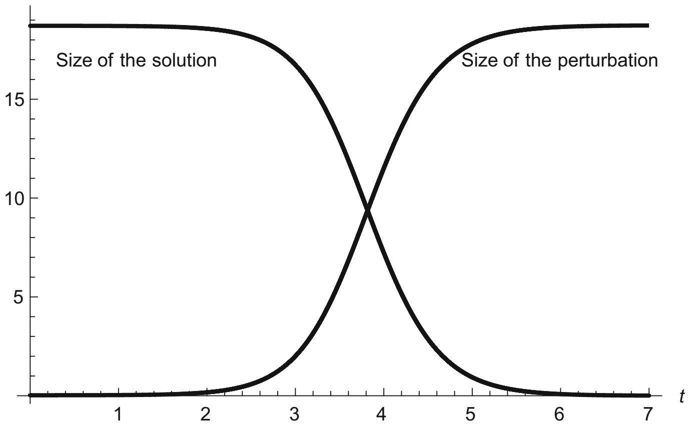

Fig. 7 Evolution of the perturbed completely passive flow shown in Fig. 5

in more detail we first calculated and graphed in Fig. 7 the size of the perturbation

$$\sqrt{\frac{1}{n}\sum_{i=1}^{n}(T(y_i,t)-T_B(y_i))^2}, \tag{15}$$

where T_B is the temperature distribution for the completely passive solution as presented in Fig. 2 and $y_i = -1 + 2i/(n+1)$. Near $t = 2.5$, the perturbations grow exponentially. From values of perturbations given by (15), we estimated the growth rate to be 2.52 which is not too far from the leading eigenvalue $\lambda_1 = 2.70885$ for the completely passive flow. We expect that the growth rate would approach λ_1 as the initial size of perturbation would go to 0 and $n \to \infty$. In Fig. 7, we also plotted the size of the solution

$$\sqrt{\frac{1}{n}\sum_{i=1}^{n}T(y_i,t)^2}. \tag{16}$$

At $t = 6.5$, we estimated that the size of the solution decays exponentially at the rate -2.464 which is very close to the leading eigenvalue $\lambda_1 = -\pi^2/4 = -2.467$ for the rest state.

5 Conclusions

The stability of dual parallel flows in a vertical porous channel has been analysed. A vertical channel with impermeable boundaries kept at a fixed temperature has been considered. Since no temperature gap is prescribed across the porous layer, the buoyant flow is entirely caused by the effect of viscous dissipation. For every given value of the vertical pressure gradient larger than a minimum, dual flows have been shown to exist. Nonlinear stability of these flows have been investigated, by limiting the type of perturbations to parallel flow modes. This class of perturbations proved to display an unstable behaviour for the upper branch of dual flows and a stable response for the lower branch. Finally, a numerical solution of the nonlinear equations of parallel flow in the channel has been employed to prove that the flows in the unstable branch can either evolve to the corresponding flow in the stable branch or undergo an explosive evolution in time. The ultimate trend of the perturbations turned out to be a consequence of the initial perturbation prescribed on the basic flow, whether its amplitude exceeds or not a given range. This analysis cannot exclude that instability of the lower branch of dual solutions can arise when perturbations of a different type are considered. Further future work is needed in this direction to check if a different outcome emerges with a fully three-dimensional analysis of perturbations.

Appendix

If we multiply Eq. (8a) by T' and integrate, we obtain

$$\frac{1}{2}\left(\frac{\mathrm{d}T}{\mathrm{d}y}\right)^2 + \frac{1}{3}\,(T - T_m - P)^3 = \frac{1}{3}\,[T(0) - T_m - P]^3\,, \tag{17}$$

after noting that $T'(0) = 0$. If we use Eq. (17) and scale with $T(0) - T_m - P$, we obtain two branches.

When $T(0) - T_m - P > 0$, one can describe T parametrically as follows:

$$T(y) = 6s^2[\varphi(s)^2 - \varphi(s|y|)^2], \tag{18}$$

where $0 \le s < s_{\max} = \int_0^\infty (3 - 3x^2 + x^4)^{-1/2}\mathrm{d}x = 2.103$ and φ is defined on $[0, s_{\max})$ by

$$\varphi' = \sqrt{3 - 3\varphi^2 + \varphi^4}, \quad \varphi(0) = 0. \tag{19}$$

In this case

$$\begin{aligned} T(0) &= 6s^2\varphi(s)^2, \quad P = -6s^2 + 6s\int_0^s \varphi(x)^2\,\mathrm{d}x, \\ T_m &= 6s^2\varphi(s)^2 - 6s\int_0^s \varphi(x)^2\,\mathrm{d}x. \end{aligned} \tag{20}$$

These solutions make up the entire upper branch in Fig. 1 and the piece between C and A on the lower branch.

When $T(0) - T_m - P < 0$ one can describe T parametrically as follows:

$$T(y) = 6s^2[\psi(s)^2 - \psi(s|y|)^2], \tag{21}$$

where $0 \le s < z_{\max} = \int_0^\infty (3 + 3x^2 + x^4)^{-1/2}\mathrm{d}x = 1.214$ and ψ is defined on $[0, z_{\max})$ by

$$\psi' = \sqrt{3 + 3\psi^2 + \psi^4}, \quad \psi(0) = 0. \tag{22}$$

In this case

$$\begin{aligned} T(0) &= 6s^2\psi(s)^2, \\ P &= 6s^2 + 6s\int_0^s \psi(x)^2\,\mathrm{d}x, \\ T_m &= 6s^2\psi(s)^2 - 6s\int_0^s \psi(x)^2\,\mathrm{d}x. \end{aligned} \tag{23}$$

These solutions make up the lower branch in Fig. 1 *without* the piece between C and A.

References

Barletta, A., Miklavčič, M.: On fully developed mixed convection with viscous dissipation in a vertical channel and its stability. Z. Angew. Math. Mech. **96**, 1457–1466 (2016)

Barletta, A., Magyari, E., Keller, B.: Dual mixed convection flows in a vertical channel. Int. J. Heat Mass Transf. **48**, 4835–4845 (2005)

Barletta, A., Magyari, E., Pop, I., Storesletten, L.: Mixed convection with viscous dissipation in a vertical channel filled with a porous medium. Acta Mech. **194**, 123–140 (2007)

Barletta, A., Lazzari, S., Magyari, E.: Buoyant Poiseuille–Couette flow with viscous dissipation in a vertical channel. Z. Angew. Math. Physik **59**, 1039–1056 (2008)

Barletta, A.: Local energy balance, specific heats and the Oberbeck–Boussinesq approximation. Int. J. Heat Mass Transf. **52**, 5266–5270 (2009)

Barletta, A.: On the thermal instability induced by viscous dissipation. Int. J. Therm. Sci. **88**, 238–247 (2015)

Barletta, A., Rees, D.A.S.: Stability analysis of dual adiabatic flows in a horizontal porous layer. Int. J. Heat Mass Transf. **52**, 2300–2310 (2009)

Barletta, A., Zanchini, E.: On the choice of the reference temperature for fully-developed mixed convection in a vertical channel. Int. J. Heat Mass Transf. **42**, 3169–3181 (1999)

Celli, M., Alves, L.S. de B., Barletta, A.: Nonlinear stability analysis of Darcy's flow with viscous heating. Proc. R. Soc. A **472**, 20160036 (2016)

Henry, D.: Geometric Theory of Semilinear Parabolic Equations. Springer, Berlin (1981)

Joseph, D.D.: Variable viscosity effects on the flow and stability of flow in channels and pipes. Phys. Fluids **7**, 1761–1771 (1964)

Joseph, D.D.: Stability of frictionally-heated flow. Phys. Fluids **8**, 2195–2200 (1965)

Miklavčič, M.: Applied Functional Analysis and Partial Differential Equations. World Scientific Publishing Co., Inc., River Edge (1998)

Miklavčič, M.: Stability analysis of some fully developed mixed convection flows in a vertical channel. Z. Angew. Math. Mech. **95**, 982–986 (2015)

Miklavčič, M., Wang, C.Y.: Completely passive natural convection. Z. Angew. Math. Mech. **91**, 601–606 (2011)

Nield, D.A., Barletta, A., Celli, M.: The effect of viscous dissipation on the onset of convection in an inclined porous layer. J. Fluid Mech. **679**, 544–558 (2011)

Nield, D.A., Bejan, A.: Convection in Porous Media, 4th edn. Springer, New York (2013)