The following exercise can be found on p. 283 of the textbook Applied Functional Analysis and Partial Differential Equations, by Milan Miklavčič, and it demonstrates nonuniqueness for semilinear parabolic equations under many definitions of a solution that are currently being used.

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10. Show that

$$u(x,t) = \left(c + \frac{\pi^2}{2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{e^{-2(n^2+m^2)t}}{n^2+m^2}\right)^{-1/4}\sum_{k=1}^{\infty}\frac{\sin kx}{k}e^{-k^2 t}$$

satisfies, for every $c \geq 0$,

$$u_t(x,t) = u_{xx}(x,t) + \left(\int_0^\pi |u_x(s,t)|^2 ds\right)^2 u(x,t) \qquad \text{for} \quad t > 0,\ 0 \leq x \leq \pi$$

$$u(0,t) = u(\pi,t) = 0 \qquad \text{for} \quad t \geq 0$$

$$\lim_{t\to 0+}\sup_{0\leq x\leq\pi}|u(x,t)| = 0.$$

(*Hint:* evaluate the Fourier sine series of $\pi - x$ and see Exercise 16 in Chapter 4.) Show that the PDE can be set as a semilinear parabolic equation in $L^2(0,\pi)$, with $\alpha = 1/2$ (see also Exercise 7) and that the uniqueness fails in this case because

$$\int_0^1 \left(\int_0^\pi |u_x(x,t)|^2 dx\right)^2 dt = \infty.$$

This example, in more abstract form, was first published by the author in *Pacific J. Math.* **118**(1985), pp. 199-214. In response to the article, Dan Henry sent to the author an example similar to the above one.