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ASYMPTOTIC PERIODICITY OF THE ITERATES OF POSITIVITY PRESERVING OPERATORS

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ABSTRACT. Assume that
(A1) X is a real Banach space.
(A2) X^+ is a closed subset of X with the following properties:
(i) if $x \in X^+$, $y \in X^+$, $\alpha \in [0, \infty)$ then $x + y \in X^+$ and $\alpha x \in X^+$;
(ii) there exists $M_0 \in (0, \infty)$ such that for each $x \in X$ there exist $x_+ \in X^+$ and $x_- \in X^+$ which satisfy

$$x = x_+ - x_-, \qquad \|x_+\| \leq M_0\|x\|, \qquad \|x_-\| \leq M_0\|x\|$$

and if $x = y_+ - y_-$ for some $y_+ \in X^+$, $y_- \in X^+$ then $y_+ - x_+ \in X^+$;
(iii) if $x \in X^+$, $y \in X^+$ then $\|x\| \leq \|x + y\|$.
(A3) B is a bounded linear operator on X.
(A4) $BX^+ \subset X^+$.
(A5) F_0 is a nonempty compact subset of X and $\lim_{n\to\infty} \operatorname{dist}(B^n x, F_0) = 0$ whenever $x \in X^+$ and $\|x\| = 1$.
Then $B^n x$ is asymptotically periodic for every $x \in X$. This, and other properties of B, are proven in the paper.

1. Introduction. It is well known that the iterates of operators with some compactness and some positivity properties are asymptotically periodic, e.g. [**1**, **3**, **4**, **5**, **7**, **8**]. This implies that the peripheral point spectrum of the operator consists of finitely many roots of unity which is quite remarkable. Assumptions A1–A5 represent results of an attempt to isolate the crucial properties of the space and of the operator which make the asymptotic periodicity possible.

It may be somewhat surprising that the space has to satisfy only assumptions A1 and A2 [**1**, p. 714]; see also [**7**], however, this is important in the statistical theory of deterministic processes [**2**, **5**]. For example, if τ is a map of the unit interval into itself and if

$$(Bf)(x) = \frac{d}{dx} \int_{\tau^{-1}[0,x]} f(y)\, dy$$

for a probability density $f \in L^1(0, 1)$, then $B^n f$ describes the evolution of densities generated by the deterministic system $\{\tau^n\}$. Assumption A5 can be verified for some τ [**5**]. In certain cases, although $B^n f$ is not eventually periodic in L^1, it is, however, eventually periodic in a space that contains Dirac-delta functions. One would expect that such situations occur typically when the sequence $\{\tau^n x\}$ is eventually periodic for almost all x. There are plenty of such spaces which satisfy also A1 and A2. For example, we can take that $X = ba(S, \mathscr{E}, \mathbf{R})$ [**1**, p. 160], the space of bounded, real-valued, finitely additive set functions on a field $\mathscr{E}$ of subsets

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of a set S and $X^+ = \{\mu \in X | \mu(E) \geq 0 \text{ for all } E \in \mathscr{E}\}$, or, if $\mathscr{E}$ is a σ-field, we can take $X = ca(S, \mathscr{E}, \mathbf{R})$ [**1**], the space of countably additive members of $ba(S, \mathscr{E}, \mathbf{R})$.

Assumption A2 seems to be intuitively clear, except possibly for "and if" part in A2ii. However, if $X = \mathbf{R}^3$, $X^+ = \{(x, y, z) | z \geq \sqrt{x^2 + y^2}\}$ and if B represents the rotation around z-axis by an angle equal to irrational multiple of π then all assumptions A1–A5 are satisfied except for "and if" part in A2ii, and of course, we do not have asymptotic periodicity in this case.

Assumption A5 was introduced in [**5**] and a simple way to verify it is given by the following:

THEOREM 1.1. *Suppose that T is a bounded linear operator on a (real or complex) Banach space W and that*

(1) $\lim_{n\to\infty}(1/n)\langle T^n x, y\rangle = 0$ *for all* $x \in W$ *and all* $y \in W^*$,

(2) $\|T^m - K\| < 1$ *for some integer* $m \geq 1$ *and some compact linear operator* K *on* W.

Then there exist $a \in (0, 1)$, $b < \infty$, *and a nonempty compact set* $F \subset W$ *such that* $\operatorname{dist}(T^n x, F) \leq ba^n$ *whenever* $n \geq 1$, $x \in W$ *and* $\|x\| \leq 1$.

We write $\langle x, y\rangle$ instead of $y(x)$ whenever $x \in W$ and $y \in W^*$ (the dual of W). This theorem is a slightly modified version of Theorem VIII.8.3 [**1**] and is proven in §6. Condition 2 of the theorem was introduced in [**4**] and has been very often used in studies of the behaviour of the iterates of T, e.g. [**1**, **7**, **8**]. Assumptions of the theorem are more restrictive than A5; for example, let $X = L^p(0, 1)$ for some $1 \leq p < \infty$ and $(Bf)(x) = xf(x)$ a.e. for $f \in X$.

If $(S, \mathscr{E}, \mu)$ is a positive measure space and if W is any of the spaces $L^1(S, \mathscr{E}, \mu)$, $ba(S, \mathscr{E}, \mathbf{R})$, $ca(S, \mathscr{E}, \mathbf{R}), \ldots$ [**1**, p. 511] then the assumption 2 of Theorem 1.1 is satisfied if T^n is a weakly compact operator on W for some $n \geq 1$. If $X = L^1(S, \mathscr{E}, \mu)$ for some σ-finite measure space $(S, \mathscr{E}, \mu)$ and if B is Markov operator on X then A5 is satisfied if there is a weakly compact set F in X such that $\lim_{n\to\infty} \operatorname{dist}(B^n x, F) = 0$ whenever $x \in X^+$ and $\|x\| = 1$ [**3**].

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2. Results. Assumptions A1 through A5 will be in effect throughout the rest of the paper. Define

$$M = \sup_{n \geq 0} \|B^n\|,$$

$$Y = \left\{ y \in X | y = \lim_{i\to\infty} B^{n_i} x \text{ for some } x \in X \text{ and some } 1 < n_1 < n_2 < \cdots \right\}.$$

THEOREM 2.1. *$M < \infty$ and Y is finite-dimensional vector space. If $Y = \{0\}$ then* $\lim_{n\to\infty} \|B^n x\| = 0$ *for every* $x \in X$.

This theorem and some other important properties of Y are proved in §3. Define $N = \dim Y \geq 0$ and let E be the set of all $x \in X^+ \cap Y$ such that $\|x\| = 1$ and if $x = y + z$ for some $y \in X^+ \cap Y$, $z \in X^+ \cap Y$ then $y = tx$ for some $t \in [0, 1]$. Some results of §4 are represented in the following theorem.

THEOREM 2.2. *E is a set of N elements and if $N \geq 1$ then E is a basis for Y. Moreover*

(1) *If $x \in E$ then $\|Bx\| \geq M^{-1}$ and $Bx = \|Bx\|y$ for some $y \in E$.*

(2) *If $\|B\| = 1$ then $\|Bx\| = 1$ for all $x \in E$.*

(3) *If $x \in E, y \in E, x \neq y$ and $h \in X^+$ are such that $x - h \in X^+$ and $y - h \in X^+$ then $\lim_{n\to\infty} \|B^n h\| = 0$, moreover, if $\|B\| = 1$ then $\|x - h\| = \|y - h\| = 1$.*

Note that part 3 is commenting on "disjoint support" of elements of E.

If $N \geq 1$ let $e_1, \ldots, e_N$ be an enumeration of E. In this case define $p\colon \{1, \ldots, N\} \to \{1, \ldots, N\}$ and $\lambda\colon \{1, \ldots, N\} \to (0, \|B\|]$ by

$$Be_i = \lambda(i) e_{p(i)}.$$

p is one-to-one (Lemma 4.6). Let m_0 be the smallest positive integer such that m_0th iterate of p is the identity map. We have that $m_0 \leq e^{N/e}$ and that $B^{m_0}x = x$ for all $x \in E$ (Lemma 4.7). Observe that if $x = e_1 + Be_1 + \cdots + B^{m_0-1}e_1$ then $x \in X^+$, $\|x\| \geq 1$ by A2iii and $Bx = x$.

THEOREM 2.3. *Suppose that $N \geq 1$. Then there exist $f_1, \ldots, f_N$ in X^* such that for all $i, j \in \{1, \ldots, N\}$*

(1) *$\lim_{n\to\infty} \|B^n(x - \sum_{k=1}^N \langle x, f_k\rangle e_k)\| = 0$ for all $x \in X$,*

(2) *$\langle e_i, f_j\rangle = \delta_{ij}$ (Kronecker delta),*

(3) *$0 \leq \langle x, f_i\rangle \leq M\|x\|$ for all $x \in X^+$,*

(4) *$\|f_i\| \leq MM_0$,*

(5) *$B^* f_{p(i)} = \lambda(i) f_i$,*

(6) *$\lim_{n\to\infty} \langle x, B^{*n}(y - \sum_{k=1}^N \langle e_k, y\rangle f_k)\rangle = 0$ for all $x \in X$ and all $y \in X^*$.*

This is our main theorem. It is proven in §5. Observe that if $M = M_0 = 1$ then f_i is actually a positive tangent functional to e_i and hence, in some spaces X, f_i is uniquely determined by e_i. The following theorem concerning the spectrum of B is also proven in §5 and it implies that if $N \geq 1$ then

$$Y = \{x \in X | B^{m_0}x = x\} = \bigcup_{n=1}^{\infty} \{x \in X | B^n x = x\}.$$

THEOREM 2.4. *Suppose that*

$$B^k x = \cos\varphi x - \sin\varphi y, \quad B^k y = \sin\varphi x + \cos\varphi y$$

for some integer $k \geq 1$, $\varphi \in [0, 2\pi)$ and some $x \in X$, $y \in X$ such that $\|x\| + \|y\| > 0$. Then $N \geq 1$, $x \in Y$, $y \in Y$ and $\varphi = 2\pi n/m_0$ for some $n \in \{0, 1, \ldots, m_0 - 1\}$.

The following theorem has applications in the study of the Boltzmann equation [**5**, **6**] and is proven in §5.

THEOREM 2.5. *For each $x \in X$ there exist $x_1 \in Y$ and a unique $x_0 \in Y$ such that*

$$\lim_{n\to\infty} \|B^n(x - x_0)\| = \lim_{n\to\infty} \left\| \frac{1}{n} \sum_{i=0}^{n-1} B^i x - x_1 \right\| = 0;$$

moreover,

$$\lim_{t\to\infty} \|e^{-t}e^{Bt}x - x_1\| = 0,$$

$Bx_1 = x_1$ *and if* $N \geq 1$ *then* $x_1 = m_0^{-1}\sum_{i=0}^{m_0-1} B^i x_0$.

For $x \in X$ define $P_0x = x_0$, $P_1x = x_1$ where x_0 and x_1 correspond to x as in the Theorem 2.5. P_0 and P_1 are projections and both commute with B. The following theorem is proven in §6.

THEOREM 2.6. *If* $\|B^m - K\| < 1$ *for some integer* $m \geq 1$ *and some compact linear operator* K *then there exist* $a \in (0,\infty)$ *and* $b \in (0,\infty)$ *such that*

$$\|B^n(I-P_0)\| \leq be^{-an}, \quad \left\|\frac{1}{n}\sum_{i=0}^{n-1} B^i - P_1\right\| \leq \frac{b}{n}, \quad \|e^{-t}e^{Bt} - P_1\| \leq be^{-at}$$

for all $n \geq 1$ *and all* $t > 0$.

3. Properties of Y. Since F_0 is a bounded set, A5 and A2ii imply that $\{\|B^n x\| | n \geq 0\}$ is bounded for every $x \in X$. The principle of uniform boundedness implies that $M < \infty$.

For $x \in X$ define

$$Q(x) = \left\{ y | y = \lim_{i\to\infty} B^{n_i}x \text{ for some } 1 < n_1 < n_2 < \cdots \right\}.$$

Note that $Y = \bigcup_{x\in X} Q(x)$ and if $y \in Q(x)$ then $By \in Q(x)$ and $\|y\| \leq M\|x\|$; hence $BY \subset Y$.

LEMMA 3.1. *If* $x \in X$ *then every sequence in* $\{B^n x | n \geq 0\}$ *has a subsequence that converges to some element in* X.

PROOF. If $x \in X^+$ and $\|x\| = 1$ then $\|B^n x - x_n\| < \text{dist}(B^n x, F_0) + 1/n$ for some $x_n \in F_0$ and all $n \geq 1$. A2ii and the fact that F_0 is compact imply the lemma.

LEMMA 3.2. *If* $x_0 \in X$, $x \in Q(x_0)$, $y \in Q(x_0)$ *then* $y \in Q(x)$ *and* $\|y\| \leq M\|x\|$.

PROOF. Pick $\varepsilon > 0$, $m \geq 1$ and note that $\|x - B^{n_1}x_0\| < \varepsilon/(1+M)$, $\|y - B^{n_2}x_0\| < \varepsilon/(1+M)$ for some $n_1 > 1$ and some $n_2 > n_1 + m$. Therefore $\|y - B^{n_2-n_1}x\| < \varepsilon$.

LEMMA 3.3. *If* $x_0 \in X$ *and* $x_1, x_2, \ldots$ *are in* $Q(x_0)$ *then there exist* $1 < n_1 < n_2 < \cdots$ *and* $x \in Q(x_0)$ *such that* $\lim_{i\to\infty} x_{n_i} = x$.

PROOF. Pick $1 < m_1 < m_2 < \cdots$ such that $\|B^{m_i}x_0 - x_i\| < 1/i$ for $i \geq 1$ and apply Lemma 3.1.

Above lemmas imply the following.

LEMMA 3.4. *If* $x \in X$ *then*

(1) $Q(x)$ *is a compact set,*

(2) $Q(x) = \{0\}$ *iff* $\inf\{\|y\| | y \in Q(x)\} = 0$ *iff* $\lim_{n\to\infty} B^n x = 0$,

(3) *if* $y \in Q(x)$ *then* $Q(y) = Q(x)$,

(4) $x \in Y$ *iff* $x \in Q(x)$,

(5) *if* $x \in X^+$ *then* $Q(x) \subset X^+$.

LEMMA 3.5. *If* $x \in Y$, $n \geq 0$, *then* $x = B^n y$ *for some* $y \in Q(x)$.

PROOF. By Lemma 3.4 $x = \lim_{i\to\infty} B^{n_i}x$ for some $n < n_1 < n_2 < \cdots$. Apply Lemma 3.1 to $B^{n_i-n}x$.

LEMMA 3.6. *If* $x \in Y$, $y \in Y$, $\alpha \in \mathbf{R}$, $\beta \in \mathbf{R}$ *then* $\alpha x + \beta y \in Y$.

PROOF. By Lemma 3.5 there exist $x_n \in Q(x)$, $y_n \in Q(y)$ such that $x = B^n x_n$, $y = B^n y_n$ for $n \geq 1$. Lemma 3.3 implies that $\lim_{i\to\infty} x_{n_i} = x_0$, $\lim_{i\to\infty} y_{n_i} = y_0$ for some $x_0 \in Q(x)$, $y_0 \in Q(y)$, $n_1 < n_2 < \cdots$; hence

$$\alpha x + \beta y = \lim_{i\to\infty} B^{n_i}(\alpha x_0 + \beta y_0).$$

LEMMA 3.7. *If* $x \in Y$, $y \in Y$, $n \geq 0$ *and* $B^n x = B^n y$ *then* $x = y$.

PROOF. Lemmas 3.6, 3.4 imply that $x - y \in Q(x - y) = \{0\}$.

LEMMA 3.8. *For each* $x \in X$ *there exists a unique* $x_0 \in Y$ *such that*

$$\lim_{n\to\infty} B^n(x - x_0) = 0;$$

moreover, $x_0 \in Q(x)$.

PROOF. Pick $n_1 < n_2 < \cdots$ and $y \in Q(x)$ such that $\lim_{i\to\infty} B^{n_i} x = y$. Let $y_n \in Q(x)$ be such that $y = B^n y_n$ for $n \geq 1$ (Lemmas 3.5, 3.4). Lemma 3.3 implies that (by renaming a subsequence) we may assume that $\lim_{i\to\infty} y_{n_i} = x_0$ for some $x_0 \in Q(x)$. If $m > n_i$ then

$$\|B^m(x - x_0)\| \leq M\|B^{n_i}x - y\| + M\|y_{n_i} - x_0\|$$

and therefore $\lim_{n\to\infty} B^n(x - x_0) = 0$. If $\lim_{n\to\infty} B^n(x - z) = 0$ for some $z \in Y$ then $B^n(z - x_0) = B^n(x - x_0) - B^n(x - z) \to 0$ as $n \to \infty$, so, $Q(z - x_0) = \{0\}$ by Lemma 3.4 and since $z - x_0 \in Q(z - x_0)$ we have $z = x_0$.

LEMMA 3.9. *If* $x \in Y$, $y \in Q(x)$ *and* $y - x \in X^+$ *then* $y = x$.

PROOF. Define $y_1 = y$, $y_0 = x$, $z_0 = y - x \in X^+ \cap Y$ and suppose that $z_0 \neq 0$. Since $z_0 \in Q(z_0)$ we have by Lemma 3.4 that $\sigma = \inf\{\|z\| | z \in Q(z_0)\} > 0$ and that $Q(z_0) \subset X^+$. Suppose that we have found $y_0, y_1, \ldots, y_{k+1}$ in $Q(x)$ and $z_0, z_1, \ldots, z_k$ in $Q(z_0)$ for some $k \geq 0$ such that $y_{i+1} = y_i + z_i$ for $0 \leq i \leq k$. By Lemma 3.2 $y_{k+1} = \lim_{i\to\infty} B^{n_i} y_k$ for some $n_1 < n_2 < \cdots$ and by choosing a subsequence we may assume (Lemma 3.1) that both $B^{n_i} y_{k+1}$ and $B^{n_i} z_k$ converge. Define $y_{k+2} = \lim_{i\to\infty} B^{n_i} y_{k+1}$, $z_{k+1} = \lim_{i\to\infty} B^{n_i} z_k$. By Lemma 3.4 $y_{k+2} \in Q(x)$ and $z_{k+1} \in Q(z_0)$ and also $y_{k+2} = y_{k+1} + z_{k+1}$. Therefore there exist $y_0, y_1, y_2, \ldots$ in $Q(x)$ and $z_0, z_1, z_2, \ldots$ in $Q(z_0)$ such that $y_{k+1} = y_0 + z_0 + \cdots + z_k$ for all $k \geq 0$. Assumption A2iii implies that if $0 \leq n < m$ then $\|y_m - y_n\| = \|z_n + \cdots + z_{m-1}\| \geq \|z_n\| \geq \sigma > 0$ and this contradicts Lemma 3.3; therefore $z_0 = 0$.

LEMMA 3.10. *If* $x \in X$, $y \in Q(x)$ *and* $y - x \in X^+$ *then* $\lim_{n\to\infty} B^n(y - x) = 0$.

PROOF. Let $z = y - x$ and pick $n_1 < n_2 < \cdots$, $z_1 \in Q(z)$, $y_1 \in Q(y)$, $x_1 \in Q(x)$ such that $B^{n_i} z \to z_1$, $B^{n_i} y \to y_1$, $B^{n_i} x \to x_1$ as $i \to \infty$. So, $x_1 \in Y$, $y_1 \in Q(x_1)$ by Lemma 3.4, $z_1 = y_1 - x_1 \in Q(z) \subset X^+$ and Lemma 3.9 implies that $z_1 = 0$. Lemma 3.4 implies that $B^n z \to 0$ as $n \to \infty$.

THEOREM 3.11. *For each* $x \in Y$ *there exist* $z_+ \in X^+ \cap Y$, $z_- \in X^+ \cap Y$ *such that*

$$x = z_+ - z_-, \quad \|z_+\| \leq MM_0\|x\|, \quad \|z_-\| \leq MM_0\|x\|$$

and if $x = y_+ - y_-$ *for some* $y_+ \in X^+ \cap Y$, $y_- \in X^+ \cap Y$ *then* $y_+ - z_+ \in X^+$.

PROOF. Take x_+, x_- as in A2ii. $x = x_+ - x_- = \lim_{i\to\infty} B^{n_i}(x_+ - x_-)$ for some $n_1 < n_2 < \cdots$. We may assume (by renaming a subsequence of n_i) that $\lim_{i\to\infty} B^{n_i} x_\pm = z_\pm$ for some $z_\pm \in Q(x_\pm)$. By Lemma 3.10

$$\lim_{n\to\infty} B^n(z_+ - x_+) = 0.$$

If $x = y_+ - y_-$ for some $y_\pm \in X^+ \cap Y$ then $g := y_+ - x_+ \in X^+$, so, $y_+ - z_+ = g + x_+ - z_+$ and since $y_+ - z_+ \in Q(y_+ - z_+)$ there are $m_1 < m_2 < \cdots$ such that $y_+ - z_+ = \lim_{i\to\infty} B^{m_i}(y_+ - z_+) = \lim_{i\to\infty}(B^{m_i}g - B^{m_i}(z_+ - x_+)) = \lim_{i\to\infty} B^{m_i} g \in X^+$.

PROOF OF THEOREM 2.1. All that we still have to show is that $\dim Y < \infty$. Let F_1 denote the closed convex hull of $F_0 \cup \{0\}$. Note that F_1 is compact, convex and $0 \in F_1$. Define

$$S^+ = \{x \in X^+ \cap Y \,|\, \|x\| \le 1\}.$$

If $x \in S^+\backslash\{0\}$ let $y = x/\|x\|$, and since $y \in Y$ there are $n_1 < n_2 < \cdots$ such that $y = \lim_{i\to\infty} B^{n_i} y$; hence $y \in F_0$ by A5 and $x = \|x\|y + (1 - \|x\|)0 \in F_1$. Therefore $S^+ \subset F_1$ and $\overline{S}^+$ is compact. Define

$$S_1 = \{x \in X \,|\, x = y - z \text{ for some } y \in \overline{S}^+, z \in \overline{S}^+\},$$
$$S = \{x \in \overline{Y} \,|\, MM_0\|x\| \le 1\}.$$

Clearly, $\overline{S}_1$ is totally bounded and hence it is compact. Theorem 3.11 implies that $S \subset \overline{S}_1$ and hence S is compact. Thus S is compact in $\overline{Y}$ and therefore $\dim Y < \infty$.

4. Properties of E. If $N = 0$ then $E = \emptyset$, so, assume $N \ge 1$ throughout this section.

LEMMA 4.1. *E is not empty and $\{\sum_{i=1}^{N+1} \alpha_i x_i \,|\, \alpha_i \in [0,\infty),\ x_i \in E$ for $i = 1, \ldots, N+1\}$ is dense in $X^+ \cap Y$.*

PROOF. Let $x_1, \ldots, x_N$ be a basis of Y. Define $T : \mathbf{R}^N \to Y$ by

$$T(\alpha_1, \ldots, \alpha_N) = \sum_{i=1}^{N} \alpha_i x_i.$$

Define $C = T^{-1}X^+$. The following properties of C will be needed.

(1) If $x \in C$, $y \in C$, $\alpha \in [0,\infty)$ then $x + y \in C$, $\alpha x \in C$.

(2) If $x \in C$ and $-x \in C$ then $x = 0$ (by A2iii).

(3) C is closed and $C \ne \{0\}$ (by Theorem 3.11 and $N > 0$).

Let C_1 be the convex hull of $\{x \in C \,|\, \|x\| = 1\}$. Note that C_1 is nonempty, compact and the property 2 of C implies that $0 \notin C_1$. Hahn-Banach theorem gives us $y_0 \in \mathbf{R}^N$ and $\gamma \in \mathbf{R}$ such that

$$0 < \gamma < (x, y_0)$$

for all $x \in C_1$. Hence $(x, y_0) \ge \gamma\|x\|$ for all $x \in C$. Define $D = \{x \in C \,|\, (x, y_0) = 1\}$. D is nonempty, compact and convex. Let E_1 be the set of extreme points of D. By verification

$$E = \left\{ \frac{1}{\|Tx\|} Tx \,\middle|\, x \in E_1 \right\}.$$

The Krein-Milman theorem implies that

$$\left\{ \sum_{i=1}^{N+1} \alpha_i x_i \,\middle|\, \alpha_i \geq 0,\ x_i \in E_1 \text{ for } i = 1, \ldots, N+1 \text{ and } \sum_{i=1}^{N+1} \alpha_i = 1 \right\}$$

is dense in D and this completes the proof.

LEMMA 4.2. *Suppose that $x_1, \ldots, x_k$ are distinct elements of E. If $(\alpha_1, \ldots, \alpha_k) \in \mathbf{R}^k$ and $\sum_{i=1}^k \alpha_i x_i \in X^+$ then $\alpha_i \geq 0$ for $i = 1, \ldots, k$.*

PROOF. Let P_k be the lemma as stated. P_1 is implied by A2iii. Assume that $k > 1$ and P_{k-1} is true. Suppose $(\alpha_1, \ldots, \alpha_k) \in \mathbf{R}^k$, $y = \sum_{i=1}^k \alpha_i x_i \in X^+$ and $\alpha_j < 0$ for some j. Since $\sum_{i \neq j} \alpha_i x_i = y - \alpha_j x_j \in X^+$, P_{k-1} implies $\alpha_i \geq 0$ whenever $i \neq j$; and A2iii implies $\alpha_m > 0$ for some m. Define $x = \alpha_m x_m - y$ and let z_+, z_- be as in Theorem 3.11; hence

$$x = \alpha_m x_m - y = -\sum_{i \neq m} \alpha_i x_i = z_+ - z_-.$$

P_{k-1} and $j \neq m$ imply $z_+ \neq 0$. Theorem 3.11 implies $\alpha_m x_m - z_+ \in X^+ \cap Y$ and $-\alpha_j x_j - z_+ \in X^+ \cap Y$. Definition of E implies $z_+ = t x_m$ and $z_+ = r x_j$ for some $t \in (0, \alpha_m]$, $r \in (0, -\alpha_j]$ and therefore $x_m = x_j$. Contradiction.

LEMMA 4.3. *E contains precisely N elements and these form a basis for Y.*

PROOF. Let $x_1, \ldots, x_k$ be distinct elements of E. Lemma 4.2 and A2iii imply that if $x = \sum_{i=1}^k \alpha_i x_i \in X^+$ then $0 \leq \alpha_i \leq \|x\|$ for $i = 1, \ldots, k$. Thus $x_1, \ldots, x_k$ are linearly independent; hence $|E| \leq N$. Assume $k = |E|$. By Lemma 4.1 $S = \{\sum_{i=1}^k \beta_i x_i | \beta_i \in [0, \infty) \text{ for } i = 1, \ldots, k\}$ is dense in $X^+ \cap Y$, and because $\sum_{i=1}^k \beta_i^2 \leq k \| \sum_{i=1}^k \beta_i x_i \|^2$ whenever $\beta_i \in [0, \infty)$ for $i = 1, \ldots, k$ we have that $S = X^+ \cap Y$. Theorem 3.11 implies $\operatorname{span}\{x_1, \ldots, x_k\} = Y$ and $k = N$.

LEMMA 4.4. *If $x \in E$ then $\|Bx\| > 0$ and $Bx = \|Bx\| y$ for some $y \in E$.*

PROOF. Lemma 3.7 implies $\|Bx\| > 0$. Let $y = \|Bx\|^{-1} Bx$ and suppose $y = u + v$ for some u, v in $X^+ \cap Y$. Lemma 3.5 implies $u = Bu_1$ and $v = Bv_1$ for some $u_1 \in Q(u) \subset X^+$, $v_1 \in Q(v) \subset X^+$. Lemma 3.7 implies $x = \|Bx\| u_1 + \|Bx\| v_1$; hence $\|Bx\| u_1 = tx$ for some $t \in [0, 1]$, so, $u = ty$ and $y \in E$.

LEMMA 4.5. *If $x \in E$, $n \geq 1$, $\lambda \in \mathbf{R}$ and $B^n x = \lambda x$ then $\lambda = 1$.*

PROOF. $|\lambda| \leq 1$ because $M < \infty$. Lemma 3.4 implies $x \in Q(x) \neq \{0\}$ and therefore $|\lambda| = 1$. A2iii implies $\lambda = 1$.

LEMMA 4.6. *p is one-to-one.*

PROOF. If $p(i) = p(j)$ then $B\lambda(i)^{-1} e_i = B\lambda(j)^{-1} e_j$ and by Lemma 3.7 $\lambda(i)^{-1} e_i = \lambda(j)^{-1} e_j$; hence $i = j$.

LEMMA 4.7. *$m_0 \leq e^{N/e}$ and $B^{m_0} x = x$ for every $x \in E$. Moreover, $M^{-1} \leq \|Bx\| \leq \|B\|$ for every $x \in E$.*

PROOF. Since p is one-to-one m_0 is well defined and $m_0 \leq e^{N/e}$. Lemma 4.5 implies $B^{m_0} x = x$ for all $x \in E$. If $x \in E$ then $1 = \|B^{m_0 - 1} Bx\| \leq M \|Bx\|$.

LEMMA 4.8. *If $x \in E$, $y \in E$, $x \neq y$ and $h \in X^+$ are such that $x - h \in X^+$ and $y - h \in X^+$ then $\lim_{n\to\infty} B^n h = 0$; moreover, if $\|B\| = 1$ then $\|x - h\| = \|y - h\| = 1$.*

PROOF. Let $f = x - h$, $g = y - h$ and note that $x = B^{nm_0} f + B^{nm_0} h$ and $y = B^{nm_0} g + B^{nm_0} h$ for $n \geq 0$. Lemma 3.1 implies that there are f_0, g_0, h_0 in $X^+ \cap Y$ and $n_1 < n_2 < \cdots$ such that $B^{n_i m_0} f \to f_0$, $B^{n_i m_0} g \to g_0$, $B^{n_i m_0} h \to h_0$ as $i \to \infty$. Therefore $x = h_0 + f_0$ and $y = h_0 + g_0$. Definition of E implies $h_0 = 0$ and by Lemma 3.4 $\lim_{n\to\infty} B^n h = 0$. Since $x \in Q(f)$ we have $1 \leq M\|f\| = M\|x - h\| \leq M$ and similarly $1 \leq M\|y - h\| \leq M$.

5. Asymptotic periodicity.

PROOF OF THEOREM 2.3. For $x \in X$ and $1 \leq i \leq N$ define $\gamma_i(x)$ by $\sum_{i=1}^N \gamma_i(x) e_i = x_0$ where x_0 is as in Lemma 3.8. Lemmas 3.8, 4.2 imply that $\gamma_i(x) \geq 0$ if $x \in X^+$ and $1 \leq i \leq N$. If $x \in X$, $y \in X$, $\alpha \in \mathbf{R}$, $\beta \in \mathbf{R}$ then

$$\left\| B^n \left(\alpha x + \beta y - \sum_{i=1}^N (\alpha\gamma_i(x) + \beta\gamma_i(y)) e_i \right) \right\| \leq |\alpha| \left\| B^n \left(x - \sum_{i=1}^N \gamma_i(x) e_i \right) \right\| + |\beta| \left\| B^n \left(y - \sum_{i=1}^N \gamma_i(y) e_i \right) \right\|$$

and therefore $\gamma_i(\cdot)$ are linear functionals. If $x \in X$ then (Lemma 3.4) for some $n_1 < n_2 < \cdots$

$$\left\| \sum_{i=1}^N \gamma_i(x) e_i \right\| = \lim_{j\to\infty} \left\| B^{n_j} \sum_{i=1}^N \gamma_i(x) e_i \right\| = \lim_{j\to\infty} \|B^{n_j} x\| \leq M\|x\|.$$

Thus, A2iii implies that $0 \leq \gamma_j(x) \leq \|\sum_{i=1}^N \gamma_i(x) e_i\| \leq M\|x\|$ for $x \in X^+$, $1 \leq j \leq N$. A2ii implies that $|\gamma_j(x)| \leq M M_0 \|x\|$ for all $1 \leq j \leq N$, $x \in X$. Define $f_i \in X^*$ by $\langle x, f_i \rangle = \gamma_i(x)$ for $x \in X$, $1 \leq i \leq N$. This proves parts 1, 2, 3, 4 of the theorem. Since

$$\begin{aligned} Bx - \sum_{i=1}^N \langle Bx, f_i \rangle e_i &= Bx - \sum_{i=1}^N \langle x, \lambda(i)^{-1} B^* f_{p(i)} \rangle \lambda(i) e_{p(i)} \\ &= B \left(x - \sum_{i=1}^N \langle x, \lambda(i)^{-1} B^* f_{p(i)} \rangle e_i \right) \end{aligned}$$

for every $x \in X$ part 5 is proven. Part 5 implies

$$\sum_{k=1}^N \langle x, f_k \rangle \langle B^n e_k, y \rangle = \sum_{k=1}^N \langle x, B^{*n} f_k \rangle \langle e_k, y \rangle$$

and

$$\left\langle x, B^{*n} \left(y - \sum_{k=1}^N \langle e_k, y \rangle f_k \right) \right\rangle = \left\langle B^n \left(x - \sum_{k=1}^N \langle x, f_k \rangle e_k \right), y \right\rangle$$

whenever $x \in X$, $y \in X^*$ and $n \geq 1$. This proves part 6.

PROOF OF THEOREM 2.4. Note that for $n \geq 1$

$$B^{nk}x = \cos(n\varphi)x - \sin(n\varphi)y,$$
$$B^{nk}y = \sin(n\varphi)x + \cos(n\varphi)y,$$
$$\inf_{\beta \geq 0}\{\|\cos\beta x - \sin\beta y\| + \|\sin\beta x + \cos\beta y\|\} > 0.$$

Theorem 2.1 implies $Y \neq \{0\}$ and hence $N \geq 1$. Pick $u \in Y$, $v \in Y$ such that $\lim_{n\to\infty} B^n(x-u) = \lim_{n\to\infty} B^n(y-v) = 0$. Note that $B^{m_0}u = u$, $B^{m_0}v = v$. Define $\alpha = m_0\varphi$ and

$$u_j := \cos(j\alpha)x - \sin(j\alpha)y = B^{jkm_0}x \to u \quad \text{as } j \to \infty,$$
$$v_j := \sin(j\alpha)x + \cos(j\alpha)y = B^{jkm_0}y \to v \quad \text{as } j \to \infty.$$

Therefore $u_{j\pm 1} = \cos\alpha u_j \mp \sin\alpha v_j$ and $v_{j\pm 1} = \cos\alpha v_j \pm \sin\alpha u_j$ for $j \geq 2$ and $\|u\| + \|v\| > 0$. This implies that $\cos\alpha = 1$, $\sin\alpha = 0$, $x = u$, $y = v$.

LEMMA 5.1. *If $m \geq 1$, $0 \leq n \leq m-1$, $t > 0$ then*

$$\left| e^{-t}\sum_{k=0}^{\infty}\frac{t^{km+n}}{(km+n)!} - \frac{1}{m}\right| \leq \frac{m-1}{m}\exp\left(-t\left(1-\cos\frac{2\pi}{m}\right)\right).$$

PROOF. If $z = \exp(2\pi i/m)$ then

$$me^{-t}\sum_{k=0}^{\infty}\frac{t^{km+n}}{(km+n)!} = \sum_{k=0}^{m-1} z^{-kn}\exp(z^k t - t).$$

PROOF OF THEOREM 2.5. Existence and uniqueness of x_0 is given in Lemma 3.8. Note that for $n \geq 1$, $t > 0$

$$\frac{1}{n}\sum_{k=0}^{n-1} B^k x = \frac{1}{n}\sum_{k=0}^{n-1} B^k(x - x_0) + \frac{1}{n}\sum_{k=0}^{n-1} B^k x_0,$$
$$e^{-t}e^{Bt}x = e^{-t}\sum_{k=0}^{\infty}\frac{t^k}{k!}B^k(x-x_0) + e^{-t}\sum_{k=0}^{\infty}\frac{t^k}{k!}B^k x_0$$

and that both first sums converge to 0 as $n \to \infty$, $t \to \infty$. Thus, if $Y = \{0\}$ take $x_1 = 0$ and if $Y \neq \{0\}$ then let $x_1 = m_0^{-1}\sum_{k=0}^{m_0-1} B^k x_0$ and observe that for $n \geq 1$

$$\left\|\frac{1}{n}\sum_{k=0}^{n-1} B^k x_0 - x_1\right\| \leq \frac{2m_0 M\|x_0\|}{n}$$

and that Lemma 5.1 implies for $t > 0$

$$\left\| e^{-t}\sum_{k=0}^{\infty}\frac{t^k}{k!}B^k x_0 - x_1\right\| \leq (m_0 - 1)M\|x_0\|\exp\left(-t\left(1-\cos\frac{2\pi}{m_0}\right)\right).$$

This completes the proof.

6. Quasi-compactness. If W is a Banach space let $\mathscr{L}(W)$ denote the set of all bounded linear maps from W into W. If W is complex Banach space and $C : W \to W$ is such that $C^2 = I$, $\|Cx\| = \|x\|$ and $C(\alpha x + \beta y) = \overline{\alpha}x + \overline{\beta}y$ for all $x \in W$, $y \in W$, $\alpha \in \mathbf{C}$, $\beta \in \mathbf{C}$ then C is called conjugation on W. The following theorem follows directly from the Theorem VIII.8.3 [**1**], see also [**8**].

THEOREM 6.1. *Suppose that W is a complex Banach space, $T \in \mathscr{L}(W)$, $\lim_{n\to\infty}\langle T^n x, y\rangle/n = 0$ for all $x \in W$ and all $y \in W^*$, and $\|T^m - K\| < 1$ for some integer $m \geq 1$ and some compact $K \in \mathscr{L}(W)$. Then there exist $K_1 \in \mathscr{L}(W)$, $V_1 \in \mathscr{L}(W)$, $a \in (0,1)$, $b < \infty$ such that*

(1) $\mathrm{Ran}(K_1)$ *is finite dimensional and* $\sup_{n\geq 0}\|K_1^n\| < \infty$,

(2) $\|V_1^n\| \leq ba^n$ *for* $n = 1, 2, \ldots$,

(3) $T^n = K_1^n + V_1^n$ *for* $n = 1, 2, \ldots$,

(4) *if C is a conjugation on W and $TC = CT$ then $K_1C = CK_1$*,

(5) *if $x_0 \in \mathrm{Ran}(K_1)$ then there exist $x_1, \ldots, x_n$ in $\mathrm{Ran}(K_1)$ and $\lambda_1, \ldots, \lambda_n$ in $\mathbf{C}$ such that $x_0 = x_1 + \cdots + x_n$, $K_1x_i = Tx_i = \lambda_i x_i$, $|\lambda_i| = 1$ for $1 \leq i \leq n$.*

PROOF OF THEOREM 1.1. If W is complex then the statement is obvious. Assume W is real. Let $Z = W \times W$, $Z_1 = W^* \times W^*$ be the usual complexifications of W, W^* defined by

$$\|(x,y)\| = \sup\{\sqrt{\|\alpha x - \beta y\|^2 + \|\beta x + \alpha y\|^2} | \alpha \in \mathbf{R}, \beta \in \mathbf{R}, \alpha^2 + \beta^2 = 1\}.$$

Define $V = T^m - K$, $T_0(x,y) = (Tx, Ty)$, $K_0(x,y) = (Kx, Ky)$, $V_0(x,y) = (Vx, Vy)$, $C_1(x,y) = (y,x)$, $C_2(x,y) = (x,-y)$ for $(x,y) \in Z$. It is easy to verify that T_0, K_0, V_0 are in $\mathscr{L}(Z), K_0$ is compact, $V_0 = T_0^m - K_0$, $\|V_0\| = \|V\| < 1$ and C_1 and C_2 are conjugations on Z and both commute with T_0. If $x \in Z$, $y \in Z^*$ then

$$\langle T_0^n x, y\rangle = \langle T^n x_r, y_r\rangle - \langle T^n x_i, y_i\rangle + i\langle T^n x_r, y_i\rangle + i\langle T^n x_i, y_r\rangle$$

for some $(x_r, x_i) \in Z$, $(y_r, y_i) \in Z_1$. Let $K_1 \in \mathscr{L}(Z)$, $V_1 \in \mathscr{L}(Z)$, $a \in (0,1)$, $b < \infty$ be as given by Theorem 6.1. Applying C_1 and C_2 one can show that $K_1(x,y) = (K_2x, K_2y)$, $V_1(x,y) = (V_2x, V_2y)$ for some $V_2 \in \mathscr{L}(W)$, some compact $K_2 \in \mathscr{L}(W)$ and all $(x,y) \in Z$; moreover, $\|V_2^n\| = \|V_1^n\| \leq ba^n$, $\|K_2^n\| = \|K_1^n\|$, $T^n = K_2^n + V_2^n$ for $n \geq 1$. Let F be the closure of $K_2\{x \in W | \|x\| \leq \sup_{n\geq 0}\|K_2^n\|\}$.

PROOF OF THEOREM 2.6. Let $Z = X \times X$ and $Y_c = Y \times Y$ be the complexifications of X and Y (as above). Define $B_0(x,y) = (Bx, By)$, $K_0(x,y) = (Kx, Ky)$ for $(x,y) \in Z$. Let $K_1 \in \mathscr{L}(Z)$, $V_1 \in \mathscr{L}(Z)$, $a \in (0,1)$, $b < \infty$ be as in Theorem 6.1 (corresponding to B_0). As above $K_1(x,y) = (K_2x, K_2y)$, $V_1(x,y) = (V_2x, V_2y)$ for some $K_2 \in \mathscr{L}(X)$, $V_2 \in \mathscr{L}(X)$ and all $(x,y) \in Z$.

Suppose $x_0 \in \mathrm{Ran}(K_1)$ and let $x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_n$ be as in Theorem 6.1. Since $|\lambda_i| = 1$ and $B_0x_i = \lambda_i x_i$ for $1 \leq i \leq n$ Theorem 2.4 implies that $x_0 \in Y_c$ and, clearly, $B_0x_0 = K_1x_0$. This implies that if $x \in \mathrm{Ran}(K_2)$ then $x \in Y$ and $K_2x = Bx$.

Pick $x \in X, n \geq 1$. Then $B^n x = K_2^n x + V_2^n x = B^{n-1}K_2x + V_2^n x$ and since $K_2x \in Y$ there exists (Lemma 3.5) $x_0 \in Y$ such that $K_2x = Bx_0$. Therefore $B^n x = B^n x_0 + V_2^n x$ and by Lemma 3.8

$$\|B^n(I - P_0)\| = \|V_2^n\| \leq ba^n.$$

If this inequality is used in the proof of Theorem 2.5 (§5) then the other two inequalities are obtained.

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