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The flow due to a rough rotating disk

M. Miklavčič and C. Y. Wang

Abstract. Von Kármán's problem of a rotating disk in an infinite viscous fluid is extended to the case where the disk surface admits partial slip. The nonlinear similarity equations are integrated accurately for the full range of slip coefficients. The effects of slip are discussed. An existence proof is also given.

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1. Introduction

The flow due to a disk rotating in a viscous fluid was originally solved by Von Kármán [1]. It is important in the study of flows on rotating bodies, centrifugal pumps, viscometers etc. The solution is also a rare three-dimensional, exact similarity solution of the Navier-Stokes equations [2].

Von Kármán considered the infinite disk rotating with constant angular velocity Ω in a fluid of kinematic viscosity ν . Let (u, v, w) be the velocity components in the cylindrical coordinates (r, θ, z) respectively. Using the similarity transform

$$u = \Omega r f'(\zeta), \quad v = \Omega r g(\zeta), \quad w = -2\sqrt{\nu\Omega} f(\zeta) \tag{1}$$

where

$$\zeta = z\sqrt{\Omega/\nu} \tag{2}$$

the continuity equation is satisfied exactly and the Navier-Stokes equations reduce to the non-linear ordinary differential equations

$$f''' + 2ff'' - f'^2 + g^2 = 0 \tag{3}$$

$$g'' + 2fg' - 2f'g = 0. \tag{4}$$

Far from the disk, the lateral velocities are zero, thus

$$f'(\infty) = 0, \quad g(\infty) = 0. \tag{5}$$

On the disk the no-slip condition applies

$$f'(0) = 0, \quad f(0) = 0, \quad g(0) = 1. \tag{6}$$

Equations (3-6) have been solved numerically by Cochran [3] and others. The properties of equations (3-6), such as existence, was discussed, notably by McLeod [4]. Extensions to the swirling flow above a fixed disk, suction on the disk, and the case of two rotating disks were reviewed by Zandbergen and Dijkstra [5].

However, the surface of the disk may be rough and not perfectly smooth as assumed. In these cases the no-slip boundary condition becomes impractical to apply exactly. But if the characteristic scale of the roughness is small compared to the boundary layer thickness on the disk, the no-slip condition may be approximated by a partial slip condition applied to the envelop of the protuberances. Navier [6] first proposed the equivalent partial slip condition for rough surfaces, relating the tangential velocity u to the local tangential shear stress τ

$$u = N\tau \tag{7}$$

where N is a slip coefficient to be determined by experiments. Equation (7) is valid for other surface conditions as well, notably rarefied gas flow [7], and porous boundary [8, 9]. On the other hand, the roughness may not be statistically isotropic. For example, it was found that for parallel, grooved surfaces the slip is larger in the direction along the grooves than the direction transverse to the grooves [10, 11]. In such a case the slip coefficient may be different in different directions.

The purpose of the present paper is to study the flow due to a rough rotating disk. We shall assume in general, that the principle directions of the roughness are radial and azimuthal, e.g. a concentrically grooved disk such as a phonograph record or a laser-etched disk. The results of course also apply to the special case of randomly rough disk.

2. Formulation and the numerical method

A generalization of Navier's partial slip condition gives, in the radial direction,

$$u|_{z=0} = N_1 \rho \nu \left. \frac{\partial u}{\partial z} \right|_{z=0} \tag{8}$$

and in the azimuthal direction

$$v|_{z=0} - r\Omega = N_2 \rho \nu \left. \frac{\partial v}{\partial z} \right|_{z=0} \tag{9}$$

where N_1, N_2 are the respective slip coefficients and ρ is the fluid density. Let

$$\lambda = N_1 \rho \sqrt{\nu \Omega}, \quad \eta = N_2 \rho \sqrt{\nu \Omega}. \tag{10}$$

For uniform roughness, $\lambda = \eta$, and for (anisotropic) concentric grooves $\lambda < \eta$. Equations (8,9) reduce to

$$f'(0) = \lambda f''(0) \tag{11}$$

$$g(0) - 1 = \eta g'(0). \tag{12}$$

If there is uniform suction of velocity W on the disk, the boundary condition is

$$f(0) = s \tag{13}$$

where

$$s = \frac{W}{2\sqrt{\nu\Omega}}. \tag{14}$$

The governing equations are still equations (3, 4). The boundary conditions at infinity are equation (5) but those on the disk are replaced by equations (11-13).

The existence proof as well as numerical solution are based on studying equations (3, 4) subject to initial conditions

$$f(0) = s,\ f'(0) = \lambda\alpha,\ f''(0) = \alpha,\ g(0) = 1 + \eta\beta,\ g'(0) = \beta \tag{15}$$

where α, β are to be found so that the boundary conditions at infinity equation (5) are satisfied.

At a given λ and η we first find α and β that minimize

$$J = f'^2 + f''^2 + g^2 + g'^2$$

at an ending point. With no clue for the values of α and β we first took the ending point to be as low as 3. This ending point was gradually increased to 30. We know from the Theorem in Section 4 that $f(\zeta)$ behaves like $k/2 + ce^{-k\zeta}$ for large ζ and that f'/g and f'/g' should tend to a constant. Using this, the values of α and β were improved by making f , f'/g and f'/g' vary as little as possible in the interval $[15, 25]$. This was done by minimization of a function that consisted of 12 squares of differences of values of f , f'/g and f'/g' between different points in the interval $[15, 25]$. This way we got α and β accurate to about 14 digits when η was not too large. For larger values of η (say 20) the decay rate is much slower hence we got only about 12 digits correct. Then $f(\infty) = k/2$ was estimated several different ways. First note that $(f' + f^2)' = 3f'^2 - g^2$ hence $y = f' + f^2$ approaches $(k/2)^2$ much faster then f . So, assuming that y approaches exponentially to a constant one can find the constant from the values of $y(6)$, $y(12)$ and $y(18)$. On the other hand, having an approximation of k one can show that

$$k/2 = f + (f'(\zeta) + (f'(\zeta)^2 + g(\zeta)^2)/(2k)^2)/k + O(e^{-3k\zeta}).$$

Evaluating this at $\zeta = 18$ gives an independent estimate of $k/2$ that differed from the previous one by at most about 10^{-14} when $\eta \leq 1$. When η increased to 20 the difference increased to up to 10^{-7} . In the no-slip case ($\lambda = \eta = 0$) we obtained the values

$$\alpha = 0.510232618867, \quad \beta = -0.615922014399, \quad k/2 = 0.442237055104.$$

The most accurate values so far was due to Rogers and Lance [12] who obtained 6 correct digits for α and β and 4 correct digits for $k/2$. Table 1 displays our results for a range of normalized slip coefficients $\lambda \le \eta$. Sparrow et al [9] studied the uniform roughness case for which $\lambda = \eta$. Their results cannot be compared here since no numerical values were given.

Table 1. Initial and final values for various λ and η. In each box the values from top are $\alpha = f''(0)$, $\beta = g'(0)$ and $k/2 = f(\infty)$ respectively.

λ	η	α	β	$k/2$
0.0	0.0	0.51023262	−0.61592201	0.4422371
0.0	0.1	0.46764344	−0.56451092	0.4295734
0.1	0.1	0.42145364	−0.60583524	0.4406821
0.0	0.2	0.43244953	−0.52202696	0.4185148
0.1	0.2	0.38880071	−0.55643671	0.4283232
0.2	0.2	0.35258101	−0.58367676	0.4369786
0.0	0.5	0.35526235	−0.42885126	0.3919663
0.1	0.5	0.31831442	−0.45080605	0.3992111
0.2	0.5	0.28813888	−0.46805858	0.4057219
0.5	0.5	0.22384821	−0.50280970	0.4211963
0.0	1.0	0.27704738	−0.33443488	0.3607866
0.1	1.0	0.24808615	−0.34708362	0.3658038
0.2	1.0	0.22464463	−0.35701609	0.3704435
0.5	1.0	0.17505362	−0.37708604	0.3819161
1.0	1.0	0.12792364	−0.39492760	0.3947386
0.0	2.0	0.19561205	−0.23613106	0.3212656
0.1	2.0	0.17574066	−0.24210702	0.3243297
0.2	2.0	0.15966284	−0.24683660	0.3272949
0.5	2.0	0.12553781	−0.25652755	0.3350748
1.0	2.0	0.09275714	−0.26534236	0.3444405
2.0	2.0	0.06101010	−0.27337013	0.3551567
0.0	5.0	0.10714640	−0.12934066	0.2628602
0.1	5.0	0.09731966	−0.13103722	0.2642169
0.2	5.0	0.08924617	−0.13241262	0.2656453
0.5	5.0	0.07170445	−0.13533256	0.2698076
1.0	5.0	0.05424057	−0.13812723	0.2754891
2.0	5.0	0.03660802	−0.14081126	0.2828201
5.0	5.0	0.01858853	−0.14338821	0.2918823
0.0	10.0	0.06242460	−0.07535521	0.2195420
0.1	10.0	0.05735922	−0.07591633	0.2202124
0.2	10.0	0.05311218	−0.07638266	0.2209750
0.5	10.0	0.04361295	−0.07740871	0.2234107
1.0	10.0	0.03376468	−0.07844109	0.2271102
2.0	10.0	0.02337647	−0.07948551	0.2324076
5.0	10.0	0.01221566	−0.08054447	0.2397020
10.0	10.0	0.00681256	−0.08103009	0.2437923

0.0	20.0	0.03454029	−0.04169496	0.1802357
0.1	20.0	0.03213500	−0.04186432	0.1805514
0.2	20.0	0.03007023	−0.04200886	0.1809417
0.5	20.0	0.02529181	−0.04233953	0.1823135
1.0	20.0	0.02009617	−0.04269109	0.1846365
2.0	20.0	0.01432685	−0.04306830	0.1883467
5.0	20.0	0.00774902	−0.04347617	0.1941045
10.0	20.0	0.00440006	−0.04367282	0.1976455
20.0	20.0	0.00236159	−0.04378846	0.1999879

3. The flow field and torque

Consider the uniform roughness case ($\lambda = \eta$). Fig. 1 shows the azimuthal velocity represented by $g(\zeta)$. Its value in general decreases as slip is increased, and the decay is exponential for large ζ. Fig. 2 shows the induced radial velocity profile $f'(\zeta)$ caused by the centrifugal forces. For the no-slip case (Von Kármán's original problem) the radial velocity starts from zero and reaches a maximum near $\zeta = 0.92$ then decays to zero. With slip the maximum velocity decreases, and its location moves towards the disk. Notice the prominent cross over of the curves near $\zeta = 4$, showing although slip decreases the velocity near the disk, it increases the velocity far from the disk. This cross over was not observed in [9], probably due to inaccurate initial values used. Fig. 3 shows the vertical velocity profile represented by $f(\zeta)$. Slip decreases the magnitude of the induced vertical suction far from the disk, but not necessarily in regions near the disk. For other combinations of λ and η the behavior is similar and is not presented here. The corresponding velocity profiles can be generated using the initial values given in Table 1.

Of physical interest is the magnitude of the constant suction velocity at infinity, given by $2\sqrt{\nu\Omega}f(\infty)$ and the resisting torque T on a disk of radius R

$$T = -\int_0^R \mu \frac{\partial v}{\partial z} 2\pi r^2 dr = -\frac{\pi\rho\Omega}{2}\sqrt{\nu\Omega}Rg'(0)$$

It is interesting to note, from Table 1, that an increase in λ increases both suction velocity and torque slightly, while an increase in η greatly decreases both suction velocity and torque.

4. Existence proof

We shall assume throughout that $s \geq 0, \lambda \geq 0, \eta > 0$ are arbitrary but fixed and we will consider α, β as real variables. The main idea of the proof of the following Thoerem is similar to the one McLeod and Serrin used [4, 13] to prove existence in the special case $\lambda = \eta = 0$. The main Lemma 5 is proved quite differently here. Our proof gives also rigorous asymptotics.

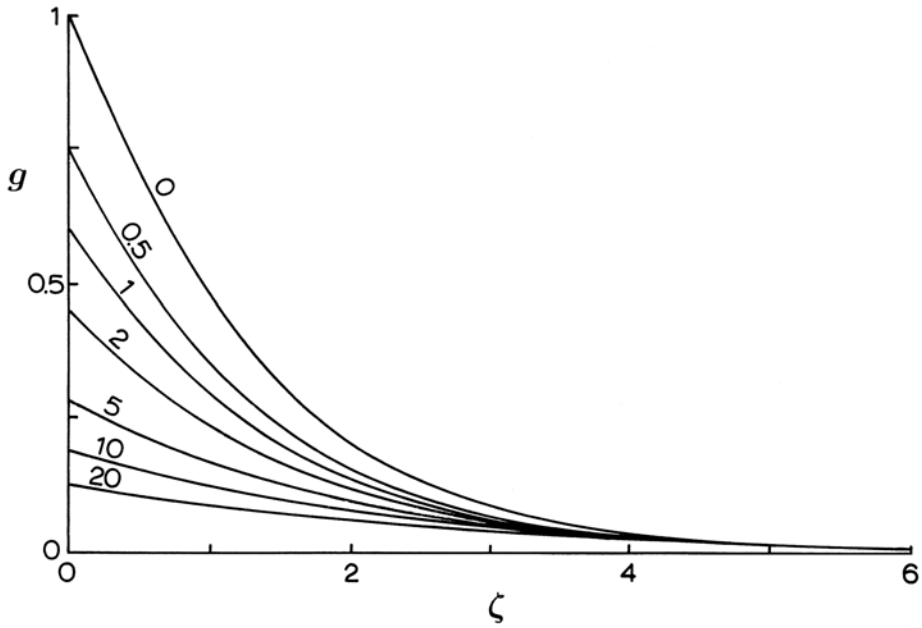

Figure 1. The azimuthal velocity profile, $\lambda = \eta$.

Theorem. *There exist* $\alpha > 0, \beta < 0$ *for which the system (3,4,15) has a solution on* $[0, \infty)$ *such that* $f' > 0$ *and* $g' < 0$ *on* $(0, \infty)$ *and for some* $k \in (0, \infty)$ *each of*

$$f'''(\zeta)e^{k\zeta}, \quad f''(\zeta)e^{k\zeta}, \quad f'(\zeta)e^{k\zeta}, \quad (k/2 - f(\zeta))e^{k\zeta}, \quad g'(\zeta)e^{k\zeta}, \quad g(\zeta)e^{k\zeta}$$

has a finite limit as $\zeta \to \infty$.

For any real α, β the system (3,4,15) has a solution on some maximal interval $[0, \ell)$ where $\ell = \ell(\alpha, \beta) \in (0, \infty]$. We shall frequently use the following obvious identities on $[0, \ell)$

$$(f''e^F)' = (f'^2 - g^2)e^F \tag{16}$$

$$(g'e^F)' = 2f'ge^F \tag{17}$$

$$(f'''e^F)' = -2gg'e^F \tag{18}$$

where $F(x) = 2\int_0^x f(t)dt$.

Define S^- to be the set of real number pairs (α, β) such that there exists $x^- \in (0, \ell)$ for which

$$g(x^-) < 0$$

$$g'(x) < 0 \quad \text{for} \quad 0 \le x \le x^-.$$

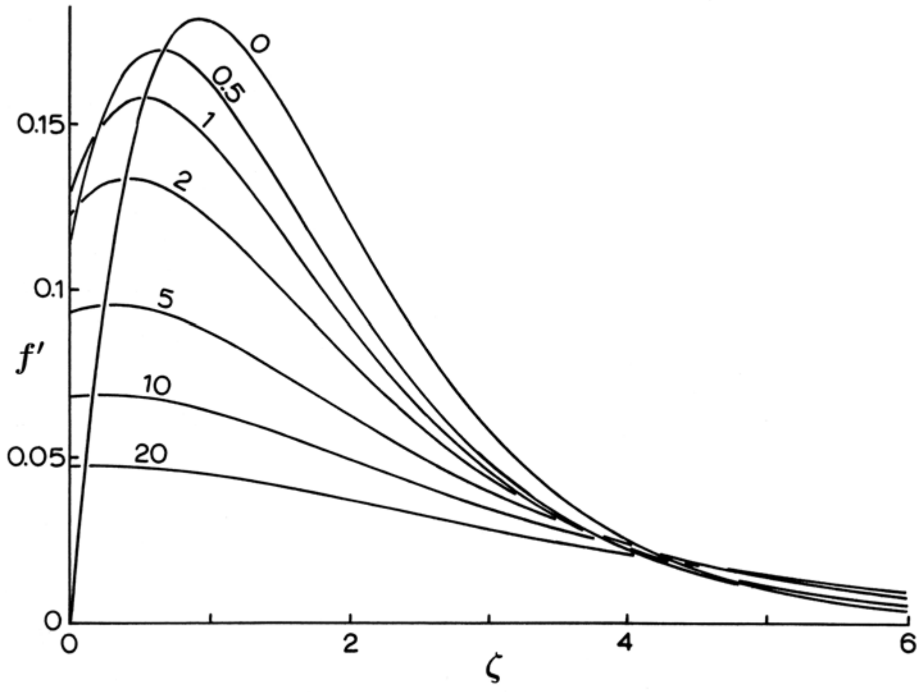

Figure 2. The radial velocity profile, $\lambda = \eta$.

Define S^+ to be the set of real number pairs (α, β) such that there exists $x^+ \in (0, \ell)$ for which

$$g'(x^+) > 0$$

$$g(x) > 0 \quad \text{for} \quad 0 \leq x \leq x^+.$$

Note that S^- and S^+ are open sets in the plane.

Lemma 1. *If* $f' \geq 0, g \geq 0, g' \leq 0$ *on* $[0, \ell)$ *then* $\ell = \infty$ *and*

$$|f'''(x)| \leq (|f'''(0)| + 2g(0)|\beta|x)e^{-F(x)} \quad \text{for } x \in [0, \infty), \tag{19}$$

Proof. Note first that there exists $\lim_{x \to \ell} g(x) \in [0, g(0)]$. $g' \leq 0$ and (17) imply that

$$\beta \leq g' e^F \leq 0 \quad \text{on} \quad [0, \ell). \tag{20}$$

This and (18) imply $|(f''' e^F)'| \leq 2g(0)|\beta|$ hence

$$|f'''(x)e^{F(x)} - f'''(0)| \leq 2g(0)|\beta|x \quad \text{for } x \in [0, \ell)$$

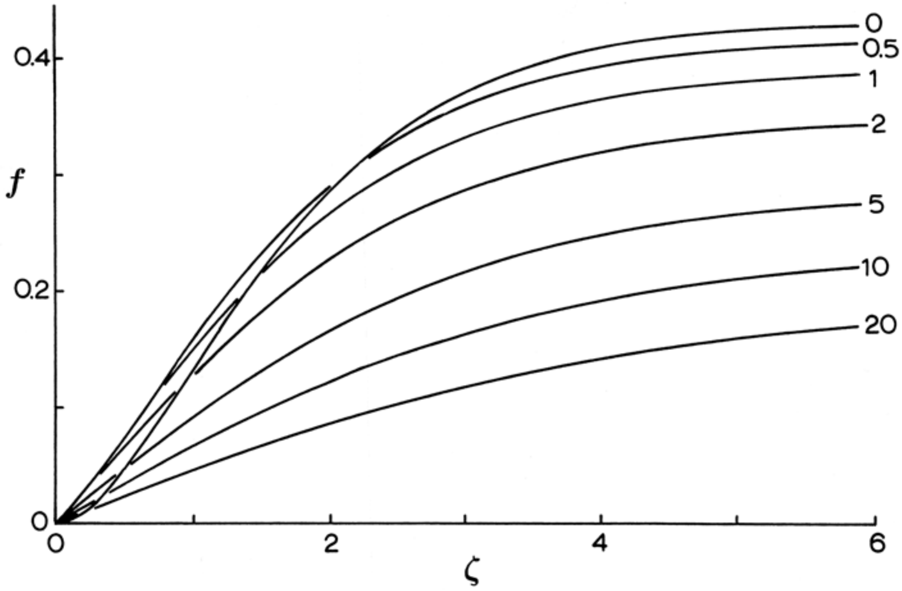

Figure 3. The vertical velocity profile, $\lambda = \eta$.

which implies (19). If $\ell < \infty$ then

$$|f'''(x)e^{2xs}| \leq |f'''(0)| + 2g(0)|\beta|x$$

and therefore f''' is bounded on $[0, \ell)$, which implies that f'', f', f have limits as $x \to \ell$. (20) implies that g' is bounded and hence g'' given by (4) is bounded, which implies that g' has limits as $x \to \ell$ and therefore the solution of (3,4,15) can be continued which is a contradiction.

Lemma 2. *If* $\alpha > 0$ *and* $\beta = 0$ *then* $(\alpha, \beta) \in S^+$.

Proof. Since $f''(0) = \alpha > 0$ and $g(0) = 1$ there exists $x^+ > 0$ such that $f'' > 0$ and $g > 0$ on $[0, x^+]$. Hence $f' > 0$ on $(0, x^+]$ and (17) implies $(g'e^F)' > 0$ on $(0, x^+]$ and therefore $g' > 0$ on $(0, x^+]$.

Lemma 3. *If* $\alpha \geq 0$ *and* $\beta = -1/\eta$ *then* $(\alpha, \beta) \in S^-$.

Proof. Since $g'(0) = \beta < 0$ there exists $x^- > 0$ such that $g' < 0$ on $[0, x^-]$; and since $g(0) = 0$ we have that $g < 0$ on $(0, x^-]$.

Lemma 4. *If* $\alpha > 0$, $\beta \in (-1/\eta, 0)$, $(\alpha, \beta) \notin S^+ \cup S^-$ *and* $f' > 0$ *on* $(0, a)$ *for some* $a \in (0, \ell)$ *, then,* $g > 0$ *on* $[0, a]$ *and* $g' < 0$ *on* $[0, a)$ *. Moreover, if in* *addition* $f'(a) = 0$ *then* $f' < 1$ *on* $[0, a]$.

Proof. If $g(b) \leq 0$ for some $b \in (0, a]$ pick the smallest such b . Hence $g(b) = 0$, $g'(b) \leq 0$ and since $g \not\equiv 0$ we have that $g'(b) < 0$. (17) implies that $g' < 0$ on $[0, b]$ implying contradiction $(\alpha, \beta) \in S^-$. Therefore $g > 0$ on $[0, a]$.

If $g'(b) \geq 0$ for some $b \in (0, a)$ then (17) implies that $g' > 0$ on (b, a) and hence $(\alpha, \beta) \in S^+$ which is a contradiction. Therefore $g' < 0$ on $[0, a)$.

Suppose now $f'(a) = 0$. Since $f''(0) > 0$ there exists $b \in (0, a)$ where f' attains a local maximum. Hence $f''(b) = 0$, $f'''(b) \leq 0$ and (3) imply that $f'(b)^2 \leq g(b)^2 < g(0)^2$ and therefore $f' < g(0) < 1$ on $[0, a]$.

Lemma 5. *If* $\alpha > 0$, $\beta \in (-1/\eta, 0)$ *and* $(\alpha, \beta) \notin S^+ \cup S^-$ *then either*
(Case 1) there exists $x \in (0, \ell)$ *such that* $f'(x) = 0$
or
(Case 2) $\ell = \infty$, $f' > 0$ *and* $g' < 0$ *on* $(0, \infty)$, $g(\infty) = g'(\infty) = 0$, $0 \leq f''(\infty) < \infty$ *and if* $f''(\infty) = 0$ *then for some* $k \in (0, \infty)$ *each of*

$$f'''(x)e^{kx}, \quad f''(x)e^{kx}, \quad f'(x)e^{kx}, \quad (k/2 - f(x))e^{kx}, \quad g'(x)e^{kx}, \quad g(x)e^{kx}$$

has a finite limit as $x \to \infty$.

Proof. Suppose that it is not Case 1.

Hence $f' > 0$ on $(0, \ell)$ and Lemma 4 implies that $g > 0$, $g' < 0$ on $[0, \ell)$.

Lemma 1 implies that $\ell = \infty$.

Since $g > 0, g' < 0$ we have that $g(\infty) \in [0, g(0)]$.

(4) implies that $g'' > 0$ on $(0, \infty)$ hence $g'(\infty) = 0$.

Exponential decay of f''' in (19) implies that

$$f''(x) = c_2 - \int_x^\infty f'''(t)dt$$

$$f'(x) = c_1 + c_2\,x - \int_x^\infty (x - t)f'''(t)dt$$

$$f(x) = \frac{k}{2} + c_1\,x + \frac{1}{2}\,c_2\,x^2 - \frac{1}{2}\int_x^\infty (x - t)^2 f'''(t)dt$$

$$F(x) = c_0 + kx + c_1\,x^2 + \frac{1}{3}\,c_2\,x^3 - \frac{1}{3}\int_x^\infty (x - t)^3 f'''(t)dt \tag{21}$$

for some finite constants c_i and k .

If $f(\infty) < \infty$ then $c_1 = c_2 = 0$ and (3) implies that $g(\infty) = 0$. (17) implies that $g'e^F$ is increasing and since it is always negative it has a limit. This and (21) imply that $g'(x)e^{kx} \to a_0$ and hence $g(x)e^{kx} \to -a_0/k$. Thus the right hand side of (18) is integrable, hence $f'''e^F$ has a finite limit and therefore $f'''(x)e^{kx} \to a_1$ as $x \to \infty$. Which then implies $f''(x)e^{kx} \to -a_1/k$, $f'(x)e^{kx} \to a_1/k^2$ $(k/2 - f(x))e^{kx} \to a_1/k^3$.

If $f(\infty) = \infty$ then either c_1 or c_2 is not zero and hence there exist $\delta > 0$ and $a > 0$ such that $xf'(x) > \delta f(x) > 0$ for $x > a$. If $g(\infty) > 0$ then (17) implies that for $x > a$

$$\begin{aligned} g'(x)e^{F(x)} - g'(a)e^{F(a)} &= 2\int_a^x f'ge^F \\ &> 2g(\infty)\delta \int_a^x \frac{f(u)}{u} e^{F(u)}\, du \\ &> \frac{g(\infty)\delta}{x}(e^{F(x)} - e^{F(a)}) \end{aligned}$$

$$g'(x) > \frac{g(\infty)\delta}{x} + \left(g'(a) - \frac{g(\infty)\delta}{x}\right) e^{F(a)-F(x)}$$

which implies contradiction $g(\infty) = \infty$ and therefore we must have $g(\infty) = 0$. If $f''(\infty) = 0$ then the exponential decay of f''' implies exponential decay of ff'' and hence (3) implies $f'(\infty) = 0$ which is a contradiction.

This completes the proof of the Lemma 5.

Lemma 6. *There exists* $\alpha_1 > 0$ *such that Case 1 in Lemma 5 holds whenever* $\alpha \in (0, \alpha_1)$.

Proof. If $\alpha = 0$, $\beta \in (-1/\eta, 0]$ then $f'(0) = f''(0) = 0$, $f'''(0) = -g(0)^2 < 0$ hence $f'(\delta) < 0$ for some $\delta > 0$ and, by continuity, the same holds in a neighborhood of (α, β). Since $(0, -1/\eta)$ belongs to the open set S^-, compactness implies that there exists $\alpha_1 > 0$ such that if $\alpha \in (0, \alpha_1)$, $\beta \in [-1/\eta, 0]$ then either $(\alpha, \beta) \in S^-$ or $f'(\delta) < 0$ for some $\delta > 0$.

Lemma 7. *There exists* $\alpha_2 < \infty$ *such that Case 2 in Lemma 5 holds whenever* $\alpha > \alpha_2$.

Proof. Suppose $\alpha > 0$, $\beta \in (-1/\eta, 0)$, $(\alpha, \beta) \notin S^+ \cup S^-$ and that for some $a > 0$ we have $f' > 0$ on $(0, a)$, $f'(a) = 0$. Lemma 4 implies $f' < 1$ on $[0, a]$. Thus, $f(x) \le s + x$ and (16) imply

$$|f''(x)e^{F(x)} - \alpha| \le \int_0^x e^{2su+u^2} du \qquad \text{for } x \in [0, a]. \tag{22}$$

If for each integer n we can choose $\alpha > n$ and $a \le 1$ then (22) implies

$$|f''e^F - \alpha| \le K \equiv \int_0^1 e^{2su+u^2} du \qquad \text{on } [0, a]$$

which implies contradiction $f''(a) > 0$ for large enough n.

Thus for some N we cannot choose $\alpha > N$ and $a \le 1$. Let

$$\alpha_2 = \max(N, 2K, 2e^{2s+1}).$$

If $\alpha > \alpha_2$ and one can find such $a > 1$ then (22) implies

$$f''e^F - \alpha \geq -K \quad \text{on } [0,1]$$

and since $\alpha > 2K$ then

$$f'' > e^{-2s-1}\alpha/2 \quad \text{on } [0,1],$$

$$f'(1) > e^{-2s-1}\alpha/2 > 1$$

which contradicts the fact that $f' < 1$ on $[0,a]$ and therefore there is no such a when $\alpha > \alpha_2$. This proves Lemma 7.

We can now apply the topological theorem of [13] to assert that there exists a continuum of values (α,β) connecting α_1 and α_2 such that the points in the continuum belong to neither S^+ nor S^-.

Lemma 8. *The subset of continuum for which Case 1 of Lemma 5 holds is open relative to continuum.*

Proof. Suppose that Case 1 holds at some (α,β). Let $a \in (0,\ell)$ be such that $f' > 0$ on $(0,a)$ and $f'(a) = 0$. If f' becomes negative after a then continuity would imply that Case 1 persists in a neighborhood of (α,β). If on the other hand $f' \geq 0$ then $f''(a) = 0$ and (3) implies $f'''(a) = -g(a)^2 = 0 < 0$ by Lemma 4 - which contradicts $f' \geq 0$.

Lemma 9. *The subset of continuum for which Case 2 of Lemma 5 holds, with $f''(\infty) > 0$, is open relative to continuum.*

Proof. Suppose (α_0,β_0) belongs to the continuum, with the corresponding solution f_0, g_0, and that Case 2 holds with $f_0''(\infty) > 0$. Then $f_0'(a) > 1$ and $f_0''(a) > 0$ for some $a > 0$ hence continuity implies that for (α,β) close enough to (α_0,β_0) we can assume that $f' > 0$ on $(0,a)$, $f'(a) > 1$ and $f''(a) > 0$. Since $f'(a) > 1$ Lemma 4 implies that Case 1 cannot happen. (16) implies that $f'' > 0$ and hence $f' > 1$ on $[a,\infty)$ hence $f'(\infty) \neq 0$ and therefore $f''(\infty) > 0$.

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M. Miklavčič
Department of Mathematics
Michigan State University
E. Lansing
MI 48824
USA
milan@math.msu.edu

C. Y. Wang
Department of Mathematics
Michigan State University
E. Lansing
MI 48824
USA
cywang@math.msu.edu

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