An Ozsváth-Szabó Floer homology invariant of knots in a contact manifold *

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Abstract

Using the knot Floer homology filtration, we define invariants associated to a knot in a three-manifold possessing non-vanishing Floer co(homology) classes. In the case of the Ozsváth-Szabó contact invariant we obtain an invariant of knots in a contact three-manifold. This invariant provides an upper bound for the Thurston-Bennequin plus rotation number of any Legendrian realization of the knot. We use it to demonstrate the first systematic construction of prime knots in contact manifolds other than \( S^3 \) with negative maximal Thurston-Bennequin invariant. Perhaps more interesting, our invariant provides a criterion for an open book to induce a tight contact structure. A corollary is that if a manifold possesses contact structures with distinct non-vanishing Ozsváth-Szabó invariants, then any fibered knot can realize the classical Eliashberg-Bennequin bound in at most one of these contact structures.

Key words: Heegaard Floer homology, contact geometry, Legendrian knot

1 Introduction

A contact structure, \( \xi \), on a closed oriented three-manifold, \( Y \), is an oriented two-dimensional sub-bundle of the tangent bundle, \( TY \), which is completely non-integrable. This means there do not exist surfaces embedded in \( Y \) whose tangent planes lie in \( \xi \) in any open subset of the surface. See [5] for an introduction. It has been known for some time that there is a dichotomy between contact structures on a three-manifold: every contact structure falls into one of two classes, overtwisted or tight. These classes are determined by the existence (in the case of overtwisted) or non-existence (in the case of tight) of an

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embedded disk whose interior is transverse to $\xi$ everywhere except one point, and whose boundary is tangent to $\xi$. A fundamental theorem of Eliashberg states that the overtwisted contact structures are classified by the homotopy type of the contact structure as a two-plane field. Tight contact structures, on the other hand, have proved to be much more difficult to understand and their classification is presently out of reach for a general three-manifold. Since the definition of overtwistedness involves the existence of a particular type of unknotted circle tangent to $\xi$ (unknotted in the sense that it bounds a disk), it may not be surprising to find that one of the ways in which tight contact structures differ from overtwisted involves knot theory. To describe this distinction, we first recall some basic definitions from the theory of Legendrian knots.

A knot which is everywhere tangent to $\xi$ is called Legendrian. Given a Legendrian knot, $K$, we can form a push-off, $K'$, of $K$ using a vector field tangent to the contact planes but orthogonal to the tangent vector field of $K$. If $K$ is null-homologous then the linking number $\text{lk}(K, K')$ is well-defined. This linking number is called the Thurston-Bennequin number of $K$ and is denoted $tb(K)$. It is immediate from the definition that $tb(K)$ is invariant under isotopy of $K$ through Legendrian knots, so-called Legendrian isotopy.

There is another easily defined integer-valued invariant of Legendrian knots. Let $K$ as above be a null-homologous Legendrian knot with Seifert surface $S$. Since the contact structure restricted to $S$, $\xi|_S$, is a real oriented two-dimensional vector bundle on a surface-with-boundary, it is necessarily trivial. Picking a trivialization, 

$$\tau : \xi|_S \xrightarrow{\cong} S \times \mathbb{R}^2,$$

the tangent vector field to $K$ yields a map $u : S^1 \to \mathbb{R}^2 \setminus \{0\}$. We define the rotation number of $K$, $\text{rot}_S(K)$, to be the winding number of this map. Note that the rotation number depends on our choice of Seifert surface, but only through its homology class $[S] \in H_2(Y - K; \mathbb{Z}) \cong H_2(Y; \mathbb{Z})$. It is straightforward to verify that the rotation number, like the Thurston-Bennequin number, is invariant under Legendrian isotopy. We refer to the Thurston-Bennequin and rotation numbers of a Legendrian knot as its “classical” invariants.

A fundamental theorem of Eliashberg [4] (first proved by Bennequin [2] for the unique tight contact structure, $\xi_{\text{std}}$, on the three-sphere) states that for tight contact structures, the classical invariants of Legendrian knots are constrained by the topology of the three-manifold:
Theorem 1 (Eliashberg-Bennequin inequality) Let $\xi$ be a tight contact structure on a three-manifold, $Y$. Then for a null-homologous knot $K \hookrightarrow Y$ and Seifert surface, $S$, we have

$$tb(\tilde{K}) + |\text{rot}_{S}(\tilde{K})| \leq 2g(S) - 1,$$

where $\tilde{K}$ is any Legendrian representative of $K$.

This is in stark contrast with overtwisted contact structures, where a given knot type has Legendrian representatives with arbitrarily large classical invariants. $^1$

Since the Eliashberg-Bennequin inequality, much work has been done to further constrain the classical invariants of Legendrian knots. However, the work has primarily addressed the special case of Legendrian knots in $(S^3, \xi_{\text{std}})$. The primary reason for the focus on knots in $(S, \xi_{\text{std}})$ is due to the fact that, for such knots, the classical invariants have a combinatorial description in terms of a particular type of projection of $K$ to $\mathbb{R}^2$, the front projection. The combinatorics of such diagrams share some properties with various combinatorially defined knot invariants, e.g. the HOMFLY and Kauffman polynomials and Khovanov homology, and the best bounds for the classical invariants of knots in $(S^3, \xi_{\text{std}})$ come from these combinatorial knot invariants.

For contact manifolds other than $(S^3, \xi_{\text{std}})$, much less is known about the classical invariants of Legendrian knots. For Stein fillable contact structures, Eliashberg’s bound was improved by Lisca and Matic [21] (see also Akbulut and Matveyev [1]) and recently Mrowka and Rollin [24] extended this to tight contact structures with non-vanishing Seiberg-Witten contact invariant. An analogous theorem was proved for the Ozsváth-Szabó contact invariant by Wu [46]. In both cases, the theorems replaced the genus of the Seifert surface by the genus of a surface properly embedded in a four-manifold bounded by the three-manifold. It is important to note that aside from $(S^3, \xi_{\text{std}})$, all known bounds for the Thurston-Bennequin and rotation numbers of Legendrian knots involve $2g(S) - 1 = -\chi(S)$, for a surface-with-boundary, $S$, and hence are necessarily greater than or equal to -1.

The primary purpose of this paper is to introduce an integer-valued invariant $\tau_{\xi}(K)$ of a quadruple $(Y, \xi, [S], K)$ which will replace $g(S)$ in the Eliashberg-Bennequin inequality. Here $(Y, K)$ is a null-homologous knot, $[S]$ a homology class of Seifert surface, and $\xi$ a contact structure. The precise definition of $\tau_{\xi}(K)$ will be given in the next section, but roughly speaking it uses the knot Floer homology filtration associated to $(Y, [S], K)$ together with the Ozsváth-

$^1$ This is known to the experts, but appears be a result of lore. A sketch can be found in Lemma 2.4.3 of [3]. For completeness, we provide a proof in Subsection 2.4 (Proposition 22)
Szabó contact invariant, $c(\xi)$. In the case that $c(\xi) \neq 0$ we will prove the following:

**Theorem 2** Let $(Y, \xi)$ be a contact three-manifold with non-trivial Ozsváth-Szabó contact invariant. Then for a null-homologous knot $K \hookrightarrow Y$ and Seifert surface, $S$ we have

$$tb(\tilde{K}) + |\text{rot}_S(\tilde{K})| \leq 2\tau_\xi(K) - 1,$$

where $\tilde{K}$ is any Legendrian representative of $K$.

**Remark 3** Note that, in general, $\text{rot}_S(K)$ and $\tau_\xi(K)$ both depend on $[S] \in H_2(Y; \mathbb{Z})$. However, it will be shown that if $S$ and $S'$ are two Seifert surfaces then

$$2\tau^S_\xi(K) - 2\tau^{S'}_\xi(K) = \langle c_1(\xi), [S - S'] \rangle,$$

where $c_1(\xi)$ is the first Chern class of the contact structure or, equivalently, its Euler class. On the other hand, it is easy to see that the rotation number depends on $[S]$ in the same way.

By a theorem of Ozsváth and Szabó, the non-vanishing of the contact invariant implies tightness of $\xi$ and it will be immediate from the definition and an adjunction inequality that $\tau_\xi(K) \leq g(S)$. Thus the bounds obtained above will be at least as good as the Eliashberg-Bennequin bound. Indeed, in an upcoming paper we will show that the above bound is also as good as that provided by Wu [46] (or Mrowka and Rollin). Unlike $g(S)$, however, $\tau_\xi(K)$ can be negative and hence provides the first general method for determining prime knot types in contact manifolds other than $S^3$ whose classical Legendrian invariants are constrained to be negative. Here, prime means that the only decomposition of $(Y, K)$ as a connected sum $(Y_1, K_1) \# (Y_2, K_2)$ is when one of the summands is $(S^3, \text{unknot})$. The primeness condition here is essential, since the combinatorial techniques described above can be adapted to the situation when we form the connected sum of a knot in $S^3$ and an unknot (i.e. a knot bounding a disk) in an arbitrary tight contact manifold. Indeed, we will show that there exist prime knots in any contact manifold with $c(\xi) \neq 0$ which have classical invariants constrained to be arbitrarily negative. More precisely, we have

**Theorem 4** Let $(Y, \xi)$ be a contact manifold with non-trivial Ozsváth-Szabó contact invariant. Then for any $N > 0$, there exists a prime knot $K \hookrightarrow Y$ such that

$$tb(\tilde{K}) + |\text{rot}_S(\tilde{K})| \leq -N,$$

for any Legendrian representative, $\tilde{K}$, of $K$, and any Seifert surface, $S$.

The proof draws on results of [16] which determine the behavior of the knot Floer homology filtration under a certain satellite operation called cabling. In particular, negative upper bounds on $\tau_\xi(K)$ of sufficiently negative cables
of any knot can easily be achieved. The precise statement of these bounds is described in Section 4.

Another application of $\tau_\xi(K)$ involves contact structures induced by open book decompositions of a given three-manifold. Recall that a fibered knot is a triple of data $(Y, S, K)$ consisting of a knot $(Y, K)$ and a surface $S$ with $\partial S = K$ for which we have the following identification:

$$Y - \nu(K) \cong \frac{S \times [0, 1]}{\{(x, 0) \simeq (\phi(x), 1)\}},$$

where $\phi$ is a diffeomorphism of $S$ fixing $\partial S$ and $\nu(K)$ is a neighborhood of $K$. The decomposition of $Y - \nu(K)$ given above produces a decomposition of $Y$:

$$Y \cong \frac{S \times [0, 1]}{\{(x, 0) \simeq (\phi(x), 1)\}} \cup D^2 \times S^1,$$

where we identify $\partial S \times \{p\}$ with $\{q \in \partial D^2 \times S^1\}$ and $\{p' \in \partial S\} \times S^1$ with $\partial D^2 \times \{q' \in S^1\}$. Such a decomposition is called an open book decomposition of $Y$. There is a well-known construction due to Thurston and Winkelnkemper [43] which associates a canonical contact structure on $Y$ to an open book decomposition. In this way, we can associate a contact structure to a fibered knot $(S, K)$. Let

$$\xi_{(S,K)} := \text{contact structure associated to the fibered knot } (S, K).$$

Given a contact manifold $(Y, \xi)$ and fibered knot $(S, K)$ one can ask whether there is a relationship between the classical invariants of $K$ in $\xi$ and the contact structure $\xi_{(S,K)}$. The following theorem indicates that such a relationship exists, and provides a sufficient condition for $\xi_{(S,K)}$ to be tight in terms of the classical invariants of Legendrian representatives of $K$ in $\xi$.

**Theorem 5** Let $(Y, \xi)$ be a contact structure with non-trivial Ozsváth-Szabó contact invariant. Let $(S, K)$ be a fibered knot which realizes the Eliashberg-Bennequin bound in $\xi$. That is, there exists a Legendrian representative, $\tilde{K}$, of $K$ such that:

$$tb(\tilde{K}) + |rots(\tilde{K})| = 2g(S) - 1.$$  \hspace{1cm} (1)

Then the contact structure associated to $(S, K)$ by the Thurston-Winkelnkemper construction, $\xi_{(S,K)}$, is tight. Furthermore, the Ozsváth-Szabó contact invariants of $\xi_{(S,K)}$ and $\xi$ are identical. That is, $c(\xi_{(S,K)}) = c(\xi)$.

Note that as a special case of the above theorem we have that a fibered knot in $S^3$ with $TB(K) = 2g(K) - 1$ induces the standard tight contact structure (here $TB(K)$ is the maximal Thurston-Bennequin number over all Legendrian representatives of $K$.) We also have the immediate corollary
Corollary 6 Let $Y$ be a three manifold and $\xi_1, \ldots, \xi_i$ be contact structures with distinct non-trivial Ozsváth-Szabó invariants. That is, $c(\xi_i) \neq c(\xi_j)$ unless $i = j$. Then, given a fibered knot $(S, K)$, the equality
\[ tb(\tilde{K}) + |\text{rot}_S(\tilde{K})| = 2g(S) - 1 \]
can hold in at most one of $\xi_j$.

It is interesting to note that a fibered knot $(S, K)$ always has a Legendrian representative in $\xi_{(S,K)}$ realizing the Eliashberg-Bennequin bound (1). Indeed, the construction of $\xi_{(S,K)}$ presents $K$ as a transverse knot satisfying $sl_S(K) = 2g(S) - 1$, and any Legendrian push-off of $K$ will satisfy (1). (Here, $sl_S(K)$ is the self-linking number of the transverse knot, taken with respect to the fiber surface, $S$. The stated equality follows from the observation that the characteristic foliation of $S$ in $\xi_{(S,K)}$ has no negative singularities.) Thus Theorem 5 and its corollary seem to indicate a surprising “preference” of a fibered knot for its own contact structure, $\xi_{(S,K)}$, at least when we restrict attention to the subset of tight contact structures distinguished by their Ozsváth-Szabó invariants. It also leads to the following

**Question:** It is known that there are tight contact structures with trivial Ozsváth-Szabó invariant [10]. However, one can ask if the conclusion of Theorem 1.4 holds if $(Y, \xi)$ is only assumed to be tight. That is, does the existence of a Legendrian representative of a fibered knot $(S, K)$ in a tight contact structure $(Y, \xi)$ satisfying Equation (1) imply that $\xi_{(S,K)}$ is tight? If so, what is the relationship between $\xi_{(S,K)}$ and $\xi$?

**Remark 7** In another direction, we expect $\tau_\xi(K)$ to provide an obstruction to a knot $(Y, K)$ arising as the boundary of a properly embedded $J$-holomorphic curve in a symplectic filling of $(Y, K)$. We will return to this point in an upcoming paper.

**Outline:** The organization of the paper is as follows. In the next section we spend a considerable amount of time setting up notation, reviewing basic properties of Ozsváth-Szabó Floer homology for three-manifolds, its refinement for null-homologous knots, and the construction and properties of the contact invariant, $c(\xi)$. The main purpose of this section is to define several invariants associated to a knot in a three-manifold possessing non-vanishing Floer (co)homology classes. The invariant $\tau_\xi(K)$ will be the special case of one of these invariants, when the Floer cohomology class is the contact invariant. Section 3 establishes key properties of the invariants which generalize analogous properties of the Ozsváth-Szabó concordance invariant. Together with the results of [16], these properties will be used in Section 4 to prove the theorems.
Acknowledgments: The original motivation for this work came from Olga Plamenevskaya’s paper [38], which established Theorem 2 for the Ozsváth-Szabó concordance invariant. I have enjoyed and benefited from conversations with many people regarding the ideas presented here, among them John Etnyre, Paolo Ghiggini, Tom Mrowka, Peter Ozsváth, András Stipsicz, and Hao Wu. Special thanks go to Tim Perutz for pointing out an algebraic oversight in an earlier version of this work, and to Tom Mark for many useful comments and suggestions, and especially for his interest and help in dealing with the aforementioned oversight.

2 Background on Ozsváth-Szabó theory

In this section we introduce and recall background on various aspects of the Floer homology package developed by Ozsváth and Szabó over the past several years. All chain complexes will be over the field $\mathbb{Z}/2\mathbb{Z}$. Due to the breadth of the theory, this section may not be sufficient for a complete understanding of the Ozsváth-Szabó machinery, but we include it here to establish notation and recall the main results and structures of the theory which will be used. Much of the section can be skipped by the reader familiar with Ozsváth-Szabó theory. However, for such a reader, we call attention to Definitions 11, 17, and 23. These are the definitions of the invariants $\tau_{[x]}(Y, K)$, $\tau^*_{[y]}(Y, K)$, and $\tau_{\xi}(K)$, respectively. The idea behind each invariant is same as that of the Ozsváth-Szabó concordance invariant or the Rasmussen invariant - a knot induces a filtration of a certain (co)chain complex and each invariant measures when the (co)homology of the subcomplexes in the filtration start to hit specific (co)homology classes. The reason for multiple invariants is that in Ozsváth-Szabó theory a knot induces a filtration on both the chain and cochain complexes associated to $Y$. Moreover, the contact invariant $c(Y, \xi)$ is really an element of the Floer cohomology of $Y$, and hence we need an invariant, $\tau^*_{[y]}(Y, K)$, associated to a knot $K$ and a Floer cohomology class, $[y]$.

Aside from these definitions, the only original material presented here is contained in Subsection 2.4. One thing we do there is prove Property 4 of the contact invariant, which is the behavior of the contact invariant under connected sums. Though this property is expected and its proof straightforward, its appearance here is the first that we know of and may be of independent interest. We also highlight the connection between the knot Floer homology invariants of a fibered knot and the Hopf invariant of the associated contact structure (Proposition 19). This is essentially contained in work of Ozsváth and Szabó, but seems not to be widely known. Towards the end of Subsection 2.4 we use this connection to give new proofs of several well-known properties of the Hopf invariant (Propositions 20 and 21). Finally, we pay a debt to the introduction and provide a proof that knot types in overtwisted contact
manifolds can have arbitrary classical invariants (in Proposition 22).

The rest of this section draws heavily on several articles of Ozsváth and Szabó [31,30,33], and in some places we have simply adapted their work with notational changes - we stress that our purpose is to collect relevant results and establish notation.

2.1 The Knot Floer homology filtration

To a closed oriented three-manifold $Y$, equipped with a Spin$^c$ structure, $s$, Ozsváth and Szabó defined several chain complexes, $CF^\infty(Y, s), CF^+(Y, s), CF^-(Y, s), \hat{CF}(Y, s)$ [27]. The homologies of these chain complexes, denoted $HF^\infty(Y, s), HF^+(Y, s), HF^-(Y, s), \hat{HF}(Y, s)$ were proved to be invariants of the pair $(Y, s)$. Associated to a null-homologous knot $K \hookrightarrow Y$, a choice of Seifert surface, $S$, and a Spin$^c$ structure, $s$, they subsequently defined filtered versions of the above chain complexes, and proved that the filtered chain homotopy types of these chain complexes are invariants of the quadruple $(Y, [S], K, s)$ (here $[S] \in H_2(Y - K; \mathbb{Z}) \cong H_2(Y; \mathbb{Z})$ is the homology class of the Seifert surface) [33,41]. We discuss the most general of these complexes, denoted $CFK^\infty(Y, [S], K, s)$. Each of the other Ozsváth-Szabó Floer chain complexes for knots and three-manifolds can be derived from this chain complex, and so we describe it first. We then discuss how to obtain some of the other invariants from it. This approach is historically backwards, but our main purpose here it set up notation and collect properties of the chain complexes we use throughout the text. For a complete discussion we refer the interested reader to [27,28,33,41] and to [34] for a survey.

Fix a doubly-pointed (admissible) Heegaard diagram $(\Sigma_g, \alpha, \beta, w, z)$ for the knot $(Y, K)$ (see Definition 2.4 of [33] and Definition 4.10 of [27]) and consider the $g$-fold symmetric product $\text{Sym}^g(\Sigma_g)$, with two Lagrangian tori $T_\alpha = \alpha_1 \times \cdots \times \alpha_g$ and $T_\beta = \beta_1 \times \cdots \times \beta_g$.

By an isotopy of the attaching curves, these tori intersect transversely in a finite number of points. In Section 2.3 of [33] Ozsváth and Szabó define a map

$$\mathcal{S} : \{T_\alpha \cap T_\beta\} \to \text{Spin}^c(Y_0(K)) \simeq \text{Spin}^c(Y) \times \mathbb{Z},$$

which assigns to each intersection point $x \in \{T_\alpha \cap T_\beta\} \subset \text{Sym}^g(\Sigma)$ a Spin$^c$ structure on the zero-surgery of $Y$ along $K$, $Y_0(K)$. The projection from $\text{Spin}^c(Y_0(K))$ to $\text{Spin}^c(Y)$ is obtained by first restricting $s$ to $Y - K$, and then

\[2\] In the original treatment [27], these tori were only known to be totally real. However, work of Perutz [40] shows that they can be taken to be Lagrangian.
uniquely extending it to \( Y \). Projection to the second factor comes from evaluation \( \frac{1}{2}\langle c_1(s), [\tilde{S}] \rangle \), where \( \tilde{S} \) denotes a surface in \( Y_0(K) \) obtained by capping off a fixed Seifert surface, \( S \), for \( K \) with the meridian disk of the solid torus glued to \( Y - K \) in the surgery. We say that a \( \text{Spin}^c \) structure on \( Y_0(K) \) extends \( s \in \text{Spin}^c(Y) \) if projection onto the factor of \( \text{Spin}^c(Y_0(K)) \) corresponding to \( \text{Spin}^c(Y) \) is equal to \( s \).

**Remark 8** More generally, to an intersection point \( x \), the map \( \tilde{s} \) assigns a relative \( \text{Spin}^c \) structure \( \tilde{g}(x) \) on the knot complement. However, for null-homologous knots relative \( \text{Spin}^c \) structures can be identified with \( \text{Spin}^c \) structures on \( Y_0(K) \).

Fix \( s \in \text{Spin}^c(Y) \) and a homology class of Seifert surface \( [S] \in H_2(Y; \mathbb{Z}) \). Now let \( s_0 \) denote the unique \( \text{Spin}^c \) structure in \( \text{Spin}^c(Y_0(K)) \) such that \( s_0 \) extends \( s \) and satisfies \( \frac{1}{2}\langle c_1(s_0), [\tilde{S}] \rangle = 0 \). The chain complex \( CFK^\infty(Y, [S], K, s) \) is then generated (as a \( \mathbb{Z}/2\mathbb{Z} \) vector space) by triples \( [x, i, j] \) satisfying the constraint

\[
\tilde{g}(x) + (i - j)\text{PD}(\mu) = s_0. \tag{2}
\]

Here \( \text{PD}(\mu) \in H^2(Y_0(K); \mathbb{Z}) \) is the Poincaré dual to the meridian of \( K \) and addition is meant to signify the action of \( H^2(Y_0(K); \mathbb{Z}) \) on \( \text{Spin}^c(Y_0(K)) \). The constraint depends on the choice of Seifert surface but only through its homology class \( [S] \in H_2(Y; \mathbb{Z}) \). Indeed, this is the only place where the Seifert surface appears in the knot Floer homology construction. Furthermore, Propositions 9 and 10 below show that the effect of varying \([S]\) can be easily understood in terms of the algebraic topology of \( Y \). Thus, when \([S]\) is clear from the context e.g. \( Y \) is a rational homology sphere and \([S] = 0\), or when it becomes notationally cumbersome, we will omit it from the discussion.

The boundary operator on \( CFK^\infty(Y, [S], K, s) \) is defined by

\[
\partial[x, i, j] = \sum_{y \in \mathcal{T}_a \cap \mathcal{T}_b} \sum_{\phi \in \pi_2(x, y)} \# \left( \frac{\mathcal{M}(\phi)}{\mathbb{R}} \right) \left[ y, i - n_w(\phi), j - n_z(\phi) \right],
\]

where \( \# \left( \frac{\mathcal{M}(\phi)}{\mathbb{R}} \right) \) denotes a count, modulo 2, of points in the moduli space of unparameterized pseudo-holomorphic Whitney disks, \( \phi \), with boundary conditions specified by \( x, y \) and \( \mathcal{T}_a, \mathcal{T}_b \). The integers \( n_w(\phi), n_z(\phi) \) are intersection numbers between the image of \( \phi \) in \( \text{Sym}^g(\Sigma) \) with the codimension one subvarieties \( \{ w \} \times \text{Sym}^{g-1}(\Sigma), \{ z \} \times \text{Sym}^{g-1}(\Sigma) \). See Sections 2 and 4 of [27] for relevant details and definitions regarding the boundary operator, and Section 3 of [27] for its analytical underpinnings.

If we define a partial ordering on \( \mathbb{Z} \oplus \mathbb{Z} \) by the rule that \((i, j) \leq (i', j')\) if \( i \leq i' \) and \( j \leq j' \), then a \( \mathbb{Z} \oplus \mathbb{Z} \)-filtered chain complex is by definition a chain complex \( C_* \) equipped with a map:

\[
\mathcal{F} : C_* \to \mathbb{Z} \oplus \mathbb{Z},
\]
such that the differential $\partial$ respects $F$ in the sense that

$$F(\partial(x)) \leq F(x) \text{ for every } x \in C_*.$$ 

From its construction, it is immediate that $\text{CFK}^\infty(Y, [S], K, s)$ is a $\mathbb{Z} \oplus \mathbb{Z}$-filtered chain complex - for a generator we simply define $F_s([x, i, j]) = (i, j)$.

More generally, for a chain $c = \sum_k [x_k, i_k, j_k]$, the filtration is given by $F_s(c) = (\max_k i_k, \max_k j_k)$.

Now the Whitney disks counted in $\# \left( \frac{M(\phi)}{\mathbb{R}} \right)$ have pseudo-holomorphic representatives, and hence the quantities $n_w(\phi)$ and $n_z(\phi)$ are necessarily positive - indeed the submanifolds $\{w\} \times \text{Sym}^{g-1}(\Sigma), \{z\} \times \text{Sym}^{g-1}(\Sigma)$ are pseudo-holomorphic and thus intersect the image of pseudo-holomorphic Whitney disks positively (see Lemma 3.2 of [27]). Hence $F_s$ equips $\text{CFK}^\infty(Y, [S], K, s)$ with a $\mathbb{Z} \oplus \mathbb{Z}$-filtration. Theorem 3.1 of [33] proved that the $\mathbb{Z} \oplus \mathbb{Z}$-filtered chain homotopy type of $\text{CFK}^\infty(Y, [S], K, s)$ is an invariant of the quadruple $(Y, [S], K, s)$. Indeed this is the primary knot invariant in Ozsváth-Szabó theory and is quite powerful - it has been shown that the filtered chain homotopy type of $\text{CFK}^\infty(Y, [S], K, s)$ determines the genus of $K$ [35], whether $K$ is fibered [9,26,14], can be used to determine the Floer homology of three-manifolds obtained by surgery along $(Y, K)$ [36,37], and has applications to determining the smooth four-genus of knots in $S^3$ [31,41]. Note that $F_s$ depends on $[S]$ through Equation (2), but only up to an overall shift which we now make precise.

**Proposition 9** Let $S, S'$ be two Seifert surfaces for a knot $K \hookrightarrow Y$. Fix $x \in T_\alpha \cap T_\beta$ and let $[x, i, j] \subset \text{CFK}^\infty(Y, [S], K, s)$ and $[x, i', j'] \subset \text{CFK}^\infty(Y, [S'], K, s)$ be generators. Then we have the relation:

$$(i - i') + (j' - j) = -\frac{1}{2} \langle c_1(s), [S - S'] \rangle,$$ 

where $[S - S'] \in H_2(Y; \mathbb{Z})$ is the difference of the homology classes of $S$ and $S'$, and $c_1(s)$ is the first Chern class of the Spin$^c$ structure, $s$.

**Proof.** Equation (2) determines the triples which generate $\text{CFK}^\infty(Y, [S], K, s)$ and $\text{CFK}^\infty(Y, [S'], K, s)$. The place in this equation where $[S]$ places a role is in the choice of $s_0 \in \text{Spin}^c(Y_0(K))$ extending $s \in \text{Spin}^c(Y)$. Now $S$ and $S'$ yield Spin$^c$ structures $s_0$ and $s'_0$, respectively, which extend $s$ and satisfy

$$\langle c_1(s_0), [\hat{S}] \rangle = 0 \text{ resp. } \langle c_1(s'_0), [\hat{S}'] \rangle = 0.$$ 

In order for both these equalities to hold, $s_0$ and $s'_0$ are forced to be related
by
\[ s'_0 - s_0 = \frac{1}{2} \langle c_1(s), [S-S'] \rangle \cdot \text{PD}(\mu) \in H^2(Y_0(K); \mathbb{Z}). \]

Now (2) requires that
\[ s(x) + (i-j) \text{PD}(\mu) = s_0 \]
\[ s(x) + (i'-j') \text{PD}(\mu) = s'_0, \]
for the respective choice of Seifert surfaces. Subtracting the second equation from the first yields the desired relation. \[ \square \]

Much of the power of the filtered chain homotopy type of \( \text{CF}_K^\infty(Y, [S], K, s) \) lies in our ability to construct new topological invariants by restricting attention to subsets \( C_s \subset \text{CF}_K^\infty(Y, [S], K, s) \) whose \( \mathcal{F}_s \)-values satisfy various numerical constraints. If the differential on \( \text{CF}_K^\infty \) restricts to a differential on the chosen subset (i.e. \( \partial|_{C_s} \) = 0) then the homology of \( C_s \) with respect to the restricted differential will be an invariant of \( (Y, [S], K, s) \). For instance, we can examine the set
\[ C_s\{i=0\} \subset \text{CF}_K^\infty(Y, [S], K, s), \]
consisting of generators of the form \([x, 0, j]\) for some \( j \in \mathbb{Z} \). This set naturally inherits a differential from \( \text{CF}_K^\infty \), since it is a subcomplex of the quotient complex \( \text{CF}_K^\infty_{C_s}\{t<0\} \). We have the isomorphism of chain complexes
\[ C_s\{i=0\} \cong \hat{CF}(Y, s), \]
(which the uninitiated reader can take as the definition of \( \hat{CF}(Y, s) \)). Thus we recover the “hat” Floer homology of \( (Y, s) \) from \( \text{CF}_K^\infty \). Furthermore, by restricting \( \mathcal{F}_s \) to \( C_s\{i=0\} \) we equip \( \hat{CF}(Y, s) \) with a \( \mathbb{Z} \)-filtration. In particular, if we denote by \( \mathcal{F}_s(Y, [S], K, m) \) the subcomplex of \( \hat{CF}(Y, s) \):
\[ \mathcal{F}_s(Y, [S], K, m) := C_s\{i=0, j \leq m\}, \]
then we have the finite sequence of inclusions:
\[ 0 = \mathcal{F}_s(Y, [S], K, -j) \hookrightarrow \mathcal{F}_s(Y, [S], K, -j+1) \hookrightarrow \ldots \hookrightarrow \mathcal{F}_s(Y, [S], K, n) = C_s\{i=0\}. \]
(Finiteness of the above sequence follows from the fact the number of intersection points \( x \in \{\mathbb{T}_\alpha \cap \mathbb{T}_\beta\} \) is finite.) Proposition 9 indicates that the dependence of this filtration on \([S]\) is given by:

**Proposition 10** Let \( S, S' \) be two Seifert surfaces for a knot \( K \hookrightarrow Y \). Let \( \mathcal{F}_s \) and \( \mathcal{F}'_s \) denote the resulting filtrations of \( \hat{CF}(Y, s) \) induced by \([S], K\) and \([S'], K\), respectively. Then, for fixed \( x \in \hat{CF}(Y, s) \), we have:
\[ \mathcal{F}'_s(x) = \mathcal{F}_s(x) - \frac{1}{2} \langle c_1(s), [S-S'] \rangle. \]
Proof. The relation follows immediately from Equation (3). Specifically, a
generator \( x \in \hat{CF}(Y, s) \) corresponds either to a triple \([x, 0, j] \in CFK^\infty(Y, [S], K, s)\) or to a triple \([x, 0, j'] \in CFK^\infty(Y, [S'], K, s)\). In terms of these triples, \( F_s(x) = j \) and \( F'_s(x) = j' \). Equation (3) now shows that \( j' - j = -\frac{1}{2} \langle c_1(s), [S - S'] \rangle \), proving the proposition.

Some particularly interesting invariants derived from the filtration of \( \hat{CF}(Y, s) \) are the homology groups of the successive quotients (the associated graded homology groups), \( H_* \left( \frac{F_s(Y, [S], K, m)}{F_s(Y, [S], K, m-1)} \right) \) which we denote by \( \hat{HF}_s(Y, [S], K, m) \). These are the so-called “knot Floer homology groups” of \((Y, [S], K, s)\). For the case of knots in the three-sphere, the weighted Euler characteristic of these groups is the classical Alexander-Conway polynomial, \( \Delta_K(T) \), of the knot (see [33, 41]):

\[
\sum_m \chi \left( \hat{HF}(S^3, K, m) \right) \cdot T^m = \Delta_K(T).
\]

In terms of the above subcomplexes, we can define a numerical invariant of a knot in a three-manifold with non-zero Floer homology class \([x] \neq 0 \in \hat{HF}(Y, s)\) as follows. Let \( I_m \) denote the map on homology induced by the inclusion:

\[
\iota_m : F_s(Y, [S], K, m) \hookrightarrow \hat{CF}(Y, s).
\]

Then, given \([x] \neq 0 \in \hat{HF}(Y, s)\), we have the following integer associated to \((Y, [S], K, s)\):

Definition 11

\[
\tau_{[x]}(Y, [S], K) = \min \{ m \in \mathbb{Z} | \ [x] \in \text{Im } \iota_m \}.
\]

Remark 12 Note that our notation suppresses \( s \in \text{Spin}^c(Y) \). In light of Proposition 10, the dependence on \([S]\) is given by:

\[
\tau_{[x]}(Y, [S], K) - \tau_{[x]}(Y, [S'], K) = \frac{1}{2} \langle c_1(s), [S - S'] \rangle.
\]

It follows immediately from the fact that the \( \mathbb{Z} \oplus \mathbb{Z} \) filtered chain homology type of \( CFK^\infty \) is an invariant of \((Y, [S], K, s)\), that \( \tau_{[x]}(Y, [S], K) \) is also an invariant of \((Y, [S], K, s)\). This paper will focus on the case when \([x] \) is the Ozsváth-Szabó contact invariant \( c(\xi) \in \hat{HF}(-Y) \), described in Subsection 2.4 below. Since \( c(\xi) \) is an element of the Floer homology of the three-manifold with reversed orientation, \( \hat{HF}_s(-Y) \), and this group can be identified with the Floer cohomology, \( \hat{HF}^s(Y) \), it will be useful to be able to “dualize” \( \tau_{[x]}(Y, [S], K) \) in an appropriate sense. For these purposes, we digress to discuss the precise behavior of (knot) Floer homology under orientation reversal of the underlying three-manifold.
2.2 Orientation Reversal of $Y$

We begin by recalling the following proposition:

**Proposition 13** (Proposition 2.5 of [28]) Let $Y$ be an oriented three-manifold equipped with a Spin$^c$ structure, $s$, and let $-Y$ denote the manifold with reversed orientation, then we have a natural chain homotopy equivalence:

$$\hat{CF}^+(Y, s) := \text{Hom}(\hat{CF}(Y, s), \mathbb{Z}/2\mathbb{Z}), \delta) \cong \hat{CF}_*(-Y, s)$$

**Remark 14** The term on the left is the dual complex associated to the chain complex $\hat{CF}(Y, s)$, hence the Floer homology of $-Y$ is isomorphic to the Floer cohomology of $Y$. Throughout, we will denote dual complexes with an upper star and, like our chain complexes, these will always be with $\mathbb{Z}/2\mathbb{Z}$ coefficients so to avoid dealing with Ext terms (note that at the time of writing, the author knows of no examples of knots or three-manifolds with torsion in the “hat” versions of Ozsváth-Szabó Floer homology, though torsion has been found in the $\pm$, $\infty$ varieties [23]).

**Proof.** If we fix a Heegaard diagram $(\Sigma, \alpha, \beta, w)$ for $Y$, a Heegaard diagram for $-Y$ is obtained by either reversing the orientation of the Heegaard surface, $(-\Sigma, \alpha, \beta, w)$, or switching the roles of the $\alpha$ and $\beta$ curves, $(\Sigma, \beta, \alpha, w)$. In either case there is an identification of intersection points $x \in T_\alpha \cap T_\beta \subset \text{Sym}^g(-\Sigma)$ (respectively $x \in T_\beta \cap T_\alpha \subset \text{Sym}^g(\Sigma)$) with those in $T_\alpha \cap T_\beta \subset \text{Sym}^g(\Sigma)$. Moreover, upon switching $\alpha$ and $\beta$, $J_s$-holomorphic Whitney disks in $\text{Sym}^g(\Sigma)$ connecting $x$ to $y$ for the chain complex coming from $(\Sigma, \alpha, \beta, w)$ are identified with $J_s$-holomorphic Whitney disks in $\text{Sym}^g(\Sigma)$ connecting $y$ to $x$ in the chain complex for $(\Sigma, \beta, \alpha, w)$. This yields the identification of the proposition. For the case where the orientation of $\Sigma$ is reversed, we can alternatively prove the proposition as follows: Fix $\phi \in \pi_2(x, y)$, and let $\overline{\phi} \in \pi_2(y, x)$ denote the homotopy class of the disk in $\text{Sym}^g(-\Sigma)$ obtained from $\phi$ by pre-composing with complex conjugation in $\mathbb{C}$. Then there is an identification of moduli spaces

$$\mathcal{M}_{J_s}(\overline{\phi}) \cong \mathcal{M}_{J_s}(\phi),$$

where $\overline{J}_s$ denotes the almost complex structure on $\text{Sym}^g(-\Sigma)$ obtained from $J_s$ by conjugation. This identification of moduli spaces provides an alternative proof of the proposition. Note that since conjugation takes place in both $\mathbb{C}$ and $\text{Sym}^g(-\Sigma)$, intersection numbers are unaffected, i.e. $n_z(\phi) = n_z(\overline{\phi})$. 

We will also have need for the behavior of the knot filtration $F_s(Y, [S], K)$ under orientation reversal of $Y$. For the present paper, the following proposition will be sufficient:
Proposition 15 (compare Proposition 3.7 of [33]) Consider the short exact sequence of chain complexes for $\mathcal{F}_s(Y, K, m)$:

$$0 \longrightarrow \mathcal{F}_s(Y, K, m) \xrightarrow{\iota_m} \widehat{CF}(Y, s) \xrightarrow{p_m} Q_s(Y, K, m) \longrightarrow 0.$$ 

There is a natural identification:

$$0 \leftarrow \mathcal{F}_s^*(Y, K, m) \leftarrow\iota^* \mathcal{F}^*(Y, s) \leftarrow p^* Q_s^*(Y, K, m) \leftarrow 0 \tag{4}$$

where the top row is the dual of the first short exact sequence and the bottom is the short exact sequence corresponding to $\mathcal{F}_s(-Y, K, -m-1)$

Remark 16 Here, and throughout, we denote by $\iota_m$ and $p_m$ the inclusion and projection maps for the short exact sequence corresponding to $\mathcal{F}_s(Y, K, m)$, and $I_m$ and $P_m$ for the corresponding maps on homology. Note that we have suppressed $[S]$ to simplify notation.

Proof. Upon dualizing, it is immediate that subcomplexes become quotient complexes, and conversely. Thus it remains to see that we can identify filtrations as stated. As in Proposition 13, we can obtain a doubly-pointed Heegaard diagram for $(-Y, K)$ from one representing $(Y, K)$ by either reversing the orientation of $\Sigma$ or switching the roles of the $\alpha$ and $\beta$ curves. In either event, the net result was that Whitney disks reversed direction (i.e. $\phi \in \pi_2(x, y)$ became a disk $\phi' \in \pi_2(y, x)$), but intersection numbers $n_z(\phi), n_w(\phi)$ were unchanged.

Now the relative filtration difference between two intersection points $x, y$ can be computed by the equation:

$$\mathcal{F}_s(x) - \mathcal{F}_s(y) = n_z(\phi) - n_w(\phi),$$

with $\phi \in \pi_2(x, y)$ any Whitney disk connecting $x$ to $y$. It follows that

$$\mathcal{F}_s(x) - \mathcal{F}_s(y) = \overline{\mathcal{F}_s(y)} - \overline{\mathcal{F}_s(x)},$$

where we temporarily use the notation $\overline{\mathcal{F}_s}$ to indicate the filtration of $\widehat{CF}(-Y)$ induced by $K$. Thus the relative $\mathbb{Z}$-filtration is reversed (changes sign) upon changing the orientation of $Y$. It follows that $Q_s^*(Y, K, m)$ is isomorphic to $\mathcal{F}_s(-Y, K, m')$ for some $m'$. It remains to see that $m' = -m - 1$. This would follow if we could show that reversing the orientation of $Y$ reverses the absolute $\mathbb{Z}$-filtration of a generator $x \in T_{\alpha} \cap T_{\beta}$, i.e. $\overline{\mathcal{F}_s(x)} = -\mathcal{F}_s(x)$. To this end, recall that the $\mathbb{Z} \oplus \mathbb{Z}$ filtration of a generator $[x, i, j] \in CFK^\infty(Y, K)$ is given by $(i, j)$ and that

$$s(x) + (i - j)PD(\mu) = s_0.$$
Evaluating the Chern class of the Spin<sup>c</sup> structure on both sides against \([\hat{S}]\) yields:
\[
\langle c_1(\mathfrak{g}(x)), [\hat{S}] \rangle + 2(i - j) = 0.
\]
Letting \(i = 0\), it follows that the absolute filtration grading of a generator \(x \in \widehat{CF}(Y)\) is given by \(j = \frac{1}{2}(c_1(\mathfrak{g}(x)), [\hat{S}])\). This number, in turn, is given by
\[
\frac{1}{2}\langle c_1(\mathfrak{g}(x)), [\hat{S}] \rangle = \frac{1}{2} \hat{\chi}(P) + n_x(P),
\]
where \(\hat{\chi}(P)\) is the Euler measure of a periodic domain, \(P\), whose homology class corresponds to \([\hat{S}] \in H_2(Y_0(K))\), and \(n_x(P)\) is the average of the local multiplicities of \(P\) near the individual intersection points on \(\Sigma\) (see Section 7 of [28], specifically Proposition 7.5, and also Section 2.3 of [33] for further explanation of these terms and the above formula). Fix a periodic domain \(P\) for \(\Sigma\) whose homology class corresponds to \([\hat{S}]\) and represent it by a map
\[
\Phi : (R, \partial R) \to (\Sigma, \alpha \cup \beta),
\]
where \((R, \partial R)\) is a surface-with-boundary. If we now realize the orientation reversal of \(Y\) by reversing the orientation of \(\Sigma\), then the map \(\Phi\) still gives rise to a periodic domain whose homology class represents \([\hat{S}]\). However, the orientation reversal of \(\Sigma\) changes the sign of the multiplicities of \(\text{Im}(\Phi)\). It follows that \(\frac{1}{2} \hat{\chi}(P)\) and \(n_x(P)\) both change sign, and hence \(\mathcal{F}_s(x) = -\mathcal{F}_s(x)\), as claimed. \(\square\)

The above propositions show that the pairings:
\[
\langle -, - \rangle : \widehat{CF}(-Y, s) \otimes \widehat{CF}(Y, s) \to \mathbb{Z}/2\mathbb{Z},
\]
\[
\langle -, - \rangle_m : Q_s(-Y, K, -m-1) \otimes \mathcal{F}_s(Y, K, m) \to \mathbb{Z}/2\mathbb{Z},
\]
defined by
\[
\langle x, y \rangle = \begin{cases} 
1 & \text{if } x = y \\
0 & \text{otherwise}
\end{cases}
\]
descend to yield pairings
\[
\langle -, - \rangle : \widehat{HF}(-Y, s) \otimes \widehat{HF}(Y, s) \to \mathbb{Z}/2\mathbb{Z},
\]
\[
\langle -, - \rangle_m : H_*(Q_s(-Y, K, -m-1)) \otimes H_*(\mathcal{F}_s(Y, K, m)) \to \mathbb{Z}/2\mathbb{Z}.
\]
(again, we momentarily suppress \([S]\).)

Thus, given a Floer class \([y] \neq 0 \in \widehat{HF}(-Y, s)\), a Seifert surface, \(S\), and a knot, \(K\), there are two natural numerical invariants associated to the triple \(([y], [S], K)\). The first invariant is simply \(\tau_{[y]}(-Y, [S], K)\) of Definition 11. The
next uses the filtration $\mathcal{F}_s(Y, [S], K)$ which $K$ induces on $\widehat{C}\mathcal{F}(Y, s)$. It measures when the filtration first starts hitting homology classes in $\widehat{H}\mathcal{F}(Y, s)$ which pair non-trivially with $[y]$.

**Definition 17**

$$\tau^*_y(Y, [S], K) = \min\{m \in \mathbb{Z} | \exists \alpha \in \text{Im } I_m \text{ such that } \langle [y], \alpha \rangle \neq 0 \}.$$  

**Remark 18**  
Like $\tau_x(Y, [S], K)$, the dependence of $\tau^*_y(Y, [S], K)$ on $[S]$ is given by:

$$\tau^*_y(Y, [S], K) - \tau^*_y(Y, [S'], K) = \frac{1}{2} \langle c_1(s), [S - S'] \rangle.$$  

**Example: The three-sphere**  
We conclude this subsection by briefly discussing the case of knots in $S^3$. In this case, $\widehat{H}\mathcal{F}(S^3) \cong \mathbb{Z}/2\mathbb{Z}$, supported in grading zero, and we have a canonical Floer homology class given by the generator $\Theta$. Further, since $-S^3 \cong S^3$, we also have $\widehat{H}\mathcal{F}(-S^3) \cong \mathbb{Z}/2\mathbb{Z}$ (in grading zero) and a canonical generator $\Omega$. Here we have equality $\tau_\Theta(S^3, K) = \tau^*_\Omega(S^3, K)$. Following Ozsváth and Szabó [31], we denote this invariant by $\tau(K)$ (this invariant was also defined and studied by Rasmussen [41]). Since its discovery, $\tau(K)$ has proved to be rich with geometric content. Indeed, the original motivation for its definition is that $\tau(K)$ is an invariant of the smooth concordance class of $K$ and furthermore provides bounds for the smooth four-ball genus:

$$|\tau(K)| \leq g_4(K).$$

Plamenevskaya [38] showed that $\tau(K)$ provides bounds on the classical invariants of Legendrian knots in $(S^3, \xi_{\text{std}})$ and work of the author [15] has shown that $\tau(K)$ detects when a fibered knot bounds a complex curve in the four-dimensional unit ball $B^4 \subset \mathbb{C}^2$ of genus equal to the Seifert genus of $K$.

### 2.3 Surgery Formula

Let $K \hookrightarrow Y$ be a knot. A framing of $K$, denoted $\lambda$, is an isotopy class of simple closed curve on $\partial Y(K)$ which intersects the meridian disk of $\nu(K)$ once, positively. Let $X_\lambda(K)$ denote the four-manifold obtained by attaching a four-dimensional two-handle to $[0, 1] \times Y$ along $K \hookrightarrow \{1\} \times Y$ with framing $\lambda$. Note

$$\partial X_\lambda(K) = -Y \sqcup Y_\lambda(K) = -(Y_\lambda(K)) \sqcup (-Y)$$

where $Y_\lambda(K)$ is the three-manifold obtained by performing $\lambda$-framed Dehn surgery on $Y$ along $K$. Thus $X_\lambda(K)$ can be thought of either as a cobordism from $Y$ to $Y_\lambda(K)$ or as a cobordism from $-Y_\lambda(K)$ to $-Y$. When adopting the latter point of view we denote the cobordism by $X_\lambda(K)$. Given a Spin$^c$
structure $t$ on $X_\lambda(K)$, there are induced maps
\[
\hat{F}_{X_\lambda(K),t} : \hat{HF}(Y, t|_Y) \rightarrow \hat{HF}(Y_\lambda(K), t|_{Y_\lambda(K)}),
\]
\[
\hat{F}_{X_\lambda(K),t} : \hat{HF}(-Y_\lambda(K), t|_{-Y_\lambda(K)}) \rightarrow \hat{HF}(-Y, t|_{-Y}),
\]
(and also maps for the other versions of Floer homology). These maps are dual to each other under the pairing of Equation (5):
\[
\langle \hat{F}_{X_\lambda(K),t}([y]), [x] \rangle = \langle [y], \hat{F}_{X_\lambda(K),t}([x]) \rangle. \tag{7}
\]
The maps are induced from corresponding chain maps obtained by counting pseudo-holomorphic triangles in $\text{Sym}^g(\Sigma)$, as explained in Section 9 of [28]. It was proved in [29] that the maps are invariants of the smooth four-manifold $X_\lambda(K)$ and Spin$^c$ structures. Note that [29] assigns maps to arbitrary Spin$^c$ cobordisms, but these will be unnecessary for the present discussion.

Section 4 of [33], describes the relationship between the knot filtration and the Ozsváth-Szábo Floer homologies of three-manifolds obtained by performing “sufficiently large” integral surgeries on $Y$ along $K$. Moreover, this relationship gives an interpretation of some of the maps induced by cobordisms in terms of the knot filtration. These results were generalized to include all rational surgeries on knots in rational homology spheres in [36,37], but the results of [33] will be sufficient for our purposes. We review these results here, and refer the reader to [33,36,37] for a more thorough treatment.

Fix a null-homologous knot $K \hookrightarrow Y$, a Spin$^c$ structure, $s \in \text{Spin}^c(Y)$, and a Seifert surface, $S$. Framings, $\lambda$, for $K$ are canonically identified with the integers via the intersection number $\lambda \cdot S$ (note that this number is independent of the choice of $S$). Further, for a given $n > 0 \in \mathbb{Z}$, there are natural affine identifications
\[
\text{Spin}^c(Y_n(K)) \cong \text{Spin}^c(Y) \times \mathbb{Z}/n\mathbb{Z}
\]
\[
\text{Spin}^c(X_n(K)) \cong \text{Spin}^c(Y) \times \mathbb{Z}
\]
where $Y_n(K)$ is the three-manifold obtained by $(-n)$-framed surgery on $Y$ along $K$, and $X_n(K)$ is the associated two-handle cobordism from $Y$ to $Y_n(K)$. To make these identifications precise, we first fix an orientation of $K$. This induces an orientation on $S$. The oriented Seifert surface can be capped off inside the two-handle to obtain a closed surface $\hat{S}$. Now a given $s' \in \text{Spin}^c(Y_n(K))$ is then identified with a pair $[s, m] \in \text{Spin}^c(Y) \times \mathbb{Z}/n\mathbb{Z}$ consisting of a Spin$^c$ structure $s$ which is cobordant to $s'$ via a Spin$^c$ structure, $t_m$ on $X_n(K)$ satisfying
\[
\langle c_1(t_m), [\hat{S}] \rangle - n = 2m. \tag{8}
\]
Furthermore, a Spin$^c$ structure $t_m \in \text{Spin}^c(X_n(K))$ is uniquely specified by the requirement that $t_m|Y = s$ and that Equation (8) be satisfied. This yields
The latter identification above. Note, however, that both identifications depend on the homology class \([S] \in H_2(Y; \mathbb{Z})\).

Theorem 4.1 of [33] shows that for each integer \(m \in \mathbb{Z}\), there is an integer \(N\) so that for all \(n \geq N\), we have the isomorphism:

\[ H_*\left(C_s\{\min(i, j - m) = 0\}\right) \cong \hat{HF}(Y_n(K), [s, m]). \]

There is a natural chain map

\[ f_m : C_s\{i = 0\} \to C_s\{\min(i, j - m) = 0\}, \]

which is defined as the inclusion on the quotient complex \(C_s\{i = 0, j \geq m\}\) and is zero for the subcomplex \(C_s\{i = 0, j < m\}\). The proof of Theorem 4.1 of [33] shows that \(f_m\) induces the map

\[ \tilde{F}_{X_n(K)} : \hat{HF}(Y, s) \to \hat{HF}(Y_n(K), [s, m]), \]

given by the two-handle addition, endowed with the unique \(\text{Spin}^c\) structure \(t_m\) restricting to \(s\) on \(Y\) and satisfying Equation (8) above (again, provided that \(n\) is sufficiently large compared to \(m\) and the genus of the knot). Note that in order for this theorem to be used as stated, the labeling of \(\text{Spin}^c\) structures on \(Y_n(K)\) and \(X_n(K)\) must be induced by the same homology class of Seifert surface as used in the definition of \(CFK^\infty(Y, [S], K, s)\). Finally, we remark that Theorem 4.1 is stronger than what we have stated. It identifies the various Ozsváth-Szabó homologies, \(HF^+(Y_n(K), [s, m])\), \(HF^-(Y_n(K), [s, m])\), and \(HF^\infty(Y_n(K), [s, m])\) with the homology of certain sub and quotient complexes of \(CFK^\infty(Y, [S], K, s)\). This level of generality, however, will not be necessary for our purposes.

2.4 Background on the Ozsváth-Szabó contact invariant

In this subsection we briefly review the definition and basic properties of the Ozsváth-Szabó contact invariant. A fundamental theorem in three-dimensional contact geometry, due to Giroux [11], states that the construction of Thurston and Winkelnkemper [43] discussed in the introduction can be reversed. Moreover, Giroux’s theorem states that there is an equivalence:

\[ \frac{\{\text{open book decompositions of } Y^3\}}{\{\text{positive Hopf stabilization}\}} \cong \frac{\{\text{contact structures on } Y^3\}}{\{\text{isotopy}\}} \]

See [6] for an exposition of this theorem.

Thus, associated to a contact structure is an equivalence class of open book decompositions of \(Y\), where any two open books are related by a sequence of plumbing and deplumbing of positive Hopf bands.
Choose then a fibered knot \((S, K)\) whose associated open book decomposition supports the contact structure \((Y, \xi)\). In Theorem 1.1 of [30], Ozsváth and Szabó show that the knot Floer homology of a fibered knot satisfies:

\[
H_*(\mathcal{F}_{s_\xi}(-Y, [S], K, -g(S))) \cong \mathbb{Z}/2\mathbb{Z},
\]

where \(s_\xi\) is the Spin\(^c\) structure on \(Y\) associated to the contact structure, \(\xi\). Furthermore, the grading of this group captures the classical Hopf invariant:

**Proposition 19** Let \((S, K')\) be a fibered knot in \(Y\) whose open book decomposition supports the contact structure \((Y, \xi)\), and suppose that \(c_1(\xi) \in H^2(Y; \mathbb{Z})\) is a torsion class. Then we have

\[
H_*(\mathcal{F}_{s_\xi}(-Y, [S], K, -g(S))) \cong \begin{cases} 
\mathbb{Z}/2\mathbb{Z} & * = h(\xi_K) \\
0 & \text{otherwise},
\end{cases}
\]

where \(h(\xi)\) is the Hopf invariant of the underlying two-plane field of \(\xi\).

**Proof.** This follows immediately from the proof of Proposition 4.6 of [30]. Strictly speaking, Proposition 4.6 is for the contact class, \(c(\xi)\), defined below. However, the cycle which generates \(H_*(\mathcal{F}_{s_\xi}(-Y, [S], K, -g(S)))\) is a cycle representative for the contact class. The proof goes through by noting that Proposition 3.1 of [30] is proved on the chain level. We refer the reader to [30] for more details.

Let \(c_0\) denote a generator of \(H_*(\mathcal{F}_{s_\xi}(-Y, [S], K, -g(S)))\).

We define:

\[
c(S, K) = T_{-g(S)}(c_0) \in \widehat{HF}(-Y, s_\xi),
\]

where \(T_{-g(S)}\), as above, is the map on homology induced by the inclusion

\[
\tau_{-g(S)} : \mathcal{F}_{s_\xi}(-Y, [S], K, -g(S)) \to \widehat{CF}(-Y, s_\xi).
\]

Ozsváth and Szabó showed that \(c(S, K)\) depends only on the contact structure induced by the open book decomposition associated to \((S, K)\) (Theorem 1.3 of [30]). Thus we have the Ozsváth-Szabó contact invariant:

\[
c(\xi) := c(S, K),
\]

where \((S, K)\) is any fibered knot whose open book supports \(\xi\).

The contact invariant enjoys the following properties:

1. (Vanishing [30]) If \(\xi\) is overtwisted then \(c(\xi) = 0\)
(2) (Non-Vanishing [35]) If \((W, \omega)\) is a strong symplectic filling of \((Y, \xi)\) then \(c(\xi) \neq 0\).

(3) (Naturality [30, 20, 39, 8]) If \((W_K, \omega)\) denotes the symplectic cobordism between \((Y, \xi)\) and \((Y_K, \xi_K)\) induced by Legendrian surgery along a Legendrian knot \(K\), then we have

\[
\widehat{F}_{W_K, t}(c(\xi_K)) = c(\xi),
\]

where \(t\) is the canonical \(\text{Spin}^c\) structure induced by the symplectic form \(\omega\). Furthermore, if \(t \neq t\) we have

\[
\widehat{F}_{W_K, t}(c(\xi_K)) = 0
\]

(4) (Product Formula) Let \((Y_1 \# Y_2, \xi_1 \# \xi_2)\) denote the contact connected sum of \((Y_1, \xi_1)\) and \((Y_2, \xi_2)\) (see [5]). Theorem 6.1 of [28] indicates that there is an isomorphism

\[
\tilde{H}F(-Y_1 \# -Y_2, \xi_1 \# \xi_2) \cong \tilde{H}F(-Y_1, \xi_1) \otimes_{\mathbb{Z}/2\mathbb{Z}} \tilde{H}F(-Y_2, \xi_2).
\]

Under this isomorphism, \(c(Y_1 \# Y_2, \xi_1 \# \xi_2) = c(Y_1, \xi_1) \otimes c(Y_2, \xi_2)\).

We expound upon Properties 3 and 4. To understand Property 3, first recall that to a contact three-manifold \((Y, \xi)\) with a Legendrian knot, \(K\), Weinstein [45] constructs a symplectic cobordism \((W_K, \omega)\) between \((Y, \xi)\) and a contact manifold \((Y_K, \xi_K)\). Topologically, \(Y_K\) is the manifold obtained by \((tb - 1)\)-framed surgery along \(K\) and \(W_K\) is the corresponding two-handle cobordism (here \(tb\) is the Thurston-Bennequin number of \(K\)). The contact structure \(\xi_K\) is constructed so that it agrees with \(\xi\) on \(Y - \nu(K)\). Gompf shows (Proposition 2.3 of [12]) that the first Chern class of the canonical \(\text{Spin}^c\) structure, \(t\) of \((W_K, \omega)\) satisfies:

\[
\langle c_1(t), [\hat{S}] \rangle = \text{rot}_S(K),
\]

and it is clear that

\[
tb(K) - 1 = [\hat{S}] \cdot [\hat{S}]
\]

where \(S\) is the Seifert surface for \(K\) defining the 0-framing, and \(\hat{S}\) is the closed surface in the cobordism obtained by capping off \(S\) with the core of the two-handle. Now we have

\[
\partial W_K = -(Y_K) \sqcup (-Y).
\]

As a cobordism from \(-Y_K\) to \(-Y\), \(W_K\) induces maps on Floer homology as described in the preceding subsection, and the naturality statement says that the contact invariants behave nicely in the presence of the symplectic structure on \(W_K\). We should mention that the naturality property was proved in increasing levels of generality by Ozsváth and Szabó (Theorem 4.2 of [30]), Lisca and Stipsicz (Theorem 2.3 of [20]) and Ghiggini (Proposition 3.3 of [8]).
In fact, Lisca and Stipsicz’s result is a naturality statement for the contact invariant under contact +1 surgery i.e. \( tb + 1 \) framed surgery, and does not make mention of the Spin\(^C\) structure on \( W_{tb+1}(K) \), but instead sums over all Spin\(^c\) structures. The statement we have included as Property 3 is nearly identical to Ghiggini’s result, but here we have stated the result for the maps induced on the “hat” version of Ozsváth-Szabó homology. Ghiggini’s result is for an analogous contact invariant \( c^+(Y, \xi) \in HF^+(-Y, \xi) \) and for the map on \( HF^+ \) induced by Weinstein’s cobordism. As stated, Property 3 follows easily from Ghiggini’s result and naturality of the long exact sequence relating \( HF^+ \) to \( \hat{HF} \) (Lemma 4.4 of [27]) with respect to maps induced by cobordisms.

To the best of our knowledge, a proof of Property 4 does not exist in the literature but is straightforward. For completeness, we spell out the details here.

**Proof of Property 4.** Let \( K_1 \hookrightarrow Y_1 \) and \( K_2 \hookrightarrow Y_2 \) be fibered knots equipped with fiber surfaces \( S_1 \) and \( S_2 \) whose associated open book decompositions induce \( (Y_1, \xi_1) \) and \( (Y_2, \xi_2) \), respectively. Then the connected sum \( K_1 \natural K_2 \hookrightarrow Y_1 \natural Y_2 \) is a fibered knot equipped with fiber surface \( S_1 \sharp S_2 \), where \( \sharp \) denotes boundary connected sum. Torisu [44] shows that the contact structure associated to the resulting open book decomposition of \( Y_1 \natural Y_2 \) is isotopic to \( (Y_1 \natural Y_2, \xi_1 \natural \xi_2) \). As for the Floer homology of \( Y_1 \natural Y_2 \), Ozsváth and Szabó proved (Proposition 6.1 of [28]) that there is an isomorphism

\[
\hat{HF}(-Y_1 \natural Y_2, s_{\xi_1} \natural s_{\xi_2}) \cong \hat{HF}(-Y_1, s_{\xi_1}) \otimes_{\mathbb{Z}/2\mathbb{Z}} \hat{HF}(-Y_2, s_{\xi_2}),
\]

induced by a chain homotopy equivalence

\[
\overline{CF}(-Y_1 \natural Y_2, s_{\xi_1} \natural s_{\xi_2}) \cong \overline{CF}(-Y_1, s_{\xi_1}) \otimes_{\mathbb{Z}/2\mathbb{Z}} \overline{CF}(-Y_2, s_{\xi_2}). \tag{9}
\]

Theorem 7.1 of [33] states that an analogous result holds in the category of filtered chain complexes when we form the connected sum of knots. More precisely, recall from Subsection 2.1 that associated to \( K_j \) we have a \( \mathbb{Z} \)-filtration of \( \overline{CF}(-Y_j, s_j) \), \( j = 1, 2 \). We denoted the subcomplexes of this filtration by \( \mathcal{F}_{s_j}(-Y_j, K_j, m) \), so that there are inclusions:

\[
\tau^K_{m_j} : \mathcal{F}_{s_j}(-Y_j, K_j, m) \hookrightarrow \overline{CF}(-Y_j, s_j)
\]

The inclusion maps \( \tau^K_{m_j} \) induce a filtration of \( \overline{CF}(-Y_1, s_1) \otimes_{\mathbb{Z}/2\mathbb{Z}} \overline{CF}(-Y_2, s_2) \) as the image of

\[
\sum_{m_1 + m_2 = m} \tau^{K_1}_{m_1} \otimes \tau^{K_2}_{m_2} : \bigoplus_{m_1 + m_2 = m} \mathcal{F}_{s_1}(-Y_1, K_1, m_1) \otimes \mathcal{F}_{s_2}(-Y_2, K_2, m_2) \hookrightarrow \overline{CF}(-Y_1, s_1) \otimes \overline{CF}(-Y_2, s_2).
\]
According to Theorem 7.1 of [33], under the chain homotopy equivalence given by Equation (9), the above filtration of $\hat{CF}(-Y_1, s_1) \otimes_{\mathbb{Z}/2\mathbb{Z}} \hat{CF}(-Y_2, s_2)$ is identified with the filtration of $\hat{CF}(-Y_1 \# -Y_2, s_1 \# s_2)$ induced by the connected sum $K_1 \# K_2$.

It follows immediately that
\[
\tau_{g(s_1 \# s_2)}^{K_1 \# K_2}(c_0(K_1 \# K_2)) = \tau_{g(s_1)}^{K_1}(c_0(K_1)) \otimes \tau_{g(s_2)}^{K_2}(c_0(K_2)),
\]
where $c_0(K_1 \# K_2)$, $c_0(K_1)$, and $c_0(K_2)$ are cycles whose homology classes generate
\[
H_{h(\xi_1) + h(\xi_2)}(\mathcal{F}_{s_1 \# s_2}(-Y_1 \# -Y_2, K_1 \# K_2, -g(S_1 \# S_2)) \cong H_{h(\xi_1)}(\mathcal{F}_{s_1}(-Y_1, K_1, -g(S_1)) \cong \mathbb{Z}/2\mathbb{Z},
\]
respectively. Property 4 now follows from the definition of the contact invariant. □

The proof of Property 4 allows us to give simple proofs of a few well-known properties of the Hopf invariant.

**Proposition 20** (Additivity under connected sums) Let $(Y_1, \xi_1)$ and $(Y_2, \xi_2)$ be two contact manifolds with $c_1(\xi_i) \in H^2(Y, \mathbb{Z})$ torsion, $i = 1, 2$. Then, for $(Y_1 \# Y_2, \xi_1 \# \xi_2)$ we have $$h(\xi_1 \# \xi_2) = h(\xi_1) + h(\xi_2).$$

**Proof.** This follows immediately from Proposition 19 and the proof of Property 4. Indeed, Proposition 19 identifies the grading of the cycle generating the contact invariant with $h(\xi)$, while the Künneth formula used in the proof of Property 4 indicates that this grading is additive under connected sums. □

We can also define the **Hopf invariant of a fibered knot**, $(S, K)$, to be the Hopf invariant of the contact structure associated to the open book of $(S, K)$. We will denote this invariant by $h(S, K)$. This Hopf invariant has been studied in various contexts, and $-h(S, K)$ is called the **enhanced Milnor number** of $(S, K)$ by Neumann and Rudolph [25,42] (they denote $-h(S, K)$ by $\lambda(K)$). We immediately recover the following property of $h(S, K)$:

**Proposition 21** (Additivity under Murasugi sums) Let $(S_1, K_1) \subset Y_1$ and $(S_2, K_2) \subset Y_2$ be two fibered knots, and let $(S_1 * S_2, \partial(S_1 * S_2)) \subset Y_1 \# Y_2$ denote any Murasugi sum of $(S_1, K_1)$ and $(S_2, K_2)$. Then
\[
h(S_1 * S_2, \partial(S_1 * S_2)) = h(S_1, K_1) + h(S_2, K_2).
\]
Proof. Theorem 1.3 of [44] shows that $\xi(S_1 \ast S_2, \partial(S_1 \ast S_2)) \simeq \xi(S_1, K_1) \# \xi(S_2, K_2)$. The proposition now follows from its predecessor.

As an aside, we find it convenient to include a proof of the following folklore theorem mentioned in introduction:

**Proposition 22** Let $K \hookrightarrow Y$ be a knot, and let $\xi$ be an overtwisted contact structure on $Y$. Then any pair $(t, r) \in \mathbb{Z}^2$ can be realized as the Thurston-Bennequin and rotation number, respectively, of a Legendrian representative of $K$ in $\xi$.

**Proof.** Since stabilizing a Legendrian knot (i.e. adding kinks to the front projection of a portion of $K$ contained in a Darboux neighborhood) decreases the Thurston-Bennequin number and changes the rotation number by $\pm 1$, it suffices to show that we can raise the Thurston-Bennequin number. More precisely, if we are given an arbitrary Legendrian representative $\tilde{K}$ of $K$, we would like to find another representative $\tilde{K}'$ satisfying

$$tb(\tilde{K}') = tb(\tilde{K}) + 1.$$

To arrange this, we form a Legendrian connected sum of $\tilde{K}$ and the boundary of an overtwisted disk in the (overtwisted) contact structure on $S^3$ with Hopf invariant zero. Connected sums in the category of Legendrian knots can easily be defined by using Darboux’s theorem to identify tubular neighborhoods of a small Legendrian arc from each knot (see Section 3 of [7] for more details). Note, though, that in the present case of overtwisted contact structures we do not claim that this operation is uniquely defined.

Now it is easy to see that such a connected sum changes neither the three-manifold, the knot type, nor the contact structure. The fact that the three-manifold and knot type are unchanged follows from the fact that the boundary of the overtwisted disk is an unknot in $S^3$. The Hopf invariant of the contact structure on $S^3$ being zero implies that the homotopy class of the two-plane field is unchanged which, by Eliashberg’s theorem, indicates that the contact structure is also preserved (see [18] and Section 2.4 of [19] for details on the homotopy classification of two-plane fields on three-manifolds). On the other hand, since the boundary of the overtwisted disk is a Legendrian knot with zero Thurston-Bennequin number, Lemma 3.3 of [7] shows that the Thurston-Bennequin number increases by one under the connected sum. 

□
2.5 Definition of $\tau_\xi(K)$

With all necessary background in place, we can define the invariant which will be our main object of study:

**Definition 23** Let $K \hookrightarrow Y$ be a knot, $S$ a Seifert surface for $K$, and $\xi$ a contact structure with $c(\xi) \neq 0$, then

$$\tau_\xi([S], K) := \tau_{c(\xi)}^*(Y, [S], K),$$

where the right-hand side is the invariant of Definition 17.

Note that our notation suppresses the Spin$^c$ structure on $Y$, but that it is implicitly specified, since $c(\xi) \in \widehat{HF}(-Y, \mathfrak{s}_\xi)$. Thus $\tau_\xi([S], K)$ is defined via the filtration of $\mathcal{CF}(Y, \mathfrak{s}_\xi)$. It is immediate from the theorems of Ozsváth and Szabó concerning the invariance of $\widehat{HF}(Y, \mathfrak{s}), \mathcal{F}_s(Y, [S], K, m)$, and $c(\xi)$, that $\tau_\xi([S], K)$ is an invariant of the quadruple $(Y, [S], K, \xi)$ (the invariance theorems alluded to are Theorem 1.1 of [27], Theorem 3.1 of [33], and Theorem 1.3 of [30], respectively).

3 Properties of $\tau_{[x]}(Y, K)$ and $\tau_{[y]}^*(Y, K)$

Fix non-vanishing Floer classes $[x] \in \widehat{HF}(Y)$ and $[y] \in \widehat{HF}(-Y)$. In this section we prove some basic properties of $\tau_{[x]}(Y, K)$, $\tau_{[y]}^*(Y, K)$, (see Definitions 11, 17). Throughout, we will suppress the Seifert surface from the notation as much as possible, calling attention to its role when there may be ambiguity. The properties here generalize properties of the Ozsváth-Szabó concordance invariant, $\tau(K)$, most of which are established in Section 3 of [31]. For the present paper we will be primarily interested in the case when $[y] = c(\xi)$, and indeed the main theorems will utilize $\tau_\xi(K) := \tau_{c(\xi)}^*(Y, K)$. We choose to discuss the more general invariants for arbitrary non-zero classes since they also contain geometric content, see for instance [13]. It will thus be useful to collect in one place the general algebraic properties of these invariants. The following section will use the properties developed here to prove the theorems stated in the introduction.

Let $W_n(K)$ be the cobordism from from $Y$ to $Y_n(K)$ induced from the two-handle attachment along $K \hookrightarrow Y$ with framing $-n < 0$. Subsection 2.3 indicates that associated to each $t \in \text{Spin}^c(W_n(K))$ there is a map:

$$\hat{F}_{W_n(K),t}: \widehat{HF}(Y, t|_Y) \to \widehat{HF}(Y_n(K), t|_{Y_n(K)}).$$
Fix $s \in \text{Spin}^c(Y)$. To simplify notation, we use $\hat{F}_{n,m}$ to denote the map $\hat{F}_{W_n(K), t_m}$ associated to the unique $t_m \in \text{Spin}^c(W_n(K))$ satisfying:

- $t_m|Y = s$
- $\langle c_1(t_m), [\hat{S}] \rangle - n = 2m$,

where $[\hat{S}]$ denotes the homology class of a fixed Seifert surface, $S$, capped off in the two-handle to yield a closed surface, $\hat{S}$. In light of the relationship between the knot Floer homology filtration and the maps on Floer homology induced by four-dimensional two-handle attachment, we have the following proposition. Roughly speaking, it says that $\tau_{[x]}(Y, K)$ controls when $\hat{F}_{n,m}$ maps $[x]$ non-trivially.

**Proposition 24** (Compare Proposition 3.1 of [31]) Let $[x] \neq 0 \in \widehat{HF}(Y, s)$ be a non-trivial Floer homology class and let $n > 0$ be sufficiently large. We have

- If $m < \tau_{[x]}(Y, K)$, then $\hat{F}_{n,m}([x]) \neq 0$
- If $m > \tau_{[x]}(Y, K)$, then $\hat{F}_{n,m}([x]) = 0$

**Remark 25** Changing $[S]$ changes $\tau_{[x]}(Y, K)$ according to Remark 12. However, changing $[S]$ also changes the labeling of $t_m \in \text{Spin}^c(W_n(K))$ according to Equation (8), and the two changes cancel.

**Proof.** Consider the following commutative diagram of chain complexes and chain maps,

$$
\begin{array}{cccccc}
0 & \rightarrow & \mathcal{F}_s(Y, K, m) & \xrightarrow{\iota_m} & C_s\{i = 0\} & \simeq \widehat{CF}(Y, s) & \xrightarrow{P_m} & Q_s(Y, K, m) & \rightarrow & 0 \\
\Pi & & \downarrow f_m & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
0 & \rightarrow & C_s\{i \geq 0, j = m\} & \rightarrow & C_s\{\min(i, j - m) = 0\} & \rightarrow & Q_s(Y, K, m) & \rightarrow & 0,
\end{array}
$$

(10)

The vertical map on the left is defined to vanish on the subcomplex $C_s\{i = 0, j \leq m - 1\}$ of $\mathcal{F}_s(Y, K, m) = C_s\{i = 0, j \leq m\}$, while the vertical map on the right is simply the identity. The middle vertical map is the chain map described in Subsection 2.3, which vanishes on the subcomplex, $C_s\{i = 0, j < m\}$ of $C_s\{i = 0\}$. Theorem 4.1 of [33] states that for $n$ sufficiently large, we have an identification

$$
C_s\{\min(i, j - m) = 0\} \simeq \widehat{CF}(Y_n(K), [s, m])
$$

under which the map $f_m$ represents the chain map inducing $\hat{F}_{n,m}$ above. Let $I_m, P_m$ denote the maps on homology induced by $\iota_m, P_m$. Now, if $m < \tau_{[x]}(Y, K)$, we have that $P_m([x]) \neq 0$ (by the long exact sequence associated to the upper short exact sequence) and hence $\hat{F}_{n,m}([x]) \neq 0$. Moreover, since
Remark 27 Here Proposition 26

Let $[y] \in \hat{H}F(-Y)$ be a non-trivial Floer homology class and let $n > 0$ be sufficiently large. Then we have

- If $m < \tau_{[y]}(Y, K)$, then for every $\alpha \in \hat{H}F(Y, s)$ such that $\langle [y], \alpha \rangle \neq 0$
  \[ \hat{F}_{n,m}(\alpha) \neq 0 \]
- If $m > \tau_{[y]}(Y, K)$, then there exists $\alpha \in \hat{H}F(Y, s)$ such that $\langle [y], \alpha \rangle \neq 0$ and $\hat{F}_{n,m}(\alpha) = 0$.

Remark 27 Here $\langle -, - \rangle : \hat{H}F(-Y) \otimes \hat{H}F(Y) \to \mathbb{Z}/2\mathbb{Z}$ is the pairing (Equation (5)) defined in Subsection 2.1.

Proof. The proof is the same as the preceding Proposition, bearing in mind the definition of $\tau_{[y]}(Y, K)$.

Given a non-vanishing Floer class $[y] \in \hat{H}F(-Y, s)$ we can use the filtrations $\mathcal{F}(Y, K)$ and $\mathcal{F}(Y, K)$ to define $\tau_{[y]}(-Y, K)$ or $\tau_{[y]}(Y, K)$, respectively. The next proposition says that the two invariants are related by a change of sign.

Proposition 28 (Compare Proposition 3.3 of [31]) Let $[y] \neq 0 \in \hat{H}F(-Y, s)$. Then

\[ \tau_{[y]}(-Y, K) = -\tau_{[y]}(Y, K) \]

Proof. Proposition 15 states that the short exact sequence corresponding to $\mathcal{F}(Y, K, -m-1)$ is naturally isomorphic to the dual of the short exact sequence coming from $\mathcal{F}(Y, K, m)$. Specifically, recall Commutative Diagram (4):

\begin{align*}
0 & \to \mathcal{F}^*(Y, K, m) \overset{\iota_m}{\to} \hat{C}F^*(Y, s) \overset{\nu_m}{\to} Q^*_s(Y, K, m) \to 0 \\
\cong & \quad \cong \quad \cong \\
0 & \to Q_s(-Y, K, -m-1) \overset{-\iota_{m-1}}{\to} \hat{C}F(-Y, s) \overset{-\nu_{m-1}}{\to} \mathcal{F}_s(-Y, K, -m-1) \to 0
\end{align*}
Thus the inclusion and projection maps $\tau_{-m-1}$ and $p_{-m-1}$ are identified with the dual maps $p^*_m$ and $\iota^*_m$, respectively. If follows that the induced maps $\overline{p}_{-m-1}$ and $I_m$ are adjoint with respect to the pairings of Equation (5) and (6) i.e. for any $[x] \in H_s(\mathcal{F}_s(Y, K, m))$ and $[y] \in \overline{HF}(\mathcal{Y})$ we have

$$(\overline{p}_{-m-1}([y]), [x])_m = ([y], I_m([x])).$$

Suppose that $\tau^*_m(Y, K) = m$. The definition then implies that there exists $\alpha = I_m(a)$ such that

$$0 \neq \langle [y], \alpha \rangle = \langle [y], I_m(a) \rangle = (\overline{p}_{-m-1}([y]), a).$$

Thus $\overline{p}_{-m-1}([y]) \neq 0$. This implies $[y] \notin \operatorname{Im}(I_{-m-1})$, by the long exact sequence coming from the lower short exact sequence in the above commutative diagram. Hence

$$\tau_{[y]}(-Y, K) \geq -m = -\tau^*_m(Y, K).$$

We wish to show that the inequality is in fact an equality. Assume, then, that

$$\tau_{[y]}(-Y, K) = k > -\tau^*_m(Y, K).$$

Then $[y] \notin \operatorname{Im}(I_{k-1})$ and hence

$$0 \neq \overline{p}_{k-1}([y]) \in H_s(Q_s(-Y, K, k-1)) \cong H^*(\mathcal{F}^*_s(Y, K, -k)).$$

Thus there exists $a \in H_s(\mathcal{F}_s(Y, K, -k))$ such that

$$0 \neq \langle \overline{p}_{k-1}[y], a \rangle = \langle [y], I_{k}(a) \rangle,$$

It follows that $\tau^*_m(Y, K) \leq -k$, contradicting the assumption. \hfill \Box

Similar to the Ozsváth-Szabó concordance invariant, both $\tau_{[x]}(Y, K)$ and $\tau^*_m(Y, K)$ satisfy an additivity property under connected sums. As in [31], this follows readily from Theorem 7.1 of [33], which explains the behavior of the knot Floer homology filtration under the connected sum of knots.

**Proposition 29** (Compare Proposition 3.2 of [31]) Let $K_1$ and $K_2$ be knots in three-manifolds $Y_1$ and $Y_2$, respectively and let $K_1\#K_2$ denote their connected sum. Then, for any pair of non-vanishing Floer classes $[x_i] \in \overline{HF}(Y_i, s_i)$, we have

$$\tau_{[x_1] \otimes [x_2]}(Y_1 \# Y_2, K_1 \# K_2) = \tau_{[x_1]}(Y_1, K_1) + \tau_{[x_2]}(Y_2, K_2),$$

where $[x_1] \otimes [x_2]$ denotes the image of $[x_1]$ and $[x_2]$ under the isomorphism

$$\overline{HF}(Y_1, s_1) \otimes_{\mathbb{Z}/2\mathbb{Z}} \overline{HF}(Y_2, s_2) \cong \overline{HF}(Y_1 \# Y_2, s_1 \# s_2).$$

Similarly, for $\tau^*$ we have

$$\tau^{*}_{[y_1] \otimes [y_2]}(Y_1 \# Y_2, K_1 \# K_2) = \tau^{*}_{[y_1]}(Y_1, K_1) + \tau^{*}_{[y_2]}(Y_2, K_2),$$

with $\tau^{*}_{[y_1]}(Y_1, K_1)$ and $\tau^{*}_{[y_2]}(Y_2, K_2)$ denoting the $\tau^*$ invariants of $Y_1$ and $Y_2$ with respect to $K_1$ and $K_2$, respectively.
for any non-vanishing \([y_i] \in \mathcal{H}F(-Y_1, s_i)\).

**Remark 30** The Seifert surface used for \(K_1\#K_2\) should, in each case, be the boundary connected sum, \(S_1\natural S_2\), of the Seifert surfaces \(S_1\) and \(S_2\) used for \(K_1\) and \(K_2\), respectively.

**Proof.** According to Theorem 7.1 of [33], the filtration of \(\mathcal{C}F(Y_1\#Y_2, s_1\#s_2)\) induced by \(K_1\#K_2\) is filtered chain homotopy equivalent to the filtration of \(\mathcal{C}F(Y_1, s_{\xi_1}) \otimes_{\mathbb{Z}/2\mathbb{Z}} \mathcal{C}F(Y_2, s_{\xi_2})\) induced by the tensor product of inclusion maps for \(K_1, K_2\):

\[
\sum_{m_1+m_2=m} I_{m_1}^{K_1} \otimes I_{m_2}^{K_2} : \bigoplus_{m_1+m_2=m} \mathcal{F}_{s_1}(Y_1, K_1, m_1) \otimes \mathcal{F}_{s_2}(Y_2, K_2, m_2) \longrightarrow \mathcal{C}F(Y_1, s_1) \otimes \mathcal{C}F(Y_2, s_2)
\]

It follows that \(\text{Im}(I_{m_1}^{K_1} \# K_2)\) contains \([x_1] \otimes [x_2]\) if and only if there is a decomposition \(m = m_1 + m_2\) such that \(\text{Im}(I_{m_1}^{K_1})\) and \(\text{Im}(I_{m_2}^{K_2})\) contain \([x_1]\) and \([x_2]\), respectively. The minimum value of \(m\) for which this occurs is clearly \(m = \tau_{[x_1]}(Y_1, K_1) + \tau_{[x_2]}(Y_2, K_2)\). Additivity of \(\tau_{[y]}^*\) follows from the additivity of \(\tau_{[x]}\) just proved, and the preceding proposition. Indeed, we have:

\[
-\tau_{[y_1] \otimes [y_2]}(Y_1\#Y_2, K_1\#K_2) = \tau_{[y_1] \otimes [y_2]}(-Y_1\#-Y_2, K_1\#K_2) = \tau_{[y_1]}(-Y_1, K_1) + \tau_{[y_2]}(-Y_2, K_2) = -\tau_{[y_1]}(Y_1, K_1) - \tau_{[y_2]}(Y_2, K_2),
\]

where the first and last equalities follow from Proposition 28, and the middle equality from the first part of the present proposition applied to the manifolds \(-Y_1, -Y_2\).

\[\square\]

### 4 Proof of Theorems

We now apply the general properties of \(\tau_{[y]}(-Y, K)\) and \(\tau_{[y]}^*(Y, K)\) established in the previous section to the special case when \([y] = c(\xi) \in \mathcal{H}F(-Y, s_{\xi})\). Recall that in this case we have denoted \(\tau_{c(\xi)}^*(Y, K)\) by \(\tau_{\xi}^*(K)\). Throughout this section, we will fix the homology class of Seifert surface once and for all, so that any invariant or identification depending on this choice will use the same \([S] \in H_2(Y; \mathbb{Z})\), unless otherwise specified. In light of this, we will simplify the notation by omitting \([S]\) whenever possible.
Proof of Theorem 2: We follow Plamenevskaya’s proof [38] of the analogous theorem for $K \hookrightarrow (S^3, \xi_{std})$ and $\tau_{\xi,\text{std}}(S^3, K) = \tau(K)$. Assume that we have a Legendrian representative $\tilde{K}$ with Thurston-Bennequin and rotation numbers $tb(\tilde{K})$ and $\text{rot}(\tilde{K})$, respectively. Since changing the orientation of the knot changes the sign of its rotation number, it suffices to prove the inequality

$$tb(\tilde{K}) + \text{rot}(\tilde{K}) \leq 2\tau_{\xi}(K) - 1,$$

for any oriented Legendrian knot. This is because $\tau_{\xi}(K)$, and indeed the $\mathbb{Z} \oplus \mathbb{Z}$ filtered chain homotopy type of $CFK_\infty(Y, K)$, is independent of the orientation on $K$ (see Proposition 3.8 of [33]). According to [45,12] there is a symplectic cobordism $(W_K, \omega)$ between $(Y, \xi)$ and $(Y_K, \xi_K)$ induced by Legendrian surgery along $\tilde{K}$ which satisfies:

$$\langle c_1(\mathfrak{k}), [\hat{S}] \rangle = \text{rot}_S(\tilde{K}),$$

$$[\hat{S}] \cdot [\hat{S}] = tb(\tilde{K}) - 1,$$

where $\mathfrak{k}$ is the canonical Spin$^c$ structure associated to $(W_K, \omega)$. The naturality property of the contact invariant (Property 3 in Subsection 2.4) indicates that $\hat{F}_{W_K, \mathfrak{k}}(c(\xi_K)) = c(\xi)$

(11)

Pick any homogeneous $\alpha \in \hat{HF}(Y, \xi)$ which pairs non-trivially with $c(\xi) \in \hat{HF}(-Y, \xi)$ under Equation (5). It follows from Equations (11) and (7) that:

$$0 \neq \langle c(\xi), \alpha \rangle = \langle \hat{F}_{W_K, \mathfrak{k}}(c(\xi_K)), \alpha \rangle = \langle c(\xi_K), \hat{F}_{W_K, \mathfrak{k}}(\alpha) \rangle,$$

and hence that $\hat{F}_{W_K, \mathfrak{k}}(\alpha) \neq 0$.

Thus every homogeneous class pairing non-trivially with $c(\xi)$ is mapped non-trivially by $\hat{F}_{W_K, \mathfrak{k}}$, and so we wish to use Proposition 26 to bound $\langle c_1(\mathfrak{k}), [\hat{S}] \rangle + [\hat{S}]^2$ in terms of $\tau_{\xi}(K)$. To carry this out, we stabilize $\tilde{K}$ (i.e. add kinks to the front projection of a portion of $\tilde{K}$ contained in a Darboux neighborhood) to decrease the Thurston-Bennequin number and increase the rotation number while keeping $tb + \text{rot}_S$ constant. Thus, without loss of generality we may assume that the framing $tb - 1 = -n$ for the Legendrian surgery is sufficiently negative for Proposition 26 to hold. This immediately yields

$$\text{rot}_S(\tilde{K}) + tb(\tilde{K}) - 1 = \langle c_1(\mathfrak{k}), [\hat{S}] \rangle - n = 2m \leq 2\tau_{\xi}(K).$$

We have shown that $tb(\tilde{K}) + \text{rot}_S(\tilde{K}) \leq 2\tau_{\xi}(K) + 1$. To achieve the stated inequality, we examine the connected sum $K \# K \hookrightarrow (Y \# Y, \xi \# \xi)$. It is straightforward to see that:

$$\max_{\tilde{K} \# K} \left[ tb(\tilde{K} \# K) + \text{rot}_S(\tilde{K} \# K) \right] \geq 2 \max_{\tilde{K}} \left[ tb(\tilde{K}) + \text{rot}_S(\tilde{K}) \right] + 1,$$

(12)
where the maximum on both sides is taken over all Legendrian representatives of $K\#K$ and $K$, respectively. Indeed, to see the inequality, simply take the connected sum of a particular representative, $\tilde{K}$, of $K$ maximizing $tb(\tilde{K})+rot_S(\tilde{K})$. Under this sum, $tb(\tilde{K}\#K) = 2tb(\tilde{K}) + 1$ and $rot_{SS}(\tilde{K}\#\tilde{K}) = 2rot_S(\tilde{K})$, establishing the inequality (in fact, work of Etnyre and Honda [7] shows that equality is always satisfied in (12)). Now Property 4 of the contact invariant in Subsection 2.4 states that $c(\xi\#\xi) = c(\xi)\otimes c(\xi)$. It then follows from the additivity of $\tau^\ast[y](Y, K)$ under connected sums (Proposition 29) that $\tau(\tilde{K}\#K) = 2\tau(\tilde{K})$. Combining this with inequality (12), we have

$$2(tb(\tilde{K}) + rot_S(\tilde{K})) + 1 \leq \max_{K\#K} [tb(K\#K) + rot_{SS}(\tilde{K}\#\tilde{K})] \leq 2\tau(\tilde{K} + 1),$$

Where $\tilde{K}$ is any Legendrian representative of $K$. In other words, we have

$$tb(\tilde{K}) + rot_S(\tilde{K}) \leq 2\tau(\tilde{K})$$

The theorem follows from the observation that $tb + rot$ is always odd since

$$rot_S(K) = \langle c_1(\mathfrak{t}), [\tilde{S}] \rangle = [\tilde{S}] \cdot [\tilde{S}] = tb(K) - 1 \mod 2,$$

which in turn follows from the fact that $c_1(\mathfrak{t})$ is characteristic.

$\square$

**Proof of Theorem 5:** The Theorem will follow from Theorem 2 and Proposition 28, together with the definitions of the contact invariant and $\tau_\xi(K)$. Assume we have a Legendrian realization $\tilde{K}$ of the fibered knot $(S, K)$ for which:

$$tb(\tilde{K}) + |rot_S(\tilde{K})| = 2g(S) - 1.$$

By Theorem 2 we have that $tb(\tilde{K}) + |rot_S(\tilde{K})| \leq 2\tau_\xi(K) - 1$. However, the adjunction inequality (Theorem 5.1 of [33]) states that

$$\hat{HFK}(Y, [\tilde{S}], K, m) = 0 \text{ if } |m| > g(S).$$

Since $H_*(\overset{\sim}{\mathcal{F}}_s(Y, K, m-1)) := \hat{HFK}(Y, [\tilde{S}], K, m)$, it follows that

$$H_*(\mathcal{F}_s(Y, K, m)) \cong H_*(\mathcal{F}_s(Y, K, m-1)) \text{ if } m > g(S).$$

Thus $H_*(\mathcal{F}_s(Y, K, g(S))) \cong \hat{HF}(Y, \mathfrak{s})$ implying that $\tau_\xi(K) \leq g(S)$.

Summarizing, we have:

$$2g(S) - 1 = tb(\tilde{K}) + |rot_S(\tilde{K})| \leq 2\tau_\xi(K) - 1 \leq 2g(S) - 1.$$
Thus \( \tau_\xi(K) = g(S) \). Now Proposition 28 tells us that

\[
\tau_{c(\xi)}(-Y, K) = -\tau_{c(\xi)}^*(Y, K) := -\tau_\xi(K),
\]

and hence that \( \tau_{c(\xi)}(-Y, K) = -g(S) \).

Recall that since \((S, K)\) is fibered,

\[
H_*(\mathcal{F}_{sS}(-Y, [S], K, -g(S))) \cong \mathbb{Z} / 2\mathbb{Z}.
\]

Here, \( sS \) is the Spin\(^c\) structure associated to the plane field coming from the open book of \((S, K)\). Furthermore, if we let \( c_0 \) be a generator, the definition of \( \tau_{c(\xi)}(-Y, K) \) (Definition 11) implies that \( \tau_{-g(S)}(c_0) = c(\xi) \) where, as usual, \( T_{-g(S)} \) is the map on homology induced by the inclusion:

\[
T_{-g(S)} : \mathcal{F}_{sS}(-Y, [S], K, -g(S)) \hookrightarrow \widehat{CF}(-Y, sS).
\]

On the other hand, \( T_{-g(S)}(c_0) = c(\xi(S,K)) \), by the definition of the contact invariant.

\( \square \)

**Proof of Theorem 4:** Theorem 4 will follow from a more precise result involving the behavior of \( \tau_\xi(K) \) under the cabling operation, which we now review. Recall that to a knot \((Y, K)\) and choice of Seifert surface, \( S \), there is a canonical identification of the boundary of a neighborhood of \( K, \nu(K) \) with a torus i.e. \( \partial \nu(K) \cong S^1 \times S^1 \). The identification is such that \( \{pt\} \times S^1 \equiv \lambda \) and \( S^1 \times \{pt\} \equiv \mu \), where \( \lambda \) is the longitude of \( K \) coming from \( S \) and \( \mu \) is the meridian of \( K \). Given this identification, we can form a new knot, the \((p, q)\) cable of \( K \). By definition the \((p, q)\) cable of \( K \) is the isotopy class of a simple closed curve on \( \partial \nu(K) \) of slope \( \frac{\lambda}{q} \) with respect to the framing of \( \partial \nu(K) \) given by \( (\lambda, \mu) \). Theorem 4 is a consequence of the following, which is contained in Theorem 2.10 of [16]:

**Theorem 31** Let \( K \rightarrow Y \) be a null-homologous knot with Seifert surface \( S \) and let \( K_{p,-pn+1} \) denote the \((p, -pn + 1)\) cable of \( K \). Then there exists a constant, \( R > 0 \) so that \( \forall \ n > R, \) and any \( [x] \neq 0 \) \( \in \widehat{HF}(Y, s) \) we have

\[
\tau_{[x]}(Y, p \cdot [S], K_{p,-pn+1}) = \begin{cases} 
p \cdot \tau_{[x]}(Y, [S], K) - \frac{(pn)(p-1)}{2} + p - 1 & \text{or} \\
p \cdot \tau_{[x]}(Y, [S], K) - \frac{(pn)(p-1)}{2}. \end{cases}
\]

Theorem 4 now follows from Theorem 2 and the above. Specifically, if we let

\[
M = \max \{ \tau_{[x]}(Y, [S], K) \mid [x] \in \widehat{HF}(Y, s_\xi) \text{ satisfies } \langle c(\xi), [x] \rangle \neq 0 \},
\]

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then Theorem 31 shows that
\[ \tau_\xi(p \cdot [S], K_{p, -pn+1}) \leq p \cdot M - \frac{(pn)(p-1)}{2} + p - 1. \]
Taking \( n \) to be large enough so that
\[ 2pM - (pn)(p-1) + 2p - 3 \leq -N \]
provides the bounds stated in Theorem 4. □

References


[40] T. Perutz. A remark on Kähler forms on symmetric products of Riemann surfaces. arxiv.org/abs/math/051547


