11/30/10 Matt Hedden HFH

Last Time:

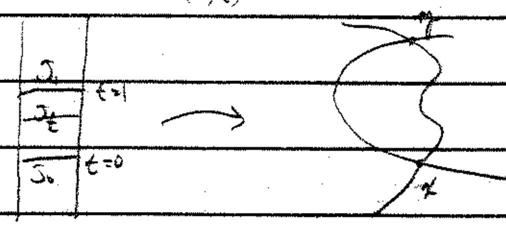
Given a path of almost complex structures J_s $s \in [0,1]$

$$J_s : T\mathrm{Sym}^g \to T\mathrm{Sym}^g \qquad {J_s}^2 = -1$$

We can assign
$$f_{J_s} : CF^\circ(J_0) \to CF^\circ(J_s)$$
$$f_{J_s}(x) = \underbrace{\# \mathcal{M}(\phi)}_{\#\, J_s\text{-holomorphic maps}} \cdot y$$

$u : [0,1] \times \mathbb{R} \to \mathrm{Sym}^g$ connecting x to y.
(u, t)

with



To a path of paths of a.c.s.

$$J_{s,\tau} \quad \tau \in [0,1] \qquad J_{s,0} = J_s, \quad J_{s,1} = J_s'$$
$$s \in [0,1]$$

Assign $H_{J_{s,\tau}} : CF^\circ_*(J_0) \to CF^\circ_{*+1}(J_1)$

$$H_{J_{s,\tau}}(x) = \# \mathcal{M}^\tau(\phi) \cdot y$$

$\# \mathcal{M}^\tau(\phi) :=$ # pts. in the moduli space
$$\mathcal{M}^\tau(\phi) = \bigsqcup_{\tau \in [0,1]} \mathcal{M}_{J_{s,\tau}}(\phi)$$

where $\phi \in \pi_2(x,y)$ has $\mu(\phi) = -1$.

Recall: $gr(x) - gr(y) = \mu(\phi) - 2\cancel{n_z}(\phi)$
$= -1$

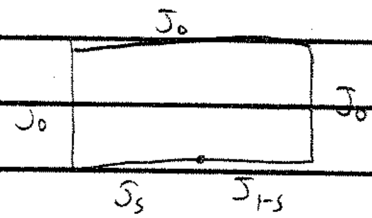
Prop. $f_{J_s} \circ \partial_{J_0} + \partial_{J_1} \circ f_{J_s} = 0$ (mod 2)

and $\partial_{J_1} \circ H_{J_s,\tau} + H_{J_s,\tau} \circ \partial_{J_0} + f_{J_s} + f_{J_s'} = 0.$

Pf. (In both cases) follows from Gromov compactness and gluing.

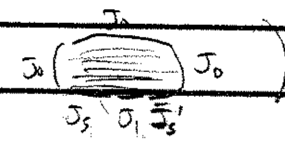
To complete invariance under choice of J,

Consider $J_{1-s} * J_s \simeq J_0$



Prop. $f_{J_s * J_{1-s}} + f_{J_0} + \partial \circ H_{s,\tau} + H_{s,\tau} \circ \partial = 0$

Lemma. $f_{J_s * J_s'} + f_{J_s} \circ f_{J_s'} + \partial \circ H + H \circ \partial = 0$

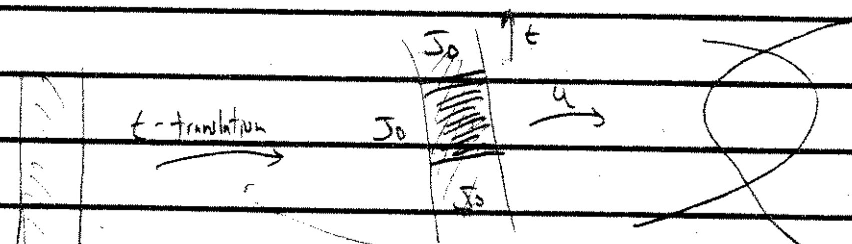
Pf. Exercise. (Assume Gromov compactness) (Hint: 

)

The lemma implies the Prop: $f_{J_s} \circ f_{J_{1-s}} + f_{J_0} + \partial H + H \partial = 0$

Lemma: $f_{J_0} \equiv Id.$

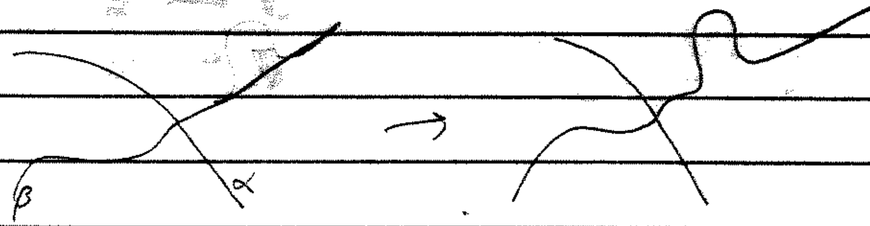
Pf. $$f_{J_0}(x) = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(x,y) \\ \mu(\phi)=0}} M_{J_0}(\phi) \cdot U^{n_z(\phi)} \cdot y$$



So here, $M(\phi)$ is negative dimensional unless $\mathbb{R}$-action isn't free, i.e. for $\phi = \text{const}$.

$\Rightarrow f_{J_0}(\vec{x}) = 1 \cdot \vec{x}$ # of constant J_0-hol disks.

Cor. $H_*\left(CF(\Sigma, \vec{\alpha}, \vec{\beta}, z)\right)$ remains unchanged under isotopies of α-curves or β-curves which don't introduce new intersection pts.



Independence

(1) A.C.S. ✓ Chain maps associated to paths J_s, inducing $\cong$

(2) Heegaard Diagram

(a) Isotopies

(i) Introducing new intersection pts.

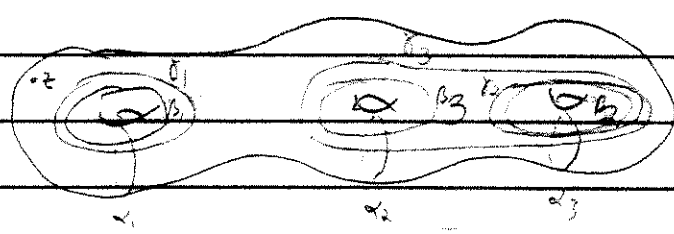
(ii) The rest ✓ Follows from (1)

(b) Handle slides

(c) Stabilization ✓ Proved for $\widehat{HF}$ via gluing thm. for disconnected domains.

Handle Slide Invariance

Idea: To a handleslide, we naturally obtain a H.D. with 3 sets of curves $(\Sigma, \vec{\alpha}, \vec{\beta}, \vec{\gamma}, z)$



Heegaard triple diagram

where $\gamma_1, \dots, \gamma_{g-1}$ are isotopic to $\beta_1, \dots, \beta_{g-1}$, *

& $\gamma_g = (\beta_g$ slid over something else$)$

Heegaard Triple Diagram $\longrightarrow$ 3 Heegaard Diagrams

$(\Sigma, \vec{\alpha}, \vec{\beta}, z) \rightsquigarrow Y$

$(\Sigma, \vec{\alpha}, \vec{\gamma}, z) \rightsquigarrow Y$

$(\Sigma, \vec{\beta}, \vec{\gamma}, z) \rightsquigarrow \#_g S^1 \times S^2$.

* This would not be an admissible diagram. So we need to wind (for the torsion spin^c structure on $\#_g S^1\times S^2$...)
(Small admissibility perturbation)

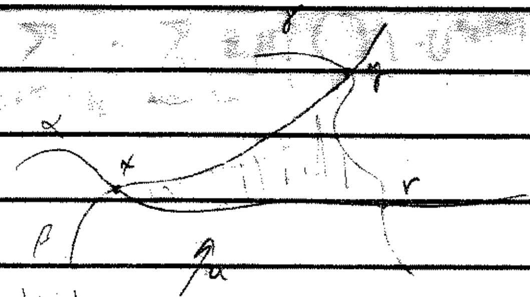
Want a chain map between $CF(\Sigma, \vec{\alpha}, \vec{\beta}, z)$ + $CF(\Sigma, \vec{\alpha}, \vec{\gamma}, z)$

We'll define chain maps by counting holomorphic triangles.

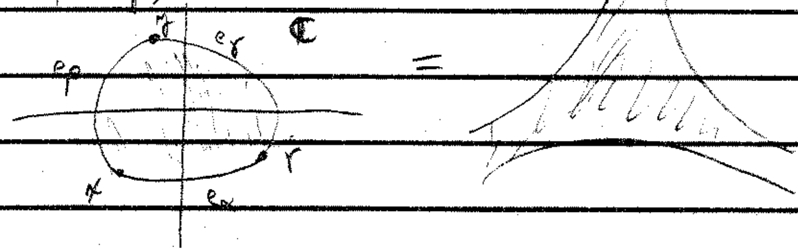
$\vec{\alpha} \longrightarrow \mathbb{T}_\alpha$

$\vec{\beta} \longrightarrow \mathbb{T}_\beta$

$\gamma \longrightarrow \mathbb{T}_\gamma$



Want to count J-holomorphic Whitney triangles.



Prop. $f_{\alpha,\beta,\gamma} : CF^0(Y_{\alpha\beta}) \otimes CF^0(Y_{\beta\gamma}) \longrightarrow CF^0(Y_{\alpha\gamma})$

$$\underset{x}{CF^0(Y_{\alpha\beta})} \quad \underset{y}{CF^0(Y_{\beta\gamma})} \quad \underset{r}{CF^0(Y_{\alpha\gamma})}$$

$$f_{\alpha\beta,\gamma}(x \otimes y) = \sum_{r \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma} \cdot \sum_{\psi \in \pi_2(x,y,r)} \# \mathcal{M}(\psi) \cdot U^{n_z(\psi)} \cdot r$$

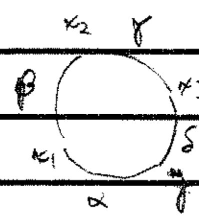
$$\pi_2(x,y,r) = \left\{ \begin{matrix} \text{Whitney triangles} \\ \text{connecting } y, x, r \end{matrix} \right\} \Big/ \underset{\sim}{\text{homotopy}}$$

$$\mu(\psi) = 0$$

is a chain map.

Note:

order matters



$$\xrightarrow{u} \left(Sym^g(\Sigma_g), \mathbb{T}_\alpha, \ldots, \mathbb{T}_\delta \right)$$

$$\psi \in \pi_2(x_1, x_2, x_3, y) = \left\{ \begin{matrix} \text{homotopy classes of} \\ \text{Whitney squares} \end{matrix} \right\}$$

$\# \mathcal{M}(\psi) = \#$ of J-holomorphic squares in homotopy class of ψ.

and so forth.. We can consider Whitney n-gons and J-holomorphic Whitney n-gons.

Prop. $f_{\vec{\alpha}^1, \dots, \vec{\alpha}^{n-1}, \beta} : \underset{\vec{x}_1}{CF^0(Y_{\vec{\alpha}^1 \vec{\alpha}^2})} \otimes \cdots \otimes \underset{\vec{x}_{n-2}}{CF^0(Y_{\vec{\alpha}^{n-2} \vec{\alpha}^{n-1}})} \otimes \cancel{CF^0(Y_{\alpha^{n-1}\beta})} \underset{\vec{x}_{n-1}}{} \to CF^0(Y_{\vec{\alpha}\vec{\beta}}) \underset{\vec{y}}{}$

$$f_{\vec{\alpha}^1, \dots, \vec{\alpha}^{n-1}, \vec{\beta}}(\vec{x}_1 \otimes \cdots \otimes \vec{x}_{n-1}) = \sum_{\vec{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \cdot \sum_{\mu(\psi) = 3-n} \# \mathcal{M}(\psi) \cdot U^{n_z(\psi)} \cdot \vec{y}$$

$\psi \in \pi_2(\vec{x}_1, \dots, \vec{x}_{n-1}, \vec{y})$

$\mu(\psi) = 3 - n$

This is explained by the fact that

$$\dim\big(\mathcal{M}(\bigcirc)\big) = n-3 \qquad (n \geq 3)$$

$\shortparallel$

Moduli space of conformal n-gons.

if $\mu(\psi) = 3-n$

$$\mathcal{M}(\psi) = \coprod_{\substack{\bigcirc \in \mathcal{M}(n\text{-gons}) \\ \dim = n-3}} \mathcal{M}\left(u : \begin{smallmatrix}\text{specific} \\ n\text{-gon}\end{smallmatrix} \to \mathrm{Sym}^g\right)$$

is a chain map.

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2 Items remain to show invariance for HFH

(a) Isotopies that introduce new intersections

(b) Handle slides.

Approach for both is to Define (chain) maps by counting J-holomorphic Whitney n-gons.

Prop. $f_{\alpha\beta\gamma} : CF(Y_{\alpha\beta}) \otimes CF(Y_{\beta\gamma}) \to CF(Y_{\alpha\gamma})$

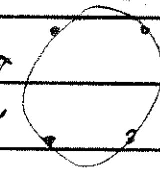
$$f_{\alpha\beta\gamma}(\vec{x} \otimes \vec{r}) = \sum_{\vec{r} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\psi \in \pi_2(\vec{x}, \vec{r}, \vec{y}) \\ \mu(\psi) = 0}} \# \mathcal{M}(\psi) \cdot U^{n_z(\psi)} \cdot \vec{y}$$

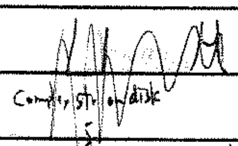
is a chain map.

Note: In general, the Maslov index tells you your expected dimension w/ a fixed domain.

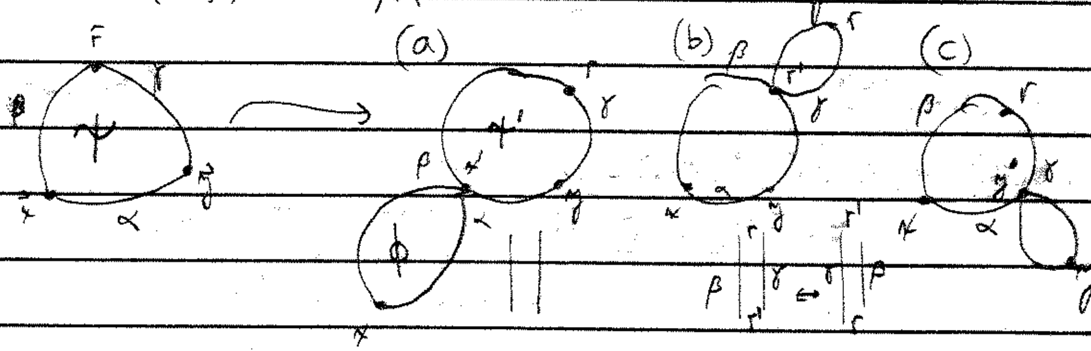
There's only one conformal map of triangle w/ marked pts.

There are more for n-gons with $n > 3$. vary distance b/w pts.





Pf. Consider $\mathcal{M}(\psi)$, where $\mu(\psi) = 1$, $\psi \in \pi_2(\vec{x}, \vec{r}, \vec{y})$.



If we have this type of degeneration then we can decompose the Whitney triangle into a Whitney strip + another Whitney △.

$$\psi = \phi * \psi'$$

But then $\mu(\phi * \psi') = \mu(\psi) = 1$

$$\mu(\phi) + \mu(\psi')$$

Since $\mu(\psi') < 0$ triangles have $\mathcal{M}(\psi') = \emptyset$

& $\mu(\phi) \leq 0$ disks have $\widehat{\mathcal{M}}(\phi) = \emptyset$ (if $\phi \neq$ const.)

$\Rightarrow$ Only cases which arise are $\mu(\psi') = 0$ and $\mu(\phi) = 1$.

Gromov Compactness + Gluing $\Rightarrow$ $\overline{\mathcal{M}(\psi)}$ is compact with $\partial \overline{\mathcal{M}(\psi)}$ = pairs of J-holomorphic maps of the form (a), (b), or (c).

Chain map: $f \circ \partial = \partial \circ f$

Such pairs are in 1-1 correspondence with

$$f_{\alpha\beta\gamma}(\partial \vec{x} \otimes \vec{r}) + f_{\alpha\beta\gamma}(\vec{x} \otimes \partial \vec{r}) + \partial f_{\alpha\beta\gamma}(\vec{x} \otimes \vec{r})$$

(no sign in $\mathbb{Z}_2$)

$$f_{\alpha\beta\gamma} \circ \partial(\vec{x} \otimes \vec{r}) = f_{\alpha\beta\gamma}(\partial\vec{x} \otimes \vec{r} + \vec{x} \otimes \partial\vec{r}) = f_{\alpha\beta\gamma}(\partial\vec{x} \otimes \vec{r}) + f_{\alpha\beta\gamma}(\vec{x} \otimes \partial\vec{r})$$

$$\overset{\text{WANT}}{=} \partial_{\text{if}} f_{\alpha\beta\gamma}(\vec{x} \otimes \vec{r})$$

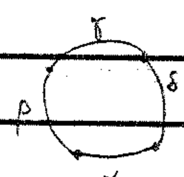
Tada!

"$\square$"

Whitney 4-gons!

$(\Sigma, \alpha, \beta, \gamma, \delta)$

$$H_{\alpha\beta\gamma\delta} : CF(Y_{\alpha\beta}) \otimes CF(Y_{\beta\gamma}) \otimes CF(Y_{\gamma,\delta}) \to CF(Y_{\alpha\delta})$$



(Note: $CF_*(Y_{\alpha\beta}) \otimes CF_*(Y_{\beta\gamma}) \otimes CF_*(Y_{\gamma,\delta}) \otimes \underset{nz}{CF_*}(Y_{\delta,\alpha}) \to \mathbb{Z}/2$)

$CF^*(Y_{\alpha,\delta})$

$$\mathrm{Hom}(C_* \otimes C_* \otimes C_* \otimes C^*, \mathbb{Z}/2) \cong \mathrm{Hom}(C_* \otimes C_* \otimes C_*, C^*)$$

$$H_{\alpha\beta\gamma\delta}(\vec{x} \otimes \vec{y} \otimes \vec{r}) = \sum_{m \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\square \in \pi_2(x,y,r,m) \\ \mu(\square) = -1}} \# \mathcal{M}(\square) \cdot U^{n_z(\square)} \cdot \vec{m}$$

What type of map is this?

Not a chain map (for squares)

Consider $\mathcal{M}(\square)$ $\quad \square \in \pi_2(\vec{x}, \vec{y}, \vec{r})$ with $\mu(\square) = 0$.

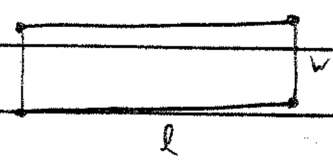
$$\mathcal{M}(\square) = \coprod_{\text{holomorphic quads}} \mathcal{M}\left((\square, j) \xrightarrow{u} \mathrm{Sym}^g\right) \quad \Big\} \text{ 1-dim'l.}$$

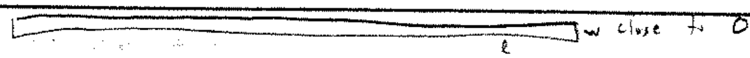
$\mathcal{M}(\square)$ has a compactification $\overline{\mathcal{M}(\square)}$

Recall: $H_{J_{s,\tau}}$ counted pts. in $\mathcal{M}^{\tau}(\phi) = \coprod_{\tau \in [0,1]} \mathcal{M}_{J_{s,\tau}}(\phi)$

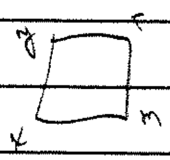
In the case at hand, then, we expect $\mathcal{M}(\square)$ to have 2 types of ends (i.e., boundary pts.) in its compactification, namely those from Gromov compactness, & those from the compactification of the space of holomorphic quads.

The moduli space of holomorphic quads. is 1-dim'l., & parametrized by $w/\ell \in \mathbb{R}$

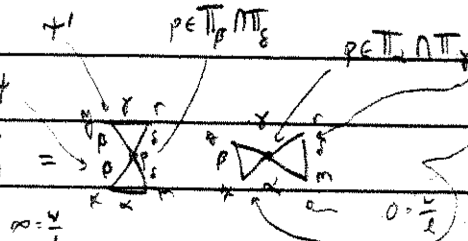




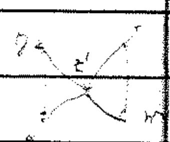
Seidel "Fukaya Categories + Picard-Lefschetz Theory" Chapter 2.



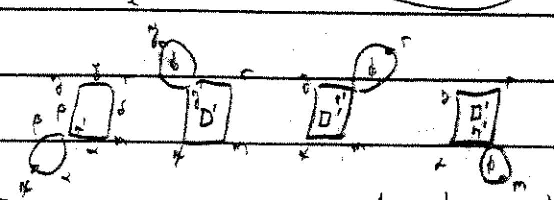
Claim (w/o Pf) : $\partial \overline{\mathcal{M}(\square)} =$



← Moduli space of quad. ends



(0, ∞)

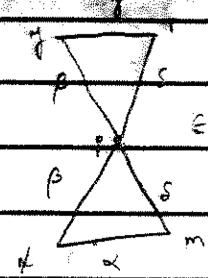


← Gromov compactness ends

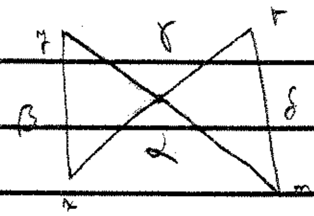
In $H_{\alpha\beta\gamma\delta} \circ \partial(x \otimes y \otimes r)$ $\qquad$ In $\partial_{\alpha\delta} \circ H_{\alpha\beta\gamma\delta}$

It must be that $\mu(\psi) = \mu(\psi) = 0$

$\mu(\phi) = 1, \quad \mu(D') = -1$

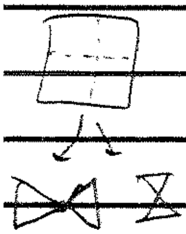


$\in f_{\alpha\beta\delta}\left(x \otimes f_{\beta\gamma\delta}(y \otimes r) \right)$

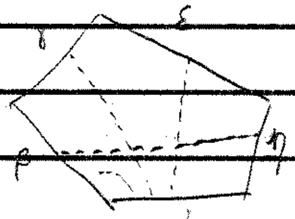


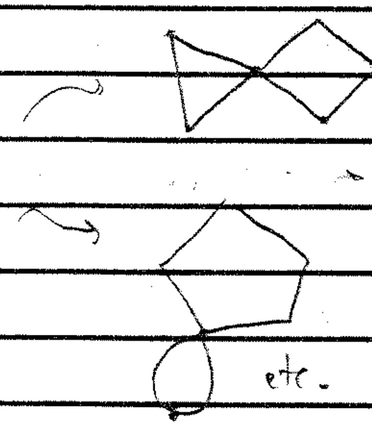
$\in f_{\alpha\gamma\delta}\left(f_{\alpha\beta\gamma}(x \otimes y) \otimes r \right)$

SO, $\quad \partial H + H \partial = f \circ f + f \circ f$



Exercise: Think about pentagons.

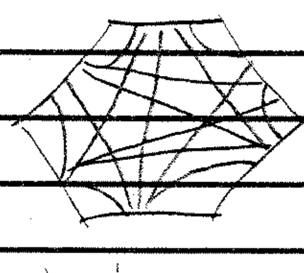




etc.

For n-gon

$\mu = (3-n) + 1$



Returning to earth

varied slightly for admissibility

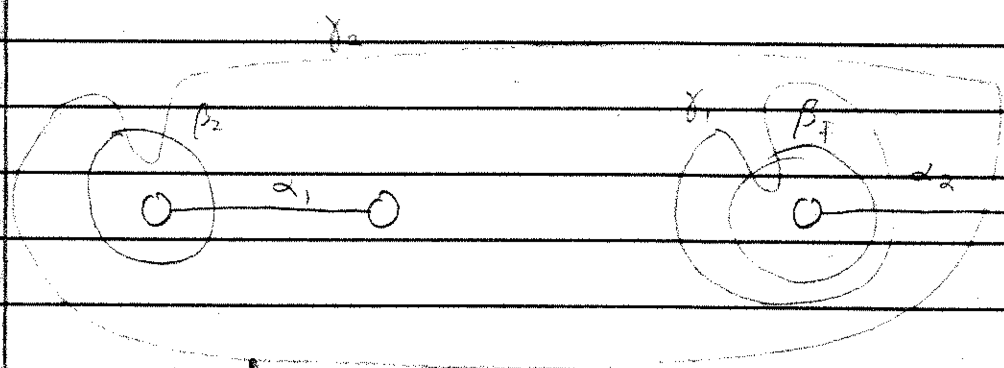
Handleslides $\longrightarrow (\Sigma, \alpha, \beta, \gamma)$, where $\gamma_i = \tilde{\beta}_i \quad \forall i \neq g$.

$\gamma_g = \beta_g$ slid over $\beta_k \quad k \neq g$.

$\to \quad f_{\alpha\beta\gamma} : CF(Y_{\alpha\beta}) \otimes CF(Y_{\beta\gamma}) \to CF(Y_{\alpha\gamma})$.

$$\begin{array}{ccc} \cong & \cong & \cong \\ Y & \#^{g} S^1\times S^2 & Y \end{array}$$

So consider $\#^{g} S^1\times S^2$



$Y_{\beta\gamma} = \#^{g} S^1\times S^2$

$$\widehat{HF}_*\left(\#^{g} S^1\times S^2, s_0\right) \cong H_*\left(T^g\right) \qquad (1)$$

$$s_0 \in Spin^c(S^1\times S^2) \quad \langle c_1(s_0), [S^2]\rangle = 0$$

Exercise! Show that (1) holds by Künneth formula for $\widehat{HF}$ of connected sums.

$$\widehat{HF}(S^1\times S^2, s_0) \cong H_*(S^1)$$

Last time

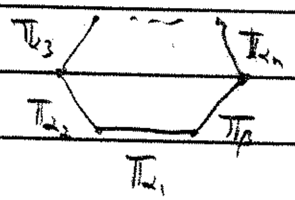
(A_∞) Associativity Properties satisfied by

$$f_{\alpha_1, \dots, \alpha_n, \beta} : CF(Y_{\alpha_1\alpha_2}) \otimes \cdots \otimes CF(Y_{\alpha_{n-1},\alpha_n}) \to CF(Y_{\alpha\beta}).$$

Defined by counting J-holomorphic Whitney $(n+1)$-gons

with $\mu(\psi) = 3-n$

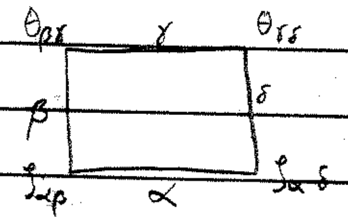
$\psi \in \pi_2(x_1, \dots, x_n, y)$



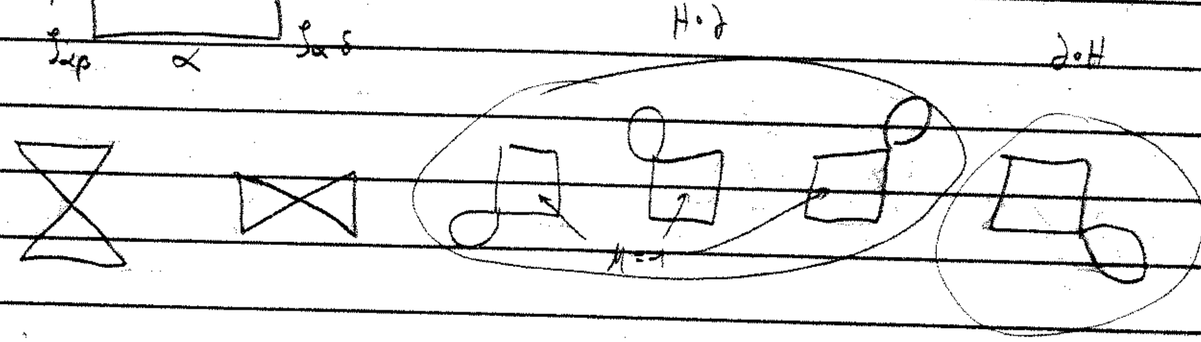
Prop. $$f_{\alpha\beta\delta}(\xi_{\alpha\beta} \circ f_{\beta\gamma\delta}(\Theta_{\beta\gamma} \otimes \Theta_{\gamma\delta}))$$

$$\sim_{\text{Chain homotopic}} f_{\alpha\gamma\delta}(f_{\alpha\beta\gamma}(\xi_{\alpha\beta} \otimes \Theta_{\beta\gamma}) \otimes \Theta_{\gamma\delta})$$

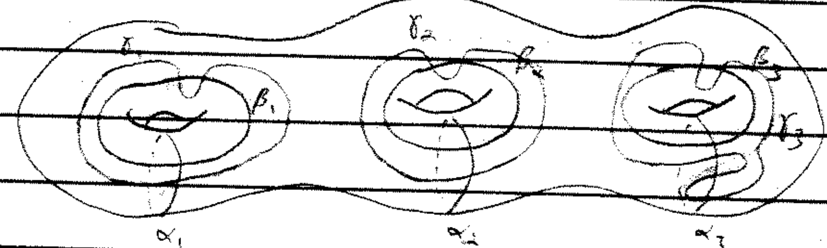
Pf. via $H_{\alpha\beta\gamma\delta} : CF(Y_{\alpha\beta}) \otimes CF(Y_{\beta\gamma}) \otimes CF(Y_{\gamma\delta}) \to CF(Y_{\alpha,\delta})$



$\mu = 0$



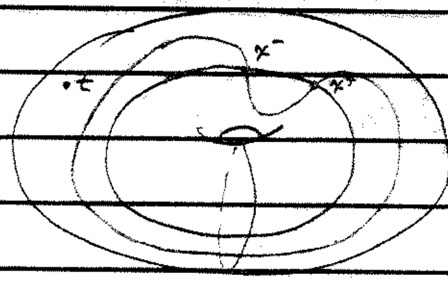
Isotopy invariance

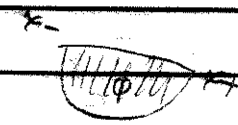


$$f_{\alpha\beta\gamma} : CF(Y_{\alpha\beta}) \otimes CF(Y_{\beta\gamma}) \to CF(Y_{\alpha\gamma})$$

$$f^{\text{Isotopy}} : CF(Y_{\alpha\beta}) \to CF(Y_{\alpha\gamma})$$
$$x \longmapsto f_{\alpha\beta\gamma}(x \otimes \Theta_{top})$$

$HF_*(Y_{\beta\gamma} = \#^{g} S^1 \times S^2) \cong H_*^{\text{singular}}(\underbrace{S^1 \times \cdots \times S^1}_{g})$





$\mu(\phi) = 1$

$\Rightarrow \; gr(x_+) - gr(x_-) = \mu(\phi) - 2n_z(\phi)$
$= 1 - 0.$

So x_+ is "higher" than x_-

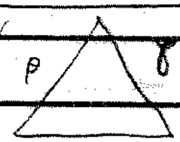
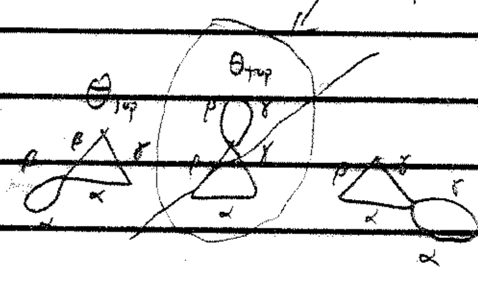
The g-tuple of x^+'s in $\#^{r} S^1 \times S^2$ gives a top graded generator.

rank top grading $= 1$

(rank top ________ ?)

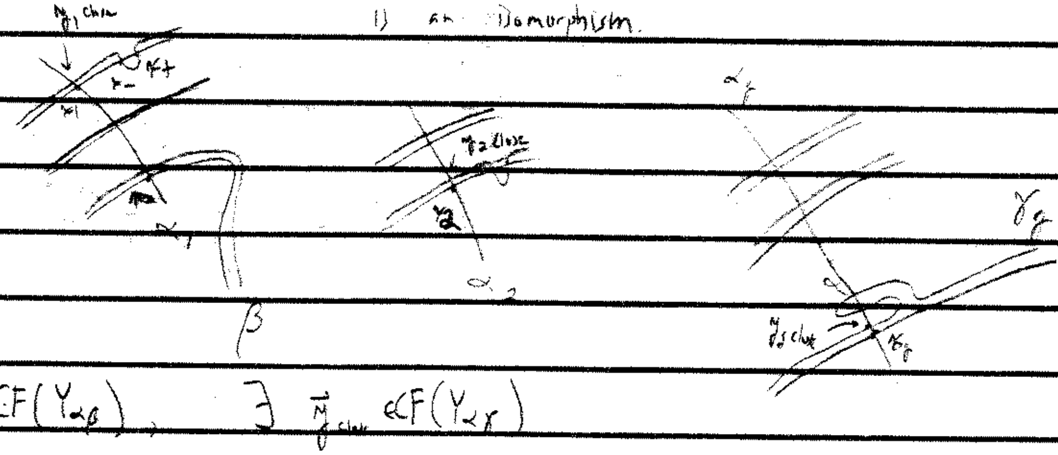
So $\Theta_{top} := (\underbrace{x_+, \ldots, x_+}_{g})$

Claim 1: f^{Isotopy} is a chain map.

Pf. $\partial \mu\Big($  $(\mu=1)\Big) \xleftrightarrow{1-1}$ 

Not possible since $\partial\Theta_{top} = 0$

Claim 2: $(f^{\text{Isotopy}})_* : HF(Y_{\alpha\beta}) \to HF(Y_{\alpha\gamma})$ is an isomorphism.

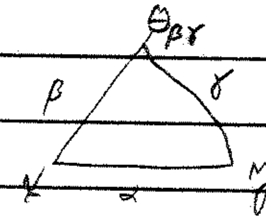


$\forall\; \vec{x} \in CF(Y_{\alpha\beta}), \quad \exists\; \vec{y}_{close} \in CF(Y_{\alpha\gamma})$

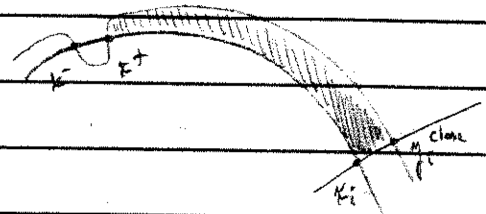
Ideally: $CF(Y_{\alpha\beta}) \xrightarrow{i} CF(Y_{\alpha\gamma})$ to be $f^{\text{Isotopy}} = i$

inclusion of **groups** (not chain maps)

$$f^{\text{Isotopy}}(\vec{x}) = f_{\alpha\beta\gamma}(\vec{x} \otimes \Theta_{\beta\gamma})$$, which counts J-holomorphic triangles

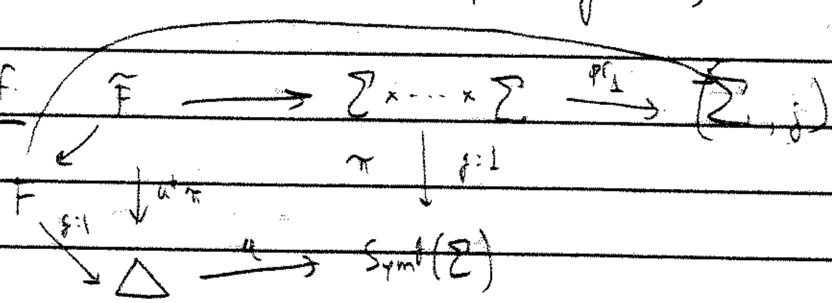


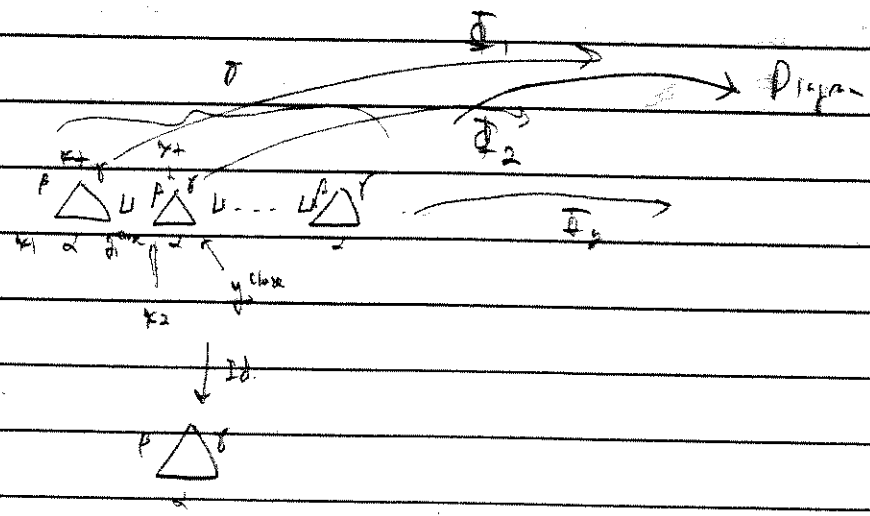
Notice: For every y_i^{close}, $\exists$ a triangle involving x^+. Take the domain of Ψ in each of these triangles, and 0 elsewhere.



Lemma. $\exists\ \Psi \in \pi_2(\vec{x}, \Theta_{\beta\gamma}, \vec{y}_{\text{close}})$ (Homotopy class)

Pf.





Lemma 2: $\#\mathcal{M}(\Upsilon) = 1$.

Pf: Apply Riemann Mapping Theorem to the g-triple of maps involved in Φ.

Thus, $\langle f^{\text{Ibsdway}}(\vec{x}), \vec{y}^{\,\text{close}} \rangle \neq 0 \quad (\gg 1 \bmod 2)$

coefficient of $\vec{y}^{\,\text{close}}$

Q: How do we know there aren't other homotopy classes?

A:

Exercise: (1) ~~Define~~ Given $(\Sigma, \alpha_1, \ldots, \alpha_n, \beta)$, a Heegaard $(n+1)$-tuple diagram, define $\varepsilon(x_1, \ldots, x_n, y)$ which characterizes when $\pi_2(\vec{x}_1, \ldots, \vec{x}_n, \vec{y}) \neq \emptyset$. $x_i \in \mathbb{T}_{\alpha_i} \cap \mathbb{T}_{\alpha_{i+1}}$, $y \in \mathbb{T}_{\alpha_1} \cap \mathbb{T}_\beta$.

(2) Where does ε live? (Some group)

(3) When $\varepsilon = 0$, draw the domain of a Whitney $(n+1)$-gon in π_2. BEWARE this notation

(4) If $\varepsilon = 0$, how many $\psi \in \pi_2$ are there?

(For disks, $\mathbb{Z} \oplus H_2 = \ker(\alpha + \beta \to H_1(\Sigma))$).

(5) Check that for the isotopy Heegaard triple diagram in question, $\pi_2(x, \theta, y) = \mathbb{Z} \oplus \pi_2(\vec{x}, \vec{y}) \oplus \pi_2(\theta, \theta) \oplus \pi_2(\vec{y}, \vec{y})$

With these exercises, we claim: $\pi_2(\vec{x}, \Theta, \vec{y}_{\text{close}}) \longleftrightarrow \mathbb{Z} \oplus$

$\Upsilon' \neq \Upsilon$.

Any other $\Upsilon' = \Upsilon * P_{\alpha\beta} * P_{\alpha\gamma} + P_{\beta\gamma}$

But admissibility guarantees that $P_{\alpha\beta}, P_{\alpha\gamma}, P_{\beta\gamma}$ have $\oplus$ and $\ominus$ coefficients.

Since Υ is so small, this ensures that the domain of $\Upsilon' \not\geq 0 \Rightarrow \mathcal{M}(\Upsilon') = \emptyset$.

Nice trick

We've shown that a certain chain map has terms which look like what we want,

+ these terms are realized by geometric objects (triangles) which are very small (in area).

Def. An $\mathbb{R}$-filtered chain complex is a chain complex (C_*, ∂)

plus a map $\mathcal{F}: C_* \to \mathbb{R}$

s.t. $\mathcal{F}(x) \geq \mathcal{F}(\partial x)$.

Def. A filtered chain map is a chain map $g : (C_*, \partial, \mathcal{F}) \to (C'_*, \partial', \mathcal{F}')$ between filtered complexes

s.t. $\mathcal{F}'(g(x)) \leq \mathcal{F}(x)$.

Lemma. Suppose $g : (C, \partial, \mathcal{F}) \to (C', \partial', \mathcal{F}')$ is a filtered chain map, and g = isomorphism + lower order terms.

i.e. $g(x) = i(x) + lo(x)$

with $\mathcal{F}'(lo(x)) < \mathcal{F}'(i(x))$ $\forall x$.

Then $g_* : H_*(C) \to H_*(C')$ is an isomorphism.

Pf. • Choose filtered bases for C and C'.

• Wrt these bases,

$$g = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{bmatrix}$$

• Then, a filtered change of basis doesn't change g_*.

Thus, we can perform column operations to change the matrix to Id. $\square$

Conclusion of Proof that f^{isotopy} induces an isomorphism (on Homology)

Pick filtrations on $CF(Y_{\alpha,\beta})$, $CF(Y_{\alpha,\gamma})$ by

$$\mathcal{F}(x_{\text{arbitrary}}) = 0, \quad \mathcal{F}(x_{\text{else}}) = -\text{Signed Area}(D(\phi))$$ or vice versa, whichever works

$\phi \in \pi_2(x_{\text{arbitrary}}, x_{\text{else}})$.

$(CF(Y_{\alpha\beta}), \partial, \mathcal{F})$ is filtered by $\left(\mathcal{M}(\phi) \neq \emptyset \Rightarrow D(\phi) \geq 0 \Rightarrow -\text{Area}(D(\phi)) \leq 0\right)$

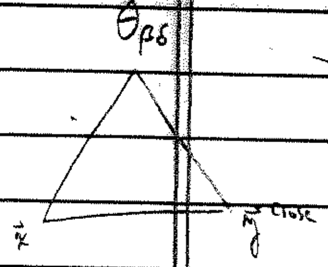
Then show the triangle map is filtered, the small triangles preserve the filtration,
+ every other triangle (having more area) decreases filtration.

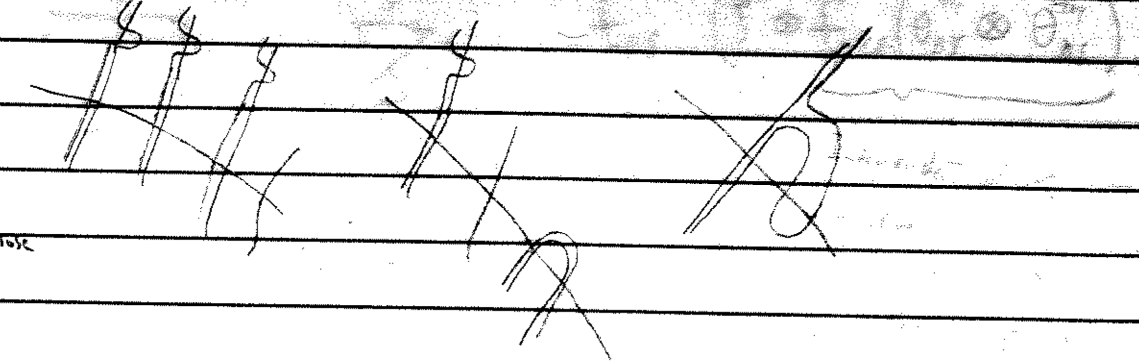
$\square$

[illegible]

Last time: Showed that for $f^{\text{Isotopy}}(\vec{x}) = f_{\alpha\beta\delta}(\vec{x} \otimes \Theta_{\beta\delta}^{\text{top}})$,

$$(f^{\text{iso}})_* : HF(Y_{\alpha\beta}) \xrightarrow{\cong} HF(Y_{\alpha\delta}).$$





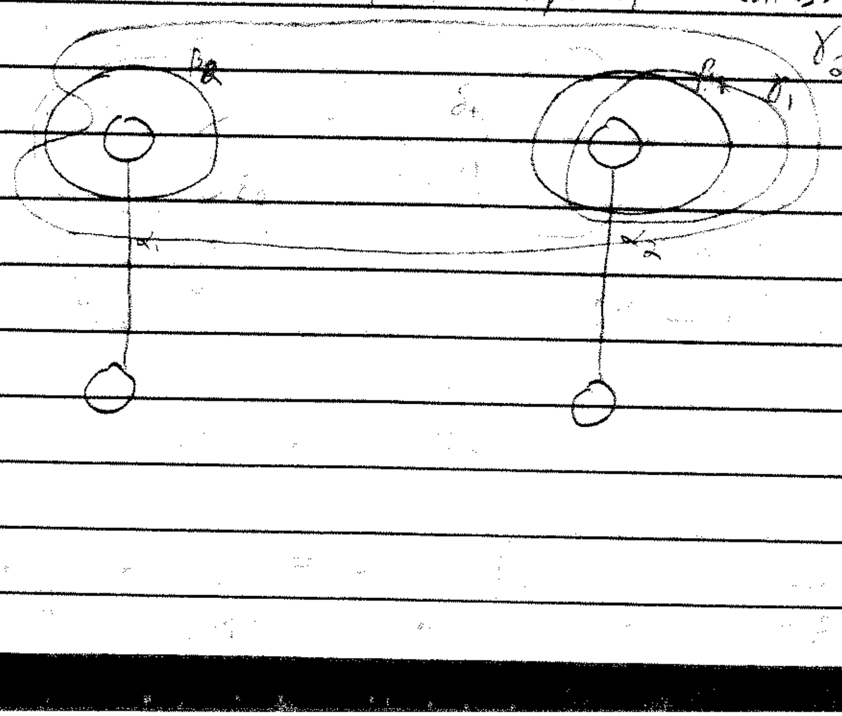
Note: $rk(CF(Y_{\alpha\beta})) = n$, $rk(CF(Y_{\alpha\delta})) = n+k$, $k \geq 0$.

Note: From last time, we showed that the chain map looks like

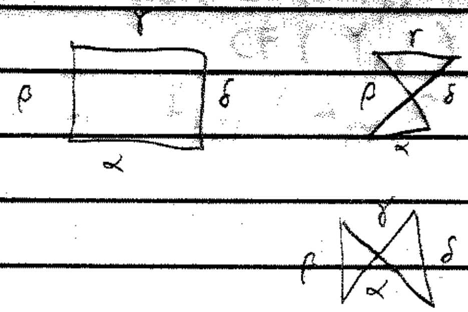
$$\underbrace{\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \\ \hline & * & \end{bmatrix}}_{n} \begin{matrix} \}\, n \\ \\ \}\, k \end{matrix} \xrightarrow[\text{basis}]{\text{Change of}} \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \\ \hline & 0 & \end{bmatrix}$$

So HF is independent of HD up to isotopy of curves.

It remains to show HF is ~~independent~~ unchanged by handleslides.



Prop. $$f_{\alpha\gamma\delta}\left(f_{\alpha\beta\gamma}\left(\xi\otimes\Theta^{top}_{\beta\gamma}\right)\otimes\Theta^{top}_{\gamma\delta}\right)\underset{\substack{\text{chain}\\ \text{homotopic}}}{\simeq} f_{\alpha\beta\delta}\left(\xi\otimes\underbrace{f_{\beta\gamma\delta}\left(\Theta^{top}_{\beta\gamma}\otimes\Theta^{top}_{\gamma\delta}\right)}\right)$$



Independent of the α-curves.

This picture of a handleslide doesn't care what manifold we're in, except by varying the α curves.

The next step is to show:
$$f_{\beta\gamma\delta}\left(\Theta^{top}_{\beta\gamma}\otimes\Theta^{top}_{\gamma\delta}\right)=\Theta_{\beta\delta}$$

Finally, $f_{\alpha\beta\delta}(\xi\otimes\Theta_{\beta\delta}) =: f^{\text{isotopy}}(\xi)$

is an isomorphism on homology (as we showed last time).

So, $f_{\alpha\gamma\delta}\left(f_{\alpha\beta\gamma}\left(\xi\otimes\Theta^{top}_{\beta\gamma}\right)\otimes\Theta^{top}_{\gamma\delta}\right)$ is an isomorphism on homology

$\Rightarrow$ $f_{\alpha\beta\gamma}(\xi\otimes\Theta_{\beta\gamma}) =: f^{\text{handleslide}}(\xi)$

is injective on the level of homology.

To show surjectivity, we will "cyclically permute the argument."

Let η be parallel (but admissible) to γ.

Look at $\boxed{\alpha\ \beta\ \gamma\ \delta}\ \eta$

$(\alpha\ \beta\ \gamma\ \delta\ \eta)$

$\delta \simeq \beta$

$$f_{\alpha\delta\eta}\left(f_{\alpha\gamma\delta}(\xi\otimes\Theta_{\gamma\delta})\otimes\Theta_{\delta\eta}\right)\underset{\text{c.h.}}{\simeq} f_{\alpha\gamma\eta}\left(\xi\otimes f_{\gamma\delta\eta}\left(\Theta_{\gamma\delta}\otimes\Theta_{\delta\eta}\right)\right)$$

And, $f_{\gamma\delta\eta}\left(\Theta_{\gamma\delta}\otimes\Theta_{\delta\eta}\right)=\Theta_{\gamma\eta}$

And $f_{\alpha\gamma\eta}(\xi\otimes\Theta_{\gamma\eta}) =: f^{\text{isotopy}}(\xi)$ is an isomorphism on homology

$\Rightarrow$ $f^{\text{handleslide}}$ is surjective (remembering that δ is isotopic to β)

What's left?

Compute $CF(Y_{\gamma\delta})$

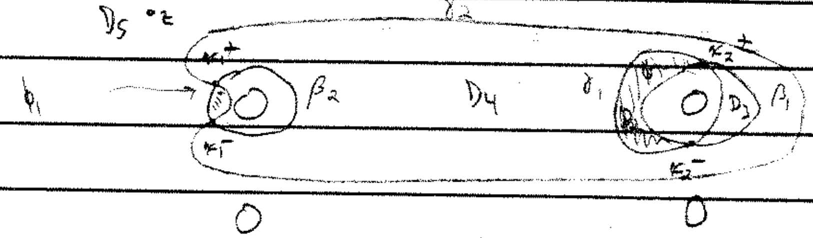
$$H_*\left(CF(Y_{\beta\delta}), \partial\right) \equiv H_*(T^?)$$

$CF(Y_{\beta\gamma})$

Compute $f_{\beta\gamma\delta}(\Theta_{\beta\gamma} \otimes \Theta_{\gamma\delta})$

Note: We only calculated homology of $\#^? S^1 \times S^2$ with the nicest diagrams.

We have to show that this is invariant for the rest of our argument to go through.



Claim: The relative grading for this diagram is the same as the previous.

$*$	$x_1^+ \times x_2^+$	
$*-1$	$x_1^+ \times x_2^-$	$x_1^- \times x_2^+$
$*-2$	$x_1^- \times x_2^-$	

$$gr(x_1^+ x_2^+,\ x_1^+ x_2^-) = \mu(\phi) - 2 n_z(\phi)$$
$$= 1$$

Can't really see the board.

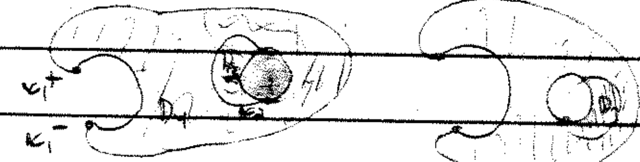
$$x_1^+ x_2^+$$

$D_2 \swarrow \phi_1 \qquad \searrow D(\phi) - D_1$

$x_1^+ x_2^- \qquad x_1^- x_2^+$

$\searrow D_1$

$$x_1^- x_2^-$$

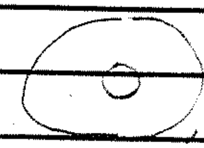


Check - All other domains differ from these two and ϕ_1 by Periodic domains w/ $\oplus$ and $\ominus$ coefficients, so these are the only 3 domains to worry about, one of which is covered by the R.M.T.

Claim: $\# \widehat{\mathcal{M}}(\phi_2) + \# \widehat{\mathcal{M}}(\phi_3) = \pm 1.$

Pf. For each domain $\exists$ 1 parameter family of holomorphic annuli, indexed by the cut length along β_1 (resp. γ_1).

If cut length along $\beta_1 = 0$, annulus is



which has no holomorphic rep.

length of interior β-arc for max cut length is greater than exterior,

$$\Leftrightarrow \quad \# \widehat{\mathcal{M}}(\phi_2) = 1.$$

Similarly, length of interior γ_1-arc for max cut length is greater than the exterior γ-arc

$$\Leftrightarrow \quad \# \widehat{\mathcal{M}}(\phi_3) = 1.$$

Thus, HF is an invariant of a closed, oriented 3-mfld.

In fact, our proof shows that $HF^{\circ}_{*}(Y, s)$ are invariants where $*$ is a relative $\mathbb{Z}$-grading if

$$\mathrm{div}(c_1(s)) := \gcd_{\alpha \in H_2(Y)} \langle c_1(s), \alpha \rangle = 0$$

$$gr(x) - gr(y) = \mu(\phi) - 2n_z(\phi)$$

In general, we showed that $HF_*(Y, s)$ is an invariant, as a $\mathbb{Z}/\mathrm{div}(c_1(t))$-graded group.

Also, the $\mathrm{spin}^c(Y)$ splitting is an invariant.

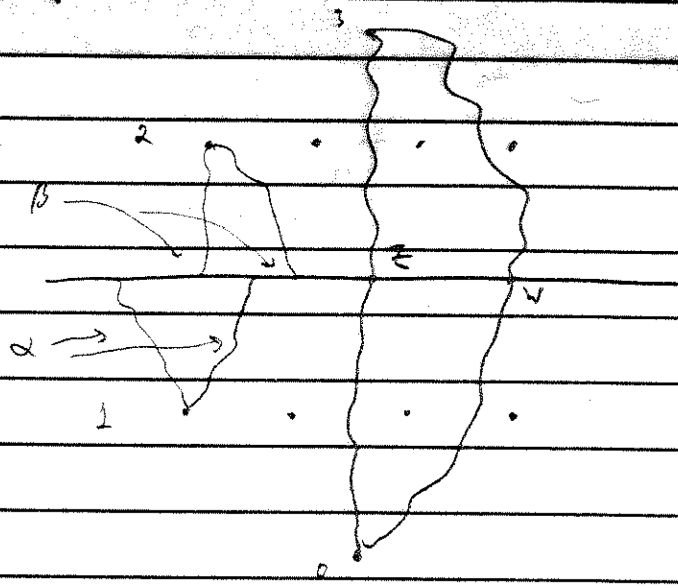
$$CF(Y_{\alpha\beta}, s) \xrightarrow{f_{\alpha\beta\gamma}} CF(Y_{\alpha\gamma}, s)$$

$$s_z(\vec{x}) = s \longmapsto s_{z'}(f_{\alpha\beta\gamma}(\vec{x})) = s.$$

$\rightsquigarrow$ 4-mfld invariants come about by studying maps on Floer homology induced by J-holo. triangles

Invariance proof considered maps induced by $Y \times I = W$.

⇝ Considering HD's w/ more sets of base points $(\Sigma, \vec{\alpha}, \vec{\beta}, z, w, \ldots)$ gives rise to knot & link invariants, & invariants of sutured mnflds.

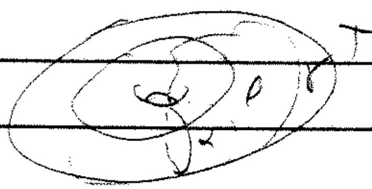


$$(\Sigma, \vec{\alpha}, \vec{\beta}, z, w)$$

$$\gamma_z \cup -\gamma_w =: K \subset Y$$

⇝ Using the A_∞-associativity of J-holomorphic pentagon, quad, & triangle counts, we can obtain l.e.s. relating HF of 3-mnflds. differing by surgery along K.

$M, \qquad \partial M = T$



$\alpha, \beta, \gamma \subset T$ s.t. $\alpha \cdot \beta = 1$, $\beta \cdot \gamma = 1$, $\gamma \cdot \alpha = 1$.

Then $(M, \alpha) \rightsquigarrow M(\alpha) = M \underset{\partial D^2 \to \alpha}{\cup} S^1 \times D^2$

$$\cdots \to \widehat{HF}^+(M(\alpha)) \to \widehat{HF}^+(M(\beta)) \to \widehat{HF}^+(M(\gamma)) \to \cdots$$