HFH · [illegible]

Plan (1) When does $\phi \in \pi_2(\vec{x}, \vec{y})$ admit J-holo rep

(2) Spin^c-structures

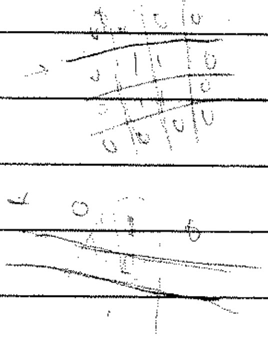
(3) Admissibility criterion for H. D.

$$\begin{tikzcd}
 & (\hat{F}, \hat{j}) \arrow[dl] \arrow[dd] & (\Sigma, j) \times \cdots \times (\Sigma, j) \arrow[r] \arrow[dd, "\pi"] & (\Sigma, j) \\
(F, j) \arrow[urrr, bend left, "\tilde{a}"] \arrow[dr, "a^* \pi / S_g"'] & & & \\
 & (\mathbb{D}^2, i) \arrow[r, "a"] & (\mathrm{Sym}^g(\Sigma), \mathrm{Sym}^g(j)) &
\end{tikzcd}$$

$$\Longleftrightarrow \left\{ \begin{tikzcd} (F^2, \tilde{j}) \arrow[r, "\Phi"] \arrow[d, "p"] & (\Sigma, j) \\ (\mathbb{D}^2, i) & \end{tikzcd} \right\}$$

$$du \circ i = \mathrm{Sym}^g(j) \circ du.$$

Prop. Suppose $\phi \in \pi_2(\vec{x}, \vec{y})$ is a Whitney disk, & $D(\phi) = \begin{cases} \text{rectangle} \rightarrow \\ \text{bigon.} \end{cases}$



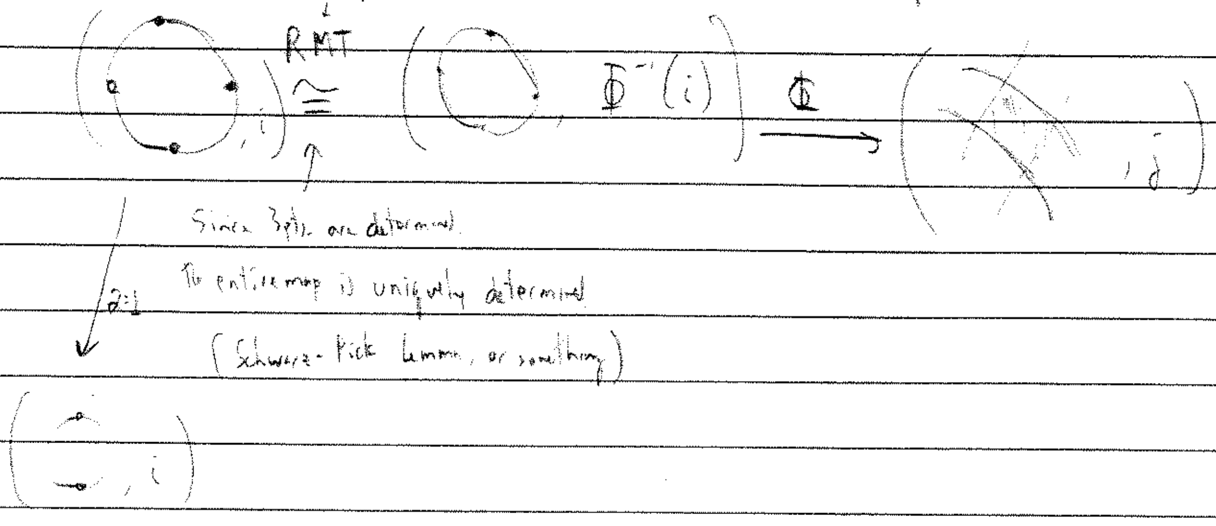
Then $\# \widehat{\mathcal{M}}(\phi) = 1$.

Pf. Assume that $\mathrm{Sym}^g(j)$ is generic, i.e. achieves transversality.

Then $\# \widehat{\mathcal{M}}(\phi) = \# \left\{ \begin{tikzcd} (F^2, j) \arrow[r, "\Phi"] \arrow[d] & (\Sigma, j) \\ \mathbb{D}^2 & \end{tikzcd} \right\} (*)$ s.t.

$$D(\Phi) = D(\phi).$$

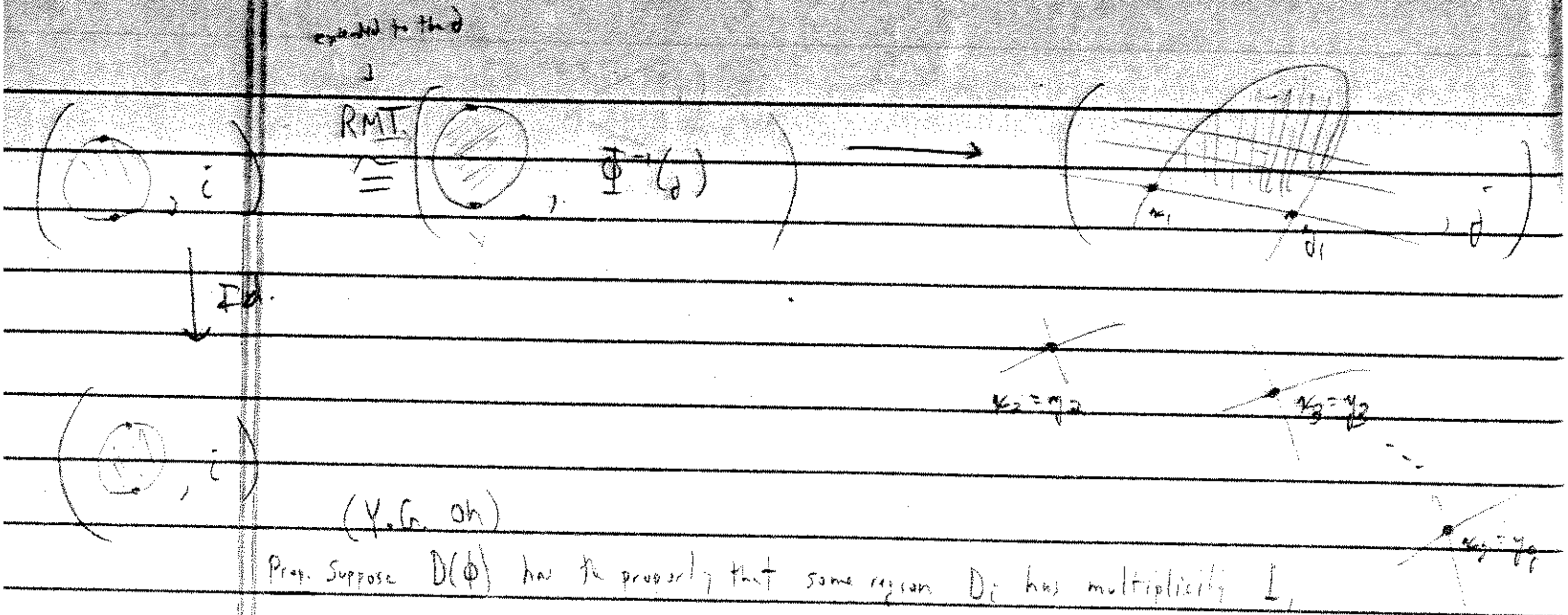
WTS: $\exists !$ holomorphic pair in $(*)$ for a rectangle or bigon.



Since 3pts are determined.

The entire map is uniquely determined

(Schwarz-Pick Lemma, or something)



(Y. Gr. Oh)

Prop. Suppose $D(\phi)$ has the property that some region D_i has multiplicity 1,

some region D_j has mult. 0,

& $\partial D_i \cap \partial D_j \neq \emptyset$

Then, after possibly performing an isotopy of α and β curves, $\mathrm{Sym}^g(j)$ is generic $\overset{\text{for } \hat{\mathcal{M}}(\phi)}{\wedge}$ for any j, complex structure on Σ

Def. Call a H.D. nice if every domain $D_i \subset \Sigma \setminus \{\vec{\alpha} \cup \vec{\beta}\}$ are either rectangles or bigons.

Exercise Show that if a H.D. is nice, then the corresponding 3-mfld. is a lens space, or S^3, or $S^1 \times S^2$

Def. A H.D. is called nearly nice if every region except one is a rectangle or a bigon and this one bad region contains the base point.

Exercise Show that, if $(\Sigma, \vec{\alpha}, \vec{\beta}, z)$ is

(A) Nice, then Every $\phi \in \pi_2(\vec{x}, \vec{y})$ with $\mu(\phi) = 1$, $D(\phi) \geq 0$ admits a unique J-holomorphic rep. $\left(\Rightarrow \widehat{CF}^{\geq 0} \text{ are algorithmically computable}\right)$

(B) Nearly nice, then Every $\phi \in \pi_2(\vec{x}, \vec{y})$ with $\mu(\phi) = 1$, $D(\phi) \geq 0$, $n_z(\phi) = 0$ admits a unique J-holomorphic rep. $\left(\Rightarrow \widehat{CF} \text{ is algorithmically computable}\right)$

This Exercise is due to S. Sarkar. He and J. Wang showed that every 3-mfld. possesses a nearly nice diagram. (i.e., $\widehat{CF}$ is algorithmically computable for every 3-mfld.)

Finger move decreases complexity of this region, & smaller cost of the compl. in rest.

Spin^c Structures

Recall: $$CF^0(Y) \cong \bigoplus_{\alpha \in H^2(Y)} CF^0(Y, \alpha)$$

because $$\underset{[\gamma_x \cup \gamma_b]}{\underset{\shortparallel}{\varepsilon(\vec{x}, \vec{y})}} \in \underset{H_1(\Sigma)/\mathrm{Span}\,\vec{\alpha} + \mathrm{Span}\,\vec{\beta}}{\underset{\cong}{H_1(Y)}} \cong H^2(Y)$$

To obtain this splitting, we must choose which $\vec{x} \longleftrightarrow 0 \in H^2(Y)$.

Spin^c Structures provide a canonical splitting of $CF^0(Y)$ into a direct sum of complexes.

Def. Two nowhere vanishing vector fields V_1, V_2 on a closed 3-mfld are homologous if they are homotopic (through nowhere vanishing vector fields (nvvf)) on $Y \setminus B^3$.

We write $V_1 \sim V_2$. [illegible]

Exercise: Show that $\sim$ is an equivalence relation.

Def. A Spin^c structure, $\mathfrak{s}$, is an equivalence class of n.v.v.f.

$$\mathrm{Spin}^c(Y) := \{\text{n.v.v.f.}\}/\sim.$$

Prop. $\mathrm{Spin}^c(Y) \xleftrightarrow{1\text{-}1} H^2(Y)$.

Pf. Pick a trivialization of TY (Obstruction Theory), $TY \xrightarrow{\phi} Y \times \mathbb{R}^3$

+ a Riemannian metric g.

Using (ϕ, g), a n.v.v.f. $V \rightsquigarrow (Y \to S^2$

$$\left(\begin{array}{ccccc} TY & \xrightarrow{\phi} & Y \times \mathbb{R}^3 & \to & S^2 \\ X & \longmapsto & (p, v_p) & \longmapsto & v_p/\|v_p\| \end{array} \right)$$

Varying V in its $\sim$ class, $V_1 \sim V_2 \Rightarrow$ $Y^3 \setminus B^3 \xrightarrow{V_2} S^2$ ~~are homotopic~~

$Y^3 \setminus B^3 \xrightarrow{V_1} S^2$

So, $$\mathrm{Spin}^c(Y) \rightsquigarrow [Y^3 \setminus B^3, S^2] \cong [Y^3 \setminus B^3, \mathbb{CP}^\infty]$$

Retracts onto a 2-complex

$$H^2(X; \mathbb{Z}) \xleftrightarrow{1\text{-}1} [X, K(\mathbb{Z}, 2)] \qquad K(\mathbb{Z},2) \cong \mathbb{CP}^\infty$$

$$\cong$$
$$H^2(Y \setminus B^3; \mathbb{Z})$$
$$\cong \quad \text{LES of pair}$$
$$H^2(Y; \mathbb{Z})$$

$$\begin{tikzcd} & Y \times \mathbb{R}^3 \\ TY \arrow[ur, "\phi_1"] \arrow[dr, "\phi_2"'] & \\ & Y \times \mathbb{R}^3 \end{tikzcd}$$

Remarks:

- Does not depend on choice of metric g, (because the space of metrics is contractible)
- Does depend on choice of trivialization $TY \to Y^3 \times \mathbb{R}^3$. How?

$\phi_1 \circ \phi_2^{-1}(p) \in SO(3)$, $p \in Y$

Prop. Let v be a nvvf, $[V_\phi] \in H^2(Y)$ denote the class coming from ϕ in the previous construction. $\phi_{12} : Y \longrightarrow SO(3)$,

Then $[V_{\phi_1}] - [V_{\phi_2}] = \phi_{12}^* w \in H^2(Y; \mathbb{Z}/2\mathbb{Z})$, where $w \in H^2(SO(3); \mathbb{Z}) \cong \mathbb{Z}/2$ is a generator.

Using this Proposition, we can define a difference $\mathfrak{s}_1 - \mathfrak{s}_2 \in H^2(Y;\mathbb{Z})$ between two Spin^c structures which depends on nothing (except $\mathfrak{s}_1$ and $\mathfrak{s}_2$)

To see this, pick any trivialization $\phi_1 : TY \to Y \times \mathbb{R}^3$ and consider

$$\left[(v_1)_{\phi_1} \right] - \left[(v_2)_{\phi_1} \right]$$

and compare to

By Prop.

$$\left[(v_1)_{\phi_2} \right] - \left[(v_2)_{\phi_2} \right] = -\phi_{12}^* w + \left[(v_1)_{\phi_1} \right] - \Big(-\phi_{12}^* w + [(v_2)_{\phi_2}$$

$$= \left[(v_1)_{\phi_1} \right] - \left[(v_2)_{\phi_1} \right]$$

$$\left[(v_1)_{\phi_2} \right] - \left[(v_2)_{\phi_2} \right]$$

$$=$$

$$\Big(\left[(V_1)_{\phi_2} \right] - \left[(v_1)_{\phi_1} \right] \Big) + \left[(v_1)_{\phi_1} \right] \Big(- \left[(v_2)_{\phi_2} \right] + \left[(v_2)_{\phi_1} \right] \Big) - \left[(v_2)\phi_1 \right]$$

$$=$$

Prop.

$$\left(\phi_{21}^* w \right) + \left[(V_1)_{\phi_1} \right] - \left(\phi_{21}^* (w) \right) - \left[(v_2)\phi_1 \right]$$

$$=$$

$$\left[(v_1)_{\phi_1} \right] - \left[(v_2)_{\phi_1} \right] \quad ✓$$

Last Time • Sufficient conditions for $\# \hat{\mathcal{M}}(\phi)=1$, i.e. if $D(\phi)$ is a rectangle or bigon. (*)

$\phi\in\pi_2(\vec{x},\vec{y})$,

$\mu(\phi)=1$

Exercise: If a diagram is nice, then all relevant $\phi\in\pi_2(\vec{x},\vec{y})$ (over all $\vec{x},\vec{y}$) are dealt with by (*)

• Spin^c structures

Spin^c structure is a homology class of n.v.v.f.

Given a trivialization $\tau: TY \to Y\times\mathbb{R}^3$ (a metric g).

v n.v.v.f. $\quad \phi^{\tau}_{v}: Y \to S^2$

$$x \mapsto \frac{\tau(v_x)}{\|\tau(v_x)\|}$$

ϕ^{τ}_{v} gives rise to a bijection $\quad \mathrm{Spin}^c(Y) \longleftrightarrow H^2(Y-B^3) \cong H^2(Y)$

$[v] \longleftrightarrow \phi^{\tau\,*}_{v}(m)$, where $m\in H^2(S^2)$ is a generator.

Prop. If τ_1,τ_2 are different trivializations, then $(\phi^{\tau_1}_{[v]})^*(m)-(\phi^{\tau_2}_{[v]})^*(m)=(\phi^{12})^*(w)$,

where $\phi^{12}: Y\to SO(3)$ obtained as the difference of τ_1,τ_2,

and $w\in H^2(SO(3);\mathbb{Z})\cong\mathbb{Z}/2$ is a generator.

Pf.

$S^2\times SO(3)\to S^2$

$(p,A)\mapsto (A\cdot p)$

$$\begin{tikzcd} \phi^{\tau_1}_{v}: Y \arrow[r] & S^2\times SO(3) \arrow[r, "\text{evaluation}"] & S^2 \\ x \arrow[r, mapsto] & \left(\frac{\tau_1(v_x)}{\|\tau_1(v_x)\|}, \mathrm{Id.}\right) \arrow[r, mapsto] & \left(\frac{\tau_1(v_x)}{\|\tau_1(v_x)\|}\right) \end{tikzcd}$$

$$\begin{tikzcd} \phi^{\tau_2}_{v}: Y \arrow[r] & S^2\times SO(3) \arrow[r] & S^2 \\ x \arrow[r, mapsto] & \left(\frac{\tau_1(v_x)}{\|\tau_1(v_x)\|}, \phi^{12}(x)\right) \arrow[r, mapsto] & \left(\phi^{12}(x)\cdot\frac{\tau_1(v_x)}{\|\tau_1(v_x)\|}\right) \end{tikzcd}$$

So $\quad \phi^{\tau_1\,*}_{v}(m) = (\phi^{\tau_1}_{v}\times \mathrm{Id.})\circ \mathrm{eval.}^*(m)$

$\phi^{\tau_2\,*}_{v}(m) = (\phi^{\tau_1}_{v}\times\phi^{12})\circ \mathrm{eval}^*(m)$.

Prop. follows by Kunneth Formula. $\square$ $\qquad \mathrm{eval}^*\big|_{SO(3)} = w, \quad \mathrm{ev}^*\big|_{S^2} = m$

$$\begin{tikzcd} S^3 \times SO(3) \arrow[r, "\iota", "\sim"'] \arrow[rr, bend right, "evl."'] & E \arrow[r, "\pi"] & S^2 \end{tikzcd}$$

$$\begin{tikzcd} SO(3) \arrow[r, hook, "i"] & E \arrow[d, "\pi"] \arrow[r, phantom, "\cong"] & S^2 \times SO(3) \\ & S^2 & \end{tikzcd}$$

$(p, A) \sim (B \cdot p, AB^{-1})$

Now, $evl.|_{SO(3)} = i^* \pi^*(m) = w$

$$SO(3) \xhookrightarrow{i} S^2 \times SO(3) \xrightarrow{evl.} S^2 \qquad i^* evl^*(m) = w$$

$$\begin{tikzcd} SO(3) \arrow[r] & E \arrow[r, "?", "\cong"'] \arrow[d, "\pi"] & S^2 \times SO(3) \arrow[d, "evl"] & (p, A) \arrow[d] \\ & S^2 & S^2 & A \cdot p \end{tikzcd}$$

$\exists f : S^2 \times SO(3) \xrightarrow{\cong} E$?? "Pun!"

Prop allows a well-defined difference of Spin^c structures to be defined:

$$([v_1], [v_2]) \longrightarrow [v_1] - [v_2] := (\phi^{\tau}_{v_1} - \phi^{\tau}_{v_2})^*(m)$$

homotopy classes of n.v.v.f.?

$\cap$

$Spin^c(Y) \times Spin^c(Y)$

$\cap$

$H^2(Y)$

for any τ.

Def. Given some Spin^c structure $[v]$, let $J([v]) := [-v] \in Spin^c(Y)$.

J is called conjugation, $J : Spin^c(Y) \to Spin^c(Y)$.

We'll denote $J\mathfrak{s} = \overline{\mathfrak{s}}$.

Def. $c_1(\mathfrak{s}) \in H^2(Y)$

$\quad \| $

$\mathfrak{s} - J(\mathfrak{s})$.

Remark: This is well-defined and independent of τ.

This is called the (first) Chern class of $\mathfrak{s}$.

Exercise: Show that $c_1(\overset{[v]}{\mathfrak{s}}) = c_1(v^{\perp})$ or equivalently $e(v^{\perp})$

Euler class

where $v^{\perp}$ is the perpendicular oriented 2-plane bundle to v.

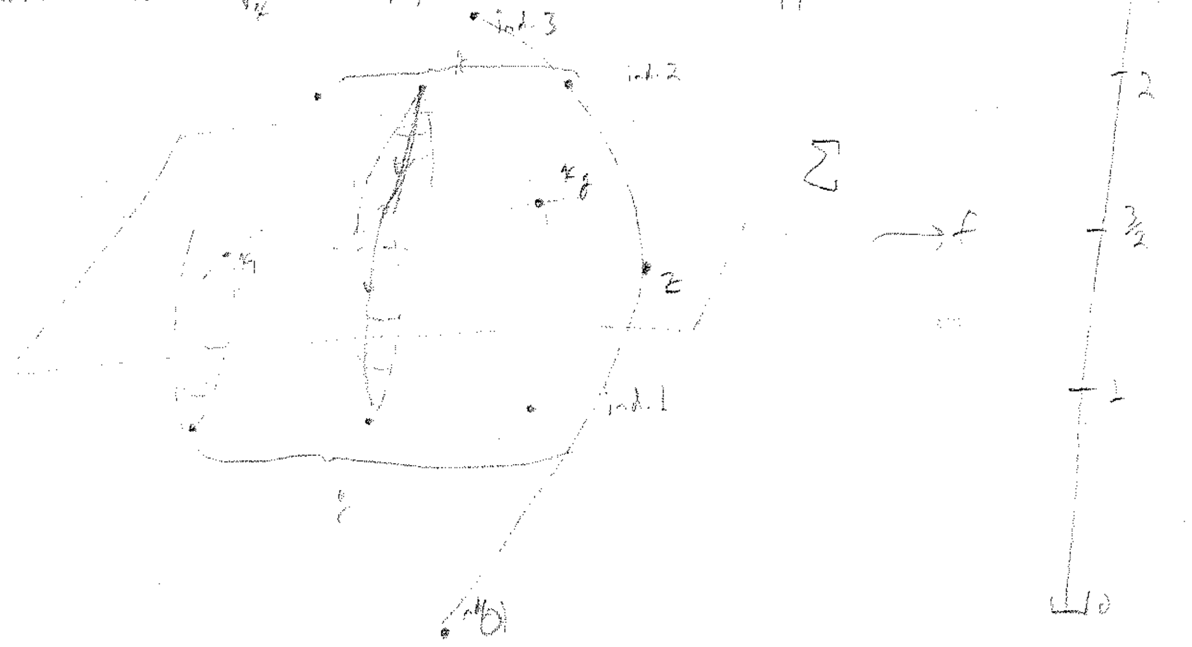
10/21/10

We've seen that $CF^0(Y) = \bigoplus_{\alpha \in H^1(Y)} CF^0(Y, \alpha)$, moreover nat'lly

but why should this splitting have anything to do with . Spin$^c(Y)$?

Given $\vec{x} = \{x_1, \ldots, x_g\} \in \{\alpha_1 \cap \beta_{\sigma(1)}, \ldots, \alpha_g \cap \beta_{\sigma(g)}\}$

We want to produce a nvvf $V_{\vec{x}}$ from $\vec{x}$, well-defined up to homology.



Start w/ $-\nabla f$. This is not nowhere vanishing.

From $\{x_1, \ldots, x_g\}$, we get a g-tuple of flow lines $\{\gamma_{x_1}, \ldots, \gamma_{x_g}\}$

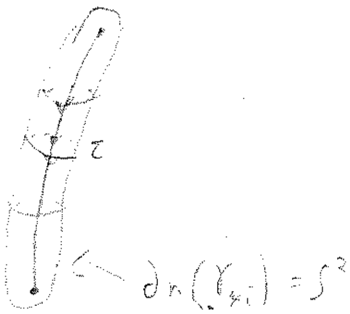
Every pt. away from $\vec{\alpha}, \vec{\beta}$ has a unique flow line passing through it connecting the ind(3) critical pt to the ind(0) crit. pt.

To $z \rightsquigarrow$ flow line $\{\gamma_z\}$

Note: $-\nabla f$ is nowhere vanishing on $Y - n(\gamma_{\vec{x}} \cup \gamma_z)$

But we want a n.v.v.f. on all of Y?

Fortunately, we can extend $-\nabla f|_{\partial n(\gamma_{x_i})}$ to $n(\gamma_{x_i})$ as a n.v.v.f.



The obstruction to extending $-\nabla f|_{S^2}$ to B^3 is its degree:

$$\phi = -\nabla f : S^2 \longrightarrow S^2$$
$$x \longmapsto -\nabla f(x) / \|\nabla f(x)\|$$

But if $-\nabla f|_{S^2}$ extends to some v.f. on B^3, (e.g. $-\nabla f$),

then $$\deg\left(-\nabla f|_{S^2}\right) = \sum_{\substack{x \in B^3 \\ s.t.\ -\nabla f(x) = 0}} \deg\left(-\nabla f_x\right) = (-1)^{\mathrm{ind}_x f}\ (-1)\ (☆)$$

Since we are considering flows connecting ind(1) & ind(2) (resp. ind(0) & ind(3)),

critical pts. ☆ $\Rightarrow$ $\deg -\nabla f|_{\partial n(\gamma_{x_i})} = 0$ and $\deg -\nabla f|_{\partial n(\gamma_z)} = 0$.

Let $s_z(\vec{x}) \in \mathrm{Spin}^c(Y)$ denote $[V^z_{\vec{x}}]$, the homology class

on any extension $V^z_{\vec{x}}$ of $-\nabla f$ to $n(\gamma_{\vec{x}} \cup \gamma_z)$.

<u>Natural Questions!</u> (1) How does this depend on z?

(2) Why is splitting along $H^2(Y)$ related to splitting along $\mathrm{Spin}^c(Y)$ of $CF^\circ(Y)$?

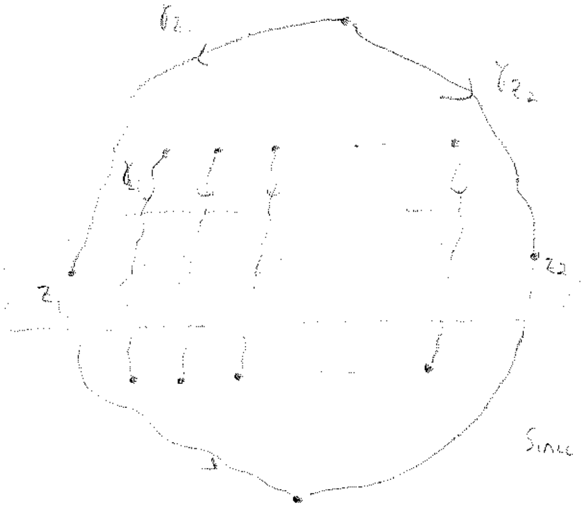
<u>Prop.</u> Let $z_1, z_2 \in \Sigma \setminus \{\vec{\alpha} \cup \vec{\beta}\}$ be two basepoints. Then $s_{z_1}(\vec{x}) - s_{z_2}(\vec{x}) = PD[\gamma_{z_1} \cup -\gamma_{z_2}] \in H^2(Y)$.

Furthermore, let $\vec{x}, \vec{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. Then $s_z(\vec{x}) - s_z(\vec{y}) = PD[\varepsilon(\vec{x}, \vec{y})] = PD[\gamma_1 \cup \gamma_2]$.

<u>Cor.</u> $s_z(\vec{x}) = s_z(\vec{y}) \Leftrightarrow \varepsilon(\vec{x}, \vec{y}) = 0 \Leftrightarrow$ [illegible]

10/21/10, p 5

Pf.



Pick $\tau : TY \to Y \times \mathbb{R}^3$ OAFA

$$\left(\phi^{\tau}_{V^{z_1}_{\vec{x}}}\right)^{*}(m) - \left(\phi^{\tau}_{V^{z_2}_{\vec{x}}}\right)^{*}(m) = ?$$

Since $V^{z_1}_{\vec{x}}$ & $V^{z_2}_{\vec{x}}$ agree outside $n(\gamma_{z_1} \cup \gamma_{z_2})$,

$S_{z_1}(\vec{x}) - S_{z_2}(\vec{x})$ is represented by a cochain with support $n(\gamma_{z_1} \cup \gamma_{z_2})$

$$S_{z_1}(\vec{x}) - S_{z_2}(\vec{x}) \in H^2_c\Big(\underbrace{\overset{\text{small}}{n}(\gamma_{z_1} \cup -\gamma_{z_2})}_{\overset{\cong}{S^1 \times D^2}}\Big)$$

$$H^2_c(n(\gamma_{z_1} \cup -\gamma_{z_2})) \overset{\text{P.D.}}{\cong} H_1(S^1 \times D^2) \cong \mathbb{Z}\langle \gamma_{z_1} \cup -\gamma_{z_2} \rangle$$

No class on [illegible]

Matt Hedden   HFH   10/28/10

$$\tau_i : TY \longrightarrow Y \times \mathbb{R}^3$$

Prop. $(\phi_v^{\tau_1})^* \mu - (\phi_v^{\tau_2})^* \mu = (\phi^{12})^* w$

$\mu \in H^2(S^2)$ generator

$\phi_v^{\tau_i} : Y \to S^2$ is map associated to τ_i, v, $[v]$ n.v.v.f.

$w \in H^2(SO(3))$ $\phi^{12} : Y \to SO(3)$

Pf. $\phi_v^{\tau_1} : Y \to SO(3) \times S^2 \xrightarrow{ev.} S^2$

$$p \longmapsto \left(Id, \frac{\tau_1(v_p)}{\|\tau_1(v_p)\|}\right) \longrightarrow Id.\left(\frac{\tau_1(v_p)}{\|\tau_1(v_p)\|}\right)$$

$\phi_v^{\tau_2} : Y \xrightarrow{(\phi^{12}, \phi_v^{\tau_1})} SO(3) \times S^2 \xrightarrow{ev.} S^2$

$$p \longmapsto \left(\phi^{12}(p), \frac{\tau_1(v_p)}{\|\tau_1(v_p)\|}\right) \longmapsto \phi^{12}(p)\left(\frac{\tau_1(v_p)}{\|\tau_1(v_p)\|}\right) = \frac{\tau_2(v_p)}{\|\tau_2(v_p)\|}$$

$$(\phi_v^{\tau_1})^* \mu - (\phi_v^{\tau_2})^* \mu$$

$$\| \quad (\phi^{12}, \phi_v^{\tau_1})^* \circ \underbrace{(ev^*)\mu}$$

$=$ $\|$ ⟵ Remains to show

(w, μ)

WTS: $SO(3) \times S^2 \xrightarrow{eval} S^2$

$ev^*(\mu) = (w, \mu)$.

Check: $S^2 \longrightarrow SO(3) \times S^2 \longrightarrow S^2$

$$x \longmapsto (Id, x) \longmapsto x$$

$$\pi_{S^2}(ev^*(\mu)) = i_{Id}^* \, ev^* \mu = \mu$$

π - projection $SO^3 \xrightarrow{i} SO(3) \times S^2 \xrightarrow{ev.} S^2$

$$A \longrightarrow (A, \text{north pole}) \longrightarrow (A(\text{north pole}))$$

$SO(3) \times S^2 \xrightarrow{ev.} S^2$ $S^2 \cong SO(3)/\mathrm{Stab}_{\text{north}}$ $\cong SO(2)$

$$\begin{tikzcd} SO(2) \arrow[r] & SO(3) \arrow[d, "\pi"] \\ & SO(3)/SO(2) \cong S^2 \end{tikzcd}$$

Since $H^2(SO(3), \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$, it suffices to show that $\pi^* : H^2(S^2) \to H^2(SO(3))$

is non-trivial.

Gysin Sequence

$$\cdots \to H^{*+1}(SO(3)) \to H^{*}(S^2) \xrightarrow{\cup e(\text{bundle})} H^{*+2}(S^2) \xrightarrow{\pi^*} H^{*+2}(SO(3)) \to \cdots$$

$$H^2(S^2) \xrightarrow{\pi^*} H^2(SO(3)) \longrightarrow \underset{\overset{\shortparallel}{0}}{H^1(S^2)} \to$$

$\Rightarrow$ π^* is surjective.

Recall: $s_z(-) : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to \mathrm{Spin}^c(Y)$

(Modify $-\nabla f$ for a Morse function defining $(\Sigma, \vec{\alpha}, \vec{\beta})$
in a nbhd. of gradient flows connecting index 2 to index 1 crit. pts.
through $x_i \in \vec{x}$, & index 3 to index 0 through z.)

Prop. $\cdot$ $s_{z_1}(\vec{x}) - s_{z_2}(\vec{x}) = PD[\gamma_{z_1} \cup -\gamma_{z_2}]$

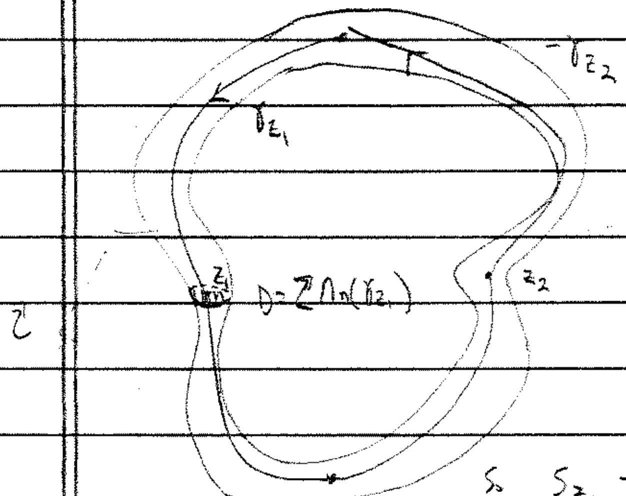
$\cdot$ $s_z(\vec{x}) - s_z(\vec{y}) = PD[\varepsilon(\vec{x}, \vec{y})]$

Pf. First Step : Observe that $s_{z_1}(\vec{x}) - s_{z_2}(\vec{x}) = \left(\phi^{\tau}_{v^{z_1}_{\vec{x}}} - \phi^{\tau}_{v^{z_2}_{\vec{x}}}\right)^* \mu \in H^2_c(\text{nbhd}(\gamma_{z_1} \cup -\gamma_{z_2}))$

$$\cong H^2(\text{nbhd}(\gamma_{z_1} \cup -\gamma_{z_2}), \partial; \mathbb{Z})$$

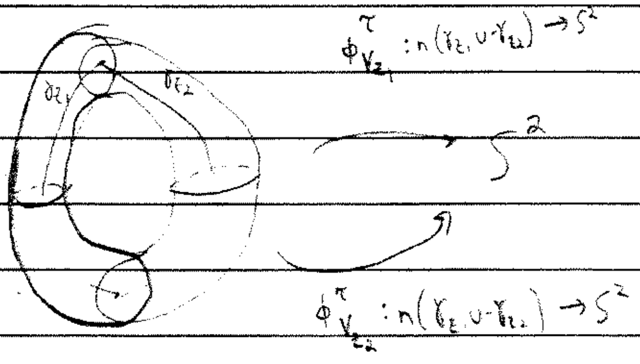
$$\cong \mathbb{Z}\langle c \rangle \qquad \cong ~~H_1(\text{nbd}(\gamma_{z_1} \cup -\gamma_{z_2}); \mathbb{Z})~~ \cong \mathbb{Z}\langle \gamma_{z_1} \cup -\gamma_{z_2} \rangle$$

where $\langle c, \underset{\underset{\underset{\mathbb{Z}}{\cong}}{H_2(S^1 \times D^2, \partial)}}{[D, \partial]} \rangle = +1$

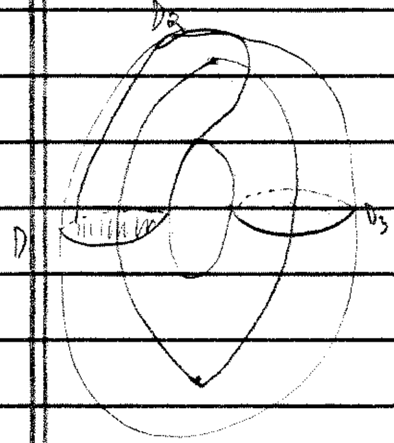


So $s_{z_1} - s_{z_2} = k\, PD[\gamma_{z_1} \cup -\gamma_{z_2}]$ for some $k \in \mathbb{Z}$.

So our class $\langle (s_{z_1} - s_{z_2})^* \mu = kc, [D, \partial D] \rangle = k$.



$$\left\langle \left(\phi^{\tau}_{v_{z_1}}\right)^*(M) - \left(\phi_{v_{z_2}}^{\tau}\right)^*(M),\ [D, \partial D] \right\rangle$$



Claim $\underbrace{\deg \Phi_{v_1}\big|_{D_1}}_{\|} - \underbrace{\deg \Phi_{v_2}\big|_{D_1}}_{\|} = \left\langle (\phi_{v_1}^* - \phi_{v_2}^*)_M, [D_1, \partial D_1] \right\rangle$

0 because v_2 is constant on D_1.

±1 because

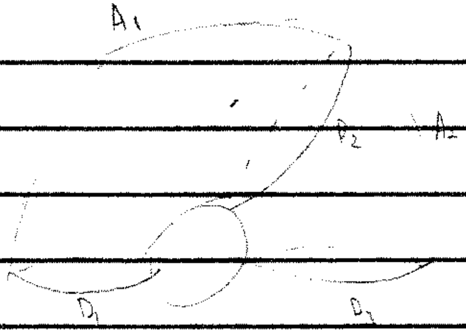
$\deg \Phi_{v_1}\big|_{D_1} - \deg \Phi_{v_1}\big|_{D_2} = -1$ because $D_1 \cup D_2$ surrounds one odd index critical pt., and that's the obstruction to extending $-\nabla f$ over the interior of $D_1 \cup D_2$

(Euler-Hopf)

and $\deg \Phi_{v_1}\big|_{D_2} = 0$ because $\deg \Phi_{v_1}\big|_{D_2} = \deg \Phi_{v_2}\big|_{D_3} = 0$.

Make-up class on M @ 4:10

$$\deg_D v = \frac{1}{2} \deg v \cup \bar{v} \left(D^2 \cup \bar{D^2} \to S^2\right)$$



$\deg v_1\big|_{D_1 \cup A_1 \cup D_2} = \deg v_1\big|_{D_1 \cup A_1 \cup A_2 \cup D_3} = -1$

$\deg v_1\big|_{D_1} - \deg v_2\big|_{D_1} =$

$\deg v_1\big|_{D_1} + \deg v_1\big|_{A_1} + \deg v_1\big|_{D_2} - \deg v_1\big|_{A_1} - \deg v_1\big|_{D_2}$

$- \deg v_2\big|_{D_1}$

=

$$\deg_{D_1}(\Phi_{v_1}) - \deg_{D_1}\overset{0}{(\Phi_{v_2})}$$
$$= -\deg_{D_2}(\Phi_{v_1}) - \deg_{D_1}(\psi_{v_2})$$
$$= -\deg_{D_2}(-\nabla f) - \deg_{D_1}(-\nabla f) = \pm 1$$

Matt Hedden HFH 11/1/10

What is Euler characteristic for $\widehat{HF}$?

Def. $\sum_{i \in \mathbb{Z}} (-1)^i \operatorname{rank}(H_i(X)) = \chi(X)$.

$\}$ Exercise

$\sum_{i \in \mathbb{Z}} (-1)^i \operatorname{rank}(C_i(X)) =$

What about grading ?

Given $\vec{x}, \vec{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, what is the grading difference $gr(\vec{x}) - gr(\vec{y})$?

Hope: $gr(\vec{x}) - gr(\vec{y}) = \mu(\phi)$

$\nwarrow$ Maslov index

$\mathbb{Z}\langle\Sigma\rangle$ $\ker(\mathrm{Span}\,\alpha + \mathrm{Span}\,\beta \to H_1(\Sigma))$

μ is only defined for a homotopy class of Whitney disks, but $\phi \in \pi_2(\vec{x}, \vec{y}) \cong \mathbb{Z} \oplus H_2(Y)$,

which is large.

$$gr(\vec{x}) - gr(\vec{y}) = \mu(\phi' = \phi * S)$$
$$= \mu(\phi) + 2. \quad \text{Not well-defined.}$$

Try: $gr(\vec{x}) - gr(\vec{y}) := \mu(\phi) - 2n_z(\phi)$.

Then $\mu(\phi') - 2n_z(\phi') = \mu(\phi) + 2 - 2n_z(\phi) - 2n_z(S)$

$$= \mu(\phi) - 2n_z(\phi) + 2 - 2$$
$$= \mu(\phi) - 2n_z(\phi).$$

What about the $H_2(Y)$ factor ?

Prop. Sps. $\phi \in \pi_2(\vec{x}, \vec{y})$ is a Whitney disk + $P \in \pi_2(\vec{x}, \vec{x}) \cong \mathbb{Z} \oplus H_2(Y)$ is a Whitney disk satisfying $n_z(P) = 0$.

Then, $\mu(\phi * P) = \mu(\phi) + \langle \underset{\overset{\uparrow}{H^2(Y)}}{\underline{c_1(s_z(\vec{x}))}}, \underset{\overset{\uparrow}{H_2(Y)}}{P} \rangle$

Def. A periodic domain is an element P of $\mathbb{Z}\langle \Sigma \setminus \{\vec{\alpha} \cup \vec{\beta}\} \rangle$ s.t. $\partial P = \sum n_i \alpha_i + \sum m_i \beta_i$.

Recall, for each periodic domain, we obtain a class $[P] \in H_2(Y)$ by capping off α + β w/ disks.

They bound, by construction

$$\underset{\overset{\uparrow}{H_2(Y)}}{\alpha} \longrightarrow \ker\{\mathrm{Span}\,\vec{\alpha} + \mathrm{Span}\,\vec{\beta} \longrightarrow H_1(\Sigma)\}$$

__Prop.__ If P is a periodic domain, then

$$\langle c_1(s_z(\vec{x})), [P] \rangle = \hat{\chi}(P) + 2n_{\vec{x}}(P).$$

But recall, $\mu(\phi) = \hat{\chi}(D(\phi)) + n_{\vec{x}}(D(\phi)) + n_{\vec{y}}(D(\phi))$

for $\phi \in \pi_2(\vec{x}, \vec{y})$.

So $n_{\vec{x}}(P) = \mu(P)$, which now proves the first Prop.

Proof of __this__ Prop. is similar to proofs we've been doing recently.

See "Holomorphic Disks & 3-Manfld. Invariants: Properties & Applications"

Section 7?, Adjunction Inequalities O-S.

__Note:__ If $c_1(s_z(\vec{x}))$ is __not__ torsion in $H^2(Y)$,

then $gr(\vec{x}) - gr(\vec{y})$ is __not__ unchanged under splicing.

__Cor.__ Suppose $c_1(s_z(\vec{x}))$ is torsion. Then

$$gr(\circ) - gr(\bullet) = \mu(\phi) - 2n_z(\phi), \quad \phi \in \pi_2(\circ, \bullet)$$

s.t. $s_z(\circ) = s_z(\bullet) = s_z(\vec{x})$ is well-defined.

__Pf.__ If $\phi, \phi' \in \pi_2(\circ, \bullet)$, then $\phi = \phi' * nS * \sum m_i [P_i]$,

where $\{[P_i]\}_{i \in I}$ is a basis for $H_2(Y)$.

Then $\mu(\phi) - 2n_z(\phi) = \mu(\phi' * nS * \sum m_i [P_i]) - 2n_z(\phi' * nS * \sum m_i [P_i])$

$$= \mu(\phi') + 2n + \sum_i \langle c_1(s_z(\circ)), [P_i] \rangle m_i$$

$$- 2n_z(\phi') - 2n - 2 \cdot 0$$

If $c_1(s_z(\vec{x}))$ has torsion, then

$$\langle c_1(s_z(\circ)), [P_i] \rangle = 0.$$

__Prop.__ $\chi(\widehat{HF}(Y, s)) = \begin{cases} \pm 1 & \text{if} \\ 0 & \text{otherwise.} \end{cases}$ ~~$c_1(s)$ is torsion~~ $|H_1(Y)|$ is finite

__Pf.__ $\chi(\widehat{HF}(Y, s)) := \sum_{i \in \mathbb{Z}} (-1)^i \operatorname{rank}(\widehat{CF}_i(Y, s))$

$gr(\circ) - gr(\bullet) = \mu(\phi) - 2n_z(\phi) \pmod 2$ is well-defined.

Because

$\mu(\phi') - 2n_z(\phi') = \mu(\phi) - 2n_z(\phi) - \sum \langle c_1(s_z(\circ)), [P_i] \rangle m_i$

Even numbers.

Claim 1. $\chi(\widehat{HF}(Y,s))$ is independent of s.

Pf. Vary the basepoint.

$$s_{z_1}(\vec{x}) - s_{z_2}(\vec{x}) = PD[\gamma_{z_1} \cup \gamma_{z_2}]$$

$\Rightarrow$ We can change which spin^c-structure we associate to $\vec{x}$.

+ $s_{z_1}(\vec{y}) - s_{z_1}(\vec{x}) = 0 \Leftrightarrow s_{z_2}(\vec{y}) - s_{z_1}(\vec{x}) = 0.$

$$\widehat{CF}(Y) = \bigoplus_{s \in Spin^c(Y)} \widehat{CF}(Y,s)$$

z_2	z_1	$(s_{z_1} - s_{z_2} = PD[\delta]$
$s_2 = s_1 - \delta$	s_1	$\vec{x}_1, \vec{x}_2, \vec{x}_3$
$s_1 = s_2 - \delta$	s_2	$\vec{y}_1, \vec{y}_2, \vec{y}_3, \vec{y}_4, \vec{y}_5$
$s_3 = s_3 - \delta$	s_3	$\vec{r}$

Note that $gr_{z_1}(\circ) - gr_{z_1}(\bullet) = gr_{z_2}(\circ) - gr_{z_2}(\bullet) \pmod 2$

So, changing basepoint simultaneously changes the Spin^c-structure associated to a summand in $\bigoplus_s \widehat{CF}(Y,s)$,

while leaving $\chi(CF(Y,s))$ invariant.

Claim 1 follows if we can realize every Spin^c-structure by a single summand via varying the basepoint.

If $|H_1(Y)| < \infty$, this is possible. (Exercise.)

If $|H_1(Y)| = \infty$, then we can do this by varying $\vec{\alpha}$ & $\vec{\beta}$ curves.

Now, we want to show that $\chi(\widehat{HF}(Y,s)) = \pm 1$ if $|H_1(Y)| < \infty$.

$$gr(\circ) - gr(\bullet) = \mu(\phi) \pmod 2$$

$$= \hat{\chi}(D(\phi)) + n_{\vec{x}}(D(\phi)) + n_{\vec{y}}(D(\phi))$$

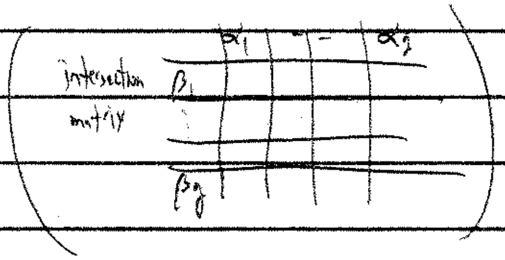
$$= \underset{\text{Alg.}}{\#}(\mathbb{T}_\alpha \cap \mathbb{T}_\beta \text{ at } \vec{x}) - \underset{\text{Alg.}}{\#}(\mathbb{T}_\alpha \cap \mathbb{T}_\beta \text{ at } \vec{y})$$

$\rightsquigarrow$ Orient $\vec{\alpha}$ & $\vec{\beta}$ curves, & orient Σ.

$$\underset{\text{Alg.}}{\#}(\mathbb{T}_\alpha \cap \mathbb{T}_\beta \text{ at } \vec{x}) = \prod_{i=1}^{g}\left((-1)^{\text{Alg. int}_{x_i} \alpha_i \cap \beta_{\sigma(i)}}\right)\underset{(-1)}{\text{sign}(\sigma)}$$

Now note that the signed number of intersection pts. $\vec{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ equals $|H_1(Y)|$.

det



"I have p things, and I divide them into p groups. There better be the same number (± 1) in each group."

If $|H_1(Y)| = \infty$, then

$\chi(\widehat{HF}(Y,s))$ is independent of s.

Finite # pts. in Heegaard digram...