Matt Hedden HFH 3/15/11

Defined Knot Floer homology & Produced

Thm. (Large Framed surgery Formula) Expresses $HF^{\circ}(Y_N(K))$ in terms of homology of sub-, quotient-, and subquotient- complexes of $CFK^{\infty}(Y,K) \leftarrow \mathbb{Z}\oplus\mathbb{Z}$

Surgery Exact Triangle: $\exists$ l.e.s.

$$\cdots \to HFK^{\circ}(Y_{\infty}(K)) \to HFK^{\circ}(Y_0(K)) \to HFK^{\circ}(Y_1(K)) \to \cdots$$

$\exists$ a surgery formula for all framings. To prove it, we need a different surgery l.e.s.

Thm. $\exists$ l.e.s. $\forall n,m \geq 0$, $K \subseteq Y$ knot

$$\bigoplus_{i=1}^{n} HF^{\circ}(Y) \longrightarrow HF^{\circ}(Y_m(K)) \to HF^{\circ}(Y_{n+m}(K))$$

Rmk: $n=1$ is the previous theorem.

To compute $HF^{\circ}(Y_m)$ for any m, it suffices to know

$$HF^{\circ}(Y_{m+n}(K)) \xrightarrow{F} \bigoplus_{i=1}^{n} HF(Y)$$

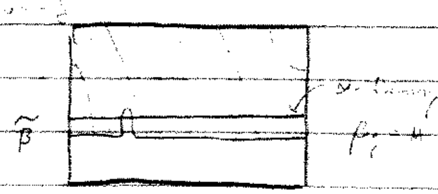
$F_{\overline{-W_{n+m}(K)}}$ natural cobordism map.

Note, for $n+m \gg 0$, we've computed $HF^{\circ}(Y_{n+m}(K))$ and $F_{\overline{-W_{n+m}(K)}}$

Exercise: Prove the general Surgery Exact Triangle Formula.

Hint: Take notes 2 Tuesdays ago and repeat.

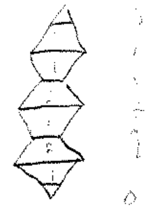
Use the Mapping Cone Lemma



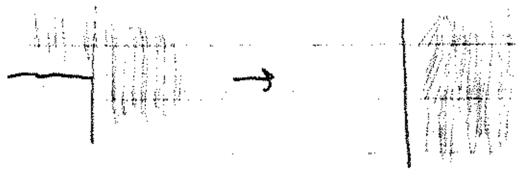
Given $CFK^{\infty}(Y,K)$. For sanity, assume $Y = \mathbb{Z}HS^3$. Interested in $HF^{+}(Y_n(K))$ for any n.

Let $A_i^{+} = C(\max\{i, j-m\} \geq 0)$ (Recall: $H_*(A_m^{+}) \cong HF^{+}(Y_N(K), s_m)$)

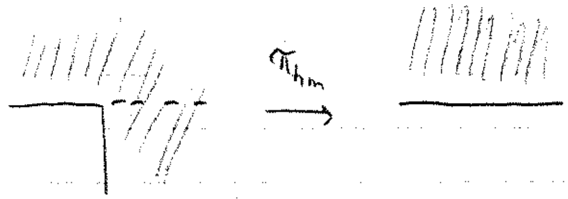
Let $B_m^{+} = C(i \geq 0)$ (Recall: $H_*(B_m^{+}) \cong HF^{+}(Y)$)



Let $v_m : A_m^+ \to B_m^+$ be the vertical projection map.
$\quad\;\; \pi_v$ ″



Let $h_m = \iota \circ \pi_{h_m}$ where $h_m : A_m^+ \to B_{m+n}^+$ is



π_{h_m} is the projection onto $C(j-m \geq 0)$

<u>Note:</u> $C(j-m \geq 0)$ is a complex for $HF^+(Y)$,
since it is a complex defined by a H.D. for Y,

$$(\Sigma, \vec{\alpha}, \vec{\beta}, z)$$

Not w, the other basepoint, on the other side of the meridian.

Let $\iota : C(j-m \geq 0) \to C(i \geq 0)$
be a chain homotopy equivalence induced by change of basepoint.

Let $\vec{A} = \bigoplus_{m=-\infty}^{\infty} A_m^+$, $\quad \vec{B} = \bigoplus_{m=-\infty}^{\infty} B_m^+$

Define $\vec{A} \xrightarrow{\vec{v}+\vec{h}} \vec{B}$ by $v\big|_{A_m} = v_m$, $\quad h\big|_{A_m} = h_m$.

<u>Thm. (O-S)</u> Given $K \subseteq Y$, then

$$HF^+(Y_n(K)) \cong H_*(M(\vec{v}+\vec{h})) \qquad \forall n \in \mathbb{Z}.$$

mapping cone of

<u>Pf.</u> Use general surgery exact sequence as $N \to \infty$.

$$\cdots \to \bigoplus_{i=1}^{N} HF^+(Y) \to HF^+(Y_n(K)) \to HF^+(Y_{n+N}(K)) \to \cdots$$

If $N \gg 0$, then Large N surgery formula produces $HF^+(Y_{n+N}(K), s_m)$

$$\cong$$

$$H_*(A_m^+).$$

Now, the $\vec{v}$'s are the maps associated to 2-handle cobordism $W_{n+N}(K)$ in Spin^c-structure t_m, i.e. unique Spin^c-structure s.t.

$$\langle c_1(t_m), [\hat{S}] \rangle + (n+N) = 2m.$$

And $h_m \equiv$ Map induced by $t_m + PD[\text{core of 2-handle}]$

Similar proof to the corresponding statement for V_m.

Grading argument tells us that although we need to compute $\bigoplus_{i=-\infty}^{\infty} F^+_{W_{n+N}(K), t_i}$

the lowest order part of the map comes from v^+ + h^+.

What does this mean?

Recall from last time, the grading shift formula:

$$gr(F_{W,t}(\xi)) - gr(\xi) = \frac{c_1^2(t) - 2\chi - 3\sigma}{4}$$

What does $c_1^2(t)$ mean?

$$H^2(W) \otimes H^2(W) \xrightarrow{\cup} H^4(W) = 0.$$

$$c_1(t) \qquad c_1(t)$$

$$Y_n(K) \not\cong \mathbb{Q}HS^3$$

$$H^2(W, \partial) \longrightarrow H^2(W) \longrightarrow H^2(\partial)$$

$$c_1(t) \mapsto c_1(t|_\partial)$$

$$\Rightarrow L \cdot c_1(t|_\partial) = 0 \quad \text{for some } L \in \mathbb{Z}$$

$\Rightarrow L \cdot c_1(t)$ lifts to $\widetilde{c_1(t)} \in H^2(W, \partial)$ by exact sequence.

Now, $$H^2(W) \otimes H^2(W, \partial) \xrightarrow{\cup} H^2(W, \partial) \cong \mathbb{Z}$$

$$(c_1(t), \widetilde{c_1(t)}) \longmapsto \frac{\langle c_1(t) \cup \widetilde{L \cdot c_1(t)}, [W, \partial] \rangle}{L} \in \mathbb{Q}.$$

Coming back to: $\mathrm{Spin}^c\left(W^4_{n+N}(K)\right) \ni t_m$, $\quad c_1^2(t_m) = \dfrac{\left(2m-(n+N)\right)^2}{-(n+N)}$

__Exercise__: Show this

__Observe__: As $|m| \to \infty$, $c_1^2 \longrightarrow -\infty$,

and $\quad t_m, \quad t_m + PD[S]$

(the Spin^c-structures we understand)

are those Spin^c-structures which shift degree the least.

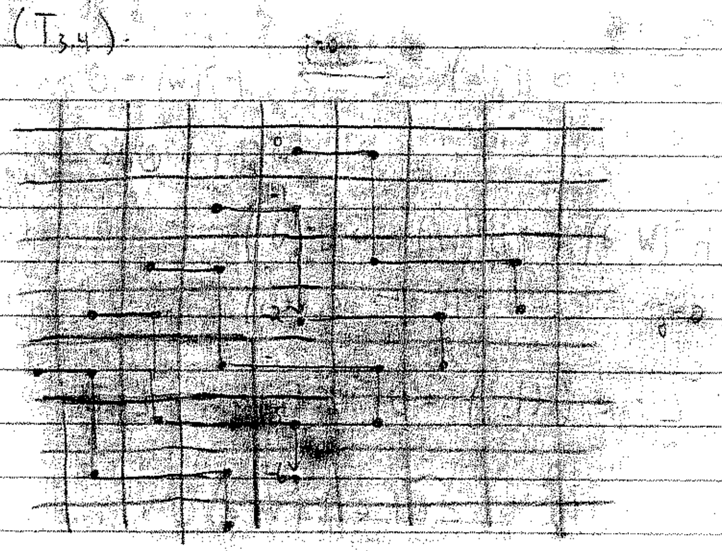
$$\langle c_1(t_\ell), [S] \rangle + (n+N) = 2\ell$$

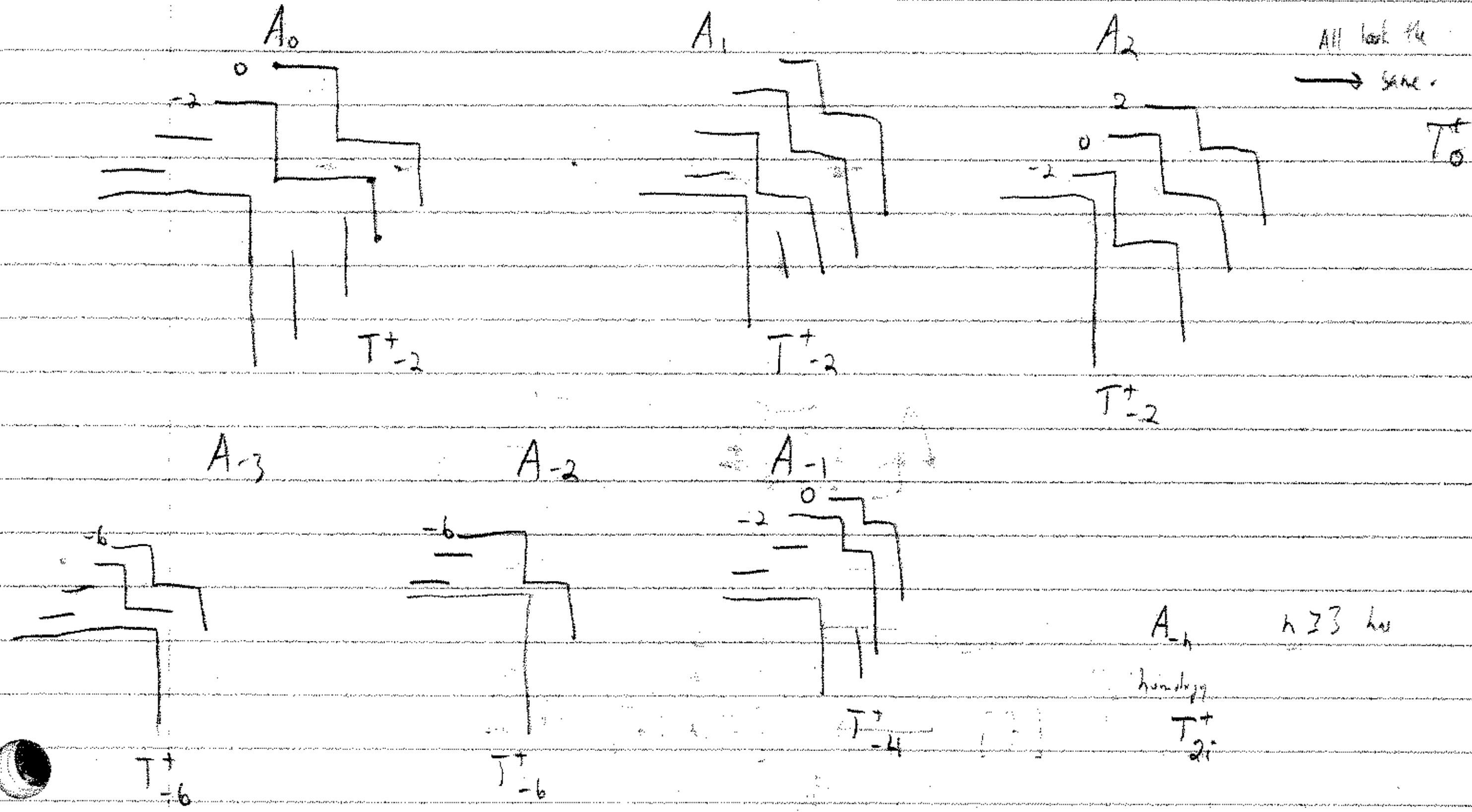
Finally, apply the Filtered Chain Map argument, together with observation that $M(\vec{v} + \vec{h})$ is finitely generated in each degree to yield the Theorem.

__Example__: Surgeries on $T_{3,4}$.

$CFK^\infty(T_{3,4})$.

__Note__: $\widehat{HFK}_*(K,m) \cong \widehat{HFK}_{*-2m}(K,-m)$





$$T^+ \cong \mathbb{Z}[U, U^{-1}] \big/ U\mathbb{Z}[U] \cong HF^+(S^3)$$

T^+_i means U with grading i instead of 0

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Potential Issue with Giroux's Theorem!!

• Proved the existence of Surgery Exact Triangles.

$$\cdots \to HF^{\circ}(Y_\infty) \to HF^{\circ}(Y_0) \to HF^{\circ}(Y_1) \to \cdots$$

$$\cdots \to \bigoplus_{i=0}^{n} HF^{\circ}(Y_\infty) \to HF^{\circ}(Y_m) \to HF^{\circ}(Y_{n+m}) \to \cdots$$

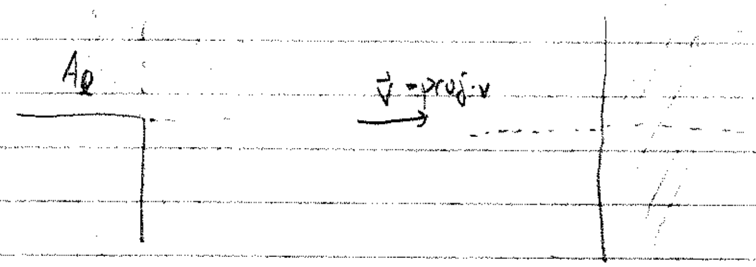
• Used latter surgery sequence to complete the surgery formula for $CFK^{\infty}(Y,K)$.

Thm.
$$M\begin{pmatrix} \bigoplus A_i & \xrightarrow{v+h^m} & \bigoplus B_i \\ \cong & & \cong \\ \bigoplus_{i=-\infty}^{\infty} CF^+(Y_N(K), s_i) & \xrightarrow{v+h^m} & CF^+(Y) \end{pmatrix} \cong HF^+(Y_m(K)) \quad \forall m.$$

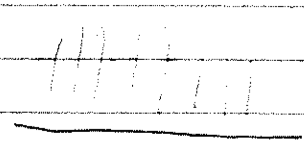
mapping cone

$v_i : A_i \to B_i$

$h_i^m : A_i \to B_{i+m}$



$\downarrow h = \tau \circ \mathrm{proj}_h$



$\tau : C(\{j \geq 0\}) \to C(\{i \geq 0\})$

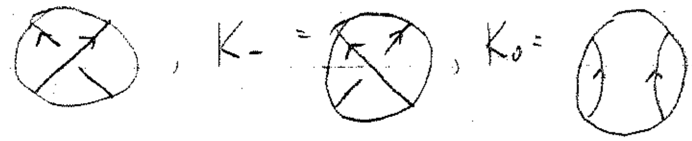
• Computed Example: Surgery on $T_{3,4}$.

As an aside, recall $\Delta_K(T)$ is characterized by

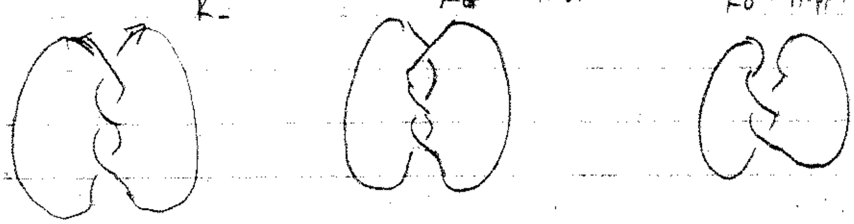
(1) $\Delta_K(1) = 1$ (Normalization)

(2) $\Delta_{K_+}(T) - \Delta_{K_-}(T) = (T^{\frac{1}{2}} - T^{-\frac{1}{2}})\Delta_{K_0}(T)$,

where $K_+ =$, $K_- =$, $K_0 =$



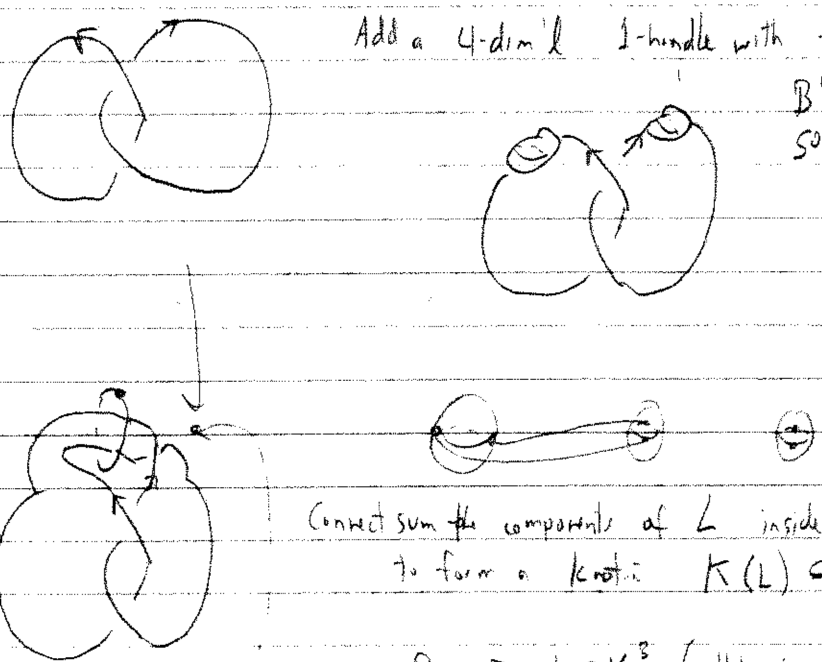
The Skein relation turns into an exact sequence for Knot Floer homology.



To define Knot Floer homology of a link, we "knotify" the link:

Add a 4-dim'l 1-handle with feet on the link.

$B^1 \times B^3$ attached along $S^0 \times B^3$.



Connect sum the components of L inside $I \times S^2$ to form a knot: $K(L) \subseteq S^1 \times S^2$.

(This twist is to preserve orientation, independent of choice of twist.)

<u>Prop.</u> To $L \subseteq Y^3$ (null homologous) link, attaching $|L|-1$ 1 handles to Y^3 along pts in L, we obtain a (null homologous) knot,

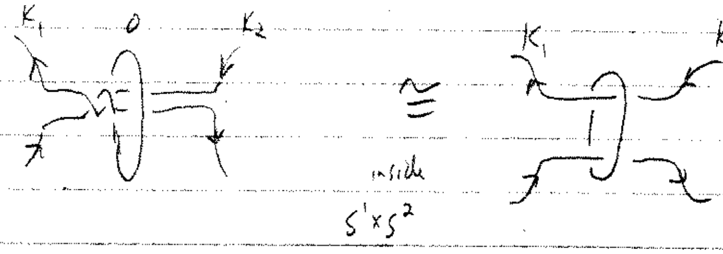
$$K(L) \subseteq Y \#^{|L|-1} S^1 \times S^2.$$

$K(L)$, up to isotopy, is independent of all choices.

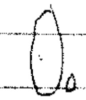
0 $-$

$0 \to \mathbb{F}_0 \to \mathbb{F}_{-\frac{1}{2}} \quad \mathbb{F}_{\frac{1}{2}} \to \mathbb{F}_0 \to 0$

Pf. 4-dim'l. Slide the 1-handles over eachother, dragging the band along, to show $K(L)$ is independent of choice of base points on L.



Because you can handle slide $K(L)$ over



$\square$

Def. $\widehat{HFK}(Y, L) := \widehat{HFK}\left(Y \#^{|L|-1} S^1 \times S^2, K(L)\right)$.

Thm. $\sum_{i \in \mathbb{Z}} \chi\left(\widehat{HFK}(Y, L, i)\right) \cdot T^i = \Delta_L(T) \cdot \left(T^{\frac{1}{2}} - T^{-\frac{1}{2}}\right)^{|L|-1}$

Alexander grading

i.e. $n_z(\phi) - n_w(\phi) = A(\vec{x}) - A(\vec{y})$

Prop. $L \subseteq Y$ a 2-component link.

$$(Y \# S^1 \times S^2)_0(K(L)) \cong (Y \setminus L)/\sim$$

$\sim$ means meridian 1 $\leftrightarrow$ meridian 2
longitude 1 $\leftrightarrow$ longitude 2.

End of Aside.

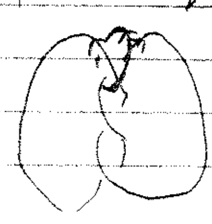
Thm. $\exists$ long exact sequence, for each $i \in \mathbb{Z}$:

$$\cdots \to \widehat{HFK}(K_+, i) \xrightarrow{\hat{F}_2} \widehat{HFK}(K_-, i) \xrightarrow{\hat{F}_0} \widehat{HFK}(K_0, i) \xrightarrow{\hat{F}_1} \cdots,$$

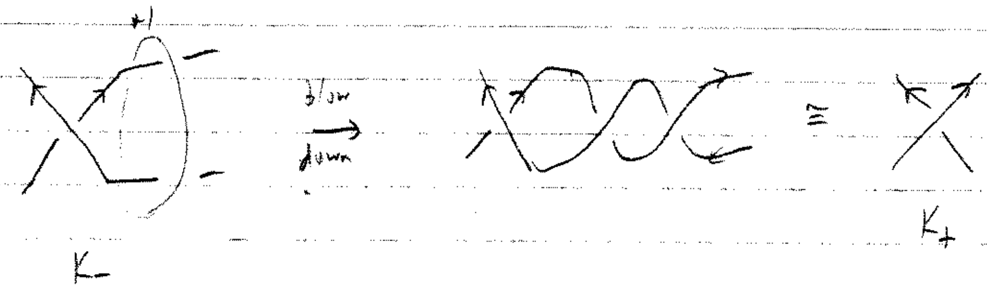
if arcs used for K_+ + K_- belong to the same component, where $\hat{F}_0$ + $\hat{F}_1$ decrease absolute grading by $\frac{1}{2}$, and $\hat{F}_2$ preserves absolute grading.

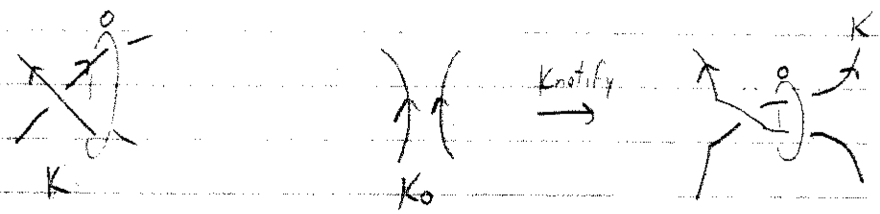
Pf Corollary to Surgery Triangle:

K_-



Consider the surgery sequence for ∞, 0, 1 surgery on the blue unknot, a crossing circle.





So $\quad (Y_\infty, K) \to (Y_0, K) \to (Y_1, K)$

$$\begin{array}{ccccc} (Y_\infty, K) & \to & (Y_0, K) & \to & (Y_1, K) \\ \cong & & \cong & & \cong \\ \cdots \to (Y, K_-) & \to & (Y \# S^1 \times S^2, K(K_0)) & \to & (Y, K_+) \to \cdots \end{array}$$

Existence of Skein Exact Sequence would follow if we know the ordinary Exact Sequence for $\infty, 0, 1$ respected the filtration induced by the knot.

One can show that

<u>Thm.</u> Let $K \cup J$ be a 2-component link with $lk(K, J) = 0$.

Then consider

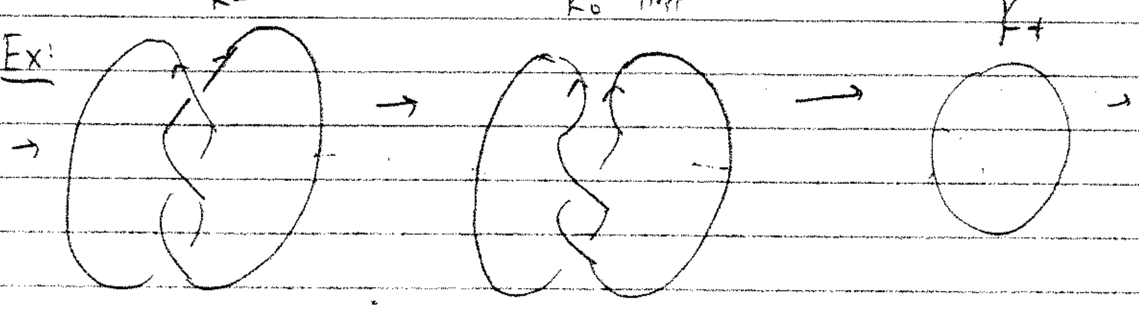
$$F^{\circ}_{W_n(J), t} : CF^{\circ}(Y) \to CF^{\circ}(Y_n(J)).$$

This chain map preserves the Alexander filtration on both sides induced by K.

In the proof $lk(K, J) = 0$ translates to information about how K intersects generator for . . .

K_- $\qquad\qquad$ $K_0 = \text{Hopf}$ $\qquad\qquad$ K_+

Ex:



$\widehat{HFK}(\text{trefoil})$		$\widehat{HFK}(\text{Hopf})$	$\widehat{HFK}(\text{unknot})$		
$\mathbb{F}_{(2)}$	$\xrightarrow{\cong}$	$\mathbb{F}_{(\frac{3}{2})}$	0		$A=1$
$\mathbb{F}_{(1)}$		$\mathbb{F}_{(\frac{1}{2})} \oplus \mathbb{F}_{(\frac{1}{2})}$	$\mathbb{F}_{(0)}$	$\longrightarrow 0$	$A=0$
$\mathbb{F}_{(0)}$	$\xrightarrow{\cong}$	$\mathbb{F}_{(-\frac{1}{2})}$	0	$\cap$	$A=-1$

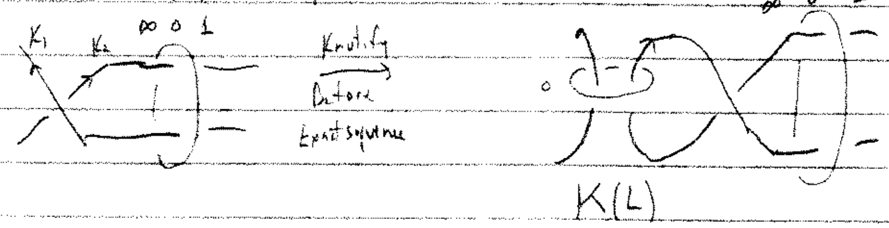
Known $\qquad\qquad\qquad$ Since $\hat{F}_2$ preserves grading.

Note: $\sum \chi(\widehat{HFK}(\text{Hopf}))\cdot T^i =$

$$\pm 1\cdot T^{1} \mp 2\cdot T^{2} \pm 1\cdot T^{-1}$$

$$= \pm\Delta_{\text{Hopf}}(T)\cdot\left(T^{\frac{1}{2}} - T^{-\frac{1}{2}}\right)$$

If the arcs in K_+ are part of different components:

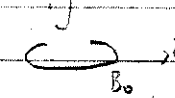


$K(L)$

$$\widehat{HFK}(Y_\infty, K(L)) \to \widehat{HFK}(Y_0, K(L)) \to \widehat{HFK}(Y_1, K(L))$$

$\qquad \| \qquad\qquad L_0 \#(\cancel{B}\ \cancel{2}\ S^1\times S^2) \qquad\qquad \|$

$K(L_-) \qquad\qquad\qquad\qquad\qquad\qquad K(L_+)$

now a knot $\widehat{HFK}\Big($  $B_0, i\Big) \cong$ $\mathbb{F}_1$ ($i=1$), $\mathbb{F}_0\oplus\mathbb{F}_0$ ($i=0$), $\mathbb{F}_{(-1)}$ ($i=-1$)

Endgame:

$$\cdots \to \widehat{HFK}(K_-, i) \to \widehat{HFK}(K_0 \# B, i) \to \widehat{HFK}(K_+, i) \to$$

if strands for K_-, K_+ belong to different components.

Exercise: Use the Skein Exact sequence to show $\sum_i \chi\left(\widehat{HFK}(K(L), i)\right) \cdot T^i$

$$\| $$

$$\Delta_L(T) \cdot \left(T^{\frac{1}{2}} - T^{-\frac{1}{2}}\right)^{|L|-1}.$$

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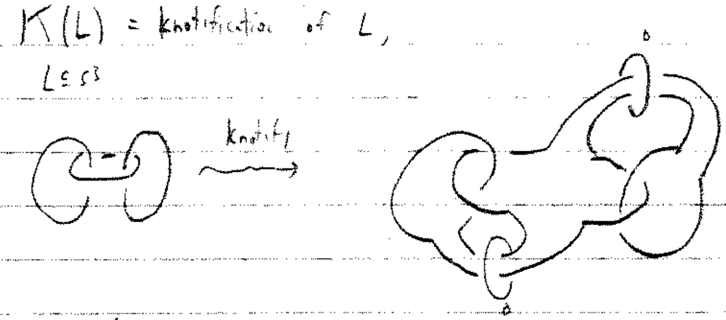
Last Time:

- Defined $\widehat{HFK}(\text{n-component link} \subseteq Y) := \widehat{HFK}(Y \#^{n-1} S^1\times S^2, K(L))$,

where $K(L)$ = knotification of L,

e.g. $L \subseteq S^3$



$K(L) \subseteq \#^2 S^1\times S^2$

- Discussed Thm. $\exists$ l.e.s.

$$\cdots \to \widehat{HFK}(K_-, i) \to \widehat{HFK}(K_+, i) \to \widehat{HFK}(K_0, i) \to \cdots \quad \forall i \in \mathbb{Z}$$

(Alexander grading)

if  strands of crossing belong to same component of K_+ (and K_-).

$$\cdots \to \widehat{HFK}(K_-, i) \to \widehat{HFK}(K_+, i) \to (\widehat{HFK}(K_0) \otimes V, i) \to \cdots$$

where $V \cong \begin{cases} \mathbb{F}_{(1)} & \text{Alex.} = 1 \\ \mathbb{F}^2_{(0)} & \text{Alex.} = 0 \\ \mathbb{F}_{(-1)} & \text{Alex.} = -1 \end{cases}$

if strands belong to different components.

Iteration of surgery exact triangle $\rightsquigarrow$ Spectral sequence.

Cor. $\exists$ spectral sequence whose

$$E_2 \cong Kh^{\text{red.}}(L) \quad + \quad E_\infty \cong \widehat{HF}(\Sigma(L))$$

$\uparrow$ Double-branched cover of S^3 branched over L.

$$\left(\pi_1(S^3 \setminus L) \to H_1(S^3 \setminus L) \cong (\mathbb{Z}\langle\mu\rangle)_{i=1}^{|L|} \to \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \right.$$

This gives $(S^3 \setminus L)_{(2)}$

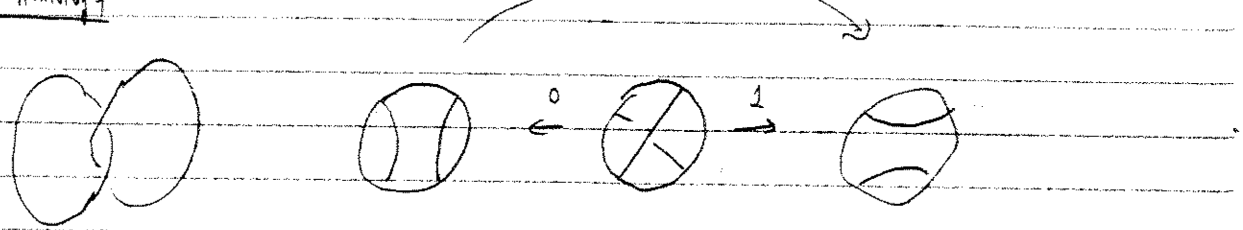
$\downarrow 2:1$ glue $S^1\times D^2$ to $\partial(S^3\setminus L)_2$ so that covering extends over $S^1\times D^2$

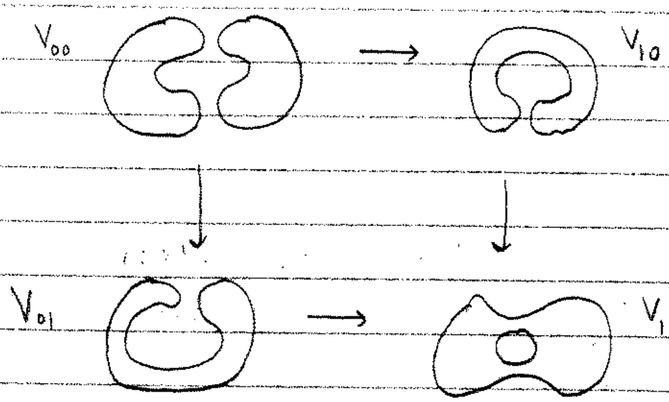
$(S^3 \setminus L)$

$\downarrow (\theta, z^2)$

$S^1\times D^2$ $\Big)$

Khovanov homology





(1+1) dim'l TQFT   i.e. Functor from Category of 1-mnflds. to Algebraic Category

$A = \mathbb{F}[x]/x^2$

$\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$

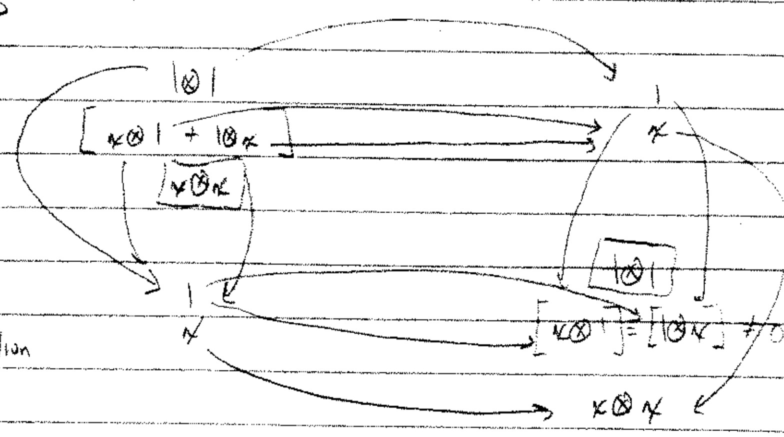
$$\begin{tikzcd} A^{\otimes 2} \arrow[r, "m"] \arrow[d, "m"'] & A \arrow[d, "s"] \\ A \arrow[r, "s"'] & A^{\otimes 2} \end{tikzcd}$$

(m or s depending on whether you merge or split circles)

$$\left.\begin{array}{l} m: A\otimes A \to A \\ 1\otimes x \mapsto x \\ x\otimes 1 \mapsto x \\ 1\otimes 1 \mapsto 1 \\ x\otimes x \mapsto 0 \end{array}\right\} \text{multiplication}$$

and

$$\left.\begin{array}{l} s: A \to A\otimes A \\ 1 \mapsto 1\otimes x + x\otimes 1 \\ x \mapsto x\otimes x \end{array}\right\} \text{comultiplication}$$



Since $\mathbb{F}=\mathbb{Z}/2\mathbb{Z}$, commutativity corresponds to compositions being 0.

So $\left(\sum \text{edge maps}\right)^2 = 0$, i.e. $\partial = \sum$ edge maps is a differential.

Denote this complex by $CKh(D)$ ← Diagram.

<u>Thm.</u> $H_*(CKh(D), \partial)$ is independent of choice of diagram for L.

$$H_*(CKh(D), \partial) \overset{..}{=} Kh(L)$$

Khovanov homology for Hopf link is rank 4.

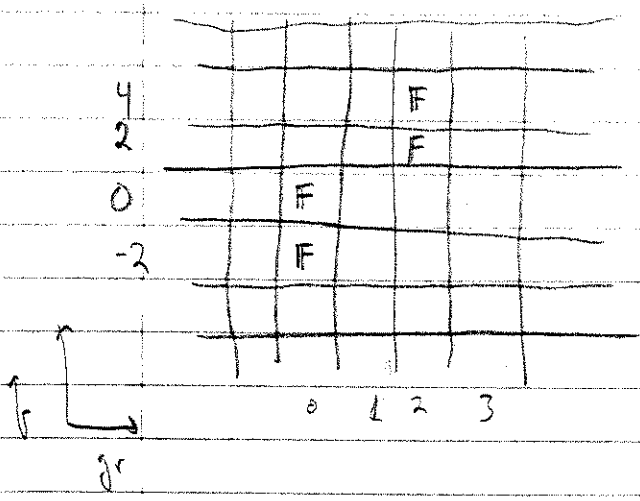
$\exists$ bigrading:

(1) Homological grading $gr(\vec{v}) = \#\{1\text{'s in vertex where } \vec{v} \text{ lives}\}$

(2) Quantum grading $q(\vec{v}) = gr(\vec{v}) + q'(\vec{v})$,

where

$$\left.\begin{array}{ll} q'(1) = 1 & q'(1\otimes 1) = 2 \\ q'(x) = -1 & q'(x\otimes 1) = q'(1\otimes x) = 0 \\ & q'(x\otimes x) = -2 \end{array}\right\} \text{ grading on } A.$$



(Note: Technically, this depends on the writhe of the diagram, though this could be accounted for.)

<u>Why should Khovanov homology have any relation at all to HFH?</u>

<u>Ex:</u> Show $\exists$ l.e.s. relating

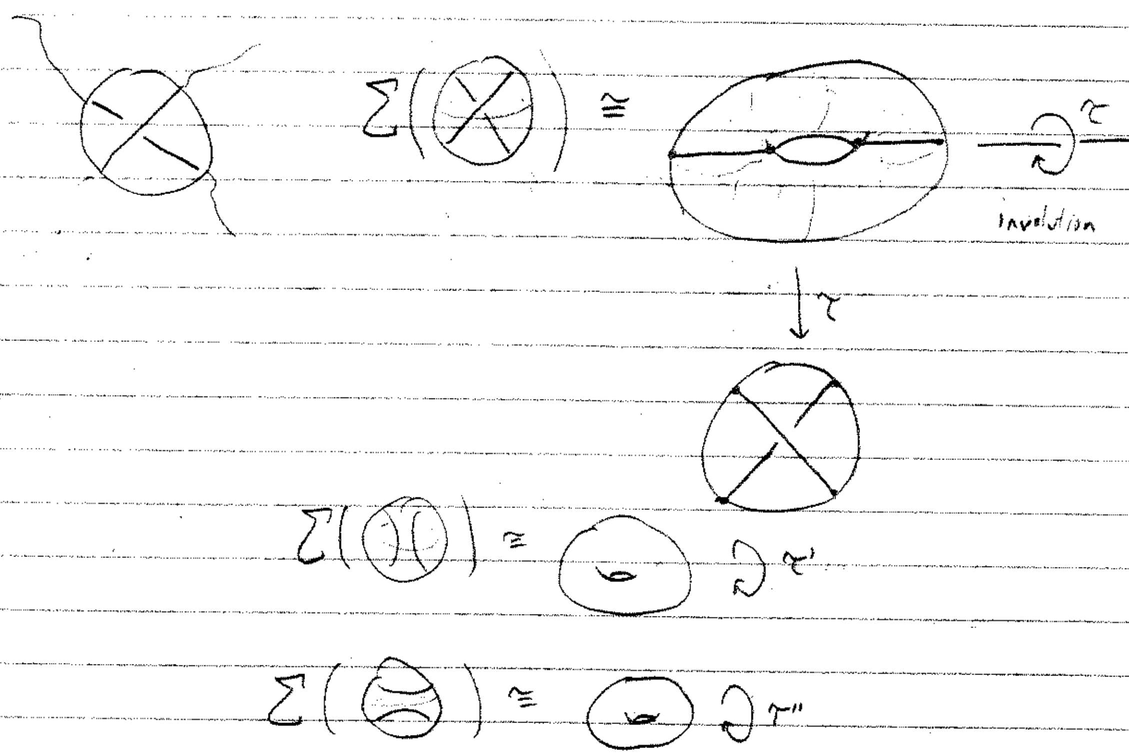
$$\cdots \to Kh(\,)(\,) \to Kh(\asymp) \to Kh(\times) \to \cdots$$

$$\left(0 \to CKh(\asymp) \to CKh(\times) \to CKh(\,)(\,) \to 0\right)$$

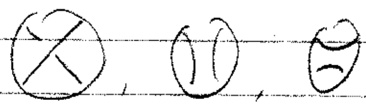
Lemma. Consider L_0, L_1, L differing by the Khovanov Skein move (aka: unoriented Skein relation)

Then $\Sigma(L_0)$, $\Sigma(L_1)$, $\Sigma(L)$ differ by surgery on a knot, & the 3 surgery curves form a triad.

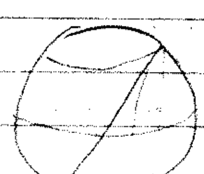
Pf.



Exercise: See that the boundary of a meridian disk for each solid torus obtained from



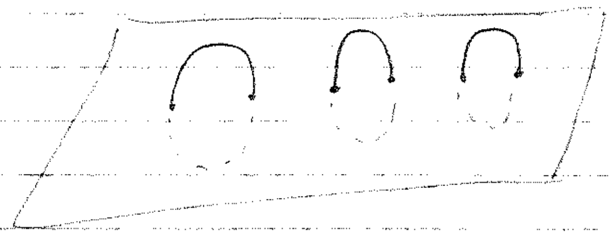
satisfy the triad condition.

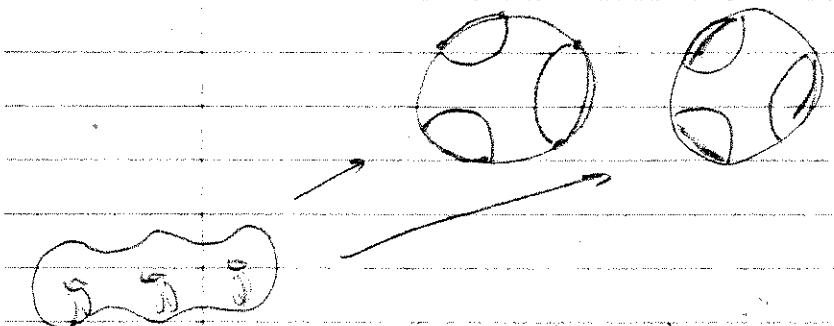


(these arcs, pushed onto the boundary, lift to meridians in the solid torus.)

Lemma. $\Sigma(\underbrace{O \cdots O}_{n}) \cong \#^{n-1} S^1 \times S^2$

Sketch:





$(S^3, L) = (B^3, arc) \cup_{Id} (B^3, arc)$

$\Sigma(B^3, arc) \cong H_{|L-1|}$

↑ Handlebody of genus $|L-1|$

So $\Sigma(S^3, L) \cong H_{|L-1|} \cup_{Id} H_{|L-1|} \cong \#^{|L-1|} S^1 \times S^2$.

Recall: $\widehat{HF}(S^1 \times S^2) \cong \begin{cases} \mathbb{F}_{\frac{1}{2}} \\ \mathbb{F}_{-\frac{1}{2}} \end{cases}$

$\widehat{HF}(\#^2 S^1 \times S^2) \cong \begin{cases} \mathbb{F}_{(1)} \\ \mathbb{F}^2_{(0)} \\ \mathbb{F}_{(-1)} \end{cases}$

Shift the grading down by $\frac{n}{2}$.

$\widehat{HF}_*(\#^n S^1 \times S^2) \cong H_*(T^n)\{-\frac{n}{2}\} \cong Kh^{q'}_{\frac{*}{2}}(\underbrace{O \cdots O}_{n})$

$Kh_0(\underbrace{O \cdots O}_{n-1}) \cong CKh(\underbrace{O \cdots O}_{n}) \cong A^{\otimes n}$

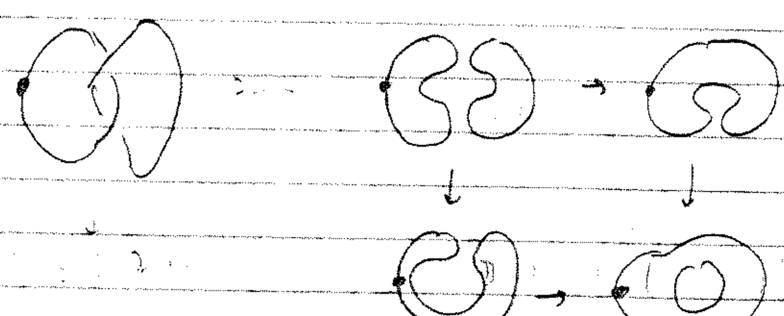
Σ (Khovanov cube for Hopf link)

$$\begin{tikzcd} \Sigma(\bigcirc\bigcirc) \arrow[r] \arrow[d] & \Sigma(\bigcirc) \arrow[d] \arrow[r, phantom, "\cong"] & \#^1 S^1\times S^2 \arrow[r] \arrow[d] & S^3 \; (= \#^0 S^1\times S^2) \\ \Sigma(\bigcirc) \arrow[r] & \Sigma(\bigcirc\bigcirc) & S^3 \arrow[r] & \#^1 S^1\times S^2 \end{tikzcd}$$

HFH

$$\begin{tikzcd} \widetilde{Kh}(\bigcirc\bigcirc) \cong A^{\otimes 1} \arrow[r] \arrow[d] & A^{\otimes 0} \cong \widetilde{Kh}(\bigcirc) \arrow[d] \\ \widetilde{Kh}(\bigcirc\bigcirc) \cong A^{\otimes 0} \arrow[r] & A^{\otimes 1} \cong \widetilde{Kh}(\bigcirc) \end{tikzcd}$$

Reduced Khovanov homology



$1\otimes 1$

$x\otimes 1 \quad 1\otimes x$

$x\otimes x$

x

x

$1\otimes 1$

$x\otimes 1 \quad 1\otimes x$

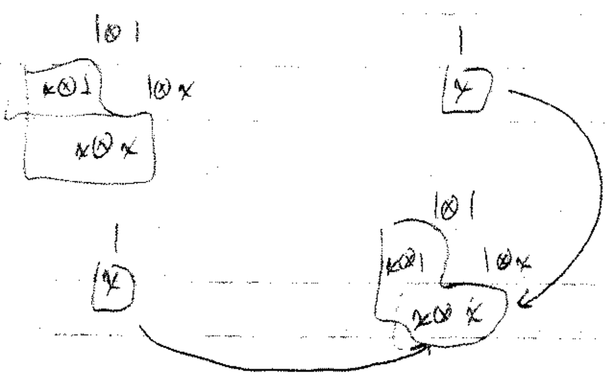
$x\otimes x$

Claim: Picking x in the tensor factor corresponding to the circle containing marked point gives rise to a sub-complex of CKh.

Def. $\widetilde{Kh}(K) := H_*($marked pt. subcomplex$)$.

Recall: The exact triangle provides chain maps from

$$\Sigma(\;\;) \xrightarrow[\Sigma(\;\;)]{} \Sigma(\;\;)$$



$$\widehat{HF}(\Sigma(0)) \cong \widehat{HF}(S^3) \cong \mathbb{F}\langle x\rangle$$

$$\downarrow \widehat{F}_{W_{2\text{-handle}}} \qquad 0$$

$$\widehat{HF}(\Sigma(0\,0)) \cong \widehat{HF}(S^1\times S^2) \cong \mathbb{F}\langle x\otimes 1\rangle$$

$$\mathbb{F}\langle x\otimes x\rangle$$

∞ -1 0

mapping cone

Exercise: Compute exact triangle for $\overset{0}{O}$ $\overset{0}{O}$

$$\begin{tikzcd}
\widehat{HF}(\Sigma(0\,0)) \arrow[r, "\widehat{F}_0"] \arrow[d, "\widehat{G}_0"] \arrow[dr, "H"] & \widehat{HF}(\Sigma(0)) \arrow[r] \arrow[d, "\widehat{G}_1"] & \widehat{HF}(\Sigma(\;\;)) \cong M(\widehat{F}_0) \arrow[d, "M(\widehat{G}_0,\widehat{G}_1)"] \\
\widehat{HF}(\Sigma(0)) \arrow[r, "\widehat{F}_1"] & \widehat{HF}(\Sigma(0\,0)) \arrow[r] & \widehat{HF}(\Sigma(\;\;)) \cong M(\widehat{F}_1) \arrow[d] \\
 & & \widehat{HF}(\Sigma(\;\;)) \cong M(\widehat{G}_0, \widehat{G}_1)
\end{tikzcd}$$

Chain homotopy equivalence, not equality on the nose

Thm. Blue box is a chain complex whose homology computes $\widehat{HF}(\Sigma(L))$

Moreover, the box is a filtered chain complex, filtered by Khovanov grading (cube grading)

E_2 term of the associated spectral sequence
$\cong \widetilde{Kh}(\overline{L})$.