Section 6.8 Indeterminate forms and l’Hospital’s Rule.

If we were to analyze the behavior of the function \( F(x) = \frac{\ln x}{x-1} \),
we would have trouble at \( x = 1 \) \( \left( \frac{0}{0} \right) \). However, we can still investigate \( F \) near 1: what is \( \lim_{x \to 1} \frac{\ln x}{x-1} \)?

This limit has the form \( \frac{0}{0} \); there are no rules (yet) that can help us find this limit.

In general, if \( f(x), g(x) \to 0 \) as \( x \to a \), then \( \lim_{x \to a} \frac{f(x)}{g(x)} \) may or may not exist, and is called an indeterminate form of type \( \frac{0}{0} \).

Note: That we have encountered these forms before:

\[
\lim_{x \to 1} \frac{x^2 - x}{x^2 - 1} = \lim_{x \to 1} \frac{x}{x + 1} = \frac{1}{2}, \quad \lim_{x \to 0} \frac{\sin x}{x}.
\]

But for these we had tricks. Now, we want a method that works all the time: l’Hospital’s Rule.

Another problem is encountered when evaluating \( \lim_{x \to \infty} \frac{\ln(x)}{x-1} \).
In general if \( f(x), g(x) \to \pm \infty \) as \( x \to a \), then
\[
\lim_{x \to a} \frac{f(x)}{g(x)} \text{ has the indeterminate form } \frac{\infty}{\infty}; \text{ the limit may or may not exist. (}a\text{ can be } \pm \infty)\n\]
we have also encountered these forms before:
\[
\lim_{x \to \infty} \frac{3x^3 - x}{2x^3 + 1} = \lim_{x \to \infty} \frac{3 - \frac{1}{x^2}}{2 + \frac{1}{x^3}} = \frac{3 - 0}{2 + 0} = \frac{3}{2}.
\]
Again, this method does not always work. So, we need L’Hospital’s Rule.

L’Hospital’s Rule: Suppose \( f \) and \( g \) are differentiable functions, and \( g'(x) \neq 0 \) on an open interval \( I \) that contains \( a \) (except possibly at \( a \)). Suppose that
\[
\lim_{x \to a} f(x) = 0 \text{ and } \lim_{x \to a} g(x) = 0, \text{ OR } \lim_{x \to a} f(x) = \pm \infty \text{ and } \lim_{x \to a} g(x) = \pm \infty\]
(i.e. we have an indeterminate form of type \( 0/0 \) or \( \infty/\infty \)). Then
\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \text{ if the Right Side limit exists (or is } \infty \text{ or } -\infty).\]
Note 1: you must always check the conditions of L'H. Rule before using it.

Note 2: L'H. Rule is valid for one-sided limits, and for limits at \( \infty \) or \(-\infty\).

Note 3: For a proof of Special Cases of L'H. Rule, see book, or come talk to me :)!

Examples

1. \( \lim_{x \to 1} \frac{\ln x}{x-1} \) has the form \( \frac{0}{0} \).

So, we can apply L'H. Rule:

\[
\lim_{x \to 1} \frac{\ln x}{x-1} = \lim_{x \to 1} \frac{(\ln x)'}{(x-1)'} = \lim_{x \to 1} \frac{1}{x} = 1
\]

2. \( \lim_{x \to \infty} \frac{e^{3x} - 1}{x^2 + x - 1} \) has the form \( \frac{\infty}{\infty} \). Use L'H. Rule

\[
\lim_{x \to \infty} \frac{e^{3x}}{2x + 1} \] also has the form \( \frac{\infty}{\infty} \); Use L'H. Rule again

\[
\lim_{x \to \infty} \frac{9e^{3x}}{2} = \infty. \text{ done.}
\]

3. \( \lim_{x \to \infty} \frac{\ln(2x)}{\sqrt{x}} \) \( \left( \frac{\infty}{\infty} \right) \). L'H. Rule \( \Rightarrow \lim_{x \to \infty} \frac{\frac{1}{2} \cdot 2}{\frac{1}{4} x^{-\frac{3}{2}}} = \lim_{x \to \infty} 4x^{-\frac{1}{2}} = 0 \)

\[
= \lim_{x \to \infty} \frac{4}{x^{\frac{1}{4}}} = 0.
\]
4. \[ \lim_{{x \to 1}} \frac{\sin \left( \frac{\pi}{2} x \right) - 1}{x - 1} \quad \left( \frac{0}{0} \right) \to \text{L'H. Rule.} \]

\[ \lim_{{x \to 1}} \frac{\frac{\pi}{2} \cos \left( \frac{\pi}{2} x \right)}{1} = \frac{\pi}{2} \cdot 0 = 0. \]

Indeterminate Products

This occurs when \( \lim_{{x \to a}} f(x) \cdot g(x) \) has the indeterminate form \( 0 \cdot \infty \).

Example 1. \[ \lim_{{x \to 0^+}} x \cdot \ln x \] has the form \( 0 \cdot \infty \).

The following trick helps with this problem: Rewrite \( x \) as \( \frac{1}{x} \).

\[ \lim_{{x \to 0^+}} x \cdot \ln x = \lim_{{x \to 0^+}} \frac{\ln x}{\frac{1}{x}} \left( \text{has the form } \frac{-\infty}{\infty} \right), \text{ so we can use L'H. Rule.} \]

\[ \lim_{{x \to 0^+}} \frac{\ln x}{\frac{1}{x}} = \lim_{{x \to 0^+}} \frac{\ln x}{-\frac{1}{x^2}} = \lim_{{x \to 0^+}} (-x) = 0. \]

(Note: you could also rewrite \( \ln x \) as \( \frac{1}{\ln x} \), but L'H. Rule will be More Complicated!)

2. \[ \lim_{{x \to -\infty}} xe^x \left( \text{form } -\infty \cdot 0 \right) = \lim_{{x \to -\infty}} x \quad \left( \text{form } \frac{\infty}{\infty} \right) \]

So, use L'H Rule. \[ \lim_{{x \to -\infty}} \frac{1}{-e^x} = \lim_{{x \to -\infty}} (-e^x) = 0. \]
Indeterminate Differences:

If \( \lim_{x \to a} [f(x) - g(x)] \) has the form \((\infty - \infty)\) or \((-\infty + \infty)\).

We must try to rewrite \( f(x) - g(x) \) as a quotient.

Example \( \lim_{x \to 0} \csc(x) - \cot(x) \) (form \(\frac{\infty}{-\infty}\))

\[
= \lim_{x \to 0} \frac{1}{\sin(x)} - \frac{\cos(x)}{\sin(x)} = \lim_{x \to 0} \frac{1 - \cos(x)}{\sin(x)} \quad \text{(form } \frac{0}{0} \text{)}
\]

(L'Hopital's Rule) \( = \lim_{x \to 0} \frac{\sin(x)}{\cos(x)} = \frac{0}{1} = 0 \).

Indeterminate powers

These are several of these:

\[ \lim_{x \to a} [f(x)]^{g(x)} \]

- Type 1: \(0^0\) \((f \to 0, g \to 0)\)
- Type 2: \(\infty^0\) \((f \to \infty, g \to 0)\)
- Type 3: \(1^{\infty}\) \((f \to 1, g \to \infty)\)

We deal with these cases by:

1. Taking the natural log: \( \ln y = g(x) \cdot \ln(f(x)) \)
2. Writing function as exponential: \( [f(x)]^{g(x)} = e^{g(x) \cdot \ln f(x)} \).
Examples 1 \[ \lim_{x \to 0^+} (1 + \sin 4x)^{\cot x} \quad \text{(form } 1^\infty \text{)} \]

Let \( y = (1 + \sin 4x)^{\cot x} \). Then \( \ln y = \cot x \cdot \ln (1 + \sin 4x) \)

Rewrite \( \ln y = \frac{\ln(1 + \sin 4x)}{\tan x} \). Let \( \lim_{x \to 0^+} \frac{\ln(1 + \sin 4x)}{\tan x} \).

So, use L’H Rule:

\[ \lim_{x \to 0^+} \ln y = \lim_{x \to 0^+} \frac{1}{1 + \sin 4x} \cdot 4 \cos 4x \cdot \tan x = \frac{4}{1} = 4. \]

So, \( \lim_{x \to 0^+} \ln y = 4 \Rightarrow \lim_{x \to 0^+} e^{\ln y} = e^4 \)

\( \Rightarrow \lim_{x \to 0^+} y = e^4 \). \text{ Done.} \]

2 \( \lim_{x \to 0^+} x^x \) has the form \( 0^0 \). Let \( y = x^x \). \( \ln y = x \cdot \ln x \).

Then \( \lim_{x \to 0^+} \ln y = \lim_{x \to 0^+} x \cdot \ln x \) \quad \text{(form } 0 \cdot (-\infty) \text{)}

\[ = \lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x}} \quad \text{(form } \frac{-\infty}{\infty} \text{)} \text{ Use L’H Rule} \]

\[ = \lim_{x \to 0^+} \frac{x \cdot \frac{1}{x} - x}{x^2} = 0 \]

So, \( \lim_{x \to 0^+} \ln y = 0 \Rightarrow \lim_{x \to 0^+} e^{\ln y} = e^0 \Rightarrow \lim_{x \to 0^+} y = 1 \).

\text{ Done.}