Section 6.6 Inverse trigonometric Functions

Recall that trig functions \((\sin x, \cos x, \ldots)\) are not one-to-one. So, we must restrict their domain in order to find their inverses.

**SINE** Let \(f(x) = \sin x\), for \(-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}\). \(f\) is one-to-one on this domain. The inverse function is called the "inverse sine function" or the "arcsine function". It is denoted by \(f^{-1}(x) = \sin^{-1}(x)\) or \(\arcsin(x)\).

**Definition:** \(\sin^{-1}(x) = y \iff \sin(y) = x\)

For \(-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}, -1 \leq \sin x \leq 1\) (Range of \(\sin\) is \([-1, 1]\))

Thus, domain of \(\sin^{-1}\) is \([-1, 1]\).

**Examples**

1. \(\sin^{-1}(\frac{1}{2}) = \frac{\pi}{6}\) Since \(\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}\)

2. \(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}\) Since \(\sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}\)

**Example.** Find \(\tan(\sin^{-1}(\frac{1}{3}))\). Let \(\Theta = \sin^{-1}(\frac{1}{3})\). Then,

\[
\sin \Theta = \frac{1}{3} \quad \Rightarrow \quad \tan \Theta = \frac{\frac{1}{3}}{\sqrt{1 - \left(\frac{1}{3}\right)^2}} = \frac{1}{\sqrt{8}} = \frac{1}{2\sqrt{2}}.
\]

**Cancellation Equations:** \(\sin(\sin^{-1}(x)) = x, \sin(\sin^{-1}(x)) = x\)

**Continuity:** \(\sin^{-1}(x)\) is continuous since \(\sin(x)\) is continuous.

Graph
Differentiation: \[ \frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}} \]

proof: Let \( y = \sin^{-1} x \); then, \( \sin(y) = x \) \( \Rightarrow \) take derivative

\[ \cos(y) \cdot \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\cos(y)} = \frac{1}{\cos(\sin^{-1}(x))} = \frac{1}{\sqrt{1-x^2}} \]

This is because \( \cos(\sin^{-1}(x)) = \sqrt{1-\sin^2(\sin^{-1}(x))} = \sqrt{1-x^2} \)

Example: Let \( f(x) = \sin^{-1}(x^2-1) \). Find \( f(x) \), and Domain of \( f \).

Domain of \( f \): \(-1 \leq x^2-1 \leq 1 \Rightarrow 0 \leq x^2 \leq 2 \Rightarrow -\sqrt{2} \leq x \leq \sqrt{2} \)

\[ f'(x) = \frac{1}{\sqrt{1-(x^2-1)^2}} \cdot \frac{d}{dx} (x^2-1) = \frac{2x}{\sqrt{1-(x^2-1)^2}} \]

Domain of \( f' \): \(-\sqrt{2} < x < \sqrt{2} \)

**COSINE:** \( \cos^{-1}(x) = y \Leftrightarrow \cos(y) = x \); Domain: \([-1,1]\), Range: \([0,\pi]\)

\[ \frac{d}{dx} \cos^{-1}(x) = \frac{-1}{\sqrt{1-x^2}} \]

**TANGENT:** Let \( f(x) = \tan x \), \(-\frac{\pi}{2} < x < \frac{\pi}{2}\). Then \( f \) is one-to-one

\[ f^{-1}(x) = \tan^{-1}(x) \text{ or } \arctan(x) \]

Range of \( \tan(x) \) is \( IR=(-\infty,\infty) \Rightarrow \text{Domain of } \tan^{-1}(x) \text{ is } IR \)

Example: Simplify \( \cos(\tan^{-1}(x)) \). Let \( \theta = \tan^{-1}(x) \Rightarrow \tan \theta = x \)

Thus \( \cos(\theta) = \frac{1}{\sqrt{x^2+1}} \)

Limits: \( \lim_{x \to -\infty} \tan^{-1}(x) = -\frac{\pi}{2} \) (Since \( \lim_{x \to -\infty} \tan(x) = -\infty \))

and \( \lim_{x \to +\infty} \tan^{-1}(x) = \frac{\pi}{2} \) (Since \( \lim_{x \to +\infty} \tan(x) = +\infty \))
Graph

Differentiation
Let \( y = \tan^{-1}(x) \) \( \Rightarrow \) \( \tan(y) = x \) \( \Rightarrow \) \( \sec^2(y) \frac{dy}{dx} = 1 ; \) Thus

\[
\frac{dy}{dx} = \frac{1}{\sec^2(y)} = \frac{1}{\sec^2(\tan^{-1}(x))} = \frac{1}{1 + \tan^2(\tan^{-1}(x))} = \frac{1}{1 + x^2}
\]

\( \tan^{-1}x = \frac{1}{1 + x^2} \)

COTANGENT

Domain of \( \cot^{-1} \) is \( \mathbb{R} \), Range is \((0, \pi)\)

Derivative \( \frac{d}{dx} \cot^{-1}x = \frac{-1}{1 + x^2} \)

SECANT

Domain of \( \sec^{-1} \) is \((\infty, -1] \cup [1, \infty) \) \( (|x| \geq 1) \)

Range of \( \sec^{-1} \) is \([0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi] \)

Examples
1. \( \sec^{-1}(2) = \frac{\pi}{3} \) since \( \cos\left(\frac{\pi}{3}\right) = \frac{1}{2} \).
2. \( \sec^{-1}\left(-\frac{2}{\sqrt{3}}\right) = \frac{7\pi}{6} \) since \( \cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2} \)

Differentiation \( \frac{d}{dx} \sec^{-1}x = \frac{1}{x \sqrt{x^2 - 1}} \)

COSSECANT

Domain of \( \csc^{-1} \) is \([1, 1]) \), Range is \((0, \frac{\pi}{2}] \cup (\pi, \frac{3\pi}{2}] \)

Differentiation \( \frac{d}{dx} \csc^{-1}x = \frac{-1}{x \sqrt{x^2 - 1}} \)