Section 6.5 Exponential Growth and Decay

Some natural quantities grow (or decay) at a rate proportional to their size. For example, if \( m(t) \) is the mass of a radioactive material at time \( t \), then \( m'(t) = k \cdot m(t) \): The material decays at a rate proportional to its mass.

In general, we have \( \frac{dy}{dt} = ky(t) \), where \( k \) is a constant.

* if \( k > 0 \), we have natural growth; if \( k < 0 \), we have natural decay

* The point is to solve the ODE \( \frac{dy}{dt} = ky \) for \( y(t) \):

\[
\frac{dy}{dt} = ky \Rightarrow \frac{dy}{y} = k \, dt \quad \text{This is a separable Equation.}
\]

\[
\int \frac{dy}{y} = \int k \, dt \quad \Rightarrow \quad \ln |y| = kt + C \quad \Rightarrow \quad |y| = e^{kt+C}
\]

Since \( y \) is a quantity, it is always positive. So \( |y| = y \).

Thus, \( y = e^{kt+C} \). How do we solve for \( C \)?

at \( t = 0 \), i.e. initial time, \( y(0) = y_0 \), initial quantity.

So, \( y_0 = y(0) = e^{k \cdot 0 + C} = e^C \). Thus \( e^C = y_0 \). This yields

\( y(t) = e^{kt+C} = e^{kt} \cdot e^C = y_0 \cdot e^{kt} \). Therefore \( y(t) = y_0 \cdot e^{kt} \).

We call \( k \) the constant relative growth (or decay) rate.


Assume growth rate is proportional to population size. What is the relative growth rate?
Let $t=0$ correspond to 1920; so $t=92$ corresponds to 2022.

$$\frac{dP}{dt} = KP \Rightarrow P(t) = P(0)e^{kt} = 106.5e^{kt}.$$ 

at $t = 92$, $P(92) = 314 = 106.5e^{92k}$. Solve for $K$.

$$K = \frac{1}{92} \ln \left( \frac{314}{106.5} \right) \approx 0.0118.$$ Relative growth of about 1.18% per year.

The final formula for $P(t)$ is $P(t) = 106.5e^{0.0118t}$.

Radioactive decay: Let $m(t) = m(0)e^{kt}, k < 0$ be the mass of a radioactive substance at time $t$ (decay).

Half-life: is the time it takes for a sample of a certain substance to lose half its value. That is, if $t_0$ is the half-life, then $m(t_0) = \frac{1}{2}m(0)$.

Example: Chernobyl residents who were relocated after the blast in 1986 had exposure levels around 350 mSv (milliSievert). If the radioactive materials of Chernobyl have a half-life of 7 years, when will the area be habitable again?

Let $t = 0$ correspond to 1986. Then $R(t) = 350e^{kt}$, where $R(t)$ is the radiation level at time $t$. Half-life = 7 years $\Rightarrow$

$$R(t) = \frac{1}{2}350 = 175 = 350e^{7k}.$$ Solve for $k$. $K = -\ln 2 \approx -0.099$

Thus $R(t) = 350e^{-0.099t}$. When will $R(t) = 2$?

Set $2 = 350e^{-0.099t}$, an solve for $t \Rightarrow t \approx 52$ years (since 1986).
Newton's Law of Cooling: Rate of cooling of an object is proportional to the temperature difference between the object and its surroundings: \( \frac{dT}{dt} = K(T - T_s) \), \( K = \text{Constant} \), \( T_s = \text{Surrounding Temp. (Constant)} \)

Example: A bottle of soda at room temperature \( (72^\circ \text{F}) \) is placed in a refrigerator \( (44^\circ \text{F}) \); a half-hour later, the bottle has cooled to \( 61^\circ \text{F} \). Find a formula for the temperature \( T(t) \).

\[
\frac{dT}{dt} = K(T - T_s) \Rightarrow \frac{dT}{T - T_s} = K dt \Rightarrow \ln |T - T_s| = Kt + C
\]

Thus, \( T - T_s = e^{Kt+C} \Rightarrow T(t) = e^C e^{Kt} + T_s \).

At \( t = 0, \) \( T(0) = T_0 = e^C + T_s \Rightarrow e^C = T(0) - T_s \)

Thus, \( T(t) = (T(0) - T_s) e^{Kt} + T_s \) \( \text{Here, } T(0) = 72, \ T_s = 44 \)

So \( T(t) = 28e^{Kt} + 44 \). \( T(0.5) = 61 = 28e^{0.5K} + 44 \)

Solve for \( K \): \( K = 2 \ln \left( \frac{17}{28} \right) \Rightarrow T(t) = 28e^{2 \ln(17/28)t} + 44 \).

Continuously compounded interest.

If $1000 is invested in an account paying 5% interest, compounded annually, then:

- After 1 year, Value = $1000(1+0.05) = 1000(1.05) = $1050
- After 2 years, Value = $1050(1.05) = 1000(1.05)^2 = $1102.5
- After t years, Value = $1000(1.05)^t = 1000(1+r)^t
If interest is compounded \( n \) times a year, for \( t \) years, then

\[
\text{Value} = A_0 \left(1 + \frac{r}{n}\right)^{n \cdot t},
\]

where \( A_0 = \text{initial investment} \).

**Question:** What if interest is compounded continuously \((n \to \infty)\)?

\[
A(t) = \lim_{n \to \infty} A_0 \left(1 + \frac{r}{n}\right)^{n \cdot t}
\]

\[
= A_0 \lim_{n \to \infty} \left[ \left(1 + \frac{r}{n}\right)^{\frac{n}{r}} \right]^{r \cdot t}
\]

\[
= A_0 \left[ \lim_{n \to \infty} \left(1 + \frac{r}{n}\right)^{\frac{n}{r}} \right]^{r \cdot t}
\]

Let \( m = \frac{r}{n} \)

\[
= A_0 \left[ \lim_{m \to 0} \left(1 + m\right)^{\frac{1}{m}} \right]^{r \cdot t}
\]

\[
= A_0 \ e^{r \cdot t}
\]

(Recall, \( e = \lim_{m \to 0} \left(1 + m\right)^{\frac{1}{m}} \))

So,

\[
A(t) = A_0 \ e^{r \cdot t}.
\]