Chapter 6: Inverse Functions

Section 6.1: Inverse Functions

Definition: A function $f$ is called one-to-one if it never takes on the same value twice; that is

$$f(x_1) \neq f(x_2) \text{ whenever } x_1 \neq x_2.$$

Graphically, $f$ must pass the horizontal line test; that is, every $y$-value corresponds to only one $x$-value.

Examples:

- $y = x^3$ is one-to-one.
- $y = x^2 - 1$ is not one-to-one: $(1)^2 - 1 = (-1)^2 - 1 = 0$.

In simple terms, a function is one-to-one if each input corresponds to one output.
Examples: $f(x) = x^3$ is one-to-one, since $x_1^3 \neq x_2^3$ whenever $x_1 \neq x_2$.

$f(x) = \sin(x)$ is not one-to-one, since $\sin(x + 2k\pi) = \sin(x)$ for any integer $k$; Graphically, $\sin(x)$ does not pass the horizontal line test.

Definition: Let $f$ be one-to-one, with domain $A$, and Range $B$. Then its inverse function $f^{-1}$ has domain $B$ and Range $A$, and is defined by $f^{-1}(y) = x \iff f(x) = y$ for any $y$ in $B$.

In simpler terms, if $f$ maps $x$ into $y$, then $f^{-1}$ maps $y$ back into $x$.

Examples: Since $x^3$ is one-to-one, it has an inverse. The inverse of $f(x) = x^3$ is $f^{-1}(x) = x^{\frac{1}{3}}$, because if $y = f(x) = x^3$, then $f^{-1}(y) = f^{-1}(x^3) = (x^3)^{\frac{1}{3}} = x$.

Cancellation Equations:

$f^{-1}(f(x)) = x$, and $f(f^{-1}(x)) = x$.

Do $f(x) = x^3$, $f^{-1}(x) = x^{\frac{1}{3}}$ as an example.
How to find the inverse of a one-to-one function $f$

1. Write $y = f(x)$
2. Solve for $x$ in terms of $y$
3. Interchange $x$ and $y$; the resulting function is $y = f^{-1}(x)$

Example: find the inverse of $f(x) = x^3 - 2$.

(Note that $f(x) = x^3 - 2$ is indeed one-to-one, since it passes the horizontal line test)

1. $y = x^3 - 2$
2. $y + 2 = x^3 \implies x = \sqrt[3]{y + 2}$
3. $f^{-1}(x) = \sqrt[3]{x + 2}$

Observe that $f(x) = y \iff f^{-1}(y) = x$ means that the point $(a, b)$ is on the graph of $f$, if and only if, the point $(b, a)$ is on the graph of $f^{-1}$.

Graphically, the point $(b, a)$ is the reflection of the point $(a, b)$ about the line $y = x$.

Therefore, the graph of $f^{-1}$ is obtained by reflecting the graph of $f$ about $y = x$. 
Example:

Sketch of $f(x) = x^3$ and its inverse function $f^{-1}(x) = x^{\frac{1}{3}}$.

The Calculus of $f^{-1}$.

Continuity: if $f$ is one-to-one, continuous, then so is $f'$.

This is because the graph of $f^{-1}$ is simply the reflection of the graph of $f$ about $y=x$, so if the graph of $f$ has no breaks in it, then neither does the graph of $f^{-1}$.

Differentiability: Recall that $f(x)$ is differentiable at $x=a$, if $f'(a)$ exist.

Theorem: If $f$ is one-to-one, differentiable, with inverse $f^{-1}$, then $(f^{-1})'(a) = \frac{1}{f''(f^{-1}(a))}$, provided $f'(f(a)) \neq 0$. 
Proof: \((f^{-1})'(a) = \lim_{x \to a} \frac{f'(x) - f'(a)}{x - a}\)

So we can rewrite

\[(f^{-1})'(a) = \lim_{y \to b} \frac{y - b}{f(y) - f(b)} = \frac{1}{\lim_{y \to b} \frac{f(y) - f(b)}{y - b}} = \frac{1}{f'(b)} = \frac{1}{f'(f^{-1}(a))}\]

Example: Restrict \(y = x^2\) on the interval \([0, \infty)\).

Then \(y = x^2\) is one-to-one (on this interval).

\[(f^{-1})'(4) = \frac{1}{f'(f^{-1}(4))} = \frac{1}{f'(2)} = \frac{1}{2.2} = \frac{1}{4}\]

On, in this case, we could solve for \(f^{-1}(x)\) explicitly.

\(y = f(x) = x^2 \to x = \sqrt{y} \to f^{-1}(x) = \sqrt{x} = x^{\frac{1}{2}}\).

\[(f^{-1})'(x) = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}} ; \quad \text{Thus} \quad (f^{-1})'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}\]

Example: Consider \(f(x) = \tan(x)\) on \((-\frac{\pi}{2}, \frac{\pi}{2})\). Then \(f\) is one-to-one \((f'(x) = \sec^2 x > 0 \Rightarrow f\) is increasing).

\[(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(\frac{\pi}{4})} = \frac{1}{\sec^2(\frac{\pi}{4})} = \frac{1}{(\sqrt{2})^2} = \frac{1}{2}\).