Section 5.4 Work

if an object moves along a straight line, with position function \( S(t) \), then the force \( F \) on the object is given by \( F = m \frac{d^2s}{dt^2} \), where \( m \) = mass of object. \( F \) is in Newtons, \( s \) in m, and \( t \) in seconds (SI).

If acceleration is constant (constant force), the work done is defined to be \( W = F \cdot d \), where \( d \) is the distance the object moves. \( W \) is in Joules, when \( F \) (N) and \( d \) (m).

\( W \) is in ft-lb, when \( F \) (lb) and \( d \) (ft).

Ex: lifting a 2Kg book off the floor 1.5 m high requires

\[ W = F \cdot d = mg \cdot d = 2 \cdot 9.8 \cdot 1.5 = 29.4 \text{ N.} \]

b) lifting a 10-lb weight 8 ft off the ground requires

\[ W = 10 \cdot 8 = 80 \text{ lb-ft} \]

Now, what happens when the force is not constant? Say \( F(t) \) is the force acting on the body moving along the x-axis.
if body moves from \( x = a \) to \( x = b \)

divide \([a, b]\) into subintervals of width \( \Delta x \).

from each subinterval \([x_{i-1}, x_i]\), pick a point \( x^*_i \).

Then \( f(x^*_i) \Delta x \) is approximately the force in \([x_{i-1}, x_i]\), and

\[
W_i = f(x^*_i) \Delta x
\]

is the work done by moving the object from \( x_{i-1} \) to \( x_i \).

So, total work is:

\[
W = \sum_{i=1}^{n} f(x^*_i) \Delta x \quad \text{if we let } n \to \infty \quad (\Delta x \to 0),
\]

Then

\[
W = \lim_{n \to \infty} \sum_{i=1}^{n} f(x^*_i) \Delta x = \int_{a}^{b} f(x) \, dx.
\]

**Example 2:** when an object is located \( x \) feet from the origin, a force of \( x^3 + 3x^2 \) acts on it. How much work is done in moving it from \( x = 1 \) to \( x = 2 \).

\[
W = \int_{1}^{2} (x^3 + 3x^2) \, dx = 10.75 \text{ ft-lb}.
\]

**Hooke's Law:** force required to maintain a spring stretched \( x \) units beyond its natural length is \( f(x) = Kx \), \( K \): spring constant.

**Example 3** A force of 50N is required to hold a spring that has been stretched from 90 cm (natural length) to 95 cm. How much work is done in stretching the spring from 95 cm to 30 cm.

\[
K = \frac{F}{x} = \frac{50}{0.05} = 1000. \quad f(x) = 1000x. \quad W = \int_{0.05}^{0.05} 1000x \, dx = 3.75 \text{ J}
\]
Example 4

A 300-ft long cable is hung vertically from the top of a tall building. How much work is required to lift the cable to the top of the building. The cable weighs 3 lbs/ft.

To lift the section $S_i$, we need $W_i = \text{weight}(S_i) \times \text{distance}$.

So, $W_i = 3 \Delta x \times x_i^*$.

Thus $W = \lim_{n \to \infty} \sum_{i=1}^{n} 3 \Delta x \times x_i^* = \int_{0}^{300} 3 \, dx = 135,000 \text{ ft-lb}$

Example 5

A tank has a shape of a circular cone, height 10 m, radius 4 m, filled with water to height of 8 m.

Water density: 1000 Kg/m$^3$.

Volume of $S_i = \pi (0.4 x_i^*)^2 \Delta x$.

Weight of $S_i = \pi (0.4 x_i^*)^2 \Delta x \times 1000 \times 9.8$.

Height to be pulled = $(10 - x_i^*)$.

Thus, $W_i = \pi (0.4 x_i^*)^2 \Delta x \times 1000 \times (10 - x_i^*) \times 9.8$

$W = \lim_{n \to \infty} \sum_{i=1}^{n} W_i = \int_{0}^{8} \pi (0.4x)^2 \times 1000 \times (10-x) \, dx = 336,9827.7977$