Section 5.2: Volumes

The goal is to calculate the volume of a 3-dimensional object, $S$. To do so, we cut $S$ into small "thin" pieces, and approximate every piece by a cylinder; finally, we add the volumes of all these pieces.

On every sub-interval $[x_i, x_{i+1}]$, we take a vertical cross section at an intermediate point $x_i^*$. We denote the area of this cross-section $A(x_i^*)$. Then, the volume of the cylinder at $x_i^*$ is $A(x_i^*) \Delta x$, where $\Delta x = x_{i+1} - x_i$.

If $\Delta x$ is small enough, then these cylinders are a good approximation to the true volume of $S$. That is

$$
\lim_{n \to \infty} \sum_{i=0}^{n-1} A(x_i^*) \Delta x = \text{Volume of } S = \int_a^b A(x) \, dx
$$

Since the sum on the left is a Riemann Sum.
Example 1. Find the volume of the solid obtained by rotating about the x-axis, the region under \( y = \sqrt{x} \) from 0 to 1.

Take a cross-section at some point \( x \) between 0 and 1; the cross-section is circular, with area \( A(x) = \pi r^2(x) \);

the radius \( r(x) \) is the distance from

the point \((x, 0)\) to the function itself; so

\[ r(x) = \sqrt{x} \]

Thus, \( A(x) = \pi (\sqrt{x})^2 = \pi x \).

Volume = \( \int_0^1 A(x) \, dx = \int_0^1 \pi x \, dx = \left[ \frac{\pi}{2} x^2 \right]_0^1 = \frac{\pi}{2} \).

Example 2. Find the volume obtained by rotating the region bounded by \( y = x^3 \), \( y = 8 \), and \( x = 2 \), about the y-axis.

Take a horizontal cross-section at a point \( y \) on the y-axis; it is circular, with area

\[ A(y) = \pi r^2(y) \], with \( r(y) = x = \sqrt[3]{y} \).

So, Volume = \( \int_0^8 A(y) \, dy = \int_0^8 \pi (\sqrt[3]{y})^2 \, dy \)

= \( \left[ \frac{\pi}{3} y^{2/3} \right]_0^8 = \frac{9}{5} \pi \).
What if the cross-section is a washer? Here's an example.

**Ex 3.** Find the volume of the solid obtained by rotating the region bounded by \( y = x \), \( y = 2x \), and \( x = 2 \), about the \( x \)-axis.

We think of this problem as the bigger volume - the smaller volume. The larger solid has a circular cross-section at \( x \) with radius \( R(x) = 2x \), whereas the smaller solid has a radius \( r(x) = x \). Thus, the volume is given by

\[
\int_0^2 \pi R^2(x) - \pi r^2(x) \, dx = \int_0^2 \pi (R^2(x) - r^2(x)) \, dx = \int_0^2 \pi (3x^2) \, dx = 8\pi.
\]

What if the region above is rotated about \( y = -1 \)?

We measure the radius always from the axis of rotation; so, \( r(x) = x+1 \), \( R(x) = 2x+1 \)

\[
V = \int_0^2 \pi (R^2(x) - r^2(x)) \, dx = \int_0^2 \pi ((2x+1)^2 - (x+1)^2) \, dx.
\]