Section 11.7: Strategy for testing Series 

1. if the series is of the form $\sum \frac{1}{n^p}$, it is a p-series.
   - Converges if $p > 1$; diverges if $p \geq 1$.

2. if the series is of the form $\sum a_n r^{n-1}$, it is a geometric series, which converges if $|r| < 1$ ($\text{sum} = \frac{a}{1-r}$),
   - and diverges if $|r| \geq 1$.

3. if the series has a form similar to a p-series or a geometric series, a comparison test should be used.
   - Direct comparison: $a_n, b_n > 0$:
     - (a) if $a_n \leq b_n$ for all $n$, and $\sum b_n$ conv, then $\sum a_n$ conv.
     - (b) if $a_n \geq b_n$ for all $n$, and $\sum b_n$ div, then $\sum a_n$ div.

   - Limit comparison test: $a_n, b_n > 0$. If $\lim_{n \to \infty} \frac{a_n}{b_n} = c$, $0 < c < \infty$, then $\sum a_n, \sum b_n$ either both conv or both div.

Remark: if $\sum a_n$ has some negative terms, we can test for absolute convergence by looking at $\sum |a_n|$.
4. if it is easy to see that \( \lim_{n \to \infty} a_n = 0 \), use nth term test for divergence. Remember that \( \lim_{n \to \infty} a_n = 0 \) does not imply convergence.

5. if the series is an alternating series (\( \sum (-1)^n b_n \)), The alternating series test is an obvious possibility:
   \[ \text{if } b_{n+1} \leq b_n \text{ for all } n, \text{ and } \lim_{n \to \infty} b_n = 0, \text{ then} \]
   \( \sum (-1)^n b_n \) converges (Here \( b_n > 0 \)).

6. Series that involve factorials and/or exponentials are conveniently tested using the Ratio test:
   \[ \text{if } \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1 \text{ Series Conv. abs } \Rightarrow \text{Converges} \]
   \[ \text{if } L > 1 \text{ or } \infty, \text{ Series diverges} \]
   \[ \text{if } L = 1, \text{ Test is inconclusive.} \]

Remark: for p-series or Rational Series, the limit is always equal to 1, So do not use the Ratio test!
7. In some cases, the root test may be used instead:

(if the series has the form $\sum (b_n)^n$):

- if $\lim_{n \to \infty} n\sqrt[n]{|a_n|} < 1 \Rightarrow$ Series converges absolutely
- if $\lim_{n \to \infty} n\sqrt[n]{|a_n|} = 1$ or $\infty$, Series diverges
- if $\lim_{n \to \infty} n\sqrt[n]{|a_n|} = 1$, Test is inconclusive.

Remark: if $\lim_{n \to \infty} n\sqrt[n]{|a_n|} = 1$, then $\lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = 1$ as well.

8. If $a_n = f(n)$ where $\int f(x)\,dx$ is easily evaluated, then the integral test may be used:

$f(x)$ is positive, continuous, decreasing on $[1, \infty)$:

- if $\int_1^\infty f(x)\,dx$ converges then $\sum a_n$ converges.
- if $\int_1^\infty f(x)\,dx$ diverges then $\sum a_n$ diverges.

Recall this powerful theorem: if a series converges absolutely (that is $\sum |a_n|$ converges), then it converges.
Examples: Test the following Series for convergence or divergence.

1. \[ \sum_{n=1}^{\infty} \frac{1}{n+3^n} \] Converges by Comparison test (compare with \( \sum \frac{1}{3^n} \)).

2. \[ \sum_{n=1}^{\infty} (-1)^n \frac{n}{n + 2} \] diverges by \( n \)th term test (\( \lim_{n \to \infty} \frac{(-1)^n n}{n+2} \) DNE).

3. \[ \sum_{n=1}^{\infty} \frac{n^2 2^{n-1}}{(-5)^n} \] Converges by Ratio test (\( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{2}{5} < 1 \)).

4. \[ \sum_{n=1}^{\infty} \frac{1}{n \sqrt{n+1}} \] diverges by Integral test (\( \int_{2}^{\infty} \frac{1}{x \sqrt{x}} \, dx = \infty \)).

5. \[ \sum_{k=1}^{\infty} k^2 e^{-k} \] Converges by Ratio test (\( \lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| = \frac{1}{e} < 1 \)).

6. \[ \sum_{n=1}^{\infty} \left( \frac{1}{n^3} + \frac{1}{3^n} \right) \] Converges by p-series and geometric Series (\( p=3 > 1, \, r=\frac{1}{3} < 1 \)).

7. \[ \sum_{n=1}^{\infty} \frac{3^n n^2}{n!} \] Converges by Ratio test (\( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 0 < 1 \)).

Test whether the following Series is absolutely convergent or not.

1. \[ \sum_{n=1}^{\infty} \frac{(-1)^n}{5n} \] No because \( \sum \frac{1}{5n} \) diverges by p-series (\( p=1 \)).

2. \[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \] Yes because \( \sum \frac{1}{n^2} \) converges by p-series (\( p=2 > 1 \)).
3. \( \sum_{n=1}^{\infty} \frac{\sin(2n)}{n^2} \): yes, because \( \sum_{n=1}^{\infty} \frac{|\sin(2n)|}{n^2} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \) \( \leq \) p-series.

4. \( \sum_{n=1}^{\infty} \frac{(n+1)(6^2 - 1)^n}{6^{2n}} = \sum_{n=1}^{\infty} (n+1) \left( \frac{35}{36} \right)^n \)

\[
\lim_{n \to \infty} \left| \frac{(n+2)(35/36)^{n+1}}{(n+1)(35/36)^n} \right| = \frac{35}{36} < 1 \text{ Series Conv absolutely by Ratio test.}
\]

5. \( \sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{n^3 + 6} \): yes, the series \( \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^3 + 6} \) converges by Comparison test.

6. \( \sum_{n=1}^{\infty} \frac{n^6}{5^n} \): yes, \( \lim_{n \to \infty} \left| \frac{(n+1)^6}{5^{n+1}} \cdot \frac{5^n}{n^6} \right| = \frac{1}{5} < 1 \text{ Conv abs by Ratio test.} \)

7. \( \sum_{n=1}^{\infty} \frac{(-1)^n 4^{n-1}}{4^{n+1} n} \): No. \( \sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{16n} \) diverges by p-series.

8. \( \sum_{n=1}^{\infty} \frac{(-1)^n n!}{n} \): No. \( \sum_{n=1}^{\infty} \frac{n!}{n} = \sum_{n=1}^{\infty} (n-1)! \) diverges by nth term test.

9. \( \sum_{n=1}^{\infty} \frac{(-1)^n}{3^n n!} \): yes. \( \sum_{n=1}^{\infty} \frac{1}{3^n n!} \) converges by Ratio test.

10. \( \sum_{n=1}^{\infty} \frac{(n+2)!}{2^n n!} \): yes. \( \sum_{n=1}^{\infty} |a_n| \) converges by Ratio test.