Section 11.2: Series

Consider an infinite sequence \( \{a_n\}_{n=1}^{\infty} \). If we try to add the terms of \( \{a_n\} \), we get an expression of the form

\[ a_1 + a_2 + a_3 + \ldots + a_n + a_{n+1} + \ldots \],

which is called an "infinite Series," or just "Series" for short. We use the notation

\[ \sum_{n=1}^{\infty} a_n \]

or \( \Sigma a_n \) to denote this infinite sum.

Does it make sense to sum infinitely many terms?

Not always! For example, the infinite sum \( 1 + 2 + 3 + 4 + \ldots \)

goes to infinity; we can see that by looking at the cumulative sums:

\[ 1, 3, 6, 10, 15, 21, \ldots \rightarrow \infty \]

On the other hand, the infinite sum \( \frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots \)

approaches the value 2. The cumulative sums are

\[ 1, 1.5, 1.75, 1.875, 1.9375, \ldots \text{ and they approach 2.} \]

We generalize this idea of "partial sums" to general series.
Consider the series \( \sum_{n=1}^{\infty} a_n \). We now consider the "partial sums"

\[
S_1 = a_1 \\
S_2 = a_1 + a_2 \\
S_3 = a_1 + a_2 + a_3 \\
\vdots \\
S_n = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n = \sum_{i=1}^{n} a_i.
\]

These partial sums form a sequence \( \{S_n\}_{n=1}^{\infty} \). This sequence may or may not have a limit.

If \( \lim_{n \to \infty} S_n = s \) exists (as a finite number), we call it the sum of the infinite series \( \sum_{n=1}^{\infty} a_n \).

So, if \( \lim_{n \to \infty} S_n = s \) exists, we call the series \( \sum_{n=1}^{\infty} a_n \) convergent, and write \( \sum_{n=1}^{\infty} a_n = s \).

If the sequence \( \{S_n\} \) is divergent, (That is, \( \lim_{n \to \infty} S_n \) is \( \pm \infty \), or DNE), we call the series \( \sum_{n=1}^{\infty} a_n \) divergent.

Examples:

1. If the sum of the first \( n \) terms of \( \sum_{n=1}^{\infty} a_n \) is

\[
S_n = \frac{2n}{3n+5},
\]

Then \( \lim_{n \to \infty} S_n = \frac{2}{3} \) and \( \sum_{n=1}^{\infty} a_n = \frac{2}{3} \).
Examples 2 Geometric Series

Consider the series

$$\sum_{n=1}^{\infty} a \cdot r^{n-1} = a + a \cdot r + a \cdot r^2 + a \cdot r^3 + a \cdot r^4 + \cdots$$

where \(a \neq 0\), and \(r\) is the "common ratio".

1. If \(r = 1\), then the partial sum \(S_n = \underbrace{a + a + \cdots + a}_{n \text{ times}} \to \infty\) as \(n \to \infty\), and \(\lim_{n \to \infty} S_n\) DNE, and the series diverges.

2. If \(r \neq 1\),

\[ S_n = a + a \cdot r + a \cdot r^2 + \cdots + a \cdot r^{n-1} \]  

\[ r \cdot S_n = a \cdot r + a \cdot r^2 + \cdots + a \cdot r^{n-1} + a \cdot r^n \]  

So,

\[ S_n - r \cdot S_n = a - a \cdot r^n \] (Subtracting 1-2), and

\[ S_n = \frac{a(1 - r^n)}{1 - r} \]

- If \(-1 < r < 1\), then \(r^n \to 0\) as \(n \to \infty\), and \(S_n \to \frac{a}{1 - r}\)
- If \(|r| > 1\), then \(r^n \to \pm \infty\), and \(\lim_{n \to \infty} S_n\) DNE.

In general, the geometric series

$$\sum_{n=1}^{\infty} a \cdot r^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if \(|r| < 1\) and its sum is \(\sum_{n=1}^{\infty} a \cdot r^{n-1} = \frac{a}{1 - r} \) (First Term \(\frac{1}{1-\text{common ratio}}\)).

If \(|r| > 1\), the geometric series is divergent.
Examples 3) Find the sum \( 5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \ldots \).

This is a geometric series with ratio \( r = -\frac{2}{3} \), and first term \( a = 5 \). Since \( |r| = \frac{2}{3} < 1 \), the series is convergent, and its sum is
\[
\frac{a}{1-r} = \frac{5}{1-(-\frac{2}{3})} = 3.
\]

4) Is the series \( \sum_{n=1}^{\infty} \frac{2^n}{3^{1-n}} \) convergent or divergent?

We can rewrite \( \sum_{n=1}^{\infty} \frac{2^n}{3^{1-n}} = \sum_{n=1}^{\infty} (\frac{2}{3})^n = \sum_{n=1}^{\infty} 4 (\frac{4}{3})^{n-1} \).

This is a geometric series with \( a = 4, \quad r = \frac{4}{3} > 1 \). So this series diverges.

5) Write the number \( 4.2\overline{679} = 4.2679679679 \ldots \) as a ratio of integers.

\[ 4.2679 = 4.2 + \frac{679}{10^4} + \frac{679}{10^7} + \frac{679}{10^{10}} + \ldots = 4.2 + \sum_{n=0}^{\infty} \frac{679}{10^4 \cdot (\frac{1}{10^3})^n} \]

This series is geometric, with \( R = \frac{1}{10^3} = \frac{1}{1000} < 1 \), and
\[ a = \frac{679}{10000} \]. So \( \sum_{n=0}^{\infty} \frac{679}{10^4} \left(\frac{1}{10^3}\right)^n = \frac{679/10000}{1-\frac{1}{1000}} = \frac{679}{9990} \).

So, \( 4.2679 = \frac{42}{10} + \frac{679}{9990} = \frac{42637}{9990} \).
6. Find the sum of the series \( \sum_{n=0}^{\infty} x^n \), where \( |x| < 1 \).

Here \( a=1 \), and \( n=x \), with \( |n| < 1 \). So the series converges, and \( \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \).

This says that the function \( \frac{1}{1-x} \) is "equal" to the infinite series \( 1 + x + x^2 + x^3 + x^4 + \cdots + x^n \) when \( |x| < 1 \).

Theorem: If the series \( \sum_{n=1}^{\infty} a_n \) is convergent, then \( \lim_{n \to \infty} a_n = 0 \) (See book for proof).

CAUTION: The converse of this theorem is not true!

If \( \lim_{n \to \infty} a_n = 0 \), we don't necessarily get a convergent series.

Example: \( \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \to \infty \) (See book for proof), but \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{n} = 0 \).

Test for divergence: If \( \lim_{n \to \infty} a_n \) DNE or \( \lim_{n \to \infty} a_n \neq 0 \), then the series \( \sum_{n=1}^{\infty} a_n \) is divergent.
Example 7: The series \( \sum_{n=1}^{\infty} \frac{2n^2}{5n^2 + 4} \) diverges since

\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{2n^2}{5n^2 + 4} = \frac{2}{5} \neq 0.
\]

8. The series \( \sum_{n=1}^{\infty} \sin(2n\pi) \) diverges since

\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} \sin(2n\pi) \text{ DNE.}
\]

NOTE! if we find \( \lim_{n \to \infty} a_n = 0 \), the series \( \sum a_n \)

might converge or if it might diverge.

Theorem: if \( \sum a_n \) and \( \sum b_n \) are convergent series, then so are the series \( \sum c \cdot a_n \), \( \sum (a_n + b_n) \), \( \sum (a_n - b_n) \), and

(i) \( \sum_{n=1}^{\infty} c \cdot a_n = c \cdot \sum_{n=1}^{\infty} a_n \)

(ii) \( \sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n \)

(iii) \( \sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n \)

All these properties follow from the limit laws

for sequences.
More examples

1. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4^n} = -4 \sum_{n=1}^{\infty} \left(\frac{-1}{4}\right)^n$. geometric, $a = -4$, $r = -\frac{1}{4}$

$|r| = \frac{1}{4} < 1 \Rightarrow$ convergent. $\sum a_n = \frac{a}{1-r} = \frac{-4}{1 + \frac{1}{4}} = \frac{4}{5}$

2. $\sum_{n=0}^{\infty} \left(\frac{1}{6}\right)^{n/2} = \sum_{n=0}^{\infty} \left[\left(\frac{1}{6}\right)^{1/2}\right]^n = \sum_{n=0}^{\infty} \left(\frac{1}{\sqrt{6}}\right)^n$. $a = 1$, $r = \frac{1}{\sqrt{6}} < 1$

Convergent geometric. $\sum a_n = \frac{1}{1 - \frac{1}{\sqrt{6}}} = \frac{\sqrt{6}}{\sqrt{6} - 1}$

3. $\sum_{n=1}^{\infty} \frac{3^{n-1} - 1}{q^n} = \sum_{n=1}^{\infty} \frac{3^{n-1}}{q^n} - \sum_{n=1}^{\infty} \frac{1}{q^n} = \frac{1}{3} \sum_{n=1}^{\infty} (3) - \sum_{n=1}^{\infty} (\frac{1}{q})^n$

$= \frac{1}{3} \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n - \sum_{n=1}^{\infty} \left(\frac{1}{q}\right)^n = \frac{1}{3} \cdot \frac{\frac{1}{3}}{1 - \frac{1}{3}} - \frac{q}{1 - \frac{1}{q}}$

4. Consider $\sum_{n=1}^{\infty} \ln\left(\frac{5n}{5n+4}\right)$. $\lim_{n \to \infty} a_n = \ln(1) = 0$. So the test for divergence does not tell us anything about the series.

5. $\sum_{n=0}^{\infty} 2\cos(3n\pi)$. $\lim_{n \to \infty} a_n = \lim_{n \to \infty} 2\cos(3n\pi)$ DNE.

The test of divergence tells us that the series diverges.
6. Let $a_n = \frac{q^n}{q_0}$. \(\lim_{n \to \infty} a_n = \infty\). So,
   - The sequence \(\{a_n\}\) diverges.
   - The series $\sum_{n=1}^{\infty} a_n$ diverges.

7. Let $a_n = \ln\left(\frac{\frac{2n}{3}}{n+1}\right)$. \(\lim_{n \to \infty} a_n = \ln\left(\frac{2}{3}\right)\). So,
   - The sequence \(\{a_n\}\) converges to $\ln\left(\frac{2}{3}\right)$.
   - The series $\sum_{n=1}^{\infty} a_n$ diverges since $\lim_{n \to \infty} a_n \neq 0$.

8. Find the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

   \[
   \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1} \quad \text{(by partial fractions)}.
   \]

   The partial sum $S_n = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$

   So, $S_n = 1 - \frac{1}{n+1}$.

   \(\lim_{n \to \infty} S_n = 1\) and so, $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$. 