Section 10.3 (1) Polar Coordinates

In Cartesian Coordinates, we represent a point in the plane by a pair of coordinates \((x, y)\).

In polar coordinates, we represent a point by its distance from the origin "r", and the angle it makes with the horizontal axis, "\(\theta\)". The ordered pair \((r, \theta)\) represents a unique point in the plane; we call "r" and "\(\theta\)" polar coordinates.

Can "r" be negative? Yes!

The point \((-r, \theta)\) is the reflection of \((r, \theta)\) about the origin.

Example: Plot the points \(A = (1, 5\pi/4)\), \(B = (2, 3\pi)\), \(C = (2, -2\pi/3)\), \(D = (-3, 3\pi/4)\).
In Cartesian Coordinates, every point has one representation \((x,y)\). This is not true in polar coordinates. For example, 
\[ D = (-3, 3\pi/4) \text{ is the same as } (3, -\pi/4) \text{ or } (3, 7\pi/4). \]

The connection between Cartesian and Polar Coordinates

We see that

\[ P(x,\theta) = P(r,\theta) \]

\[ \cos\theta = \frac{x}{r}, \quad \sin\theta = \frac{y}{r} \]

and so

\[ x = r \cos\theta, \quad y = r \sin\theta \]

These equations are valid for all values of \(r\) and \(\theta\).
We also have \[ r^2 = x^2 + y^2, \] and \[ \tan \theta = \frac{x}{y}. \]

**Remark.** If \( r = 0 \), that is, we have the point \((0, \theta)\), then for any value of \( \theta \), we are at the origin. \( \theta = 0 \) corresponds to the \( x \)-axis.

**Example 2** Convert from polar to Cartesian:

\[(2, \frac{\pi}{3}) : \quad r = 2, \quad \theta = \frac{\pi}{3}, \quad x = r \cos \theta = 2 \cdot \frac{1}{2} = 1 \]

\[ y = r \sin \theta = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3} \quad \Rightarrow \quad (x, y) = (1, \sqrt{3}). \]

\[ \text{Example 3} \quad \text{Convert from Cartesian to polar} \]

\[(1, -1) : \quad x = 1, \quad y = -1 \quad \Rightarrow \quad r = \sqrt{x^2 + y^2} = \sqrt{2}; \quad \Rightarrow \quad (r, \theta) = \]

\[
\sin \theta = \frac{y}{r} = -\frac{\sqrt{2}}{2}, \quad \cos \theta = \frac{x}{r} = \frac{\sqrt{2}}{2} \quad \Rightarrow \quad \left( \sqrt{2}, \frac{7\pi}{4} \right) \text{ on} \]

\[ (\sqrt{2}, -\frac{\pi}{4}) \]

**Example 4** Consider the Polar equation \( r \sin \theta = 7 \).

What is the corresponding Cartesian equation?

We know \( \sin \theta = \frac{y}{r} \) \( \Rightarrow \) \( r \sin \theta = y = 7 \). So, \[ y - 7 = 0. \]
**Polar Curves**: In Cartesian coordinates, we looked at functions of the form $y = f(x)$. In polar coordinates, we consider polar equations of the form $r = f(\theta)$.

**Examples**

1. What curve is represented by the polar equation $r = 2$?

   $r$ represents the distance to the origin, if the distance is always 2, regardless of the angle $\theta$, then the shape is a circle, of radius 2, centered at the origin. **OR:**
   
   \[ r = \sqrt{x^2 + y^2} \Rightarrow r^2 = x^2 + y^2 = 4 \leq \text{Circle: Center (0,0), Radius 2} \]

2. What curve is $\theta = 1$?

   Now the angle is fixed for all values of $r$. So we get a line through the origin, at an angle 1 radian with the x-axis. **OR:**
   
   \[ \tan \theta = \frac{y}{x} \Rightarrow y = \tan \theta \cdot x = (\tan 1) \cdot x \leq \text{line} \]
   
   through (0,0), with slope $\tan(1)$. 

$\theta = 1$
Examples 7 Write the following equation in Cartesian Coordinates:

\[ r^2 \sin 2\theta = 9. \quad \text{Hint: } \sin 2\theta = 2 \sin \theta \cos \theta. \]

Examples 8 Sketch the following Inequalities:

(a) \( 0 \leq r \leq 2 \)
(b) \( \frac{\pi}{3} \leq \theta < \frac{2\pi}{3} \)
(c) \( 1 \leq r \leq 2 \)
(d) \( -2 \leq r \leq -1, \quad \frac{3\pi}{4} < \theta \leq \frac{7\pi}{4} \)
(e) \( r > 1, \quad \theta = \frac{5\pi}{4} \)

Examples 9 Consider the Regions graphed below. Give inequalities for \( r \) and \( \theta \) which describe the Regions in polar Coordinates.
(a) \( \theta \) increases from \( \sqrt{1^2 + \left(\frac{1}{3}\right)^2} = \sqrt{1 + \frac{1}{3}} \) to \( \sqrt{3^2 + \left(\frac{3}{\sqrt{3}}\right)^2} = \sqrt{9 + 3} \)

Thus, \( \frac{2}{\sqrt{3}} \leq \theta \leq 2\sqrt{3} \)

on the line \( y = \frac{x}{\sqrt{3}} \), \( \tan \theta = \frac{y}{x} = \frac{\sqrt{3}}{3} \Rightarrow \theta = \tan^{-1}\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{6} \)

So, \( \frac{\pi}{6} \leq \theta \leq \frac{\pi}{2} \).

(b) \( \theta \) increases between 0 and \( \frac{\pi}{2} \); so \( 0 \leq \theta \leq \frac{\pi}{2} \).

The line \( x = 5 \) in \( x = \rho \cos \theta = 5 \Rightarrow \rho = \frac{5}{\cos \theta} \)

Thus, \( \rho \) increases from 3 to \( \frac{5}{\cos \theta} \); \( 3 \leq \rho \leq \frac{5}{\cos \theta} \).
Tangents to polar curves.

Consider a polar curve given by \( r = f(\theta) \). Regard \( \theta \) as a parameter. So,

\[
X = r \cos \theta = f(\theta) \cos \theta, \quad Y = r \sin \theta = f(\theta) \sin \theta.
\]

These then are parametric equations for \( x \) and \( y \)!

We know

\[
\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{d}{d\theta}(r \sin \theta)
\]

\[
= \frac{\frac{dr}{d\theta} \sin \theta + r \cos \theta}{\frac{dr}{d\theta} \cos \theta - r \sin \theta}
\]

(\text{Remember: } r = f(\theta))

Example 1: Find the equation in \( x \) and \( y \) for the line tangent to the polar curve \( r = 2 \cos 2 \theta \) at the value \( \theta = \frac{\pi}{2} \).

\( r = f(\theta) = 2 \cos 2 \theta \), at \( \theta = \frac{\pi}{2} \), \( f(\frac{\pi}{2}) = 2 \cos \pi = -2 \).

So the point is \((r, \theta) = (-2, \frac{\pi}{2})\). In Cartesian, \((x, y) = (0, -2)\).

\[
\frac{dy}{dx} = \frac{(2 \cos 2 \theta)' \sin \theta + (2 \cos 2 \theta) \cos \theta}{(2 \cos 2 \theta)' \cos \theta - (2 \cos 2 \theta) \sin \theta} = \frac{-4 \sin 2 \theta \cos \theta + 2 \cos 2 \theta \cos \theta}{-4 \sin 2 \theta \cos \theta - 2 \cos 2 \theta \sin \theta}
\]

\[
\left. \frac{dy}{dx} \right|_{\theta = \frac{\pi}{2}} = 0 \quad \text{at} \quad \theta = \frac{\pi}{2}
\]

Tangent line: \( y - (-2) = 0 \cdot (x - 0) \)

or \( y = -2 \).