Chapter 10: Parametric Equations & Polar Coordinates

Section 10.1: Curves defined by Parametric Equations

Imagine a particle moving along a curve C as in the figure on the left. It would be impossible to represent the curve C as \( y = f(x) \) (as a function of \( x \)), because the curve fails the Vertical Line Test. But the \( x \) and \( y \)-coordinates of the particle are functions of time, \( t \). So we can write:

\[ x = f(t), \quad y = g(t); \]

We call \( t \) the "parameter". For every value of \( t \), we get a value for \( x \) and \( y \):

\( (x, y) = (f(t), g(t)) \), which represents a unique point on the \( x-y \) plane.

Note: It doesn't necessarily represent time, but in the context of a particle, it often is time.
The equations \(x = f(t), \ y = g(t)\) are called "parametric equations". The curve \(C\) is called a "parametric curve".

Example 1 Sketch and identify the curve defined by the parametric equations \(x = t^2 + 1, \ y = t\).

Let's calculate a few values for \(x\) and \(y\) for different values of \(t\), and plot them:

- \(t\) \( \quad \) \(x\) \( \quad \) \(y\)
- \(-2\) \( \quad 5\) \( \quad -2\)
- \(-1\) \( \quad 2\) \( \quad -1\)
- \(0\) \( \quad 1\) \( \quad 0\)
- \(1\) \( \quad 2\) \( \quad 1\)
- \(2\) \( \quad 5\) \( \quad 2\)

It looks like the curve traced out by the particle is a Parabola. Indeed, if \(x = t^2 + 1\), and \(y = t\), then we have \(x = y^2 + 1\), which is the equation of a parabola in the horizontal direction.

Note: That here, we did not place a Restriction on \(t\).
Sometimes, we restrict $t$ to lie in a finite interval; for instance we could write $x = t^2 + 1$, $y = t$, $0 \leq t \leq 2$.

In this case, we only get a finite piece of the curve C:

\[ \begin{array}{c}
\text{Example 2: What curve is represented by the parametric equations } x = \cos t, \ y = \sin t, \ 0 \leq t \leq 2\pi ? \\
\text{Observe that } x^2 + y^2 = \cos^2 t + \sin^2 t = 1. \text{ Thus the point } (x, y) \text{ moves on the unit circle } x^2 + y^2 = 1, \text{ as } t \text{ increases. We can interpret } t \text{ as the angle; if } t \text{ changes from } 0 \text{ to } 2\pi, \text{ the particle moves along the unit circle only once! (In the counterclockwise direction).}
\end{array} \]
Example 3: What curve is represented by the parametric equations \( x = \sin 2t, \quad y = \cos 2t, \quad 0 \leq t \leq 2\pi \)?

We still have \( x^2 + y^2 = \sin^2 2t + \cos^2 2t = 1 \), so, a Unit Circle; But 2 things are different:

1. the angle is \( 2t \); for \( 0 \leq t \leq 2\pi \), we have \( 0 \leq 2t \leq 4\pi \); So we move along the Unit Circle TWICE!

2. We now start at the point \((\sin 0, \cos 0) = (0,1)\)), and move along the circle in a Clockwise direction!

Example 4: How do we represent a circle centered at \((h, k)\) with radius \( r \), using parametric equations?

We want to find \( x = f(t), \quad y = g(t), \) where \( 0 \leq t \leq 2\pi \).

Suppose for a moment the center is \((0,0)\). Then \( x^2 + y^2 = r^2 \); So, let \( x = r \cos t, \quad y = r \sin t \).

Then \( r^2 \cos^2 t + r^2 \sin^2 t = r^2 \), as needed.
if the center is \((h, k)\), we simply translate the circle at the origin \(h\) units in the \(x\)-direction and \(k\) units in the \(y\)-direction. So, we get

\[ x = h + r \cos t, \quad y = k + r \sin t, \quad 0 \leq t \leq 2\pi. \]

Example 3: Find a parametrization of the line segment starting at the point \((x_0, y_0)\) and ending at the point \((x_1, y_1)\), where \(0 \leq t \leq 1\); use affine functions \(x(t)\) and \(y(t)\):

\[ x(t) \quad \text{and} \quad y(t); \]

\[ (f \text{ is affine if } f(t) = at + b, \; a,b \text{ constants}) \]

We want \(x = f(t) = at + b\), where \(f(0) = x_0, \; f(1) = x_1\);

So, \(x = f(t) = x_0 + t(x_1 - x_0)\).

We want \(y = g(t) = ct + d\), where \(g(0) = y_0, \; g(1) = y_1\).

So, \(y = g(t) = y_0 + t(y_1 - y_0)\). Therefore,

\[ (x, y) = (x_0 + t(x_1 - x_0), \; y_0 + t(y_1 - y_0)), \quad 0 \leq t \leq 1. \]
Example 6: Find a parametrization of the bottom part of the curve \( \frac{x^2}{4} + \frac{y^2}{9} = 1 \), which starts at \((x, y) = (2,0)\) and ends at \((x, y) = (-2, 0)\); here \(0 \leq t \leq \pi\).

Observe that \( \frac{x^2}{4} + \frac{y^2}{9} = 1 \) is the equation of an ellipse centered at \((0, 0)\), with a horizontal radius \(a = \sqrt{4} = 2\), and vertical radius \(b = \sqrt{9} = 3\). Here's a plot:

Think of an ellipse as a generalized circle. We still use trigonometric functions for \(x\) and \(y\). We have two choices:

1. \(x = 2 \sin t, \quad y = 3 \cos t \) \quad in both cases, \( \frac{x^2}{4} + \frac{y^2}{9} = 1 \)
2. \(x = 2 \cos t, \quad y = 3 \sin t \) \quad By the Pythagorean identity.

In Case 1, we start at \((2 \sin(0), 3 \cos(0)) = (0, 3)\) Incorrect.
In Case 2, we start at \((2 \cos(0), 3 \sin(0)) = (2, 0)\) and end at \((2 \cos(\pi), 3 \sin(\pi)) = (-2, 0)\). Also, at \(t = \frac{\pi}{2}\), we are at the point \((2 \cos(\pi/2), 3 \sin(\pi/2)) = (0, 3)\). This is a problem.
The point (0,3) is not in the bottom part of the ellipse.

We can fix this easily. Since at \( \frac{\pi}{2} \) we want to be at (0,-3),

let \( x = 2 \cos t, \quad y = -3 \sin t, \quad 0 \leq t \leq \pi \). Then we are done.