READ THE FOLLOWING INSTRUCTIONS.

- Do not open your exam until told to do so.
- No calculators, cell phones or any other electronic devices can be used on this exam.
- Clear your desk of everything except pens, pencils and erasers.
- If you need scratch paper, use the back of the previous page.
- Without fully opening the exam, check that you have pages 1 through 8.
- Fill in your name, etc. on this first page.

- Show all your work. Write your answers clearly! Include enough steps for the grader to be able to follow your work. Don’t skip limits or equal signs, etc. Include words to clarify your reasoning.
- Do first all of the problems you know how to do immediately. Do not spend too much time on any particular problem. Return to difficult problems later.
- If you have any questions please raise your hand and a proctor will come to you.
- There is no talking allowed during the exam.
- You will be given exactly 90 minutes for this exam.

I have read and understand the above instructions: ______________________

SIGNATURE
1. For each of the following series tell whether the indicated test determines whether the series converges or diverges. Be careful as no partial credit is given.

(a) (5 points) \( \sum_{n=2}^{\infty} \frac{2}{n(\ln n)^2} \)

A. The integral test concludes that the series converges.
B. The integral test concludes that the series diverges.
C. The integral test hypotheses are not met by this series, so it cannot be applied.
D. The integral test hypotheses are met by this series however the test is inconclusive.

(b) (5 points) \( \sum_{n=0}^{\infty} \frac{\sqrt{n}}{(n+3)(n+1)} \)

A. The comparison test concludes that the series converges.
B. The comparison test concludes that the series diverges.
C. The comparison test hypotheses are not met by this series, so it cannot be applied.
D. The comparison test hypotheses are met by this series however the test is inconclusive.

(c) (5 points) \( \sum_{n=1}^{\infty} \sin(n) \)

A. The integral test concludes that the series converges.
B. The integral test concludes that the series diverges.
C. The integral test hypotheses are not met by this series, so it cannot be applied.
D. The integral test hypotheses are met by this series however the test is inconclusive.

(d) (5 points) \( \sum_{n=1}^{\infty} \frac{1}{n} \)

A. The \( n^{th} \) term test concludes that the series converges.
B. The \( n^{th} \) term test concludes that the series diverges.
C. The \( n^{th} \) term test hypotheses are not met by this series, so it cannot be applied.
D. The \( n^{th} \) term test hypotheses are met by this series however the test is inconclusive.
1. (continued) For each of the following series tell whether the indicated test determines whether the series converges or diverges. Be careful as no partial credit is given.

(e) \( \sum_{n=0}^{\infty} \frac{4^n}{n!3^n} \)

A. The ratio test concludes that the series converges.
B. The ratio test concludes that the series diverges.
C. The ratio test hypotheses are not met by this series, so it cannot be applied.
D. The ratio test hypotheses are met by this series however the test is inconclusive.

(f) \( \sum_{n=0}^{\infty} \frac{n}{n+1} \)

A. The nth term test concludes that the series converges.
B. The nth term test concludes that the series diverges.
C. The nth term test hypotheses are not met by this series, so it cannot be applied.
D. The nth term test hypotheses are met by this series however the test is inconclusive.

2. (10 points) Determine whether the following sequences converge or diverge. If a sequence converges, find the limit. If the series sequence write “DNE”.

(a) \( \lim_{n \to \infty} \frac{n^2}{e^{2n}} = 0 \)

Solution: Apply L’Hopitals twice.

(b) \( \lim_{n \to \infty} \frac{n^2 + 2}{3 - 2n^2} = -1/2 \)

Solution: Apply L’Hopitals twice.
3. (18 points) Find the length of the curve $x = y^{3/2}$ from $y = 0$ to $y = 4$.

Solution:

$$L = \int_{0}^{4} \sqrt{1 + \left(\frac{3y^{1/2}}{2}\right)^{2}} \ dy$$

$$= \int_{0}^{4} \sqrt{1 + \frac{9y}{4}} \ dy$$

$$= \left[ \frac{8(1 + \frac{9y}{4})^{3/2}}{27} \right]_{0}^{4}$$

$$= \left[ \frac{8(10)^{3/2}}{27} \right] - \left[ \frac{8}{27} \right] = \frac{8}{27}(10^{3/2} - 1)$$

4. (14 points) Find the sum of the series $\sum_{n=0}^{\infty} \frac{4(-1)^n(3)^n}{(5)^n}$

Solution:

$$\sum_{n=0}^{\infty} \frac{4(-1)^n(3)^n}{(5)^n} = 4 \sum_{n=0}^{\infty} \left(\frac{-3}{5}\right)^n$$

$$= 4 \left(\frac{1}{1 + \frac{3}{5}}\right)$$

$$= 4 \left(\frac{5}{5 + 3}\right) = \frac{5}{2}$$
5. (20 points) Determine whether the following series converge or diverge. You must justify your answer with work and explicitly state which test(s) you use!

(a) \[ \sum_{n=1}^{\infty} (-1)^n \left( \frac{5}{2 + \sqrt{10}} \right)^n \]

Solution: Consider the ratio test:

\[ \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{\left( \frac{5}{2 + \sqrt{10}} \right)^{n+1}}{\left( \frac{5}{2 + \sqrt{10}} \right)^n} \cdot \frac{\frac{1}{2 + \sqrt{10}}}{\frac{5}{2 + \sqrt{10}}} \right| = \left( \frac{5}{2 + \sqrt{10}} \right) < 1 \]

So therefore the series converges. Another notable test that works is the Alternating Series Test.

(b) \[ \sum_{n=1}^{\infty} \frac{1}{(1 + n^{-1})^n} \]

Solution: Fractions over fractions are not so pretty so we re-write as:

\[ \frac{1}{(1 + n^{-1})^n} = \left( \frac{n}{n + 1} \right)^n \]

Now let’s consider the \( n \text{th} \) term test and the adjusted sequence: \( \lim_{n \to \infty} \ln \left( \frac{n}{n + 1} \right)^n \). We have:

\[ \lim_{n \to \infty} \ln \left( \frac{n}{n + 1} \right)^n = \lim_{n \to \infty} n \ln \left( \frac{n}{n + 1} \right) = \lim_{n \to \infty} \ln n - \ln(n + 1) = \lim_{n \to \infty} \frac{\ln n - \ln(n + 1)}{1/n} = \lim_{n \to \infty} \frac{1/n}{1/n - 1/(n + 1)} = \lim_{n \to \infty} \frac{n^2 - n(n + 1)}{n + 1} = \lim_{n \to \infty} \frac{-n}{n + 1} = -1 \]

Giving us

\[ \lim_{n \to \infty} \left( \frac{n}{n + 1} \right)^n = e^{\lim_{n \to \infty} \ln \left( \frac{n}{n + 1} \right)^n} = e^{-1} \neq 0 \]

So by the \( n \text{th} \) term test the series diverges. Alternatively clever folks may realize that

\[ \frac{1}{(1 + n^{-1})^n} = \left[ (1 + n^{-1})^{-1} \right]^n \]

and since \( 1/x \) is continuous when \( x > 0 \) we know that

\[ \lim_{n \to \infty} \left[ (1 + n^{-1})^{-1} \right]^{-1} = \left[ \lim_{n \to \infty} (1 + n^{-1})^n \right]^{-1} = [e]^{-1} \]

Giving the same result quicker.
6. (14 points) Find the Maclaurin series for \( f(x) = \frac{x^2}{2 + x} \) (Hint: Do not differentiate \( f(x) \)).

**Solution:** Consider rewriting as

\[
\frac{x^2}{2 + x} = \frac{x^2}{2} \left( \frac{1}{1 - (-x/2)} \right) = \frac{x^2}{2} \sum_{n=0}^{\infty} \left( \frac{-x}{2} \right)^n = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+2}}{2^{n+1}}
\]

7. (14 points) Find all values of \( x \) for which the series \( \sum_{n=1}^{\infty} \frac{(2x + 1)^n}{n} \) converges. 

(Hint: Make sure to test end points of the interval.)

**Solution:** Consider the ratio test:

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}/a_n} \right| = \lim_{n \to \infty} \left| \frac{(2x + 1)^{n+1}}{n + 1} \cdot \frac{n}{(2x + 1)^n} \right| = \lim_{n \to \infty} \left| (2x + 1) \frac{n}{n + 1} \right| = |2x + 1| < 1
\]

which is true on the interval \((-1, 0)\). Now test endpoints

at \( x = -1 \)

\[
\sum_{n=1}^{\infty} \frac{(2x + 1)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}
\]

Which converges because of the alternating series test.

at \( x = 0 \)

\[
\sum_{n=1}^{\infty} \frac{(2x + 1)^n}{n} = \sum_{n=1}^{\infty} \frac{1^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}
\]

Which diverges by the \( p \)-series test.

So therefore the series converges on \([-1, 0]\).
8. (14 points) Find the radius of convergence of the power series \[ \sum_{n=1}^{\infty} \frac{x^n}{n\sqrt{n}3^n} \]

**Solution:** Consider the ratio test:
\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{x^{n+1}}{(n+1)\sqrt{n+1}3^{n+1}} \cdot \frac{n\sqrt{n}3^n}{x^n} \right| \\
= \lim_{n \to \infty} \left| \frac{x}{3} \cdot \frac{n\sqrt{n}}{(n+1)\sqrt{n+1}} \right| \\
= \left| \frac{x}{3} \right| < 1
\]
Which is equivalent to \(|x| < 3\) giving us the solution \(R = 3\).

9. (16 points) Find the Taylor polynomial of degree 3 generated by \(f(x) = \sqrt{x}\) centered at \(a = 4\).

**Solution:** Calculate
\[
f(x) = \sqrt{x} \quad \implies f(4) = 2 \\
f'(x) = \frac{1}{2}x^{-1/2} \quad \implies f'(4) = \frac{1}{2}(4)^{-1/2} = 1/4 \\
f''(x) = -\frac{1}{4}x^{-3/2} \quad \implies f''(4) = -\frac{1}{4}(4)^{-3/2} = -1/32 \\
f'''(x) = \frac{3}{8}x^{-5/2} \quad \implies f'''(4) = \frac{3}{8}(4)^{-5/2} = 3/256
\]
Now we put them together to get:
\[
T_3(x) = 2 + \frac{(x-4)}{4} - \frac{(x-4)^2}{32 \cdot 2} + \frac{3(x-4)^3}{256 \cdot 6} \\
= 2 + \frac{(x-4)}{4} - \frac{(x-4)^2}{64} + \frac{(x-4)^3}{512}
\]
Congratulations you are now done with the exam! Go back and check your solutions for accuracy and clarity. Make sure your final answers are BOXED.

When you are completely happy with your work please bring your exam to the front to be handed in. Please have your MSU student ID ready so that is can be checked.

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