READ THE FOLLOWING INSTRUCTIONS.

• Do not open your exam until told to do so.

• No calculators, cell phones or any other electronic devices can be used on this exam.

• Clear your desk of everything except pens, pencils and erasers.

• If you need scratch paper, use the back of the previous page.

• Without fully opening the exam, check that you have pages 1 through 8.

• Fill in your name, etc. on this first page.

• Show all your work. Write your answers clearly! Include enough steps for the grader to be able to follow your work. Don’t skip limits or equal signs, etc. Include words to clarify your reasoning.

• Do first all of the problems you know how to do immediately. Do not spend too much time on any particular problem. Return to difficult problems later.

• You will be given exactly 90 minutes for this exam.

• If you have any questions please raise your hand and a proctor will come to you.

I have read and understand the above instructions: ____________________________

SIGNATURE
Fill in the Blanks. No work needed. No partial credit available.

1. (5 points) Evaluate: \( \lim_{n \to \infty} \sqrt[n]{5n} = 1 \)

2. (5 points) Evaluate as a simple formula: \( \sum_{n=0}^{\infty} e^{-4n} = \frac{1}{1-e^{-4}} = \frac{e^4}{e^4-1} \)

3. (21 points) For each series below, circle whether it converges or diverges, and fill in the blank with the name of a test which justifies this. The available tests for \( \sum a_n \) are:

- nth-Term Test (\( \lim_{n \to \infty} a_n \neq 0 \))
- Standard Series (\( \sum a_n \) is equal to a geometric or standard \( p \)-series)
- Integral Test
- Comparison Test (Direct or Limit, comparing to a standard series)
- Ratio Test

(a) \( \sum_{n=1}^{\infty} \frac{5 - 2n}{3 + 5n} \) converges / diverges because of nth-Term Test

(b) \( \sum_{n=2}^{\infty} \frac{1}{n \ln^2(n)} \) converges / diverges because of Integral Test

(c) \( \sum_{n=1}^{\infty} \frac{(n+1)2^n}{32n} \) converges / diverges because of Comparison Test

3 points for each correct answer on converge/diverge. 4 points for each correct reason. Note on (c) the ratio test also works.

4. (7 points) Is \( \sum_{n=1}^{\infty} (-1)^n \frac{2^n}{2n+1_n} \) absolutely convergent? Yes / No

because of Comparison Test

Extra Work Space.
5. (14 points) Set up an integral that represents the length of the curve \( y = \tan\left(\frac{1}{2}x\right) \) from the point \((0, 0)\) to the point \(\left(\frac{\pi}{2}, 1\right)\) (DO NOT EVALUATE THIS INTEGRAL.)

Solution:

\[
L(C) = \int_{a}^{b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx
\]

\[
= \int_{0}^{\frac{\pi}{2}} \sqrt{1 + \left(\frac{1}{2} \sec^2\left(\frac{1}{2}x\right)\right)^2} \, dx
\]

(Find \(dy/dx\))

\[
= \int_{0}^{\frac{\pi}{2}} \sqrt{1 + \sec^4\left(\frac{x}{2}\right)} \, dx
\]

(Put in endpoints)

\[
= \int_{0}^{\frac{\pi}{2}} \sqrt{1 + \frac{\sec^4\left(\frac{x}{2}\right)}{4}} \, dx
\]

(Simplify)

6. For the following series, determine if it converges or diverges. State which method you use, why it applies, and the computations needed to apply it.

(a) (12 points) \( \sum_{n=1}^{\infty} \frac{1}{n\sqrt{2n+1}} \)

Solution: \( \frac{1}{n\sqrt{2n+1}} \leq \frac{1}{n\sqrt{2n}} = \frac{1}{\sqrt{2} n^{3/2}}. \) Since the terms are positive we can apply the **Direct Comparison Test**. To compare to the series \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{2} n^{3/2}} \) which is convergent because it is a \( p \)-series with \( p = 3/2 > 1 \). Therefore \( \sum_{n=1}^{\infty} \frac{1}{n\sqrt{2n+1}} \text{ Converges.} \) (limit comparison test can also be applied)

(b) (12 points) \( \sum_{n=1}^{\infty} \frac{2n(n+1)}{n!} \)

Solution: By the **Ratio Test**

\[
\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{2(n+1)(n+2)/(n+1)!}{2n(n+1)/n!}\right| = \frac{(n+2)/(n+1)}{n} = \frac{n+2}{n(n+1)}
\]

Therefore the \( \lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| = 0 < 1 \) so the series **converges.**
7. Consider the power series:

\[-3 + 9(x+5) - 27(x+5)^2 + 81(x+5)^3 - \cdots\]

(a) (8 points) Write this series in sigma-notation as \(\sum_{n=0}^{\infty} a_n\), giving an explicit formula for \(a_n\).

**Solution:** It should be clear that we need the factors \((-3)^{n+1}\) and \((x+5)^n\) giving us our final solution

\[
\sum_{n=0}^{\infty} (-3)^{n+1}(x+5)^n
\]

(b) (4 points) What is the center point of this power series?

**Solution:** By the definition the center is \([a = -5]\).

(c) (12 points) Determine the interval of convergence for this power series.

**Solution:** By the ratio test:

\[
\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-3)^{n+2}(x+5)^{n+1}}{(-3)^{n+1}(x+5)^n}\right| = |-3(x+5)|
\]

Therefore for this power series to converge we need

\[
\lim_{n \to \infty} |-3(x+5)| < 1 \\
\lim_{n \to \infty} |x+5| < 1/3
\]

Giving us:

\[-1/3 < x + 5 < 1/3 \]
\[-16/3 < x < -14/3\]

Finally we check the endpoints to see that both yield \(\sum_{n=0}^{\infty} (-3)^{n+1} \left(\pm \frac{1}{3}\right)^n\) which does not converge in either case.

So we get our final answer \([-16/3, -14/3]\)
8. Consider the function \( f(x) = \frac{3}{2 + x} \).

(a) (8 points) Express \( f \) as a power series in sigma-notation.

**Solution:**

\[
\frac{3}{2 + x} = 3 \left( \frac{1}{2 + x} \right) \\
= \frac{3}{2} \left( \frac{1}{1 - \left( -\frac{x}{2} \right)} \right) \\
= \frac{3}{2} \sum_{n=0}^{\infty} \left( -\frac{x}{2} \right)^n
\]

(b) (6 points) What is the open interval of convergence for this power series? (You do not need to test the endpoints)

**Solution:**

\[
\left| -\frac{x}{2} \right| < 1 \\
|x| < 2
\]

So the open interval of convergence is \((-2, 2)\).
9. Consider the function \( g(x) = \frac{3}{(1-x)^3} \).

(a) (8 points) Express \( g \) as a power series in sigma-notation.

**Solution:**

\[
\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n
\]

\[
\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1} \quad \text{(Differentiate)}
\]

\[
\frac{2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1)x^{n-2} \quad \text{(Differentiate)}
\]

\[
\frac{3}{(1-x)^3} = \frac{3}{2} \sum_{n=2}^{\infty} n(n-1)x^{n-2} \quad \text{(Multiply by 3/2.)}
\]

(b) (6 points) What is the open interval of convergence for this power series?
(You do not need to test the endpoints)

**Solution:** We know that the radius of convergence for \( \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \) is 1. Therefore the radius of convergence for \( \frac{2}{(1-x)^3} = \sum_{n=0}^{\infty} n(n-1)x^{n-2} \) is also 1 giving us the radius of convergence for our solution \( \frac{3}{2} \sum_{n=0}^{\infty} n(n-1)x^{n-2} \) is 1. Since the center is \( x = 0 \) our final solution is \((-1, 1)\).
10. (22 points) Find the Taylor polynomial of degree 3 for the function \( f(x) = \sqrt{x + 7} \) around the point \( x = -3 \).

**Solution:** First let's collect the components

\[
\begin{align*}
c_0 &= f(-3) = \sqrt{4} = 2 \\
c_1 &= f'(-3) = \left. \frac{1}{2(x + 7)^{1/2}} \right|_{x=-3} = \frac{1}{4} \\
c_2 &= f''(-3)/2 = \left. \frac{-1}{8(x + 7)^{3/2}} \right|_{x=-3} = \frac{-1}{64} \\
c_3 &= f'''(-3)/6 = \left. \frac{1}{16(x + 7)^{5/2}} \right|_{x=-3} = \frac{1}{512}
\end{align*}
\]

Now we combine correctly to get the final answer

\[
T_3(x) = 2 + \frac{1}{4}(x + 3) - \frac{1}{64}(x + 3)^2 + \frac{1}{512}(x + 3)^3
\]
Congratulations you are now done with the exam!
Go back and check your solutions for accuracy and clarity. Make sure your final answers are BOXED.

When you are completely happy with your work please bring your exam to the front to be handed in. Please have your MSU student ID ready so that is can be checked.

DO NOT WRITE BELOW THIS LINE.

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