On normalizers of nilpotent subgroups

Bernd Baumann
Mathematisches Institut, Justus-Liebig-Universität,
Arndtstr. 2, 35392 Gießen, Deutschland
Bernd.Baumann@math.uni-giessen.de

Ulrich Meierfrankenfeld
Department of Mathematics, Michigan State University
East Lansing, MI 48840, USA
meier@math.msu.edu

Introduction

Let G be a group and Γ a collection of nilpotent subgroups of G satisfying:

(C) *$P^g \in \Gamma$ for $P \in \Gamma$ and $g \in G$.*
(I) *$P \cap Q \in \Gamma$ for $P, Q \in \Gamma$.*
(P) *$N_P(Q) \cdot N_Q(P) \in \Gamma$ for $P, Q \in \Gamma$.*
(MM) *The minimum and the maximum condition hold for Γ (i.e. each non empty subset of Γ contains a minimal and a maximal element with respect to inclusion of sets).*

Then we call Γ a *nilpotent subgroup system* of G (NSS for short) and the members of Γ we call Γ-*subgroups* of G (here $P^g := \{x^g \mid x \in P\}$, where $x^g := g^{-1}xg$, is a conjugate of P and $N_X(Y)$ is the normalizer of Y in X).

The set of all nilpotent subgroups of a group is an example of a system satisfying **(C)**, **(I)** and **(P)**. Examples of NSS's are the set of p-subgroups of a finite group (p a prime), the set of closed unipotent subgroups of an algebraic group, and the set of maximal cyclic subgroups plus the trivial group in a free group.

To state our main theorem we introduce a good portion of the notations used in this paper. Let Σ be a set of subgroups of G.

Σ^* is the set of maximal elements of Σ (with respect to inclusion). The elements of Γ^* are called *maximal* Γ-*subgroups*.

Σ_* is the set of minimal non-trivial elements of Σ. The elements of Γ_* are called *minimal* Γ-*subgroups*.

If U is a subgroup of G set $\Sigma U := \{A \in \Sigma \mid A \leq U\}$.

$\mathrm{R}(\Gamma) := \bigcap_{P \in \Gamma^*} P$ is called the *radical* of Γ.

If $\mathrm{R}(\Gamma) = 1$ the NSS Γ is called *reduced*.

Let $P \in \Gamma$. Then $\Gamma_P := \{T \in N_\Gamma(P) \mid TP \in \Gamma\}$ is the *residue* of P in Γ. It turns out that Γ_P is an NSS for $N_G(P)$, see Proposition 2.8(1).

Set $P^\circ := \mathrm{R}(\Gamma_P)$ and call P *closed* if $P = P^\circ$.

Note that by **(MM)** any chain of Γ-subgroups is finite. Let $\mathrm{rank}(\Gamma)$ be the supremum of the lengths of chains

$$P_0 < P_1 < \ldots < P_n$$

of closed Γ-subgroups. (The length of such a chain is n).

$\Omega(P) := \langle \Gamma_* P \rangle$ is the subgroup of P generated by the minimal Γ-subgroups of P.

P is called *decomposable* if $P = \Omega(P)$.

$\mu(P)$ is the length of a maximal chain in ΓP. By Proposition 5.2 this is well defined. $\mu(P)$ is called the *measure* of P. If $Q \in \Gamma P$, then $\mu(P/Q) = \mu(P) - \mu(Q)$. By Proposition 5.4(1), this is the length of any maximal Γ-chain from Q to P.

Let $A \in \Gamma_P$. If $[[P, A]A] = 1$, we say that A *acts quadratically* on P. If A and P both are decomposable abelian Γ-subgroups, $[P, A] \neq 1$ and

$$\mu(P/C_P(A)) \leq \mu(A/C_A(P))$$

then A is called a non-trivial Γ-*offender* on P. Note here that by Proposition 4.7 both $C_P(A)$ and $C_A(P)$ are Γ-subgroups.

Let V be a normal Γ-subgroup of G with $V \leq \Omega(Z(\mathrm{R}(\Gamma))$ and put $W = V/C_V(\langle \Gamma \rangle)$. We say that W is a *natural SL_2-module* for Γ provided that

(i) W is the set of points and $\{wC_W(S) \mid S \in \Gamma^*\}$ is the set of lines of an affine Moufang plane;

(ii) For each $S \in \Gamma^*$, $C_S(W) = \mathrm{R}(\Gamma)$ and S induces the group of shears on W with axis $C_W(S)$; and

(iii) $\langle \Gamma \rangle$ induces on W the subgroup of a point stabilizer (of the point 1) generated by all shears.

We say that $N \in \Gamma$ is *large* in Γ provided that N is closed and $C_P(N) \leq N$ for all $P \in \Gamma_N$.

A theorem of Glauberman's [5, Theorem 2] characterizes finite two dimensional special linear groups as groups acting on p-groups with certain features. The object of the present paper is to prove the following generalization of Glauberman's Theorem:

Theorem A *Let G be a group with an NSS Γ. Assume:*

(a) $\mathrm{rank}(\Gamma) = 1$.

(b) V is a normal Γ-subgroup of G with $V \leq \Omega(Z(\mathrm{R}(\Gamma)))$.

(c) $S \in \Gamma^*$ and $[C_G(V), S] \leq \mathrm{R}(\Gamma)$.

(d) S contains a non-trivial Γ-offender on V.

(e) $\mathrm{R}(\Gamma)$ *is large in* Γ.

Then $V/C_V(\langle\Gamma\rangle)$ is a natural SL_2-module for Γ.

It is well known that an affine Moufang plane is isomorphic to a plane whose point set consists of the ordered pairs of an alternative field or a skew field K and whose lines are the point sets $L(a,b) := \{(x, x \cdot a + b) \mid x \in K\}$ and $L(c) := \{(c, y) \mid y \in K\}$. Then for example shears with axis $L(0)$ are the mappings $(x, y) \mapsto (x, x \cdot d + y)$ (see [4] page 128 ff. and the literature quoted there).

For the proof of Theorem A see section 6 and 7 and for other main results of this paper see section 8.

We would like to thank G. Glauberman for pointing out an error in an earlier version of this paper.

1 Preliminaries

In this section we collect some elementary results about nilpotent groups. We start with some well known commutator properties (see for instance [6]).

Proposition 1.1 *Let a, b, c be elements, A, B, C subgroups and N a normal subgroup of a group. Then*

(1) $[a, bc] = [a, c][a, b][[a, b], c] = [a, c][a, b]^c$

(2) $[ab, c] = [a, c][[a, c], b][b, c] = [a, c]^b[b, c]$

(3) $[a, b] = [b, a]^{-1} = [b, a^{-1}][[b, a^{-1}], a]$

(4) $ab[b, a] = ba$

(5) $[[B, C], A] \subseteq N$ *and* $[[C, A], B] \subseteq N$ *imply* $[[A, B], C] \subseteq N$.

(6) $[A, B]$ *is a normal subgroup of* $\langle A, B\rangle$.

Proposition 1.2 *Let G be a group, V an abelian normal subgroup of G, U a subgroup of V and $g \in G$ with $[[V, g], g] = 1$. Then the following hold:*

(1) $\{[u, g] \mid u \in U\}$ *is a subgroup of* V

(2) $UU^g = U[U, g]$

(3) $UU^g = U \times [U, g]$ *if and only if* $U \cap U^g = C_U(g)$

(4) $C_{UU^g}(g) = C_{U\cap U^g}(g)[U, g]$

Proof. These properties are applications of Proposition 1.1. □

Proposition 1.3 *Let A and B be subgroups and let N be a normal subgroup of the group G. Then*

$$[N, \langle A, B\rangle] = \langle [N, A], [N, B]\rangle.$$

Proof. Obviously the right hand side is contained in the left hand side. Conversely, by Proposition 1.1(6) $M := \langle [N, A], [N, B]\rangle$ is a normal subgroup of $\langle A, B, N\rangle$ contained in N, as N is a normal subgroup. Now N/M is centralized by $\langle A, B\rangle$, whence $[N, \langle A, B\rangle] \subseteq M$. □

Let A be a group acting on a group D. We say that A acts *nilpotently* on D if $[D, A, k] = 1$ for some k (where $[D, A, 0] := D$ and $[D, A, i + 1] := [[D, A, i], A]$). The minimal such k is called the *nilpotence length of A on D*. For a group G let $L_0(G) = G$ and $L_{i+1}(G) = [L_i(G), G]$.

Lemma 1.4 *(1) Suppose A acts nilpotently on D. Then $A/C_A(D)$ is nilpotent.*

(2) Suppose G acts on $D, A \leq G$ and $B \leq N_G(A)$. If A and B act nilpotently on D, so does AB.

(3) Let N be normal in G. Then G is nilpotent if and only if G/N is nilpotent and G acts nilpotently on N.

(4) Let $G = AB$, where A and B are nilpotent subgroups of G, and A is normal in G. Assume N is a normal subgroup of G with $N \leq A \cap B$ such that G/N is nilpotent. Then G is nilpotent.

(5) Let A, B be normal in G such that G/A and G/B are nilpotent. Then $G/A \cap B$ is nilpotent.

Proof. (1) See [7, Corollary to Theorem 3.8]

(2) By induction on the nilpotency length of A on D $[[D, A], AB, i] = 1$ for some i. Also if $[D, B, j] = 1$, then $[D, AB, j] \subseteq [D, A]$ and so $[D, AB, i+j] = 1$.

(3) One direction is obvious. So suppose G/N is nilpotent and G acts nilpotently on N. Then $L_k(G) \leq N$ for some k and $[N, G, i] = 1$ for some i. Thus $L_{k+i}(G) = 1$.

(4) Since $N \leq A \cap B$, both A and B act nilpotently on N. By (2) G acts nilpotently on N and so (4) follows from (3).

(5) Let k be the maximum of the nilpotency classes of G/A and G/B. Then $L_k(G) \leq A \cap B$. □

Proposition 1.5 *Let P and Q be nilpotent subgroups of the group G with $Q \subseteq PC_G(P)$. Then PQ is a nilpotent subgroup of G.*

Proof. Clearly P is normal in PQ and PQ acts nilpotently on P. Also $PQ/P \cong Q/Q \cap P$ and so PQ/P is nilpotent. Hence the lemma follows from Lemma 1.4(3). □

Proposition 1.6 *Let X be a proper subgroup of the nilpotent group G.*

(1) X is contained in a proper normal subgroup of G.

(2) X is a proper subgroup of $N_G(X)$.

(3) If $N_G(X) = N_G(N_G(X))$, then X is normal in G.

(4) $\langle X^G \rangle$ is a proper subgroup of G.

Proof. Well-known. □

Proposition 1.7 *Let H be a nilpotent group of class k and $x, y \in H$, where x is an element of order p, p a prime. Then $[x, y^{p^{k+1}}] = 1$.*

Proof. By induction on k , $[x, y^{p^k}] \in Z(H)$. Then by Proposition 1.1

$$1 = [x^p, y^{p^k}] = [x, y^{p^k}]^p = [x, y^{p^{k+1}}].$$

□

Proposition 1.8 *Let X be a subgroup of the group G and let U and A be subsets of G with $U \subseteq X$. Then $(UA) \cap X = U(A \cap X)$.*

Proof. Let $u \in U$ and $a \in A$ with $ua \in X$. Then $a \in A \cap X$, hence $(UA) \cap X \subseteq U(A \cap X)$. If $d \in A \cap X$ then $ud \in (UA) \cap X$ as $U(A \cap X) \subseteq X$. Thus $U(A \cap X) \subseteq (UA) \cap X$ and Proposition 1.8. □

2 Basic Properties of NSS's

In this section G is a group with an NSS Γ with $1 \in \Gamma$.

We remark that **(MM)** allows us to prove statements about Γ by induction. Namely suppose given a statement $\mathcal{S}$ about Γ-subgroups. Suppose also that if $P \in \Gamma$ and $\mathcal{S}$ is true for all $Q \in \Gamma$ with $Q < P$, then $\mathcal{S}$ is also true for P. Then $\mathcal{S}$ must be true for all $P \in \Gamma$. Indeed the set of Γ-subgroups for which $\mathcal{S}$ is false, does not have a minimal element and so is empty.

Note also that **(I)** and **(MM)** imply, that arbitrary intersections of Γ-subgroups are Γ-subgroups.

Lemma 2.1 *Let $P, Q \in \Gamma$. Then $N_P(Q) \in \Gamma$.*

Proof. Note that $N_Q(P) \cap P \subseteq Q \cap P \subseteq N_P(Q)$ and so by Proposition 1.8

$$(N_P(Q)N_Q(P)) \cap P = N_P(Q)(N_Q(P) \cap P) = N_P(Q).$$

By **(P)** and **(I)** the left hand side of this equation is in Γ. $\square$

Proposition 2.2 *Let $P, Q \in \Gamma$ such that Q is a minimal element of $\{T \in \Gamma \mid P < T\}$ or that P is a maximal element of $\{T \in \Gamma \mid T < Q\}$. Then P is normal in Q.*

Proof. Note that the two conditions are actually equivalent. So suppose the first. By Lemma 2.1 $P < N_Q(P) \in \Gamma$ and so $Q = N_Q(P)$ by minimality of Q. $\square$

Proposition 2.3 *(1) If Δ is a nonempty subset of Γ, then $\bigcap_{X\in\Delta} X \in \Gamma$ and $\bigcap_{X\in\Delta} X = \bigcap_{X\in\Delta_0} X$ for some finite subset Δ_0 of Δ.*

(2) If Δ is a set of normal Γ- subgroups of G, then $\langle\Delta\rangle \in \Gamma$.

(3) If U is a subgroup of G, then ΓU is an NSS of U.

(4) $P^\circ \in \Gamma$ for all $P \in \Gamma$. In particular, $\mathrm{R}(\Gamma)$ is a normal Γ-subgroup of G.

(5) If $S \in \Gamma$ and $P \in \Gamma S \backslash \{S\}$, then $P \subset \langle \Gamma_P S\rangle$.

(6) If $\langle\Delta\rangle$ is nilpotent for $\Delta \subseteq \Gamma$, then $\langle\Delta\rangle \in \Gamma$.

(7) $\mathrm{R}(\Gamma) = \langle \bigcap_{T\in\Gamma^*} N_\Gamma(T)\rangle$.

(8) Let $S \in \Gamma^*$ and $A \in \Gamma(SC_G(S))$. Then $A \leq S$.

(9) Let $S \subseteq G$ be nilpotent and put $A = \langle \Gamma S\rangle$. Then $A \in \Gamma$, $\Gamma^* S = \{A\}$ and A is normal in $N_G(S)$.

Proof.

(1) By **(I)** intersections of the members of finite subsets of Δ are elements of Γ. Then (1) follows from the minimal condition for Γ applied to the set of intersections of the members of finite subsets of Γ.

(2) If N and M are normal Γ-subgroups then $NM \in \Gamma$ by **(P)**. Hence finite products of elements of Δ lie in Γ, and (2) follows from the maximal condition for Γ.

(3) is obvious by the definition of an NSS.

(4) is a consequence of (1).

(5) By Proposition 1.6(2) $P < N_S(P)$ and by Lemma 2.1 $N_S(P) \in \Gamma$.

(6) Let $S = \langle \Delta \rangle$ and without loss $\Delta = \Gamma S$. Let $P \in \Delta^*$. If P is not normal in S, then Proposition 1.6(c) there exists $x \in N_S(N_S(P))$ with $P \neq P^x$. By **(C)** and **(P)** we get $PP^x \in \Gamma S$, a contradiction to the maximality of P. So P is normal in S. Thus by (2) $S = \langle \Delta \rangle = \langle \Delta^* \rangle \in \Gamma$.

(7) Let

$$\Lambda := \bigcap_{T \in \Gamma^*} N_\Gamma(T) = \{A \in \Gamma \mid A \leq N_G(T) \, \forall T \in \Gamma^*\}.$$

We claim that $|\Lambda^*| = 1$. Indeed, let $X_1, X_2 \in \Lambda^*$ and pick $T_i \in \Gamma^*$ with $X_i \leq T_i$. By (6), $\langle \Lambda T_i \rangle \in \Gamma$ and so the definition of Λ implies $\langle \Lambda T_i \rangle \in \Lambda$. The maximality of X_i implies $X_i = \langle \Lambda T_i \rangle$. Hence $X_1 \leq N_G(T_2) \leq N_G(\langle \Lambda T_2 \rangle) \leq N_G(X_2)$. So X_1 normalizes X_2 and X_2 normalizes X_1. Thus by **(P)**, $X_1 X_2 \in \Gamma$. Hence also $X_1 X_2 \in \Lambda$ and $X_1 = X_2$.

So indeed $|\Lambda^*| = 1$. Let N be the unique element in Λ^*. Then N is normal in G. Let $T \in \Gamma^*$. The definition of Λ implies that N normalizes T. So by **(P)**, $NT \in \Gamma$. Thus $N \leq T$ and $N \leq \mathrm{R}(\Gamma)$. Clearly $\mathrm{R}(\Gamma) \leq N$ and (7) holds.

(8) Obviously S is contained in the right hand side of this equation. Let $P \in \Gamma(SC_G(S))$. Then SP is nilpotent by Proposition 1.5 and therefore $SP \in \Gamma$ by (6). Hence $P \subseteq S$ because S is maximal.

(9) By (6) we get $A \in \Gamma$, which implies $\Gamma^* S = \{A\}$, and by (C) A is normal in $N_G(S)$.

$\square$

Definition. A subset Δ of Γ is called a *sub-NSS* of Γ and we write $\Delta \leq \Gamma$ provided that:

(Suba) If $A \in \Gamma$ and $B \in \Delta$ with $A \subseteq B$ then $A \in \Delta$.

(Subb) If $A, B \in \Delta$ with $\langle A, B \rangle \in \Gamma$ then $\langle A, B \rangle \in \Delta$.

(Subc) If $A, B \in \Delta$ then $A^B \subseteq \Delta$.

Lemma 2.4 *Let $\Delta \leq \Gamma$, then Δ is an NSS for $\langle \Delta \rangle$.*

Proof. **(C)** follows from (Subc). Let $P, Q \in \Delta$. Then since **(I)** holds for Γ, $P \cap Q \in \Gamma$. So by (Suba), $P \cap Q \in \Delta$. So **(I)** holds. By Lemma 2.1, $N_P(Q)$ and $N_Q(P)$ are Γ-subgroups. So by (Suba), they are also Δ-subgroups. By **(P)** for Γ, $N_P(Q)N_Q(P) \in \Gamma$ and so by (Subb), $N_P(Q)N_Q(P) \in \Delta$. Thus **(P)** holds. **(MM)** follows from **(MM)** for Γ. $\square$

Lemma 2.5 *Let $\Delta \leq \Gamma$.*

(1) $\mathrm{R}(\Delta) \in \Delta$, *and* $\mathrm{R}(\Delta)$ *is normal in* $\langle \Delta \rangle$

(2) If $A \in \Delta$ then $A\,\mathrm{R}(\Delta) \in \Delta$.

(3) $\Delta \leq \Gamma_{\mathrm{R}(\Delta)}$.

(4) Let $\Lambda \leq \Delta$. Then

(i) $\mathrm{R}(\Delta) \cap S = \mathrm{R}(\Delta) \cap \mathrm{R}(\Lambda)$ *for all* $S \in \Lambda^*$.

(ii) $\mathrm{R}(\Lambda) \cap \mathrm{R}(\Delta)$ *is the unique maximal Λ-subgroup of* $\mathrm{R}(\Delta)$.

(iii) Λ-subgroups of $\mathrm{R}(\Delta)$ *are contained in* $\mathrm{R}(\Lambda)$.

(5) Let $\Lambda \leq \Delta$ with $\mathrm{R}(\Delta) \in \Lambda$. *Then* $\mathrm{R}(\Delta) \leq \mathrm{R}(\Lambda)$.

(6) Suppose that $\Lambda \leq \Delta \leq \Gamma_{\mathrm{R}(\Lambda)}$ and $\mathrm{R}(\Delta) \in \Lambda$. *Then* $\mathrm{R}(\Lambda) = \mathrm{R}(\Delta)$.

(7) $\mathrm{R}(\Delta)$ *is closed in Γ if and only if* $\mathrm{R}(\Delta)^\circ \in \Delta$.

Proof.
(1) follows from Proposition 2.3(4) applied to the NSS Δ.

(2) By **(MM)** there exists $S \in \Delta^*$ with $A \subseteq S$. By Proposition 2.3(6) $A\,\mathrm{R}(\Delta) \in \Gamma$ and so by (Subb), $A\,\mathrm{R}(\Delta) \in \Delta$.

(3) Follows from (1) and (2).

(4) Let $S, T \in \Lambda^*$. By (2), $T\,\mathrm{R}(\Delta) \in \Delta$ and so by Proposition 2.3(6) also $T(\mathrm{R}(\Delta) \cap S) \in \Delta$. By **(I)** and (Suba), $\mathrm{R}(\Delta) \cap S \in \Gamma$ and so (Subb)

implies $T(\mathrm{R}(\Delta) \cap S) \in \Lambda$. Thus by maximality of T, $\mathrm{R}(\Delta) \cap S \subseteq T$. So $\mathrm{R}(\Delta) \cap S \subseteq \mathrm{R}(\Lambda)$. So (i) holds. (ii) and (iii) follow from (i).

(5) Follows from (4).

(6) By (5) $\mathrm{R}(\Delta) \leq \mathrm{R}(\Lambda)$. Note that $\mathrm{R}(\Lambda) \in \Lambda \leq \Delta$. Thus $\mathrm{R}(\Lambda)$ is a Δ-subgroup of $\mathrm{R}(\Gamma_{\mathrm{R}(\Lambda)})$ and so by (4)(iii) applied to $\Delta \leq \Gamma_{\mathrm{R}(\Lambda)}, \mathrm{R}(\Lambda) \leq \mathrm{R}(\Delta)$.

(7) If $\mathrm{R}(\Delta) = \mathrm{R}(\Delta)^\circ$, then $\mathrm{R}(\Delta)^\circ \in \Delta$ by (1). So suppose $\mathrm{R}(\Delta)^\circ \in \Delta$. Then by (5) applied to $\Delta \leq \Gamma_{\mathrm{R}(\Delta)}$, $\mathrm{R}(\Delta)^\circ \leq \mathrm{R}(\Delta)$. So $\mathrm{R}(\Delta)$ is closed. □

Lemma 2.6 *Let $P \in \Delta \leq \Gamma$ such that $P = \mathrm{R}(\Gamma_P \cap \Delta)$. Then*

(1) $\mathrm{R}(\Delta) \subseteq P$.

(2) If $\Gamma_P^* \cap \Delta \neq \emptyset$, then $\mathrm{R}(\Delta)$ is closed.

Proof. Let $T = \mathrm{R}(\Delta)$.

(1) Since $P \in \Delta$, Lemma 2.5(2) implies $PT \in \Delta$. Hence by Lemma 2.1, $N_T(P) \in \Delta$. Let $S \in (\Gamma_P \cap \Delta)^*$. Then again by Lemma 2.5(2), $ST \in \Delta$. Hence by Proposition 2.3(6), $N_T(P)S \in \Gamma_P \cap \Delta$. By maximality of S, $N_T(P) \subseteq S$. Thus $N_T(P) \leq \mathrm{R}(\Gamma_P \cap \Delta) = P$. Since TP is nilpotent we conclude $T \subseteq P$.

(2) By Lemma 2.5(3), $\Delta \leq \Gamma_T$. Thus

$$(*) \quad \Gamma_P \cap \Delta \leq \Gamma_P \cap \Gamma_T \leq \Gamma_P.$$

Let $Q = \mathrm{R}(\Gamma_P \cap \Gamma_T)$. By assumption there exists $S \in \Gamma_P^* \cap \Delta$. Then $S \in (\Gamma_P \cap \Gamma_T)^*$ and so $Q \subseteq S$. Hence by (Suba), $Q \in \Gamma_P \cap \Delta$. By (*) we can apply Lemma 2.5(6) (with $\Lambda = \Gamma_P \cap \Delta$ and $\Delta = \Gamma_P \cap \Gamma_T$) Thus $Q = \mathrm{R}(\Gamma_P \cap \Delta) = P$. So by (1) (applied to Γ_T in place of Δ), $T^\circ \subseteq P$ and thus $T^\circ \in \Delta$. By Lemma 2.5(7), $T = \mathrm{R}(\Delta)$ is closed. □

Corollary 2.7 *Suppose that $N \in \Gamma$ is closed and $\Gamma_N \leq \Delta \leq \Gamma$. Then $\mathrm{R}(\Delta) \leq N$ and $\mathrm{R}(\Delta)$ is closed.*

Proof. Since N is closed and $\Gamma_N = \Gamma_N \cap \Delta$ we have $N = \mathrm{R}(\Gamma_N \cap \Delta)$. Also $\Gamma_N^* \subseteq \Gamma_N \subseteq \Delta$ and so $\Gamma_N^* \cap \Delta \neq \emptyset$. Thus the Corollary follows from Lemma 2.6. $\square$

Definition. If Q is a normal Γ-subgroup of G contained in $\mathrm{R}(\Gamma)$ we define

$$\Gamma/Q := \{PQ/Q \mid P \in \Gamma\}.$$

Note that $\Gamma/Q = \{P/Q \mid Q \leq P \in \Gamma\}$.

Proposition 2.8 *Let $L \in \Gamma$. Then the following hold:*

(1) Γ_L resp. Γ_L/L is an NSS of $N_G(L)$ resp. $N_G(L)/L$.

(2) $L \leq L^\circ$.

(3) $\Gamma = \Gamma_{\mathrm{R}(\Gamma)}$.

(4) $\mathrm{R}(\Gamma_L/L) = \mathrm{R}(\Gamma_L)/L$.

(5) $\Gamma/\mathrm{R}(\Gamma)$ is reduced.

(6) L is closed in Γ if and only if 1 is closed in Γ_L/L.

(7) If L is closed then $L = \bigcap\{S \in \Gamma^* \mid L \subseteq S\}$.

(8) $\Gamma_L \subseteq \Gamma_{L^\circ}$.

(9) If $M \in \Gamma$ with $\Gamma_L \leq \Gamma_M$, then $N_M(L) \leq L^\circ$. If in addition $L^\circ \leq M$, then $L^\circ = N_M(L)$.

(10) There is some (not necessarily uniquely determined) closed Γ -subgroup M with $L \subseteq M, \Gamma_L \subseteq \Gamma_M$ and $L^\circ = N_M(L)$.

(11) $L^\circ = N_{L^{\circ\circ}}(L)$.

(12) Let $S \in \Gamma^*$ and L be a normal Γ-subgroup of S. Then $L^\circ = L^{\circ\circ}$ is closed.

Proof.
(1) Let $P, Q \in \Gamma_L$. Then $(P \cap Q)L \subseteq PL \cap QL \in \Gamma$ by **(I)**. Hence $(P \cap Q)L \in \Gamma$ by Proposition 2.3(6) and $P \cap Q \in \Gamma_L$. Similarly $N_P(Q)N_Q(P)L \subseteq N_{PL}(QL)N_{QL}(PL) \in \Gamma$ by **(P)** and therefore $N_P(Q)N_Q(P)L \in \Gamma$ implying $N_P(Q)N_Q(P) \in \Gamma_L$. Condition **(MM)** is satisfied for Γ_L as $\Gamma_L \subseteq \Gamma$, and **(C)** follows for Γ_L as **(C)** holds for Γ and thus $P^g L \in \Gamma$ if $P \in \Gamma_L$ and $g \in N_G(L)$. Thus Γ_L and Γ_L/L are NSS's.

(2) and (3) are obvious.

(4) follows from $(\Gamma_L/L)^* = \Gamma^*/L := \{S/L \mid S \in \Gamma_L^*\}$.

(5) is a consequence of (4).

(6) is clear by (5) and (2).

(7) Put $D := \bigcap\{S \in \Gamma^* \mid L \subseteq S\}$. Let $T \in \Gamma_L^*$ and pick $S \in \Gamma^*$ with $T \subseteq S$. Then $T \subseteq N_S(L) \in \Gamma_L$ by Lemma 2.1 and so $T = N_S(L)$. Since $D \subseteq S$ we conclude $N_D(L) \subseteq T$. As this is true for all $T \in \Gamma_L^*$, $N_D(L) \subseteq \mathrm{R}(\Gamma_L) = L$. Since $L \subseteq D$ and D is nilpotent, $L = D$.

(8) If $P \in \Gamma_L$ then there is $Q \in \Gamma_L^*$ with $P \subseteq Q$, hence $PL^\circ \in \Gamma Q \subseteq \Gamma$ by Proposition 2.3(6), and $P \in \Gamma_{L^\circ}$.

(9) Note that $N_M(L) \in \Gamma_L$ and $N_M(L) \leq M \leq \mathrm{R}(\Gamma_M)$. Thus by Lemma 2.5(4), $N_M(L) \leq \mathrm{R}(\Gamma_L) = L^\circ$. If $L^\circ \leq M$, then $L^\circ \leq N_M(L) \leq L^\circ$ and so $L^\circ = N_M(L)$.

(10) Let M in Γ be maximal with respect to $L^\circ \leq M$ and $\Gamma_L \subseteq \Gamma_M$. Note that by (2) and (8) such an M exists. By (2) and (8) applied to M, $L^\circ \leq M \leq M^\circ$ and $\Gamma_L \subseteq \Gamma_M \subseteq \Gamma_{M^\circ}$. Thus the maximal choice of M implies $M = M^\circ$. So M is closed. By (9), $N_M(L) = L^\circ$ and all parts of (10) are verified.

(11) Follows from (2),(8) and (9).

(12) As $S \in \Gamma^*$ we get $S \in \Gamma_L^*$. It follows that L° is normal in S and thus $L^{\circ\circ} \leq S$. Hence L is normal in $L^{\circ\circ}$. So by (12) $L^\circ = N_{L^{\circ\circ}}(L) = L^{\circ\circ}$.

$\square$

Lemma 2.9 *Let $N \in \Gamma$ and $P, Q \in \Gamma_N$. If $[C_P(N), \langle P, Q\rangle] \subseteq N$ then $N_Q(P)P \in \Gamma$.*

Proof. By Lemma 2.1 we may assume that $Q = N_Q(P)$. So Q normalizes P. Since PN and QN are in Γ they are both nilpotent. So P and Q act nilpotently on N. By Lemma 1.4(2) PQ acts nilpotently on N. Thus by Lemma 1.4(1), $PQ/C_{PQ}(N)$ is nilpotent. Also $PQ/P \cong Q/Q \cap P$ is nilpotent and so by Lemma 1.4(5) $PQ/C_P(N)$ is nilpotent. Since $[C_P(N), PQ] \subseteq N$ we get that PQ acts nilpotently on $C_P(N)$. Thus the assertion follows from Lemma 1.4(3). $\square$

Proposition 2.10 *Let $G = \langle A, B\rangle$, where A and B are nilpotent subgroups of G. Assume $A \in \Gamma$, N is a normal subgroup of G, $N \subseteq A \cap B$ and G/N is nilpotent. Then G is nilpotent.*

Proof. By **(MM)** A can be chosen maximal fulfilling the assumptions of the Proposition. Then by nilpotency of G/N and **(P)** A is normal in G and Proposition 2.10 follows from Lemma 1.4(4). $\square$

Proposition 2.11 *Let $S \in \Gamma^*$ be fixed.*

(1) Let $T \in \Gamma^* \backslash \{S\}$ such that $S \cap T$ is maximal. Then $S \cap T$ is closed.

(2) Let $T \in \Gamma^* \backslash \{S\}$, then there exists a closed Γ-subgroup P with $S \cap T \leq P < S$.

Proof. (1) Set $P := S \cap T$ Then $P^\circ N_S(P) \in \Gamma$ by definition of Γ_P and 1.10(6). Therefore there is $X \in \Gamma^*$ with $P^\circ N_S(P) \subseteq X$. By maximality of S, $S \nleq T$ and so $P < S$. Hence by Proposition 1.6(3), $P < N_S(P) \leq X \cap S$. By maximality of P, $X = S$. Thus $P^\circ \leq S$. Note also that $N_T(P)P^\circ \leq Y$ for some $Y \in \Gamma^*$. Since $P < N_T(P)$, $N_T(P) \nleq S$ and so $Y \neq S$. Thus by maximality of P, $Y \cap S = P$. Since $P^\circ \leq Y \cap S$ we get $P^\circ = P$ and P is closed.

(2) Let $T^* \in \Gamma^* \backslash \{S\}$ with $S \cap T \leq S \cap T^*$ and $S \cap T^*$ maximal. Then $S \cap T^*$ is closed by (1). $\square$

The following statement is a variant of Baer's famous theorem [1].

Theorem 2.12 *Let $X \in \Gamma$ such that $\langle X, X^g\rangle \in \Gamma$ for all $g \in G$, then $\langle X^G\rangle \in \Gamma$.*

Proof. Set $\Delta := X^G$ and assume $\langle \Delta \rangle \notin \Gamma$. Then there are $Q = \langle \Delta Q \rangle \in \Gamma$ and $R = \langle \Delta R \rangle \in \Gamma$ with $\langle Q, R \rangle \notin \Gamma$. Choose Q and R such that $D := \langle \Delta(Q \cap R) \rangle$ is maximal. Suppose that $\Delta N_Q(D) = \Delta(Q \cap R)$. Then

$$N_Q(N_Q(D)) \leq N_Q(\langle \Delta N_Q(D) \rangle) = N_Q(D)$$

and so by Proposition 1.6(3), $N_Q(D) = Q$. But $\Delta Q \neq \Delta(Q \cap R)$, a contradiction. Thus there exists $A \in \Delta$ with $A \not\leq N_Q(D)$ and $A \leq D$. Similarly there exists $B \in \Delta$ with $B \leq N_R(D)$ and $B \not\leq D$. By assumption $\langle A, B \rangle \in \Gamma$. By Proposition 2.10, applied with AD,BD and D in place of A, B and N, $P := \langle A, B, D \rangle$ is nilpotent. Since $D < AD \leq Q \cap P$, the maximality of D implies $\langle Q, P \rangle \in \Gamma$. Similarly $\langle R, P \rangle \leq \Gamma$. But $\langle R, P, Q \rangle \notin \Gamma$ and $D < P \leq \langle R, P \rangle \cap \langle Q, P \rangle$. This contradiction to the maximality of D completes the proof of Theorem 2.12. □

3 NSS' of rank 1 and 2

As in the previous section let G be a group with an NSS Γ.

Theorem 3.1 *Suppose $|\Gamma^*| > 1$. Then following properties are equivalent:*

(a) $\mathrm{rank}(\Gamma) = 1$.

(b) $S \cap T = \mathrm{R}(\Gamma)$ *for* $S, T \in \Gamma^*$ *with* $S \neq T$.

(c) $S \cap S^g = \mathrm{R}(\Gamma)$ *for* $S \in \Gamma^*$ *and* $g \in G \backslash N_G(S)$.

Proof. Suppose (a) holds. Let $S, T \in \Gamma^*$ with $P = S \cap T$ maximal. Then P is closed by Proposition 2.11. and so $\mathrm{R}(\Gamma) \leq P < S$ is a chain of closed Γ-subgroups. Since Γ has rank 1, we get $P = \mathrm{R}(\Gamma)$. Thus $S \cap T = \mathrm{R}(\Gamma)$ for all $S \neq T \in \Gamma^*$ and so (b) holds.

From **(C)** we get $S^g \in \Gamma^*$ for $S \in \Gamma^*$ and $g \in G$. Thus (b) implies (c).

Suppose that (c) holds. Let P be a closed Γ subgroup. We will show that $P = \mathrm{R}(\Gamma)$ or $P \in \Gamma^*$ and note that this implies (a).

Assume that $|\Gamma_P^*| = 1$. Since P is closed we get $P \in \Gamma_P^*$. Let $P \leq S \in \Gamma^*$. Then $P \leq N_S(P) \in \Gamma_P$ and so $P = N_S(P)$ and $P = S$.

Suppose next that $|\Gamma_P^*| > 1$ and let $Q \neq T \in \Gamma_P^*$. By **(P)** applied to the NSS Γ_P, we may assume that T does not normalize Q. Let $Q \leq S \in$

Γ^*. Then $Q \leq N_S(P) \in \Gamma_P$ and so by maximality of Q, $Q = N_S(P)$. Thus $N_G(P) \cap N_G(S) \leq N_G(Q)$. Since T normalizes P but not Q we get $T \not\leq N_G(S)$. Pick $g \in T$ with $S \neq S^g$. Then $P \leq S \cap S^g = \mathrm{R}(\Gamma)$ and so $P \leq \mathrm{R}(\Gamma)$. By Corollary 2.7 $\mathrm{R}(\Gamma) \leq P$ and so $P = \mathrm{R}(\Gamma)$. □

Lemma 3.2 *Suppose that N is large in Γ and $P, Q \in \Gamma_N$*

(1) $N_Q(P)P \in \Gamma_N$.

(2) If $P \in \Gamma_N^*$, then $N_Q(P) \leq P$ and $\Gamma_N \cap \Gamma N_G(P) = \Gamma P$.

Proof. (1) By definition of large, $C_P(N) \leq N$. Hence $[C_P(N), \langle P, Q\rangle] \leq N$ and (1) follows from Lemma 2.9.

(2) By (1) and maximality of P^*, $N_Q(P) \leq P$. The second statement in (2) just rephrases the first. □

Lemma 3.3 *Suppose that $N \leq P \in \Gamma$, N is large and P is closed. Then P is large.*

Proof. Let $P \leq T \in \Gamma_P$. Then $C_T(P) \leq N_T(N) \in \Gamma_N$ and since N is large, $C_T(P) \leq N_T(N) \cap C_G(N) \leq N \leq P$. Thus P is large. □

Lemma 3.4 *Let Γ be an NSS of rank 1 and $P \in \Gamma$ with $P \not\leq \mathrm{R}(\Gamma)$.*

(1) P is contained in a unique maximal Γ-subgroup P^*.

(2) Suppose $\mathrm{R}(\Gamma)$ is large and $x \in Q \in \Gamma$. If $\langle P, P^x\rangle \in \Gamma$ then $x \in P^*$.

Proof. (1) By Theorem 3.1, $S \cap T = \mathrm{R}(\Gamma)$ for all $S \neq T \in \mathrm{R}(\Gamma)$.

(2) By (1) $P^* = \langle P, P^x\rangle^* = P^{x*} = P^{*x}$. Thus $x \in N_Q(P^*)$. By Lemma 3.2(2) $N_Q(P^*) \leq P^*$ and (2) holds. □

Lemma 3.5 *Let Γ be an NSS of rank 1 and $S \in \Gamma^*$. Define $\Pi = \bigcup_{g \in G} \Gamma S^g$. Then $\Pi \leq \Gamma$, Π has rank at most one and $\Pi^* = S^G \subseteq \Gamma^*$. If in addition $\mathrm{R}(\Gamma)$ is large then Π has rank 1 and $\mathrm{R}(\Pi) = \mathrm{R}(\Gamma)$.*

Proof. Clearly Π fulfils (Suba) and (Subc). Now let $A, B \in \Pi$ with $\langle A, B\rangle \in \Gamma$. If $A \leq \mathrm{R}(\Gamma)$, then $\langle A, B\rangle \leq B\,\mathrm{R}(\Gamma) \in \Pi$ and so also $\langle A, B\rangle \in \Pi$. So suppose $A \not\leq \mathrm{R}(\Gamma)$ and $B \not\leq \mathrm{R}(\Gamma)$. Then by Lemma 3.4(1)

$$A^* = \langle A, B\rangle^* = B^*.$$

Thus $\langle A, B\rangle \leq A^*$ and $\langle A, B\rangle \in \Pi$. Thus $\Pi \leq \Gamma$. Clearly $\Pi^* = S^G \subseteq \Gamma^*$.

Suppose first that $|\Pi^*| > 1$. By Theorem 3.1, $A \cap B = \mathrm{R}(\Gamma)$ for all $A, B \in \Pi^*$ and $\mathrm{R}(\Pi) = \mathrm{R}(\Gamma)$. Hence by Theorem 3.1 , Π has rank 1. So the lemma holds in this case.

Suppose next that $|\Pi^*| = 1$. Then $\Pi^* = \{S\}$, S is normal in G and Π has rank 0. So we may now assume that that $\mathrm{R}(\Gamma)$ is large. Since S is normal in G, Lemma 3.2 implies $PS \in \Gamma$ for all $P \in \Gamma$. But then $P \leq S$ by maximality of S and $\Gamma^* = S$, a contradiction to $\mathrm{rank}(\Gamma) = 1$. □

Lemma 3.6 *Suppose that Γ has rank 1. Let $K \leq G$ with $\langle \Gamma K\rangle \notin \Gamma$ and $P \in \Gamma K$ with $P \not\leq \mathrm{R}(\Gamma)$. Then $\langle P, P^x\rangle \notin \Gamma$ for some $x \in K$.*

Proof. Since $\langle \Gamma K\rangle \not\leq \Gamma$, Proposition 2.3(6) implies $Q \not\leq P^*$ for some $Q \in \Gamma K$. Let $x \in Q \backslash P^*$. Then by Lemma 3.4(2), $\langle P, P^x\rangle \notin \Gamma$. □

Proposition 3.7 *Let $N \in \Gamma$ be closed of co-rank 1, (here the co-rank of N is the supremum of the lengths of chains of closed Γ-subgroup starting with N).*

(1) Let $N \leq S_1 \cap S_2$ with $S_1 \neq S_2 \in \Gamma^*$. Then $N = S_1 \cap S_2$.

(2) Γ_N has rank 1.

(3) Let $N < P \in \Gamma$. Then P lies in a unique maximal Γ-subgroup P^*. Moreover, $N_G(P) \leq N_G(P^*)$,

(4) Let $P, S \in \Gamma$ with $S \in \Gamma^*$ and $N < S \cap P$. Then $P \subseteq S$.

Proof. (1) By Proposition 2.11(2) $N \leq S_1 \cap S_2 \leq T < S$ for some closed $T \in \Gamma$ and some $S \in \Gamma^*$. Since N has co-rank 1 we conclude that $N = T$ and so $N = S_1 \cap S_2$.

(2) Let $Q_1 \neq Q_2 \in \Gamma^* P$ and $Q_i \leq S_i \in \Gamma^*$. Since $\langle Q_1, Q_2 \rangle \notin \Gamma$, $S_1 \neq S_2$. So by (1) and Theorem 3.1, Γ_N has rank at most 1. Suppose that Γ_N has rank 0. Then since N is closed $\{N\} = \Gamma_N$. Let $N \leq S \in \Gamma^*$. Then $N \leq N_S(N) \in \Gamma_N$ and so $N = N_S(N)$. Hence $S = N$, a contradiction to $N \notin \Gamma^*$.

(3) and (4) are easy consequences of (1) and (2). □

Theorem 3.8 *If Γ is reduced of rank 2 then one of the following holds:*

1. *There are $S \in \Gamma^*$ and closed $P, Q \in \Gamma S \backslash \{S, 1\}$ such that $\Gamma\langle \Gamma_P, \Gamma_Q \rangle$ is reduced.*

2. *There is an reduced NSS Δ of G with $\mathrm{rank}(\Delta) = 1$ and $\Delta \leq \Gamma$.*

Proof. Suppose first that there are $S \in \Gamma^*$ and closed $P, Q \in \Gamma S \backslash \{1, S\}$ with $P \neq Q$. Let $N := \mathrm{R}(\Gamma\langle \Gamma_P, \Gamma_Q \rangle)$. By Corollary 2.7, $N \subseteq P \cap Q$ and N is closed. Since $\mathrm{rank}(\Gamma) = 2$ we get $N = 1$ and 1. holds.

Suppose next that for all $S \in \Gamma^*$ there is at most one closed $P \in \Gamma S$ with $1 \neq P \neq S$. If such a P exists we denote it by $P(S)$. Otherwise let $P(S) = 1$. We will show that

$$(^*) \qquad P(S) = P(T) \neq 1 \text{ for all } S, T \in \Gamma^* \text{ with } S \cap T \neq 1.$$

If $S \cap T$ is closed, $P(S) = S \cap T = P(T)$. So we may assume that $S \cap T$ is not closed. Then by Proposition 2.8(10) there exists a closed $M \in \Gamma$ with $S \cap T \subset M$ and $\Gamma_{S \cap T} \subseteq \Gamma_M$. By Lemma 2.5(2) $N_S(S \cap T)M \in \Gamma$. So there exists $\widetilde{S} \in \Gamma^*$ with $N_S(S \cap T)M \subseteq \widetilde{S}$ and similarly choose $\widetilde{T}$. Then $S \cap T \subset N_S(S \cap T) \subseteq S \cap \widetilde{S}$, $S \cap T \subset M \subseteq \widetilde{S} \cap \widetilde{T}$ and $S \cap T \subset N_T(S \cap T) \subseteq T \cap \widetilde{T}$. So by downwards induction on $S \cap T$, $P(S) = P(\widetilde{S}) = P(\widetilde{T}) = P(T) \neq 1$. Thus $(^*)$ holds.

Put $\Delta = \bigcup \{\Gamma P(S) \mid S \in \Gamma^*\}$. We claim that $\Delta \leq \Gamma$. (Suba) and Sub(c) are obvious from the definition of Δ. Let $A, B \in \Delta$ and $S, T \in \Gamma^*$ with $A \subseteq P(S)$ and $B \subseteq P(T)$.

To show (Subc) we assume $A \neq 1 \neq B$ and $\langle A, B \rangle \leq \Gamma$. Pick $Q \in \Gamma^*$ with $\langle A, B \rangle \leq Q$. Then $A \leq S \cap Q$ and $B \leq Q \cap T$ and $(^*)$ implies $P(S) = P(Q) = P(T)$. Thus $\langle A, B \rangle \leq P(Q)$ and $\langle A, B \rangle \in \Delta$. Thus (Subb) holds. Thus $\Delta \leq \Gamma$ and by Lemma 2.4, Δ is an NSS.

Suppose that $|\Delta^*| > 1$. Let $A, B \in \Gamma^*$ with $A \cap B \neq 1$ and let S, T be as above. Then by (*), $A = P(S) = P(T) = B$ and by Theorem 3.1, Δ is reduced of rank 1. Thus 2. holds in this case.

Suppose that $|\Delta^*| = 1$ and let A be the unique member of Δ^*. Assume that $A = 1$. Then $P(S) = 1$ for all $S \in \Gamma^*$ and so Γ has rank 1, a contradiction. Thus $A \neq 1$. Let $\Lambda = \Gamma \setminus \Gamma_A \cup \{1\}$. We claim that $\Lambda \leq \Gamma$. Let $P \leq Q \leq S$ with $1 \neq P \in \Gamma$, $Q \in \Lambda$ and $S \in \Gamma^*$. Since $\Gamma_A \leq \Gamma$ and $Q \notin \Gamma_A$, $S \notin \Gamma_A$. Thus $S \in \Lambda$. Suppose that $P \in \Gamma_A$. Then $PA \in \Gamma$. Put $PA \cap S \neq 1$ and (*) implies $A \leq S$. Thus $S \in \Gamma_A$, a contradiction. So $P \in \Lambda$ and we conclude that (Suba) holds for Λ. Clearly (Subb) and (Subc) hold.

We proved $\Lambda \leq \Gamma$. Since $A \not\leq \mathrm{R}(\Gamma)$, $\Lambda \neq \{1\}$. Suppose that Λ has a unique maximal element B. Then $B \in \Gamma^*$ and by (*), $B \cap A = 1$. Since both A and B are normal in G, $[A, B] = 1$. Thus AB is nilpotent and $AB \in \Gamma$, a contradiction to $B \notin \Gamma_A$. Thus $|\Lambda^*| > 1$. By (*) $X \cap Y = 1$ for any two maximal members of Λ and so Theorem 3.1 implies that Λ is a reduced NSS of rank 1. Thus 2. holds for Λ in place of Δ. □

4 Minimal Γ-subgroups

In this section we continue to assume that a G is group with an NSS Γ and $1 \in \Gamma$. We consider elements $X \in \Gamma_*$. Recall that this just means that X is a minimal non-trivial element of Γ. In particular for two different elements $X, Y \in \Gamma_*$ we have $X \cap Y = 1$.

Proposition 4.1 *Assume $P \in \Gamma$ and $X, Y \in \Gamma P_*$ with $X \neq Y$. If $N_X(Y) \neq 1$ or $[x, y] = 1$ for some $x \in X^\#$ and $y \in Y^\#$, then $\langle X, Y \rangle = X \times Y$.*

Proof. If $[x, y] = 1$, then $y \in Y \cap Y^x$ and so $Y = Y^x$. So we may assume $N_X(Y) \neq 1$. Using Lemma 2.1 we get $X = N_X(Y)$. Since $XY \subseteq P$, XY is nilpotent. As Y is normal in XY, $C_Y(X) \neq 1$. Hence $N_Y(X) \neq 1$ and $Y = N_Y(X)$. So $[X, Y] \leq X \cap Y = 1$. □

Proposition 4.2 *Let $P \in \Gamma$ with $P = \langle X, Y \rangle$ where $X, Y \in \Gamma P_* \setminus \{P\}$. If $X' \neq 1$ then X is a normal subgroup of P (here $X' := [X, X]$ is the commutator subgroup of X).*

Proof. Consider a counterexample with P minimal. Then there is $y \in Y$ with $X^y \neq X$. Set $E := \langle X, X^y \rangle$. So by Proposition 2.3(6) and Proposition 1.6, $E \in \Gamma$ and $E < P$. Of course $X, X^y \neq E$ and by minimality of P, X and X^y are normal in E. Therefore by Proposition 4.1 $E = X \times X^y$. Let $Q := \langle Y^P \rangle$. Then Q is a proper Γ-subgroup of P by Proposition 2.3(6) and Proposition 1.6. Since $P = \langle X, Y \rangle$, $X \not\leq Q$ and $Q \cap X = 1$ as $X \in \Gamma_*$ and $Q \cap X \in \Gamma$ by **(I)**. Now $[X, y] \leq E \cap Q$ and $E \cap Q$ is normal in E. Since $X \times X^y = X[X, y]$ we have $1 \neq [X^y, X^y] = [X^y, [X, y]] \leq X^y \cap Q$. Hence also $X \cap Q \neq 1$, a contradiction. □

Corollary 4.3 *Let $P \in \Gamma$ and $\Delta := \{X \in \Gamma P_* \mid X' \neq 1\}$. Then Δ is finite and $\langle \Delta \rangle = X_1 \times \ldots \times X_n$, where $\Delta = \{X_1, \ldots, X_n\}$.*

Proof. Let $X \neq Y \in \Delta$. Then by Proposition 4.2 and Proposition 4.1, $[X, Y] = 1$. Let $Z = \langle \Delta \setminus \{X\} \rangle$. Then $Z \in \Gamma$ and $[X, Z] = 1$. Thus $X \cap Z$ is a proper Γ-subgroup of X and so $X \cap Z$. Thus the Corollary holds by the definition of the direct product. (Note also that Δ is finite by **(MM)**) □

Define an elementary abelian p-group to be an an abelian group so that all non-trivial elements have order p. Note that this makes sense for p a prime or $p = \infty$. Indeed, an elementary abelian ∞-group is just a torsion free abelian group.

Proposition 4.4 *Let $X, Y \in \Gamma_*$, $X \neq Y$, $H := \langle X, Y \rangle \in \Gamma$ and $[X, Y] \neq 1$. Then X and Y are both elementary abelian p-groups, $p = \infty$ or a prime.*

Proof. Suppose first Y is not elementary abelian. Let $M \in \Gamma$ maximal with respect to $X \leq M < H$. Then by Proposition 2.2, M is normal in H. Also $Y \not\leq M$. Since $Y \cap M \in \Gamma$ and $Y \in \Gamma_*$, $Y \cap M = 1$. Let $1 \neq x \in X$. By Proposition 4.1, $N_X(Y) = 1$ and so $Y \neq Y^x$. Hence by Proposition 4.2, Y is abelian. Since $\langle Y, Y^x \rangle \neq H$ we get by induction that $[Y, Y^x] = 1$. Let $D = YY^x \cap M$. Since $[Y, x] \subseteq D$, $YY^x = YD = Y^x D$. Let $E \in \Gamma_* D$. Then $1 \neq Y \cap (EY^x) \in \Gamma$ and so $Y \subseteq EY^x$. Thus $E = D$. Note that D is isomorphic to Y and $\langle D, X \rangle \leq M$. In particular, Y is not elementary abelian and so by induction $[D, X] = 1$. Since $[Y, x] \leq D$ we get $[Y, x] \leq Z(\langle Y, x \rangle)$. Let $y \in Y$ has order p, p a prime. Then by Proposition 1.1, $[y, x^p] = [y^p, x] = 1$ and so by Proposition 4.1 $x^p = 1$. Hence for all $z \in Y, [z^p, x] = [z, x^p] = 1$ and so by Proposition 4.1 $z^p = 1$.

Hence Y is an elementary abelian p-group and by symmetry X is an elementary abelian q-group. To show $p = q$ we may assume $p \neq \infty$. Then by Proposition 1.7 $[y, x^{p^k}] = 1$ for some positive integer k. So by Proposition 4.1, $x^{p^k} = 1$ and $q = p$. $\square$

Proposition 4.5 *Let A_1 be a Γ-subgroup of the decomposable abelian Γ-subgroup A. Then there is a decomposable Γ-subgroup A_2 of A with $A = A_1 \times A_2$.*

Proof. Let K be a decomposable Γ-subgroup maximal with $A_1K = A_1 \times K$. If $A = A_1K$ we are done. So suppose $A_1K < A$. Since A is decomposable, there exists $X \in \Gamma_* A$ with $X \not\leq A_1K$. Then $A_1K \cap X = 1$ and $A_1KX = (A_1 \times K) \times X = A_1 \times (K \times X)$. But $K < KX$ and we obtain a contradiction to the maximal choice of K. $\square$

Proposition 4.6 *Γ-subgroups of decomposable abelian Γ-subgroups are decomposable.*

Proof. Let A be a decomposable abelian Γ-subgroup and B a Γ-subgroup of A. By Proposition 4.5 there exists $D \in \Gamma A$ with $A = \Omega(B) \times D$. By Proposition 1.8 $B = \Omega(B) \times (B \cap D)$. Also $\Omega(B \cap D) \leq \Omega(B) \cap D = 1$ and since $B \cap D \in \Gamma$, $B \cap D = 1$ and $B = \Omega(B)$. $\square$

Proposition 4.7 *Let $A, B \in \Gamma$ such that A is decomposable abelian and B is generated by abelian Γ-subgroups. If $\langle A, B\rangle \in \Gamma$, then $C_A(B)$ is a decomposable abelian Γ-subgroup.*

Proof. Since $C_A(B) = \bigcap\{C_A(E) \mid E \in \Gamma B, E\,\text{abelian}\}$ we may by **(I)** assume that B is abelian. By Proposition 4.6 we only need to show $C_A(B) \in \Gamma$. By Proposition 2.3(1) we get $C_A(B) \leq D := \bigcap_{b\in B} A^b \in \Gamma$. Note that $C_A(B) = C_D(B)$ and that B normalizes D. By Proposition 4.6 D is decomposable.

If $D = C_D(B) = C_A(B)$ we are done. So suppose $[D, B] \neq 1$. Since DB is nilpotent, there exists $d \in D$ with $1 \neq [d, B] \leq C_D(B)$. Then $B^d \leq C_D(B)B \leq C_G(B)$. Thus BB^d is abelian and $BB^d \in \Gamma$. Thus

$$1 \neq [d, B] \leq BB^d \cap D \leq C_D(B).$$

Put $E := BB^d \cap D$. Then E is a non-trivial Γ subgroup of $C_D(B)$. By Proposition 4.5, $D = E \times F$ for some decomposable Γ subgroup F of D. Then $C_D(B) = E \times C_F(B)$. Since $F < A$, induction on A shows $C_F(B) \in \Gamma$. Hence also $C_D(B) \in \Gamma$ and the Proposition is proved. $\square$

Proposition 4.8 *Let $A, B \in \Gamma$ such that A is decomposable, B an abelian Γ-subgroup and $A \in \Gamma_B$. Then $[B, A]$ is a Γ-subgroup of G.*

Proof. Since $[B, A] = \langle [B, E] \mid E \in \Gamma_* A \rangle$ we may by Proposition 2.3(6) assume that $A \in \Gamma_*$. If $A \leq B$, then since B is abelian $[A, B] = 1 \in \Gamma$. We therefore may assume $A \not\subseteq B$ and so $A \cap B = 1$ by minimality of A. Note that $\langle A^B \rangle = A[B, A]$ and so

$$\langle A^B \rangle \cap B = [B, A](A \cap B) = [B, A].$$

By Proposition 2.3(6) $\langle A^B \rangle \in \Gamma$ and so by (I), $[B, A] \in \Gamma$. $\square$

5 Measure and the Thompson subgroup

G continues to be a group with an NSS Γ with $1 \in \Gamma$. We define a measure function and use it to state and prove a variant of the Thompson Replacement Theorem.

Proposition 5.1 *Let $X, Y \in \Gamma$ with $XY \in \Gamma$. Let*

$$X = X_0 < X_1 < \ldots < X_r = XY$$

be any maximal chain of Γ-subgroups from X to XY. Then

$$X \cap Y = X_0 \cap Y < X_1 \cap Y < \ldots X_r \cap Y = Y$$

is a maximal chain of Γ-subgroups from $X \cap Y$ to Y.

Proof. Let A be a Γ subgroup with $X_i \cap Y \leq A \leq X_{i+1} \cap Y$. Since $X \leq X_i \leq XY$, Proposition 1.8 implies $X_i = X(X_i \cap Y)$. Thus

$$X_i \leq AX \leq X_{i+1}.$$

X_i is a maximal Γ-subgroup of X_{i+1} and so by Proposition 2.2 X_i is normal in X_{i+1}. Thus $AX = AX_i$ is a subgroup of X_{i+1}. Since X_{i+1} is nilpotent, Proposition 2.3(6) implies $AX \in \Gamma$. By the maximality of the X_i-chain, $AX = X_k$ for some $k \in \{i, i+1\}$. Thus $X_k \cap Y = AX \cap Y = A(X \cap Y) = A$. $\square$

Proposition 5.2 *Let $X \in \Gamma$. Then there exists a maximal chain of Γ-subgroups from 1 to X and any two such chains have the same length. We denote this common length by $\mu(X)$.*

Proof. The existence of a maximal Γ-chain from 1 to X follows from **(MM)**. Let A and B be maximal Γ-subgroups of X. By induction any maximal Γ-chain from 1 to X through A has unique length $\mu(A) + 1$. It remains to show that $\mu(A) = \mu(B)$. Without loss $A \neq B$. By maximality of A and B, A is normal in X, $AB \in \Gamma$ and $AB = X$. Note that $A \leq X$ is a maximal chain from A to X and so by Proposition 5.1, $A \cap B < B$ is a maximal chain from $A \cap B$ to B. Thus $\mu(B) = \mu(A \cap B) + 1 = \mu(A)$. $\square$

Abusing the term we call μ of Proposition 5.2 a *measure function* on Γ and $\mu(A)$ is called the *measure* of A.

Proposition 5.3 *$\mu(P) = \mu(P^g)$ for all $P \in \Gamma$ and $g \in G$.*

Proof. This follows from **(C)** and Proposition 5.2. $\square$

Proposition 5.4 *Assume $X, Y \in \Gamma$.*

(1) Suppose $X \leq Y$, then any maximal Γ-chain from X to Y has length $\mu(Y/X) := \mu(Y) - \mu(X)$.

(2) Suppose $XY \in \Gamma$. Then $\mu(XY) = \mu(X) + \mu(Y) - \mu(X \cap Y)$.

Proof. (1) follows from Proposition 5.2.

(2) By (1) $\mu(XY/X) = \mu(XY) - \mu(X)$. By (1) and Proposition 5.1, $\mu(XY/X) = \mu(Y/X \cap Y)$. Again by (1) $\mu(Y/X \cap Y) = \mu(Y) - \mu(X \cap Y)$. Thus (2) holds.

$\square$

Definition. For $P \in \Gamma$ let $\mathcal{A}(P)$ be the set of all decomposable abelian Γ-subgroups of P with maximal measure. Let $J(P) := \langle \mathcal{A}(P) \rangle$, the *Thompson-subgroup* of P (compare with the introduction of [5]).

Then $J(P)$ is a Γ-subgroup of P by Proposition 2.3(6).

Proposition 5.5 *Let V be a decomposable abelian Γ-subgroup of G and $A \in \Gamma_V$ with $A \in \mathcal{A}(AV)$. Then $C_V(A) = V \cap A$ and $\mu(A/C_A(V)) \geq \mu(V/C_V(A))$*

Proof. By Proposition 4.7 and **(P)**, $C_V(A)A$ is an decomposable abelian Γ-subgroup of P. The maximality of $\mu(A)$ implies $C_V(A) \leq A$ and thus $C_V(A) = V \cap A$. Thus $V \cap C_A(V) = V \cap A = C_V(A)$ and by maximality of A, Proposition 4.7 and Proposition 5.4:

$$\begin{aligned} \mu(A) \geq \mu(VC_A(V)) &= \mu(VC_A(V)/C_A(V)) + \mu(C_A(V)) \\ &= \mu(V/C_V(A)) + \mu(C_A(V)). \end{aligned}$$

□

The next lemma is our version of the Thompson Replacement Theorem.

Lemma 5.6 *Let A, V be decomposable abelian Γ-groups with $A \in \Gamma_V \cap \mathcal{A}(AV)$. Let $x \in N_V(N_V(A)A)$ and define*

$$D = ((AA^x) \cap V)(A \cap A^x).$$

Then

(1) $D \in \mathcal{A}(AV)$ and $\langle x \rangle N_V(A)A \subseteq N_G(D)$.

(2) If $[V, A] \neq 1$, then $[V, D] \neq 1$.

Proof. (1) Let $P = N_V(A)A$. Since x normalizes P, both A and A^x are normal in P. Thus $AA^x = \langle A, A^x \rangle$. Since A is abelian, $A \cap A^x \subseteq Z(AA^x)$. By Proposition 4.6 both $AA^x \cap V$ and $A \cap A^x$ are decomposable Γ-groups and so D is an abelian decomposable Γ-group. Also $[x, A] \subseteq V \cap (AA^x) \subseteq D$ and so $x \in N_G(D)$. Note that

(*) $\quad \mu(AA^x) = \mu(A) + \mu(A^x) - \mu(A \cap A^x) = 2\mu(A) - \mu(A \cap A^x).$

Also $AA^x \subseteq VA$ and so $AA^x = AA^x \cap VA = A(V \cap AA^x) = AD$. Moreover, $D \cap A = (V \cap A)(A \cap A^x)$ and $V \cap A \subseteq C_A(x) \subseteq A \cap A^x$. Thus

$D \cap A = A \cap A^x$. Hence $\mu(AA^x) = \mu(DA) = \mu(D) + \mu(A) - \mu(A \cap A^x)$. Comparing with (*) we obtain $\mu(A) = \mu(D)$ and so $D \in \mathcal{A}(AV)$.

(2) Suppose that $[V, D] = 1$. Then $A \cap A^x \leq C_A(V)$ and so by Proposition 5.5, $A \cap A^x = A \cap V$. Hence $D \leq V$. Since $D \in \mathcal{A}(AV)$ we get $V = D \subseteq P \subseteq N_G(A)$. Thus $A = A^x = A \cap A^x \leq D$ and $[V, A] = 1$. Thus (2) holds.

□

Proposition 5.7 *Let V be a decomposable abelian Γ-subgroup of G and $P \in \Gamma_V$ with $V \subseteq P$ and $J(P) \not\leq C_G(V)$. Then there exists $A \in \mathcal{A}(P)$ such that $[[V, A], A] = 1 \neq [V, A] \leq A$.*

Proof. Since $J(P) \not\leq C_G(V)$ there exists $A \in \mathcal{A}(P)$ with $1 \neq [V, A]$. Choose such an A with $N_V(A)$ maximal.

Suppose that V does not normalize A. Then $V \not\leq N_V(A)A$ and so by Proposition 1.6(2) there exists $x \in N_V(N_V(A)A)$ with $x \notin N_V(A)$. Let D be defined as in Lemma 5.6. Then $D \in \mathcal{A}(AV)$, $[V, D] \neq 1$ and $\langle x \rangle N_V(A) \leq N_V(D)$, contradiction to the maximal choice of $N_V(A)$.

Thus V normalizes A, $[V, A] \leq A$ and $[[V, A], A] = 1$. □

Lemma 5.8 *Let A, B be abelian Γ-subgroups with $[A, B] \leq A \cap B$. Let $a \in A$. Suppose that B is decomposable and $C_B(a) \in \Gamma$. Then $[a, B] \in \Gamma$ and $\mu([a, B]) = \mu(B/C_B(a))$.*

Proof. By Proposition 4.5 there exists a Γ-subgroup D of B with $B = C_B(a) \times D$. Then $[D, a] \leq A \cap B \leq C_A(D)$ and so by Proposition 1.2(2) $DD^a = D[D, a] \in \Gamma$. Moreover, $DD^a \cap A = (D \cap A)[D, a]$ and $D \cap A \leq C_D(a) = 1$. $[D, a] = DD^a \cap A \in \Gamma$. In particular, $D \cap [D, a] = 1$ and so by Proposition 1.2(3), $D \cap D^a = C_D(a) = 1$. Thus $2\mu(D) = \mu(DD^a) = \mu(D) + \mu([D, a])$ and

$$\mu([a, B]) = \mu([a, D]) = \mu(D) = \mu(B/C_B(a)).$$

□

6 Glauberman's Theorem, Part I

In this section we begin the proof of Theorem A stated in the introduction. Assume G, V, S, A, Γ have the meaning and the properties mentioned there.

Proposition 6.1 *Set* $\Pi := \bigcup_{g\in G} \Gamma S^g$. *Let* $T \in \Pi^*$.

(1) Π *is an NSS of rank* 1.

(2) $\Pi^* = S^G \subseteq \Gamma^*$.

(3) $\mathrm{R}(\Pi) = \mathrm{R}(\Gamma)$.

(4) $[C_G(V), \langle\Pi\rangle] \leq \mathrm{R}(\Gamma)$.

(5) Let $P \in \Gamma(TC_G(V))$, *then* $P \leq T$.

(6) $\mathrm{R}(\Gamma) = C_T(V)$.

Proof. By (a) (that is assumption (a) of Theorem A), Γ has rank 1. By (e) $\mathrm{R}(\Gamma)$ is large. So (1),(2) and (3) follow from Lemma 3.5.

(4) By (c) $[C_G(V), S] \leq \mathrm{R}(\Gamma)$. Thus (4) follows by conjugation.

(5) By (4) $[P, T] \leq [TC_G(V), T] \leq T\,\mathrm{R}(\Gamma) \leq T$. Thus $P \leq N_G(T)$. By (e), $\mathrm{R}(\Gamma)$ is large and so by Lemma 3.2 $PT \in \Gamma$. Since $T \in \Pi^* = S^G \subseteq \Gamma^*$, $P \leq T$.

(6) Let $R \in \Pi^*$. By (4) $[T \cap C_G(V), R] \leq \mathrm{R}(\Gamma) \leq R$. Thus $C_T(V) \leq E := \bigcap_{R\in\Pi^*} N_T(R)$. By **(I)** and **(MM)** $E \in \Gamma$ and by Proposition 2.3(8), $E \leq \mathrm{R}(\Pi) = \mathrm{R}(\Gamma)$. □

Lemma 6.2 *There exists a non-trivial quadratic* Γ*-offender* E *in* S *on* V *with* $C_V(E) = V \cap E$.

Proof. By (d) there exists a non-trivial Γ-offender A in S on V. Since $V \leq \mathrm{R}(\Gamma)$, $A \in \Gamma_V$. Let $B = C_A(V)V$ and $D = C_V(A)A$. By Proposition 4.7, $B \in \Gamma$. Since A is an offender on V, $\mu(V/C_V(A)) \leq \mu(A/C_A(V))$. But this is equivalent to $\mu(B) \leq \mu(D)$. We will show that

(*) $\quad J(AV) \not\leq C_{AV}(V)$.

Since $AV = AB$, $C_{AV}(V) = C_A(V)B = B$. From $\mu(D) \geq \mu(B)$, we get $D \in \mathcal{A}(AB)$ and $D \not\leq B$, since $1 \neq [A, V] \leq [B, D]$.

Thus (*) holds. The existence of E now follows from Proposition 5.7 and Proposition 5.5. $\square$

Notation.

$\Delta := E^G$, where E is as in Lemma 6.2

$E, F \in \Delta$ such that $\mu([V, E][V, F])$ is minimal with respect to $\langle E, F\rangle \notin \Gamma$.

$W := [V, E][V, F]$.

$H := \langle E, F\rangle\, \mathrm{R}(\Gamma)$.

$Z := [V, E] \cap [V, F]$.

$\Lambda := \Delta H$.

$q := \mu(A/C_A(V))$ for $A \in \Delta$.

$m := \mu([V, A])$.

Notice that $C_A(V), [V, A]$, W and Z are decomposable abelian subgroups by Proposition 4.5 - Proposition 4.8, **(P)** and **(I)**. Hence the measure of these groups is defined. Note that by Proposition 5.4 and the choice of E, F, $\mu(Z)$ is maximal with respect to $\langle E, F\rangle \notin \Gamma$. The existence of $F \in \Delta$ with $\langle E, F\rangle \notin \Gamma$ is guaranteed by Lemma 3.6. In view of Lemma 3.4(1) we denote by D^* the unique member of Γ^* which contains D provided $D \in \Gamma$ with $D \not\leq \mathrm{R}(\Gamma)$. Observe that by Proposition 6.1(6), $A \notin \mathrm{R}(\Gamma)$ for all $A \in \Delta$.

Proposition 6.3 *Let $A, B \in \Delta$*

(1) $\langle A, B\rangle \notin \Gamma$ if and only if $A^* \neq B^*$.

(2) If $A^* \neq B^*$, then $\langle A, A^b\rangle \notin \Gamma$ for all $b \in B\backslash R(\Gamma)$.

(3) If $A^* \neq B^*$, then $[V, A] \neq [V, B]$.

Proof. (1) If $\langle A, B\rangle \in \Gamma$ then $A^* = \langle A, B\rangle^* = B^*$. If $A^* = B^*$ then $\langle A, B\rangle \in \Gamma$ by Proposition 2.3(6). Hence (1).

(2) Let $b \in B$ with $\langle A, A^b\rangle \in \Gamma$. Then by Lemma 3.4(2), $b \in A^* \cap B^*$. So by Theorem 3.1(2), $b \in \mathrm{R}(\Gamma)$.

(3) Assume $[V, A] = [V, B]$. By (c), A and B are quadratic and so $[[V, A], B] = [[B, V], A] = 1$. Thus $[[A, B], V] = 1$ by Proposition 1.1 (5)

and $[A,B] \subseteq C_G(V)$. Let $b \in B \setminus \mathrm{R}(\Gamma)$. Then $A^b \in AC_G(V) \leq A^* C_G(V)$ and so by Proposition 6.1(5), $A^b \leq A^*$, a contradiction to (2). $\square$

Proposition 6.4 *Let $A \in \Lambda$ and $a \in A \setminus \mathrm{R}(\Gamma)$. Then*

(1) $[V,A] = Z \times [W,a] = C_W(a) = C_W(A)$.

(2) $\langle B, B^a \rangle \notin \Gamma$ *and* $[V,B] \cap C_V(a) = [V,A] \cap [V,B] = Z$ *for all* $B \in \Lambda$ *with* $\langle A, B \rangle \notin \Gamma$.

Proof. By Lemma 3.6 there is $B \in \Lambda$ with $\langle A, B \rangle \notin \Gamma$. So by Proposition 6.3 $\langle B, B^a \rangle \notin \Gamma$. Hence $W = [V,A][V,B] = [V,B][V,B^a]$ by minimality of $\mu(W)$. Put $D := C_{[V,B]}(a)$. Then by Proposition 5.4, Proposition 1.2, and quadratic action,

$$C_W(a) = D \times [[V,B],a] = [V,A] = C_W(A).$$

Also $[W,a] = [[V,B],a]$ and

$$D = [V,B] \cap C_W(a) = [V,B] \cap [V,A] = C_W(B) \cap C_W(A) = C_W(\langle A, B \rangle).$$

Since H centralizes Z, $Z \leq D$. The maximality of $\mu(Z)$ now implies $Z = D$. $\square$

Proposition 6.5 *Let $A, B \in \Lambda$ with $\langle A, B \rangle \notin \Gamma$, $w \in [V,B] \setminus Z$, and $a \in A \setminus \mathrm{R}(\Gamma)$. Then*

(1) $W = [V,B] \times [w,A]$.

(2) $V = WC_V(A)$.

(3) $q = \mu(V/C_V(A)) = \mu(W/C_W(A)) = \mu(A/C_A(V)) = \mu([w,A]) = \mu([V,a])$.

Proof. (1) By Proposition 4.8, $W \in \Gamma$. Since V is decomposable, also W is decomposable and by Proposition 4.7 $C_W(A) \in \Gamma$. Hence also $C_W(A)A \in \Gamma$ and we can assume $[V,A] = C_W(A) \subseteq A$. Then $A \cap W = [V,A]$. By Proposition 6.4(2), $[w,a] \neq 1$ for all $a \in A \setminus \mathrm{R}(\Gamma)$ and so $C_A(w) = A \cap R(\Gamma) \in \Gamma$. From Lemma 5.8 we conclude $[w,A] \in \Gamma$ and

(*) $\mu([w, A]) = \mu(A/C_A(V)) = q$.

Note that $\mu([V, B]) = m = \mu([V, A]) = \mu(C_W(A))$ and so

(**) $\mu(V/C_V(A)) \geq \mu(W/C_W(A)) = \mu(W/[V, B])$.

By (*),(**) and since A is an offender, $\mu([w, A]) \geq \mu(W/[V, B])$. By Proposition 6.4 $[w, A] \cap [V, B] \leq [w, A] \cap Z = 1$ and we conclude that $\mu([w, A]) = \mu(W/[V, B])$ and $W = [V, B] \times [w, A]$. So (1) holds.

(2) We also conclude that the inequalilty in (**) actually is an equality. So $\mu(V/C_V(A)) = \mu(W/C_W(A))$. Hence (2) holds.

(3) By Proposition 6.4, $C_W(a) = C_W(A) \in \Gamma$. So by Lemma 5.8, $[W, a] \in \Gamma$ and $\mu([W, a]) = \mu(W/C_W(A)) = q$. By (2), $[V, a] = [W, a]$ and all parts of (3) are proved. □

Proposition 6.6 *Let $A, B \in \Lambda$ with $\langle A, B\rangle \notin \Gamma$ and $\Sigma := [V, A]^H$. Then:*

(1) If $B \in \Lambda$ with $\langle A, B\rangle \notin \Gamma$ then $\Sigma = \{[V, A]\} \cup [V, B]^A$.

(2) If $M, N \in \Sigma$ with $M \neq N$ then $M \cap N = Z$.

(3) $W = \bigcup_{M\in\Sigma} M$.

(4) For $D \in \Delta$ put $\hat{D} = D\,\mathrm{R}(\Gamma)$. Let $D \in \Lambda$ with $D^* = A^*$, then $\hat{D} = \hat{A}$.

(5) Let $\hat{\Lambda} = \{\hat{B} \mid B \in \Lambda\}$. Then $\hat{\Lambda} = \{\hat{A}\} \cup \{\hat{B}^A\}$.

(6) $H = \langle C, D\rangle\,\mathrm{R}(\Gamma)$ for all $C, D \in \Delta H$ with $\hat{C} \neq \hat{D}$.

(7) $V = C_V(H)W$ and $V = \bigcup_{D\in\Lambda} C_V(D)$.

Proof. Let $A, B \in \Lambda$ with $\langle A, B\rangle \notin \Gamma$. Then $W = [V, A][V, B] = [V, B] \times [w, A]$ for $w \in [V, B]\backslash Z$ by Proposition 6.5 and $[V, A] \cap [V, B] = Z$ by Proposition 6.4. Therefore $w^A Z = w[V, A]$, which shows

$$(^*) \quad W = [V, A] \cup \bigcup_{a\in A} [V, B]^a.$$

By Lemma 3.6 we can apply (*) to an element of A^H in the role of B and so (3) holds.

Also $[V, B]^{a_1} \cap [V, B]^{a_2} = Z$ for $a_1, a_2 \in A$ with $a_1 a_2^{-1} \notin \mathrm{R}(\Gamma)$ by Proposition 6.4. Let $C \in \Lambda$. Then there is $D \in \{A\} \cup B^A$ with $[V, C] \cap [V, D] \supset Z$.

Hence by maximality of $\mu(Z)$, $\langle C, D\rangle \in \Gamma$, $C^* = D^*$ and $\langle C, K\rangle \notin \Gamma$ for $K \in (\{A\} \cup B^A)\backslash\{C\}$. But then by (*)

$$[V, C]\backslash Z \subseteq W\backslash \bigcup\{[V, K] \mid D \neq K \in \{A\} \cup B^A\} \subseteq [V, D]$$

and $[V, C] = [V, D]$. Thus (1) and (2) hold.

(4) Let $d \in D$. By (1) $[V, B]^{da} = [V, B]$ for some $a \in A$. Thus $da \in N_G(B^*)$ and so $da \in \mathrm{R}(\Gamma)$. Hence $d \in A\,\mathrm{R}(\Gamma) = \hat{A}$. Thus (4) holds.

(5) Let $C \in \Lambda$ with $\hat{C} \neq \hat{A}$. By (4), $C^* \neq A^*$ and by Proposition 6.3(3), $[V, C] \neq [V, A]$. So by (1), $[V, C] = [V, B]^a$ for some $a \in A$. By Proposition 6.3(3), $C^* = B^{a*}$ and so by (4) $\hat{C} = \hat{B}^a$.

(6) By (5), H is doubly transitive on $\hat{\Lambda}$. Since $H = \langle \hat{E}, \hat{F}\rangle$, (6) holds.

(7) Since $H = \langle A, B\rangle\,\mathrm{R}(\Gamma)$, we have $C_V(H) = C_V(A) \cap C_V(B)$. Since $\mu(V/C_V(A)) = q$ we get $\mu(V/C_V(H)) \leq 2q$. Since $\mu(W/C_W(H)) = 2q$, the first part of (7) holds.

Let $v \in V$. Then $v = cw$ with $c \in C_V(H)$ and $w \in W$. By (1), $w \in [V, C]$ for some $C \in \Lambda$. So $v \in C_V(H)[V, C] \leq C_V(C)$. □

Lemma 6.7 *Let $t \in G$ and $B \in \Delta$. Suppose that one of the following holds:*

1. *$t \in A \in \Delta$ and $[V, t] \cap C_V(B) \neq 1$.*
2. *$\mu(C_V(B)/(C_V(B) \cap C_V(B)^t)) < q$.*

Then $\langle B, B^t\rangle \in \Gamma$.

Proof. Suppose that 1. holds. Then by Proposition 1.2(2) $C_V(B^t) \subseteq [V, t]C_V(B)$. By Proposition 6.5(3), $\mu([V, t]) = q$ and so 1. implies 2.

So we may assume that 2. holds. Then

$$\mu([V, B]/([V, B] \cap C_V(T))) < q.$$

Since $[V, B] \cap C_V(t) \leq [V, B] \cap [V, B]^t$ and $\mu([V, B]/Z) = q$, the maximality of $\mu(Z)$ implies $\langle B, B^t\rangle \in \Gamma$.

Lemma 6.8 *Let $A \in \Delta$. Then $A \subseteq B\,\mathrm{R}(\Gamma) \subseteq H$ for some $B \in \Lambda$.*

Proof. Let $a \in A \backslash C_A(V)$. By Proposition 6.6(7) there exists $B \in \Lambda$ with $[V, a] \cap C_V(B) \neq 1$. By Lemma 6.7, $\langle B, B^a \rangle \in \Gamma$. Thus by Lemma 3.4(2), $a \in B^*$. Hence $A \subseteq B^*$ and $A^* = B^*$. Since $\mathrm{R}(\Gamma) \leq C_G(V)$, Proposition 6.6(5) implies $[V, a] \subseteq C_V(B)$. By Lemma 6.7 $a \in B^*$ and so $A^* = B^*$. So by Proposition 6.6(5), $B\,\mathrm{R}(\Gamma)$ is independent from the choice of a. Hence $[V, A] \subseteq C_V(B)$. Let $A = B^g$ for $g \in G$. Then $A \in \Lambda^g$ and so by symmetry $[V, B] \leq C_V(A)$. Thus $[V, A][V, B] \subseteq C_V(AB)$.

Let $D \in \Lambda \backslash \Lambda B^*$.

Put $T = \langle A, B, D \rangle$, $U = [V, T] = [V, A][V, B][V, D]$ and $Y = [V, A][V, B] \cap C_V(D)$. Then Y is centralized by A, B and D and so $Y \subseteq C_U(T)$. Since $\mu(V/C_V(D)) = q = \mu([V, B]C_V(D)/C_V(D)), [V, A][V, B] = [V, B]Y$. Let $a \in A$ and $w \in [V, D] \backslash Z$. Note that $Z \subseteq [V, D] \cap C_V(B) \leq Y$. By Proposition 6.5(1), $[w, B]Z = [V, B]$ and so $[V, A][V, B] = [w, B]Y$. Hence $[w, a]Y = [w, b]Y$ for some $b \in B$. Let $t = b^{-1}a$. Then $w^tY = wY$. Since $wY \subseteq [V, D]Y \subseteq C_V(D)$, $wY = w^tY \subseteq C_V(D^t)$. Hence $Z < \langle w \rangle Z \subseteq [V, D] \cap C_V(D^t)$ and so $\mu([V, D]/([V, D] \cap C_V(D^t))) < q$.

Thus by Lemma 6.7, $\langle D, D^t \rangle \in \Gamma$ and by Lemma 3.2(2), $t \in D^*$. Hence $t \in D^* \cap B^* = \mathrm{R}(\Gamma)$. So $a = bt \in B\,\mathrm{R}(\Gamma) \subseteq H$. □

Theorem 6.9 $\langle \Gamma \rangle = H$ *and* $\Gamma^* = \{A\,\mathrm{R}(\Gamma) \mid A \in \Delta\}$.

Proof. Let $P \in \Gamma^*$. By Lemma 6.8, $H = \langle \Delta \rangle\,\mathrm{R}(\Gamma)$ and so H is normal in G. So P normalizes $W = [V, H]$ and $Z = C_W(H)$. As PV is nilpotent, P centralizes some $1 \neq wY$ in W/Z. By Proposition 6.6(3), $w \in [V, A]$ for some $A \in \Lambda$. Thus $P \subseteq N_G([V, A])$. By Proposition 6.3(3) $P \subseteq N_G(A^*)$ and so by Lemma 3.2, $P = A^*$. By Proposition 6.6(5), A acts transitively on $\hat{\Lambda} \setminus \hat{A}$, whence $P = AN_P(\hat{B})$ for $B \in \Lambda$ with $\langle A, B \rangle \notin \Gamma$. But $N_P(\hat{B}) \leq N_P(\hat{B}^*) = P \cap B^* = \mathrm{R}(\Gamma)$ and so $P = A\,\mathrm{R}(\Gamma)$. □

7 Glauberman's Theorem, Part II

In this section we complete the proof of Theorem A. We continue to use the notations from the previous section. In addition we define:

$V_0 = W/Z$, written additively.

$V_1 = [V, E]/Z$ and $V_2 = [V, F]/Z$.

We view V_0 as a left module over the endomorphism ring $End(V_0)$. In particular if $\alpha, \delta \in End(V_0)$ and $v \in V_0$, then $(\alpha\delta)(v) = \alpha(\delta(v))$. For $h \in H$ define $\sigma_h \in End(V_0)$ by $\sigma_h(wZ) = w^h Z$ for $w \in W$. Note that $\sigma_{hh'} = \sigma_{h'}\sigma_h$. From Proposition 6.6 we obtain:

(i) $V_1 = C_{V_0}(E) = [V_0, E] = [V_0, a]$ for all $a \in E \backslash C_E(V_0)$.

(ii) $V_2 = C_{V_0}(F) = [V_0, F] = [V_0, b]$ for all $b \in F \backslash C_F(V_0)$.

(iii) $V_0 = V_1 \oplus V_2$.

(iv) For $g \in H$ with $\sigma_g(V_1) \neq V_1$ there is $a \in E$ with $\sigma_{ga}(V_1) = V_2$.

Take $b \in F$ fixed such that $\sigma_b(V_1) \neq V_1$ and set $\beta := \sigma_b - 1 \in End(V_0)$. Similarly set $\chi_a = \sigma_a - 1$. Moreover for $i = 1, 2$ let π_i be the projection from V_0 on V_i according to the direct sum decomposition $V_0 = V_1 \oplus V_2$.

Proposition 7.1 *The following equations hold, where $a, c \in E$:*

(1) $\sigma_b = \pi_1 + \pi_2 + \beta$.

(2) $\sigma_a = \pi_1 + \pi_2 + \chi_a$.

(3) $\chi_a\pi_1 = \pi_2\chi_a = \beta\pi_2 = \pi_1\beta = 0$.

(4) $\beta\pi_1 = \pi_2\beta = \beta$ *and* $\pi_1\chi_a = \chi_a\pi_2 = \chi_a$.

(5) $\beta^2 = \chi_a\chi_c = 0$.

(6) $\pi_1\sigma_a = \pi_1 + \chi_a$.

(7) $\chi_{ac} = \chi_a + \chi_c$ *and* $\chi_{a^{-1}} = -\chi_a$.

Proof. Straightforward.. □

Proposition 7.2 *There exists $a_1 \in E$ such that $(\chi_{a_1}\beta)\mid_{V_1} = id_{V_1}$.*

Proof. By (iv) there exists $a \in E$ such that $\sigma_{ba}(V_1) = V_2$. Now Proposition 7.1 affords

$$\pi_1\sigma_{ba} = (\pi_1\sigma_a)\sigma_b = (\pi_1 + \chi_a)(\pi_1 + \pi_2 + \beta) = \pi_1 + \chi_a + \chi_a\beta$$

and $0 = \pi_1\sigma_{ba}\mid_{V_1} = id_{V_1} + (\chi_a\beta)\mid_{V_1}$. Let $a_1 = a^{-1}$. □

Proposition 7.3 *For every $a \in E \backslash C_E(V_0)$ there exists $\widehat{a} \in A$ such that $(\chi_{\widehat{a}}\beta) |_{V_1} = ((\chi_a\beta) |_{V_1})^{-1}$.*

Proof. Let $g = b^{-1}ab$. A straightforward calculation shows

$$(^*) \quad \sigma_g = (\pi_1 + \chi_a - \chi_a\beta) + (\pi_2 + \beta\ \chi_a - \beta\chi_a\beta).$$

By Proposition 6.6 $\sigma_g(V_1) \neq V_1$. Hence by (iv) there is $c \in E$ such that $\sigma_{gc}(V_1) = V_2$. Then $\pi_1\sigma_{gc} = (\pi_1\sigma_c)\sigma_g = (\pi_1 + \chi_c)\sigma_g$. Using $(^*)$ we compute

$$0 = \pi_1\sigma_{gc} |_{V_1} = id_{V_1} - (\chi_a\beta) |_{V_1} - (\chi_c\beta\chi_a\beta) |_{V_1}\ .$$

Multiplying this equation with $((\chi_a\beta) |_{V_1})^{-1}$ from the right we obtain

$$(\chi_c\beta) |_{V_1} = ((\chi_a\beta) |_{V_1})^{-1} - id_{V_1}.$$

By Proposition 7.2 there exists $a_1 \in E$ such that $(\chi_{a_1}\beta) |_{V_1} = id$. Let $\widehat{a} = ca_1$. Then $\chi_{\widehat{a}} = \chi_c + \chi_{a_1}$ we compute $(\chi_{\widehat{a}}\beta) |_{V_1} = ((\chi_a\beta) |_{V_1})^{-1}$. □

In Proposition 7.4 and Proposition 7.5 we pick a fixed $v_1 \in V_1$ with $v_1 \neq 0$.

Proposition 7.4 *Let $a, a' \in E$. Define*

$$\overline{\chi}_a = (\sigma_a - 1)\beta \in End(V_0) \quad \text{and} \quad x_a := \overline{\chi}_a(v_1).$$

There is a unique coset $a''C_E(V_0)$ with $\overline{\chi}_{a'}(x_a) = x_{a''}$. Define

$$x_a + x_{a'} := x_{aa'} \quad \text{and} \quad x_a \cdot x_{a'} := x_{aa''}.$$

Set $D := \{x_a \mid a \in E\}$. Then $(D, +, \cdot)$ is a Cayley-Dickson-Division-Algebra or a skew field with $(D, +) \simeq E/C_E(V_0)$.

Proof. For each $v \in V_1^{\#}$ we have $\overline{\chi}_E(v) := \{\overline{\chi}_a(v) \mid a \in E\} = V_1$ by Proposition 6.5. As elements of $\overline{\chi}_A$ are not singular we get $a^{-1}a' \in C_E(V_0)$ if $\chi_a(v) = \overline{\chi}_{a'}(v)$. Hence for $v, v' \in V_1 \backslash \{0\}$ there is a unique coset $a''C_E(V_0)$ with $\overline{\chi}_{a''}(v) = v'$. Thus the product $x_a \cdot x_{a'}$ for $a, a' \in E$ is well defined. Now the proof of Glauberman [5, (IX) on page 7 f] shows that $(D, +, \cdot)$ is an alternative division ring or a skew field. Thus Proposition 7.4 follows from [2]. □

Proposition 7.5 *Let D be as in Proposition 7.4. Then $\{V_1\} \cup V_2^E$ is a congruence partition of an affine plane over D.*

Proof. By Proposition 6.6 $\{V_1\} \cup V_2^E$ is a congruence partition. Let $a_0 \in E$. Then $(\overline{\chi}_{a_0}\overline{\chi}_a)(v_1) = -(\beta\overline{\chi}_a)(v_1) + (\sigma_{a_0}\beta\overline{\chi}_a)(v_1)$ for $a \in E$. Hence $\sigma_{a_0}(V_2) = \{(\overline{\chi}_{a_0}\overline{\chi}_a)(v_1) + (\beta\overline{\chi}_a)(v_1) \mid a \in E\}$. Now $\overline{\chi}_a(v_1) \leftrightarrow aC_E(V_0) \leftrightarrow (\beta\overline{\chi}_a)(v_1)$ define bijective maps between $D \simeq E/C_E(V_0), V_1$ and V_2 which induce a bijective map between V_0 and $D \times D$. Then $\sigma_{a_0}(V_2)$ is mapped on $\{(\overline{\chi}_{a_0}\overline{\chi}_a, \overline{\chi}_a) \mid a \in E\}$ and we get Proposition 7.5 (see [4, page 131 f]). $\square$

Proposition 7.6 *By Proposition 7.5 we may view V_0 as an affine plane over D. Then E induces the group of shears with axis V_1 on V_0 and $H = L$ induces the subgroup of a point-stabilizer of V_0 generated by all shears.*

Proof. Since E is transitive on all lines through 0 different from V_1 by Proposition 6.6, E contains all shears by [4, page 122]. As H is transitive on the lines through 0 we get Proposition 7.6. $\square$

Theorem A now follows from Proposition 7.6 and Theorem 6.9.

8 Strong NSS's

We say that an NSS Γ is *strong* provided that

(Z) $\Omega(Z(N)) \neq 1$ for all $1 \neq N \in \Gamma$.

Throughout this section we assume that G is a group with a reduced strong NSS Γ with $1 \in \Gamma$. In addition to our previous notations we let

$$\Theta := \{N \in \Gamma \backslash \Gamma^* \mid N \text{ is large in } \Gamma\}.$$

Lemma 8.1 *Let* $\operatorname{rank}\Gamma = 2$, *$N \in \Theta$, $V = \Omega(Z(N))$, $P \in \Gamma_N^*$ and $Z \in \Gamma$ with Z normal in P. Then:*

(1) Let $1 \neq D \in \Gamma$ be normal in $N_G(N)$. Then $D^\circ = N$ and $N_G(D) = N_G(N)$

(2) $N_G(V) = N_G(N)$.

(3) If $\mathrm{R}(\Gamma_N \cap \Gamma_Z) \neq N$ *then* $(\Gamma_N \cap \Gamma_Z)^* = \{P\}$.

(4) If $Z \subseteq V$ *and* $\mathrm{R}(\Gamma_N \cap \Gamma_Z) \neq N$, *then* $[C_G(V), P] \subseteq C_P(V) = N$.

Proof. (1) Note that $\Gamma_N \leq \Gamma_D$. So by Corollary 2.7, D° is closed and contained in N. Since $1 \neq D \leq D^\circ \leq N < S \in \Gamma^*$ and Γ is reduced of rank 2, $D^\circ = N$. So $N_G(D) \subseteq N_G(N)$. By assumption $N_G(N) \leq N_G(D)$ and (1) holds.

(2) follows from (1) applied to $D = V$.

(3) Put $T = \mathrm{R}(\Gamma_N \cap \Gamma_Z)$ and suppose $T \not\leq N$. Since $N \leq T$ we get $N < T$. Since Γ_N has rank 1, Proposition 3.7(3) implies $(\Gamma_N \cap \Gamma_Z)^* = \{P\}$.

(4) Since $Z \subseteq V$, $C_G(V) \subseteq N_G(N) \cap N_G(Z)$ and so $C_G(V) \subseteq N_G(((\Gamma_N)_Z)^*) = N_G(P)$. Thus
$$[C_G(V), P] \subseteq C_P(V).$$

Suppose that $N_G(N) \leq N_G(P)$ and let $Q \in \Gamma_N$. By definition of Θ, $C_P(N) \leq N$ and thus $[C_P(N), \langle P, Q\rangle] \leq N$. So by Lemma 2.9, $QP \in \Gamma$. Thus $Q \in \Gamma_P$ and $\Gamma_N \leq \Gamma_P$. Corollary 2.7 implies $P \leq P^\circ \leq N$, a contradiction. Thus $N_G(N) \not\leq N_G(P)$.

Let $g \in N_G(N) \setminus N_G(P)$. Then $C_P(V) \subseteq N_P(N) \cap C_G(Z^g) \subseteq N_P(P^g)$, whence $C_P(V) C_P(V)^g \subseteq N_P(P^g) N_{P^g}(P) \in \Gamma$. Pick $Q \in \Gamma_N^*$ with
$$N_P(P^g) N_{P^g}(P) \subseteq Q.$$
If $Q \neq P$, then $C_P(V) \subseteq P \cap Q \subseteq N$, by Proposition 3.7(1).

If $Q = P$, then $C_P(V)^g \subseteq P^g \cap P = N$, again by Proposition 3.7(1). Since $N = N^g$ we get $C_P(V) \leq N$. □

Theorem 8.2 *Let G be a group with a reduced strong NSS Γ of rank 2. Let $N \in \Theta$, $S \in \Gamma_N^*$, $V := \Omega(Z(N))$ and $Z := C_V(J(S))$. Then $1 \neq Z \in \Gamma$. Moreover,*

(1) If $J(S) \leq N$, then $J(S)^\circ = N$ and $N_G(J(S)) = N_G(N)$.

(2) If $J(S) \not\leq N$ and $N = Z^\circ$, then $N = \Omega(Z(P))^\circ$ for any $P \in \Gamma^*$ with $S \leq P$.

(3) If $N \neq Z^\circ$, then $V/C_V(\langle \Gamma_N \rangle)$ is a natural SL_2-module for Γ_N and $S = J(S)N$.

Proof. Since $V \neq 1$ and $VJ(S)$ is nilpotent, $Z \neq 1$. By Proposition 4.7 $Z \in \Gamma$.

(1) Follows from Lemma 8.1(1).

(2) Suppose that $(\Gamma C_G(Z))^* = \{T\}$ for some T. Then T is normal in $N_G(Z)$. Since $N = Z^\circ$ is large, we get from Lemma 3.2 that $T \leq Q$ for all $Q \in \Gamma_Z^*$. Thus $T \leq Z^\circ = N$, a contradiction since $J(S) \leq T$ and $J(S) \neq N$.

Thus there exist $L, Q \in \Gamma C_G(Z)^*$ with $L \neq Q$. Then $\langle L, Q\rangle \notin \Gamma$. Put $M = \Omega(Z(P))^\circ$. Note that $N \subseteq L \cap Q$, and both $LN_M(N)$ and $QN_M(N)$ are in Γ. Thus $NN_M(N) \subseteq LN_M(N) \cap QN_M(N) \subseteq N$, by Proposition 3.7(1). Thus $1 \neq M \subseteq N$. Since $M \trianglelefteq P$, Proposition 2.8(12) implies M is closed and as $\operatorname{rank} G = 2$, $M = N$.

(3) From $\operatorname{rank}(\Gamma) = 2$, Proposition 3.7(1) and Theorem 3.1 we get $P \cap Q = N$ and $\operatorname{rank}(\Gamma_N) = 1$ for $P, Q \in \Gamma_N^*$ with $P \neq Q$. Suppose $N = \mathrm{R}(\Gamma_N \cap \Gamma_Z)$. Then by Lemma 2.6 (applied with $\Lambda = \Gamma_N \cap \Gamma_Z$, $\Delta = \Gamma_Z$ and $P = N$), $Z^\circ = \mathrm{R}(\Gamma_Z) \subseteq N$ and Z° is closed. Thus

$$1 \neq Z \subseteq Z^\circ \leq N \notin \Gamma^*.$$

Since $\operatorname{rank}(\Gamma) = 2$ we get $N = Z^\circ$, a contradiction. Therefore $N \neq \mathrm{R}((\Gamma_N \cap \Gamma_Z))$. In particular, $\Gamma_N \cap \Gamma_Z \neq \Gamma_N$ and so $\Gamma_N \nsubseteq \Gamma_Z$.

Moreover, by Lemma 8.1(4) $[C_G(V), S] \subseteq N = C_S(V)$.

Assume $J(S) \subseteq N$. Then $Z = V$ and $\Gamma_N \subseteq \Gamma_Z$, a contradiction. Thus $J(S) \nsubseteq N = C_S(V)$. Pick $A \in \mathcal{A}(S)$ with $A \nleq C_S(V)$. Then by Proposition 5.5 A is a non-trivial Γ-offender on V. By Proposition 3.7(2) Γ_N has rank 1. By definition of Θ, N is large in Γ_N.

We verified that all the the assumptions of Theorem A are satisfied for $N_G(N), \Gamma_N$, S,A and V. Hence $V/C_V(\langle\Gamma_N\rangle)$ is a natural SL_2-module for Γ_N. By Theorem 6.9, $S = A\,\mathrm{R}(\Gamma_N) = AN$ and so $S = J(S)N$. □

Theorem 8.3 *Suppose* $\operatorname{rank}(\Gamma) = 2$*,* $N \in \Theta$ *and* $S \in \Gamma_N^*$ *with* $N_G(S) \nsubseteq N_G(N)$*. Put* $V = \Omega(Z(N))$*. Then* $V/C_V(\langle\Gamma_N\rangle)$ *is a natural* SL_2*-module for* Γ_N*.*

Proof. Suppose that $J(S) \leq N$. Then using Theorem 8.2(1)

$$N_G(S) \leq N_G(J(S)) \leq N_G(J(S)^\circ) = N_G(N),$$

a contradiction to the assumptions.

Hence $J(S) \not\subseteq N$. Set $Z := C_V(J(S))$. Suppose that $Z^\circ = N$. By Proposition 3.7(3) S lies in a unique maximal Γ-subgroup P. Then by Theorem 8.22, $N_G(S) \subseteq N_G(P) \leq N_G(\Omega(Z(P))^\circ = N_G(N)$, a contradiction.

Hence $Z^\circ \neq N$ and Theorem 8.3 follows from Theorem 8.2(3) $\square$

Theorem 8.4 *Suppose* $\mathrm{rank}(\Gamma) = 2$*,* $S \in \Gamma^*$ *and* $|\Theta S| > 2$*. Then there is* $N \in \Theta S$ *such that* $V/C_V(\langle \Gamma_N \rangle)$ *is a natural* SL_2*-module for* Γ_N*, where* $V = \Omega(Z(N))$

Proof. Let $N \in \Theta S$. By Proposition 3.7(3), S is the unique maximal Γ-subgroup containing $N_S(N)$. Hence $N_S(N) \in \Gamma^*_N$. If $N_G(N_S(N)) \not\leq N_G(N)$ we are done by Theorem 8.3.

So we may assume that $N_G(N_S(N)) \leq N_G(N)$ for all $N \in \Theta S$. In particular $N_S(N_S(N)) \leq N_S(N)$ and so $N_S(N) = S$. Thus $S \in \Gamma_N$ and $N_G(S) \leq N_G(N)$.

Since $|\Theta S| \geq 3$ there exists $N \in \Theta S$ with $N \neq J(S)^\circ$ and $N \neq \Omega Z(S)^\circ$. Thus by Theorem 8.2 $J(S) \not\leq N$ and $N \neq Z^\circ$. So Theorem 8.4 follows from Theorem 8.2(c) $\square$

The following theorem deals with a situation which had been considered more detailed for finite groups in [3].

Theorem 8.5 *Let* $\mathrm{rank}(\Gamma) = 2, S \in \Gamma^*$ *and* $M, N \in \Theta S$ *with* $M \neq N$*. Assume there is* $P \in \Gamma^*_M \cap \Gamma^*_N$ *with*

$$(^*) \qquad Z \cap Z^g = 1 \text{ for all } g \in G \backslash N_G(P)$$

where $Z := \Omega(Z(J(P)))$*. Then* N *is a natural* SL_2*-module for* Γ_N*. Moreover* $P = MN$ *and* P *is of nilpotency class 2.*

Proof. For $L \in \{M, N\}$ set $V_L := \Omega(Z(L))$. As $\langle M, N \rangle \subseteq P \cap S$ and $\mathrm{rank}(\Gamma) = 2$ we get $P \subseteq S$ by Proposition 3.7(4).

Since $\mathrm{rank}(\Gamma) = 2$, $\langle M, N \rangle \notin \{M, N\}$. Thus by Lemma 3.2(2),

$$\Gamma_L \cap \Gamma N_G(P) = \Gamma P.$$

Suppose that $J(P) \subseteq L$. Then $V_L \subseteq Z$ and so by $(^*)$ $N_G(L) \subseteq N_G(P)$. Thus $\Gamma_L \subseteq \Gamma_L \cap \Gamma N_G(P) = \Gamma P$ and $L = P$, a contradiction.

Thus $J(P) \not\subseteq L$. Let $X = \Omega(Z(P))$. Then

$$1 \neq X \leq Z \cap V_L \subseteq V_L.$$

By (*) $N_G(X) \subseteq N_G(P)$, and so $\Gamma_L \cap \Gamma_X \subseteq \Gamma_L \cap \Gamma N_G(P)$ and $\mathrm{R}(\Gamma_L \cap \Gamma_X) = P$. Thus by Lemma 8.1(4), $C_S(V_L) \subseteq L$. So we can apply Theorem 8.2(c) and $V_L/C_{V_L}(\langle \Gamma_L \rangle)$ is a natural SL_2-module for Γ_L.

Let $\{K, L\} = \{M, N\}$. By Theorem 6.9 $KL = P = AL$ and $L = C_P(V_L)$ for all $A \in \mathcal{A}(P)$ with $A \not\leq L$. Moreover $X = C_{V_L}(A) = V_L \cap V_K = C_{V_L}(\langle \Gamma_L \rangle)[V_L, A]$. Since $X \cap X^g = 1$ for $g \in G \backslash N_G(P)$ we conclude $C_{V_L}(\langle \Gamma_L \rangle) = 1$. Thus by Proposition 6.5(3)

$$q := \mu(X) = \frac{1}{2}\mu(V_L) = \mu(X) = \mu(A/A \cap L).$$

In particular, $\mu(V_L(A \cap L) = \mu(A)$ and so $V_L(A \cap L) \in \mathcal{A}(L) \cap \mathcal{A}(P)$. Using this and symmetry in K and L, $\mathcal{A}(K) \cup \mathcal{A}(L) \subseteq \mathcal{A}(P)$. Suppose that $\mathcal{A}(K) = \mathcal{A}(L)$, then $\Gamma_L \cup \Gamma_K \subseteq \Gamma_{J(K)}$. Thus by Corollary 2.7 $\mathrm{R}(\Gamma_{J(K)})$ is closed and contained in $L \cap K$, a contradiction to $\mathrm{rank}(\Gamma) = 2$. So $\mathcal{A}(K) \neq \mathcal{A}(L)$ and interchanging K and L if necessary we assume $\mathcal{A}(K) \not\subseteq \mathcal{A}(L)$.

So we can choose $A \in \mathcal{A}(K)$.

Suppose for a contradiction that $[V_K, V_L] = 1$. Then $V_K V_L \leq K \cap L$. As $AL = P \not\subseteq C_G(V_K)$ we get $[V_K, L] = X$. Let $W \in V_K^{\langle \Gamma_L \rangle} \backslash \{V_K\}$. Then $[V_K, W] \subseteq (V_K \cap V_L) \cap (W \cap V_L) = 1$. Since A normalizes $[A \cap W, L]$ and $[A \cap W, L] \leq W \cap V_L$ we get $[A \cap W, L] = 1$ and so $A \cap W \leq A \cap W \cap V_L = 1$. Now $\mu(W) = q = 2\mu(A/A \cap L)$ implies $\mu(W(A \cap L)) > \mu(A)$. Thus $[A \cap L, W] \neq 1$ and so by Proposition 6.5(3) applied to $(A \cap L)V_L$, $[A \cap L, W] = V_L \cap W$ and $\mu(A)/\mu(C_A(W)) = \mu(W) = 2\mu(X)$. Thus $C_A(W)W \in \mathcal{A}(L)$.

Let $a \in A \backslash L$. Since W centralizes $C_A(W)$, also W^a centralizes $C_A(W)$. Since $[W, W^a] \subseteq V_L \cap W \cap W^a = 1$ we conclude that $C_A(W)WW^a$ is a decomposable abelian Γ-subgroup. Since $C_A(W)W \in \mathcal{A}(P)$, $C_A(W)W = C_A(W)W^a$. Thus

$$V_L \cap W = [A \cap L, W] = [A \cap L, W^a] = V_L \cap W^a,$$

a contradiction to $V_L \cap W \cap W^a = 1$.

Therefore $[V_K, V_L] \neq 1$ and so $V_N \not\leq M$ and $V_M \not\leq N$.

Let $h \in \langle \Gamma_M \rangle \backslash N_G(P)$. Note that $M = V_M(N \cap N^h)$. Hence $\Omega Z(N \cap N^h) \leq \Omega Z(M) = V_M$. But V_N centralizes N and so

$$\Omega Z(N \cap N^h) \leq C_{V_M}(V_N) \cap C_{V_M}(N^h) = V_M \cap V_N \cap V_N^h = 1.$$

By the assumptions of this section, Γ is strong and so $N \cap N^h = 1$. Thus $M = V_M, N = V_N$ and $P = V_M V_N = MN$. Now $P' = X = M \cap N$ and P has class 2. $\square$

Theorem 8.6 *Suppose that Π is a G-invariant subset of Θ such that*

(i) $\bigcap_{g\in G} A^g = 1$ *for all* $A \in \Pi$.

(ii) If $S \in \Gamma^*$ *with* $|\Theta S \cap \Pi| \geq 2$, *then* $|\Theta S| = 2$.

(iii) Whenever $X, Y \in \Pi$ *with* $X \in Y^G$ *and* $X \neq Y$ *then* $\mathrm{R}(\Gamma\langle X, Y\rangle) \in \Pi$.

Let Π_p *be an arbitrary orbit for* G *on* Π *and define* $\check{\Pi}_p = \{\mathrm{R}(\Gamma\langle A, B\rangle) \mid A, B \in \Pi_p$ *with* $A \neq B\}$. *Then*

(1) $\check{\check{\Pi}}_p = \Pi_p$.

(2) Π_p *is the set of points,* $\check{\Pi}_p$ *is the set of lines of a projective Moufang plane* π *and* $\langle \Pi_p \rangle = \langle \check{\Pi}_p \rangle$ *induces the group generated by all the elations on* π.

(3) $C_G(\pi) \leq C_G(\langle \Pi_p \rangle)$.

We remark that using knowledge of the automorphism group of a Moufang plane it should not be to difficult to show that G only has two orbits on Π.

Proof. From (i) we get

(1.) $N \ntrianglelefteq G$ *for all* $N \in \Pi$.

We say $X, Y \in \Pi$ are *incident* if $X \neq Y$ and $\langle X, Y\rangle \in \Gamma$. We show next

(2.) *If* X, Y *are incident then* $X \in \Gamma_Y$ *and* $Y \in \Gamma_X$.

Indeed by (ii) $\Theta\langle X, Y\rangle = \{X, Y\}$ and so X and Y are normal in $\langle X, Y\rangle$.

For $X, Y \in \Pi$ with $X \neq Y$ write $\widehat{XY} := R(\Gamma\langle X, Y\rangle)$.

(3.) $F \not< E$ *for all* $E, F \in \Pi$.

Otherwise let $g \in G \setminus N_G(E)$. Then $|\Theta E\widehat{EE^g}| = 2$ and so by (iii) $F = \widehat{EE^g} = F^g$. Since $\{F\} = \Theta E \setminus \{E\}$, $F^g = F$ for all $g \in G$. Thus $F \trianglelefteq G$, a contradiction to (1.), proving (3.)

Let $A \in \Pi$. By (1.) there exists $B \in A^G$ with $A \neq B$. By (iii) $\widehat{AB} \in \Theta$. Let $D = \widehat{AB} \in \Gamma$.

Suppose that A and B are incident. Then $\langle A, B\rangle \in \Gamma$ and $D = \langle A, B\rangle$. By (ii), $|\Theta D| = 2$. Since $A \neq B$ we may assume $A = D$. Hence $B \leq A$. Since A and B are conjugate $\mu(A) = \mu(B)$ and we conclude that $A = B$, a contradiction.

We proved

(4.) *No two distinct conjugate elements of* Π *are incident.*

Suppose $C \in \Pi$ is incident with A and B, Then $\langle A, B\rangle \leq \Gamma_C$ and so $C \leq D$. Since $\Theta AD = \{A, D\}$ and $A \not\leq D$, $C = D$. Thus

(5.) $\widehat{AB}$ *is the unique element of* Π *incident with* A *and* B.

Let $\Sigma(A) = \Pi \cap \Gamma_A \setminus \{A\}$, the set of elements of Π incident with A. Let $\Xi_A := \bigcup\{\Gamma AE \mid E \in \Sigma(A)\}$.

(6.) *Let* $A < X \in \Xi_A$. *Then there exists a unique* $X^* \in \Gamma^*$ *with* $X \leq X^*$ *and a unique* $E \in \Sigma(A)$ *with* $\Theta(X^*) = \{E, A\}$ *and* $X \leq AE$.

Pick $E \in \Sigma(A)$ with $X \leq EA$ and $P \in \Gamma^*$ with $EA \leq P$. Suppose there exists $Q \in \Gamma^*$ with $X \leq Q$ but $Q \neq P$. Choose such a Q with $P \cap Q$ maximal. Then by Proposition 2.11(1), $P \cap Q$ is closed. Since $A \leq P \cap Q$, Lemma 3.3 implies that $P \cap Q$ is large. So $P \cap Q \in \Theta P$. But $\Theta P = \{A, E\}$ and we conclude that $E = P \cap Q$, but then $A < E$, a contradiction to (3.).

(7.) $\Xi_A \leq \Gamma_A$, Ξ_A *is an NSS of rank 1 for* $N_G(A)$, $\Xi_A^* = \{AE \mid E \in \Sigma(A)\}$ *and* $\mathrm{R}(\Xi_A) = A$.

Clearly $\Xi_A^* = \{AE \mid E \in \Sigma(A)\}$, and (Suba) and (Subc) are fulfilled. Let $X, Y \in \Xi_A$ with $\langle X, Y\rangle \in \Gamma$. We need to show that $\langle X, Y\rangle \in \Xi_A$. If $X \leq A$ or $Y \leq A$ this is obvious. We may assume $A < X$ and $A < Y$. Pick $Q \in \Gamma^*$ with $\langle X, Y\rangle \leq Q$. Let $E, F \in \Sigma(A)$ with $X \leq EA$ and $Y \leq FA$. By (6.), $\Theta(Q) = \{A, E\} = \{A, F\}$ and so $E = F$. Thus $\langle X, Y\rangle \leq EA$ and $\langle X, Y\rangle \in \Xi_A$.

By Theorem 3.1 it remains to show that $|\Xi_A^*| > 1$. Otherwise we conclude that $\Sigma_A = \{K\}$ for some K, and $K = \widehat{AA^g}$ for all g with $A \neq A^g$ and then $K \trianglelefteq G$, a contradiction to (1.). This completes the proof of (7.).

(8.) *Let* $1 \neq X \in \Gamma A$. *Then* $N_\Gamma(X) \subseteq N_\Gamma(A)$.

Suppose not and pick $Q \in N_\Gamma(X)$ with $Q \not\leq N_G(A)$. Pick $g \in Q$ with $A \neq A^g$. Let $E \in \Sigma(A)$.

Suppose that $X \not\leq L$ for some $L \in \Sigma(A)$.

If A^g is incident with L, then both A and A^g are incident with L and so $L = \widehat{AA^g}$. By (6) applied to L in place of A, $X \leq A \cap A^g \leq L$.

Thus neither A^g nor $A^{g^{-1}}$ are incident with L. In particular $L \neq L^g$. Note that X normalizes L and L^g and so also $F := \widehat{LL^g}$. By Lemma 3.2(1) applied to L in the place of N, we get $XLF \in \Gamma$. By $X \not\leq L$ and (6.), XL lies in a unique maximal Γ-subgroup of G. Hence $\langle AL, XLF\rangle \in \Gamma$ and (ii) implies $A = F$. Thus L^g is incident with A and so $A^{g^{-1}}$ is incident with L, a contradiction.

Thus $X \leq L$ for all $L \in \Sigma(A)$. Let $Y := \bigcap \Sigma(A)$. Then $X \leq Y$ and so $Y \neq 1$. Since $N_G(A) \leq N_G(Y)$ we have $N_G(Y) \not\leq N_G(L)$.The claim we just proved applied to (Y, L) in place of (X, A) yields $Y \leq K$ for all $K \in \Sigma(L)$ and all $L \in \Sigma(A)$. Thus $Y \leq A^g$ for all $g \in G$ and (i) implies $Y = 1$, a contradiction.

(9.) *Let $E \in \Sigma(A)$ and $V_A = \Omega(Z(A))$. Then $[C_G(V_A), EA] \leq A$.*

Let $F \in \Sigma(A)$ and put $X = \Omega(Z(AF))$. Since A and F are large, $X \leq A \cap F$. Since Γ is strong, $X \neq 1$. By (8.) applied to F in place of A, $N_G(X) \leq N_G(F)$. Since $X \leq V_A$ we conclude that $C_G(V_A) \leq N_G(F)$ for all $F \in \Sigma(A)$. So by (6.) $C_G(V_A) \leq N_G(P)$ for all $P \in \Xi_A^*$. Define $U := \bigcap\{N_{EA}(P) \mid P \in \Xi_A^*\}$. Then $U \in \Xi_A$ and by Proposition 2.3(8), $U = \mathrm{R}(\Xi_A) = A$. But $[C_G(V_A), EA] \leq C_G(V_A) \cap EA \leq U$ and so (9.) holds.

(10.) *Let $E \in \Sigma(A)$. Then $J(EA) \not\leq E \cap A$.*

If $J(EA) \leq E \cap A$ then $J(E) = J(A)$. Then $N_G(E) \leq N_G(J(A))$ and so $N_G(J(A)) \not\leq N_G(A)$, a contradiction to (8.)

By (10.) and interchanging A and E if necessary

(11.) *we can choose $A \in \Pi$ and $E \in \Sigma(A)$ with $J(EA) \not\leq A$.*

By (8.) and Proposition 5.5 there exists a non-trivial offender in EA on V_A. Let $H_A =: \langle \Xi_A \rangle$. Note that $C_{V_A}(H_A) \leq C_{V_A}(E) \leq E$ and so by (8.) $C_{V_A}(H_A) = 1$. We conclude that the Hypothesis of Theorem A holds for $N_G(A), \Xi_A, V_A$ and EA. So V_A is a natural SL_2-module for Ξ_A. In particular,

(12.) *E acts transitively on $\Sigma(A) \setminus \{E\}$, $N_G(A)$ acts transitively on $\Sigma(A)$ and $C_{EA}(V_A) = A$.*

By (8.), $V_A \not\leq E$. Let $X \in \mathcal{A}(AE)$. By Proposition 6.5(3), $(X \cap A)V_A \in \mathcal{A}(AE)$ and we reestablish symmetry in A and E. Let $R := \langle V_E, V_E^h \rangle$ for some $h \in H_A$ with $V_E \neq V_E^h$. Then $A \cap E \cap E^h \leq C_A(R)$ and so by (8.), $A \cap E \cap E^h = 1$. It follows that $N = V_N$ and so N is a natural SL_2-module for Ξ_N for all $N \in \Pi$.

It follows from (12.) that $\widehat{AA^g} \in E^G$ and $\widehat{EE^g} \in A^G$ for all $g \in G$. Thus A^G and E^G form a projective plane.

By (12.) we have (N, N)-transitivity. Then [4, page 130] shows that we have got a projective Moufang plane.

Let C_A be the kernel of the action of G on A^G. Then clearly C_A also acts trivially on E^G. Moreover $[C_A, A] \leq A \cap C_A \leq C_A(E) \leq A \cap E$ and (8.) implies $[C_A, A] = 1$. $\square$

References

[1] R. Baer: Engelsche Elemente noetherscher Gruppen. Math. Ann. 133, 256 - 270 (1957)

[2] R. H. Bruck, E. Kleinfeld: The structure of alternative division rings. Proc. Nat. Acad. Sci. USA 37, 88-90 (1951)

[3] A.Delgado, D. Goldschmidt, B. Stellmacher: Groups and graphs: new results and methods. DMV-Seminar, Bd. 6. Birkhäuser, 1985.

[4] P. Dembowski: Finite geometries. Ergebnisse der Mathematik, Bd. 44. Springer, 1968.

[5] G. Glauberman: Weakly closed elements of Sylow subgroups. Math. Zeitschr. 107, 1-20(1968)

[6] P. Hall: A contribution to the theory of groups of prime power orders. Proc. London Math. Soc. 36, 29-95 (1933).

[7] P. Hall: The Edmonten Notes on Nilpotent Groups, 1957. Reproduced by Queen Mary College, 1969. See also: The Collected Works of Philip Hall, Compiled by K. W. Gruenberg and J. E. Roseblade, Clarendon Press, Oxford, 1988, 415 - 462.