On normalizers of nilpotent subgroups

Bernd Baumann
Mathematisches Institut, Justus-Liebig-Universität,
Arndtstr. 2, 35392 Gießen, Deutschland
Bernd.Baumann@math.uni-giessen.de

Ulrich Meierfrankenfeld
Department of Mathematics, Michigan State University
East Lansing, MI 48840, USA
meier@math.msu.edu

Introduction

Let $G$ be a group and $\Gamma$ a collection of nilpotent subgroups of $G$ satisfying:

(C) $P^g \in \Gamma$ for $P \in \Gamma$ and $g \in G$.
(I) $P \cap Q \in \Gamma$ for $P, Q \in \Gamma$.
(P) $N_P(Q) \cdot N_Q(P) \in \Gamma$ for $P, Q \in \Gamma$.
(MM) The minimum and the maximum condition hold for $\Gamma$ (i.e. each non empty subset of $\Gamma$ contains a minimal and a maximal element with respect to inclusion of sets).

Then we call $\Gamma$ a nilpotent subgroup system of $G$ (NSS for short) and the members of $\Gamma$ we call $\Gamma$-subgroups of $G$ (here $P^g := \{x^g \mid x \in P\}$, where $x^g := g^{-1}xg$, is a conjugate of $P$ and $N_X(Y)$ is the normalizer of $Y$ in $X$).

The set of all nilpotent subgroups of a group is an example of a system satisfying (C), (I) and (P). Examples of NSS’s are the set of $p$-subgroups of a finite group ($p$ a prime), the set of closed unipotent subgroups of an algebraic group, and the set of maximal cyclic subgroups plus the trivial group in a free group.

To state our main theorem we introduce a good portion of the notations used in this paper. Let $\Sigma$ be a set of subgroups of $G$.

$\Sigma^*$ is the set of maximal elements of $\Sigma$ (with respect to inclusion). The elements of $\Gamma^*$ are called maximal $\Gamma$-subgroups.

$\Sigma_*$ is the set of minimal non-trivial elements of $\Sigma$. The elements of $\Gamma_*$ are called minimal $\Gamma$-subgroups.
If $U$ is a subgroup of $G$ set $\Sigma_U := \{ A \in \Sigma \mid A \leq U \}$.

$R(\Gamma) := \bigcap_{P \in \Gamma}^\ast P$ is called the radical of $\Gamma$.

If $R(\Gamma) = 1$ the NSS $\Gamma$ is called reduced.

Let $P \in \Gamma$. Then $\Gamma_P := \{ T \in N_\Gamma(P) \mid TP \in \Gamma \}$ is the residue of $P$ in $\Gamma$.

It turns out that $\Gamma_P$ is an NSS for $N_G(P)$, see Proposition 2.8(1).

Set $P^o := R(\Gamma_P)$ and call $P$ closed if $P = P^o$.

Note that by (MM) any chain of $\Gamma$-subgroups is finite. Let $\text{rank}(\Gamma)$ be the supremum of the lengths of chains

$$P_0 < P_1 < \ldots < P_n$$

of closed $\Gamma$-subgroups. (The length of such a chain is $n$).

$\Omega(P) := \langle \Gamma \ast P \rangle$ is the subgroup of $P$ generated by the minimal $\Gamma$-subgroups of $P$.

$P$ is called decomposable if $P = \Omega(P)$.

$\mu(P)$ is the length of a maximal chain in $\Gamma P$. By Proposition 5.2 this is well defined. $\mu(P)$ is called the measure of $P$. If $Q \in \Gamma P$, then $\mu(P/Q) = \mu(P) - \mu(Q)$. By Proposition 5.4(1), this is the length of any maximal $\Gamma$-chain from $Q$ to $P$.

Let $A \in \Gamma_P$. If $[[P, A], A] = 1$, we say that $A$ acts quadratically on $P$. If $A$ and $P$ both are decomposable abelian $\Gamma$-subgroups, $[P, A] \neq 1$ and

$$\mu(P/C_P(A)) \leq \mu(A/C_A(P))$$

then $A$ is called a non-trivial $\Gamma$-offender on $P$. Note here that by Proposition 4.7 both $C_P(A)$ and $C_A(P)$ are $\Gamma$-subgroups.

Let $V$ be a normal $\Gamma$-subgroup of $G$ with $V \leq \Omega(Z(R(\Gamma)))$ and put $W = V/C_V(\Gamma)$. We say that $W$ is a natural $SL_2$-module for $\Gamma$ provided that

(i) $W$ is the set of points and $\{ wC_W(S) \mid S \in \Gamma^* \}$ is the set of lines of an affine Moufang plane;

(ii) For each $S \in \Gamma^*$, $C_S(W) = R(\Gamma)$ and $S$ induces the group of shears on $W$ with axis $C_W(S)$; and

(iii) $\langle \Gamma \rangle$ induces on $W$ the subgroup of a point stabilizer (of the point 1) generated by all shears.

We say that $N \in \Gamma$ is large in $\Gamma$ provided that $N$ is closed and $C_P(N) \leq N$ for all $P \in \Gamma_N$.
A theorem of Glauberman’s [5, Theorem 2] characterizes finite two-dimensional special linear groups as groups acting on $p$-groups with certain features. The object of the present paper is to prove the following generalization of Glauberman’s Theorem:

**Theorem A** Let $G$ be a group with an NSS $\Gamma$. Assume:

(a) $\text{rank}(\Gamma) = 1$.
(b) $V$ is a normal $\Gamma$-subgroup of $G$ with $V \leq \Omega(Z(R(\Gamma)))$.
(c) $S \in \Gamma^*$ and $[C_G(V), S] \leq R(\Gamma)$.
(d) $S$ contains a non-trivial $\Gamma$-offender on $V$.
(e) $R(\Gamma)$ is large in $\Gamma$.

Then $V/C_V(\langle \Gamma \rangle)$ is a natural $SL_2$-module for $\Gamma$.

It is well known that an affine Moufang plane is isomorphic to a plane whose point set consists of the ordered pairs of an alternative field or a skew field $K$ and whose lines are the point sets $L(a, b) := \{(x, x \cdot a + b) \mid x \in K\}$ and $L(c) := \{(c, y) \mid y \in K\}$. Then for example shears with axis $L(0)$ are the mappings $(x, y) \mapsto (x, x \cdot d + y)$ (see [4] page 128 ff. and the literature quoted there).

For the proof of Theorem A see section 6 and 7 and for other main results of this paper see section 8.

We would like to thank G. Glauberman for pointing out an error in an earlier version of this paper.

1 Preliminaries

In this section we collect some elementary results about nilpotent groups. We start with some well known commutator properties (see for instance [6]).

**Proposition 1.1** Let $a, b, c$ be elements, $A, B, C$ subgroups and $N$ a normal subgroup of a group. Then

(1) $[a, bc] = [a, c][a, b][[a, b], c] = [a, c][a, b]^c$
(2) \([ab, c] = [a, c][a, c]b = [a, c]b b\)

(3) \([a, b] = [b, a]^{-1} = [b, a]^{-1}[b, a^{-1}, a]\)

(4) \(ab[a, a] = ba\)

(5) \([[B, C], A] \subseteq N\) and \([[C, A], B] \subseteq N\) imply \([[A, B], C] \subseteq N\).

(6) \([A, B]\) is a normal subgroup of \(\langle A, B \rangle\).

**Proposition 1.2** Let \(G\) be a group, \(V\) an abelian normal subgroup of \(G\), \(U\) a subgroup of \(V\) and \(g \in G\) with \([[V, g], g] = 1\). Then the following hold:

1. \(\{[u, g] \mid u \in U\}\) is a subgroup of \(V\)
2. \(UU^g = U[U, g]\)
3. \(UU^g = U \times [U, g]\) if and only if \(U \cap U^g = C_U(g)\)
4. \(C_{UU^g}(g) = C_{U \cup U^g}(g)[U, g]\)

**Proof.** These properties are applications of Proposition 1.1. \(\Box\)

**Proposition 1.3** Let \(A\) and \(B\) be subgroups and let \(N\) be a normal subgroup of the group \(G\). Then

\([N, \langle A, B \rangle] = \langle [N, A], [N, B] \rangle\).

**Proof.** Obviously the right hand side is contained in the left hand side. Conversely, by Proposition 1.1(6) \(M := \langle [N, A], [N, B] \rangle\) is a normal subgroup of \(\langle A, B, N \rangle\) contained in \(N\), as \(N\) is a normal subgroup. Now \(N/M\) is centralized by \(\langle A, B \rangle\), whence \([N, \langle A, B \rangle] \subseteq M\). \(\Box\)

Let \(A\) be a group acting on a group \(D\). We say that \(A\) acts nilpotently on \(D\) if \([D, A, k] = 1\) for some \(k\) (where \([D, A, 0] := D\) and \([D, A, i+1] := [[D, A, i], A]\)). The minimal such \(k\) is called the nilpotence length of \(A\) on \(D\). For a group \(G\) let \(L_0(G) = G\) and \(L_{i+1}(G) = [L_i(G), G]\).

**Lemma 1.4** (1) Suppose \(A\) acts nilpotently on \(D\). Then \(A/C_A(D)\) is nilpotent.
(2) Suppose $G$ acts on $D$, $A \leq G$ and $B \leq N_G(A)$. If $A$ and $B$ act nilpotently on $D$, so does $AB$.

(3) Let $N$ be normal in $G$. Then $G$ is nilpotent if and only if $G/N$ is nilpotent and $G$ acts nilpotently on $N$.

(4) Let $G = AB$, where $A$ and $B$ are nilpotent subgroups of $G$, and $A$ is normal in $G$. Assume $N$ is a normal subgroup of $G$ with $N \leq A \cap B$ such that $G/N$ is nilpotent. Then $G$ is nilpotent.

(5) Let $A, B$ be normal in $G$ such that $G/A$ and $G/B$ are nilpotent. Then $G/A \cap B$ is nilpotent.

Proof. (1) See [7, Corollary to Theorem 3.8]

(2) By induction on the nilpotency length of $A$ on $D$ $[[D, A], AB, i] = 1$ for some $i$. Also if $[D, B, j] = 1$, then $[D, AB, j] \subseteq [D, A]$ and so $[D, AB, i + j] = 1$.

(3) One direction is obvious. So suppose $G/N$ is nilpotent and $G$ acts nilpotently on $N$. Then $L_k(G) \leq N$ for some $k$ and $[N, G, i] = 1$ for some $i$. Thus $L_{k+i}(G) = 1$.

(4) Since $N \leq A \cap B$, both $A$ and $B$ act nilpotently on $N$. By (2) $G$ acts nilpotently on $N$ and so (4) follows from (3).

(5) Let $k$ be the maximum of the nilpotency classes of $G/A$ and $G/B$. Then $L_k(G) \leq A \cap B$. \hfill \Box

Proposition 1.5 Let $P$ and $Q$ be nilpotent subgroups of the group $G$ with $Q \subseteq PC_G(P)$. Then $PQ$ is a nilpotent subgroup of $G$.

Proof. Clearly $P$ is normal in $PQ$ and $PQ$ acts nilpotently on $P$. Also $PQ/P \cong Q/Q \cap P$ and so $PQ/P$ is nilpotent. Hence the lemma follows from Lemma 1.4(3). \hfill \Box

Proposition 1.6 Let $X$ be a proper subgroup of the nilpotent group $G$.

(1) $X$ is contained in a proper normal subgroup of $G$.

(2) $X$ is a proper subgroup of $N_G(X)$. 5
(3) If \( N_G(X) = N_G(N_G(X)) \), then \( X \) is normal in \( G \).

(4) \( (X^G) \) is a proper subgroup of \( G \).

Proof. Well-known. \( \square \)

**Proposition 1.7** Let \( H \) be a nilpotent group of class \( k \) and \( x, y \in H \), where \( x \) is an element of order \( p \), \( p \) a prime. Then \( [x, y^{p^{k+1}}] = 1 \).

Proof. By induction on \( k \), \( [x, y^p] \in Z(H) \). Then by Proposition 1.1

\[
1 = [x^p, y^p] = [x, y^p]^p = [x, y^{p^{k+1}}].
\]

\( \square \)

**Proposition 1.8** Let \( X \) be a subgroup of the group \( G \) and let \( U \) and \( A \) be subsets of \( G \) with \( U \subseteq X \). Then \( (UA) \cap X = U(A \cap X) \).

Proof. Let \( u \in U \) and \( a \in A \) with \( ua \in X \). Then \( a \in A \cap X \), hence \( (UA) \cap X \subseteq U(A \cap X) \). If \( d \in A \cap X \) then \( ud \in (UA) \cap X \) as \( U(A \cap X) \subseteq X \). Thus \( U(A \cap X) \subseteq (UA) \cap X \) and Proposition 1.8. \( \square \)

2 Basic Properties of NSS’s

In this section \( G \) is a group with an NSS \( \Gamma \) with \( 1 \in \Gamma \).

We remark that (MM) allows us to prove statements about \( \Gamma \) by induction. Namely suppose given a statement \( S \) about \( \Gamma \)-subgroups. Suppose also that if \( P \in \Gamma \) and \( S \) is true for all \( Q \in \Gamma \) with \( Q < P \), then \( S \) is also true for \( P \). Then \( S \) must be true for all \( P \in \Gamma \). Indeed the set of \( \Gamma \)-subgroups for which \( S \) is false, does not have a minimal element and so is empty.

Note also that (I) and (MM) imply, that arbitrary intersections of \( \Gamma \)-subgroups are \( \Gamma \)-subgroups.

**Lemma 2.1** Let \( P, Q \in \Gamma \). Then \( N_P(Q) \in \Gamma \).
Proof. Note that $N_Q(P) \cap P \subseteq Q \cap P \subseteq N_P(Q)$ and so by Proposition 1.8

$$(N_P(Q)N_Q(P)) \cap P = N_P(Q)(N_Q(P) \cap P) = N_P(Q).$$

By (P) and (I) the left hand side of this equation is in $\Gamma$. □

**Proposition 2.2** Let $P, Q \in \Gamma$ such that $Q$ is a minimal element of $\{T \in \Gamma \mid P < T\}$ or that $P$ is a maximal element of $\{T \in \Gamma \mid T < Q\}$. Then $P$ is normal in $Q$.

Proof. Note that the two conditions are actually equivalent. So suppose the first. By Lemma 2.1 $P < N_Q(P) \in \Gamma$ and so $Q = N_Q(P)$ by minimality of $Q$. □

**Proposition 2.3**

1. If $\Delta$ is a nonempty subset of $\Gamma$, then $\bigcap_{X \in \Delta} X \in \Gamma$ and $\bigcap_{X \in \Delta_0} X$ for some finite subset $\Delta_0$ of $\Delta$.

2. If $\Delta$ is a set of normal $\Gamma$-subgroups of $G$, then $\langle \Delta \rangle \in \Gamma$.

3. If $U$ is a subgroup of $G$, then $\Gamma U$ is an NSS of $U$.

4. $P^\circ \in \Gamma$ for all $P \in \Gamma$. In particular, $R(\Gamma)$ is a normal $\Gamma$-subgroup of $G$.

5. If $S \in \Gamma$ and $P \in \Gamma S \setminus \{S\}$, then $P \subset \langle \Gamma P S \rangle$.

6. If $\langle \Delta \rangle$ is nilpotent for $\Delta \subseteq \Gamma$, then $\langle \Delta \rangle \in \Gamma$.

7. $R(\Gamma) = \langle \cap_{T \in \Gamma^*} N_{\Gamma}(T) \rangle$.

8. Let $S \in \Gamma^*$ and $A \in \Gamma(\text{SC}_G(S))$. Then $A \leq S$.

9. Let $S \subseteq G$ be nilpotent and put $A = \langle \Gamma S \rangle$. Then $A \in \Gamma$, $\Gamma^*S = \{A\}$ and $A$ is normal in $N_G(S)$.

Proof.

(1) By (I) intersections of the members of finite subsets of $\Delta$ are elements of $\Gamma$. Then (1) follows from the minimal condition for $\Gamma$ applied to the set of intersections of the members of finite subsets of $\Gamma$. 

7
(2) If $N$ and $M$ are normal $\Gamma$-subgroups then $NM \in \Gamma$ by (P). Hence finite products of elements of $\Delta$ lie in $\Gamma$, and (2) follows from the maximal condition for $\Gamma$.

(3) is obvious by the definition of an NSS.

(4) is a consequence of (1).

(5) By Proposition 1.6(2) $P < N_S(P)$ and by Lemma 2.1 $N_S(P) \in \Gamma$.

(6) Let $S = \langle \Delta \rangle$ and without loss $\Delta = \Gamma S$. Let $P \in \Delta^*$. If $P$ is not normal in $S$, then Proposition 1.6(c) there exists $x \in N_S(N_S(P))$ with $P \neq Px$. By (C) and (P) we get $PP^x \in \Gamma S$, a contradiction to the maximality of $P$. So $P$ is normal in $S$. Thus by (2) $S = \langle \Delta \rangle = \langle \Delta^* \rangle \in \Gamma$.

(7) Let

$$\Lambda := \bigcap_{T \in \Gamma^*} N_G(T) = \{ A \in \Gamma \mid A \leq N_G(T) \forall T \in \Gamma^* \}.$$  

We claim that $|\Lambda^*| = 1$. Indeed, let $X_1, X_2 \in \Lambda^*$ and pick $T_i \in \Gamma^*$. By (6), $\langle AT_i \rangle \in \Gamma$ and so the definition of $\Lambda$ implies $\langle AT_i \rangle \in \Lambda$. The maximality of $X_i$ implies $X_i = \langle AT_i \rangle$. Hence $X_1 \leq N_G(T_2) \leq N_G(\langle AT_2 \rangle) \leq N_G(X_2)$. So $X_1$ normalizes $X_2$ and $X_2$ normalizes $X_1$. Thus by (P), $X_1X_2 \in \Gamma$. Hence also $X_1X_2 \in \Lambda$ and $X_1 = X_2$.

So indeed $|\Lambda^*| = 1$. Let $N$ be the unique element in $\Lambda^*$. Then $N$ is normal in $G$. Let $T \in \Gamma^*$. The definition of $\Lambda$ implies that $N$ normalizes $T$. So by (P), $NT \in \Gamma$. Thus $N \leq T$ and $N \leq R(\Gamma)$. Clearly $R(\Gamma) \leq N$ and (7) holds.

(8) Obviously $S$ is contained in the right hand side of this equation. Let $P \in \Gamma(SC_G(S))$. Then $SP$ is nilpotent by Proposition 1.5 and therefore $SP \in \Gamma$ by (6). Hence $P \subseteq S$ because $S$ is maximal.

(9) By (6) we get $A \in \Gamma$, which implies $\Gamma^*S = \{ A \}$, and by (C) $A$ is normal in $N_G(S)$.

\[\square\]

**Definition.** A subset $\Delta$ of $\Gamma$ is called a *sub-NSS* of $\Gamma$ and we write $\Delta \leq \Gamma$ provided that:

(Suba) If $A \in \Gamma$ and $B \in \Delta$ with $A \subseteq B$ then $A \in \Delta$.

(Subb) If $A, B \in \Delta$ with $\langle A, B \rangle \in \Gamma$ then $\langle A, B \rangle \in \Delta$.  

8
Lemma 2.4 Let \( \Delta \leq \Gamma \), then \( \Delta \) is an NSS for \( \langle \Delta \rangle \).

Proof. (C) follows from (Subc). Let \( P, Q \in \Delta \). Then since (I) holds for \( \Gamma \), \( P \cap Q \in \Gamma \). So by (Suba), \( P \cap Q \in \Delta \). So (I) holds. By Lemma 2.1, \( N_P(Q) \) and \( N_Q(P) \) are \( \Gamma \)-subgroups. So by (Suba), they are also \( \Delta \)-subgroups. By (P) for \( \Gamma \), \( N_P(Q)N_Q(P) \in \Gamma \) and so by (Subb), \( N_P(Q)N_Q(P) \in \Delta \). Thus (P) holds. (MM) follows from (MM) for \( \Gamma \). \( \square \)

Lemma 2.5 Let \( \Delta \leq \Gamma \).

(1) \( R(\Delta) \in \Delta \), and \( R(\Delta) \) is normal in \( \langle \Delta \rangle \).

(2) If \( A \in \Delta \) then \( AR(\Delta) \in \Delta \).

(3) \( \Delta \leq \Gamma_{R(\Delta)} \).

(4) Let \( \Lambda \leq \Delta \). Then

   (i) \( R(\Delta) \cap S = R(\Delta) \cap R(\Lambda) \) for all \( S \in \Lambda^* \).

   (ii) \( R(\Lambda) \cap R(\Delta) \) is the unique maximal \( \Lambda \)-subgroup of \( R(\Delta) \).

   (iii) \( \Lambda \)-subgroups of \( R(\Delta) \) are contained in \( R(\Lambda) \).

(5) Let \( \Lambda \leq \Delta \) with \( R(\Delta) \in \Lambda \). Then \( R(\Delta) \leq R(\Lambda) \).

(6) Suppose that \( \Lambda \leq \Delta \leq \Gamma_{R(\Lambda)} \) and \( R(\Delta) \in \Lambda \). Then \( R(\Lambda) = R(\Delta) \).

(7) \( R(\Delta) \) is closed in \( \Gamma \) if and only if \( R(\Delta)^\circ \in \Delta \).

Proof.

(1) follows from Proposition 2.3(4) applied to the NSS \( \Delta \).

(2) By (MM) there exists \( S \in \Delta^* \) with \( A \subseteq S \). By Proposition 2.3(6) \( AR(\Delta) \in \Gamma \) and so by (Subb), \( AR(\Delta) \in \Delta \).

(3) Follows from (1) and (2).

(4) Let \( S, T \in \Lambda^* \). By (2), \( TR(\Delta) \in \Delta \) and so by Proposition 2.3(6) also \( T(R(\Delta) \cap S) \in \Delta \). By (I) and (Suba), \( R(\Delta) \cap S \in \Gamma \) and so (Subb)
implies $T(R(\Delta) \cap S) \in \Lambda$. Thus by maximality of $T$, $R(\Delta) \cap S \subseteq T$. So $R(\Delta) \cap S \subseteq R(\Lambda)$. So (i) holds. (ii) and (iii) follow from (i).

(5) Follows from (4).

(6) By (5) $R(\Delta) \leq R(\Lambda)$. Note that $R(\Lambda) \in \Lambda \leq \Delta$. Thus $R(\Lambda)$ is a $\Delta$-subgroup of $R(\Gamma_{R(\Lambda)})$ and so by (4)(iii) applied to $\Delta \leq \Gamma_{R(\Lambda)}$, $R(\Lambda) \leq R(\Delta)$.

(7) If $R(\Delta) = R(\Delta)^\circ$, then $R(\Delta)^\circ \in \Delta$ by (1). So suppose $R(\Delta)^\circ \in \Delta$. Then by (5) applied to $\Delta \leq \Gamma_{R(\Delta)}$, $R(\Delta)^\circ \leq R(\Delta)$. So $R(\Delta)$ is closed.

Lemma 2.6 Let $P \in \Delta \leq \Gamma$ such that $P = R(\Gamma_P \cap \Delta)$. Then

(1) $R(\Delta) \subseteq P$.

(2) If $\Gamma_P \cap \Delta \neq \emptyset$, then $R(\Delta)$ is closed.

Proof. Let $T = R(\Delta)$.

(1) Since $P \in \Delta$, Lemma 2.5(2) implies $PT \in \Delta$. Hence by Lemma 2.1, $N_T(P) \in \Delta$. Let $S \in (\Gamma_P \cap \Delta)^\ast$. Then again by Lemma 2.5(2), $ST \in \Delta$. Hence by Proposition 2.3(6), $N_T(P)S \in \Gamma_P \cap \Delta$. By maximality of $S$, $N_T(P) \subseteq S$. Thus $N_T(P) \leq R(\Gamma_P \cap \Delta) = P$. Since $TP$ is nilpotent we conclude $T \subseteq P$.

(2) By Lemma 2.5(3), $\Delta \leq \Gamma_T$. Thus

\[ \Gamma_P \cap \Delta \leq \Gamma_P \cap \Gamma_T \leq \Gamma_P. \]

Let $Q = R(\Gamma_P \cap \Gamma_T)$. By assumption there exists $S \in \Gamma_P \cap \Delta$. Then $S \in (\Gamma_P \cap \Gamma_T)^\ast$ and so $Q \subseteq S$. Hence by (Suba), $Q \in \Gamma_P \cap \Delta$. By (*) we can apply Lemma 2.5(6) (with $\Lambda = \Gamma_P \cap \Delta$ and $\Delta = \Gamma_P \cap \Gamma_T$) Thus $Q = R(\Gamma_P \cap \Delta) = P$. So by (1) (applied to $\Gamma_T$ in place of $\Delta$), $T^\circ \subseteq P$ and thus $T^\circ \in \Delta$. By Lemma 2.5(7), $T = R(\Delta)$ is closed.

Corollary 2.7 Suppose that $N \in \Gamma$ is closed and $\Gamma_N \leq \Delta \leq \Gamma$. Then $R(\Delta) \leq N$ and $R(\Delta)$ is closed.
Proof. Since $N$ is closed and $\Gamma_N = \Gamma_N \cap \Delta$ we have $N = R(\Gamma_N \cap \Delta)$. Also $\Gamma^*_N \subseteq \Gamma_N \subseteq \Delta$ and so $\Gamma^*_N \cap \Delta \neq \emptyset$. Thus the Corollary follows from Lemma 2.6. □

Definition. If $Q$ is a normal $\Gamma$-subgroup of $G$ contained in $R(\Gamma)$ we define

$$\Gamma/Q := \{PQ/Q \mid P \in \Gamma\}.$$  

Note that $\Gamma/Q = \{P/Q \mid Q \leq P \in \Gamma\}$.

Proposition 2.8 Let $L \in \Gamma$. Then the following hold:

1. $\Gamma_L$ resp. $\Gamma_L/L$ is an NSS of $N_G(L)$ resp. $N_G(L)/L$.
2. $L \leq L^\circ$.
3. $\Gamma = \Gamma_{R(\Gamma)}$.
4. $R(\Gamma_L/L) = R(\Gamma_L)/L$.
5. $\Gamma/R(\Gamma)$ is reduced.
6. $L$ is closed in $\Gamma$ if and only if $1$ is closed in $\Gamma_L/L$.
7. If $L$ is closed then $L = \bigcap\{S \in \Gamma^* \mid L \subseteq S\}$.
8. $\Gamma_L \subseteq \Gamma_{L^\circ}$.
9. If $M \in \Gamma$ with $\Gamma_L \leq \Gamma_M$, then $N_M(L) \leq L^\circ$. If in addition $L^\circ \leq M$, then $L^\circ = N_M(L)$.
10. There is some (not necessarily uniquely determined) closed $\Gamma$-subgroup $M$ with $L \subseteq M, \Gamma_L \subseteq \Gamma_M$ and $L^\circ = N_M(L)$.
11. $L^\circ = N_{L^\circ}(L)$.
12. Let $S \in \Gamma^*$ and $L$ be a normal $\Gamma$-subgroup of $S$. Then $L^\circ = L^{\circ\circ}$ is closed.
Proof.
(1) Let $P, Q \in \Gamma_L$. Then $(P \cap Q)L \subseteq PL \cap QL \in \Gamma$ by (I). Hence $(P \cap Q)L \subseteq \Gamma$ by Proposition 2.3(6) and $P \cap Q \in \Gamma_L$. Similarly $N_P(Q)L \subseteq N_{PL}(QL)N_{QL}(PL) \subseteq \Gamma$ by (P) and therefore $N_P(Q)L \in \Gamma$ implying $N_P(Q)L \subseteq \Gamma_L$. Condition (MM) is satisfied for $\Gamma_L$ as $\Gamma_L \subseteq \Gamma$, and (C) follows for $\Gamma_L$ as (C) holds for $\Gamma$ and thus $P^gL \in \Gamma$ if $P \in \Gamma_L$ and $g \in N_G(L)$. Thus $\Gamma_L$ and $\Gamma_L/L$ are NSS’s.

(2) and (3) are obvious.

(4) follows from $(\Gamma_L/L)^* = \Gamma^*/L := \{S/L \mid S \in \Gamma^*_L\}$.

(5) is a consequence of (4).

(6) is clear by (5) and (2).

(7) Put $D := \bigcap \{S \in \Gamma^* \mid L \subseteq S\}$. Let $T \in \Gamma^*_L$ and pick $S \in \Gamma^*$ with $T \subseteq S$. Then $T \subseteq N_S(L) \in \Gamma_L$ by Lemma 2.1 and so $T = N_S(L)$. Since $D \subseteq S$ we conclude $N_D(L) \subseteq T$. As this is true for all $T \in \Gamma^*_L$, $N_D(L) \subseteq R(\Gamma_L) = L$. Since $L \subseteq D$ and $D$ is nilpotent, $L = D$.

(8) If $P \in \Gamma_L$ then there is $Q \in \Gamma^*_L$ with $P \subseteq Q$, hence $PL^o \in \Gamma Q \subseteq \Gamma$ by Proposition 2.3(6), and $P \in \Gamma_L^o$.

(9) Note that $N_M(L) \in \Gamma_L$ and $N_M(L) \leq M \leq R(\Gamma_M)$. Thus by Lemma 2.5(4), $N_M(L) \leq R(\Gamma_L) = L^o$. If $L^o \leq M$, then $L^o \leq N_M(L) \leq L^o$ and so $L^o = N_M(L)$.

(10) Let $M$ in $\Gamma$ be maximal with respect to $L^o \leq M$ and $\Gamma_L \subseteq \Gamma_M$. Note that by (2) and (8) such an $M$ exists. By (2) and (8) applied to $M$, $L^o \leq M \leq M^o$ and $\Gamma_L \subseteq \Gamma_M \subseteq \Gamma_M^o$. Thus the maximal choice of $M$ implies $M = M^o$. So $M$ is closed. By (9), $N_M(L) = L^o$ and all parts of (10) are verified.

(11) Follows from (2),(8) and (9).

(12) As $S \in \Gamma^*$ we get $S \in \Gamma^*_L$. It follows that $L^o$ is normal in $S$ and thus $L^{oo} \leq S$. Hence $L$ is normal in $L^{oo}$. So by (12) $L^o = N_{L^{oo}}(L) = L^{oo}$.

\[ \square \]

Lemma 2.9 Let $N \in \Gamma$ and $P, Q \in \Gamma_N$. If $[C_P(N), \langle P, Q \rangle] \subseteq N$ then $N_Q(P)P \in \Gamma$.
Proof. By Lemma 2.1 we may assume that \( Q = N_Q(P) \). So \( Q \) normalizes \( P \). Since \( PN \) and \( QN \) are in \( \Gamma \) they are both nilpotent. So \( P \) and \( Q \) act nilpotently on \( N \). By Lemma 1.4(2) \( PQ \) acts nilpotently on \( N \). Thus by Lemma 1.4(1), \( PQ/C_{PQ}(N) \) is nilpotent. Also \( PQ/P \cong Q/Q \cap P \) is nilpotent and so by Lemma 1.4(5) \( PQ/C_P(N) \) is nilpotent. Since \([C_P(N), PQ] \subseteq N\) we get that \( PQ \) acts nilpotently on \( C_P(N) \). Thus the assertion follows from Lemma 1.4(3).

Proposition 2.10 Let \( G = \langle A, B \rangle \), where \( A \) and \( B \) are nilpotent subgroups of \( G \). Assume \( A \in \Gamma \), \( N \) is a normal subgroup of \( G \), \( N \subseteq A \cap B \) and \( G/N \) is nilpotent. Then \( G \) is nilpotent.

Proof. By (MM) \( A \) can be chosen maximal fulfilling the assumptions of the Proposition. Then by nilpotency of \( G/N \) and (P) \( A \) is normal in \( G \) and Proposition 2.10 follows from Lemma 1.4(4).

Proposition 2.11 Let \( S \in \Gamma^* \) be fixed.

(1) Let \( T \in \Gamma^* \setminus \{S\} \) such that \( S \cap T \) is maximal. Then \( S \cap T \) is closed.

(2) Let \( T \in \Gamma^* \setminus \{S\} \), then there exists a closed \( \Gamma \)-subgroup \( P \) with \( S \cap T \leq P < S \).

Proof. (1) Set \( P := S \cap T \) Then \( P^o N_S(P) \in \Gamma \) by definition of \( \Gamma_P \) and 1.10(6). Therefore there is \( X \in \Gamma^* \) with \( P^o N_S(P) \subseteq X \). By maximality of \( S \), \( S \nsubseteq T \) and so \( P < S \). Hence by Proposition 1.6(3), \( P < N_S(P) \leq X \cap S \). By maximality of \( P \), \( X = S \). Thus \( P^o \leq S \). Note also that \( N_T(P)P^o \leq Y \) for some \( Y \in \Gamma^* \). Since \( P < N_T(P) \), \( N_T(P) \nsubseteq S \) and so \( Y \neq S \). Thus by maximality of \( P \), \( Y \cap S = P \). Since \( P^o \leq Y \cap S \) we get \( P^o = P \) and \( P \) is closed.

(2) Let \( T^* \in \Gamma^* \setminus \{S\} \) with \( S \cap T \leq S \cap T^* \) and \( S \cap T^* \) maximal. Then \( S \cap T^* \) is closed by (1).

The following statement is a variant of Baer’s famous theorem [1].

Theorem 2.12 Let \( X \in \Gamma \) such that \( \langle X, X^g \rangle \in \Gamma \) for all \( g \in G \), then \( \langle X^G \rangle \in \Gamma \).
Proof. Set $\Delta := X^G$ and assume $\langle \Delta \rangle \notin \Gamma$. Then there are $Q = \langle \Delta Q \rangle \in \Gamma$ and $R = \langle \Delta R \rangle \in \Gamma$ with $\langle Q, R \rangle \notin \Gamma$. Choose $Q$ and $R$ such that $D := \langle \Delta (Q \cap R) \rangle$ is maximal. Suppose that $\Delta N_Q(D) = \Delta (Q \cap R)$. Then

$$N_Q(N_Q(D)) \leq N_Q(\langle \Delta N_Q(D) \rangle) = N_Q(D)$$

and so by Proposition 1.6(3), $N_Q(D) = Q$. But $\Delta Q \neq \Delta (Q \cap R)$, a contradiction. Thus there exists $A \in \Delta$ with $A \notin N_Q(D)$ and $A \leq D$. Similarly there exists $B \in \Delta$ with $B \leq N_R(D)$ and $B \notin D$. By assumption $\langle A, B \rangle \in \Gamma$. By Proposition 2.10, applied with $AD, BD$ and $D$ in place of $A$, $B$ and $N$, $P := \langle A, B, D \rangle$ is nilpotent. Since $D < AD \leq Q \cap P$, the maximality of $D$ implies $\langle Q, P \rangle \in \Gamma$. Similarly $\langle R, P \rangle \leq \Gamma$. But $\langle R, P, Q \rangle / \Gamma$ and $D < P \leq \langle R, P \rangle \cap \langle Q, P \rangle$. This contradiction to the maximality of $D$ completes the proof of Theorem 2.12. \hfill $\Box$

3 NSS’ of rank 1 and 2

As in the previous section let $G$ be a group with an NSS $\Gamma$.

**Theorem 3.1** Suppose $|\Gamma^*| > 1$. Then following properties are equivalent:

(a) rank($\Gamma$) = 1.

(b) $S \cap T = R(\Gamma)$ for $S, T \in \Gamma^*$ with $S \neq T$.

(c) $S \cap S^g = R(\Gamma)$ for $S \in \Gamma^*$ and $g \in G \setminus N_G(S)$.

Proof. Suppose (a) holds. Let $S, T \in \Gamma^*$ with $P = S \cap T$ maximal. Then $P$ is closed by Proposition 2.11. and so $R(\Gamma) \leq P < S$ is a chain of closed $\Gamma$-subgroups. Since $\Gamma$ has rank 1, we get $P = R(\Gamma)$. Thus $S \cap T = R(\Gamma)$ for all $S \neq T \in \Gamma^*$ and so (b) holds.

From (C) we get $S^g \in \Gamma^*$ for $S \in \Gamma^*$ and $g \in G$. Thus (b) implies (c).

Suppose that (c) holds. Let $P$ be a closed $\Gamma$ subgroup. We will show that $P = R(\Gamma)$ or $P \in \Gamma^*$ and note that this implies (a).

Assume that $|\Gamma^*_P| = 1$. Since $P$ is closed we get $P \in \Gamma^*_P$. Let $P \leq S \in \Gamma^*$. Then $P \leq N_S(P) \in \Gamma_P$ and so $P = N_S(P)$ and $P = S$.

Suppose next that $|\Gamma^*_P| > 1$ and let $Q \neq T \in \Gamma^*_P$. By (P) applied to the NSS $\Gamma_P$, we may assume that $T$ does not normalize $Q$. Let $Q \leq S \in \Gamma^*_P$. Then
Γ∗. Then \(Q \leq N_S(P) \in \Gamma_P\) and so by maximality of \(Q\), \(Q = N_S(P)\). Thus \(N_G(P) \cap N_G(S) \leq N_G(Q)\). Since \(T\) normalizes \(P\) but not \(Q\) we get \(T \not\leq N_G(S)\). Pick \(g \in T\) with \(S \neq S^g\). Then \(P \leq S \cap S^g = R(\Gamma)\) and so \(P \leq R(\Gamma)\). By Corollary 2.7 \(R(\Gamma) \leq P\) and so \(P = R(\Gamma)\). \(\Box\)

**Lemma 3.2** Suppose that \(N\) is large in \(\Gamma\) and \(P, Q \in \Gamma_N\)

1. \(N_Q(P)P \in \Gamma_N\).

2. If \(P \in \Gamma_N\), then \(N_Q(P) \leq P\) and \(\Gamma_N \cap \Gamma N_G(P) = \Gamma P\).

**Proof.** (1) By definition of large, \(C_P(N) \leq N\). Hence \([C_P(N), \langle P, Q \rangle] \leq N\) and (1) follows from Lemma 2.9.

(2) By (1) and maximality of \(P^*\), \(N_Q(P) \leq P\). The second statement in (2) just rephrases the first. \(\Box\)

**Lemma 3.3** Suppose that \(N \leq P \in \Gamma\), \(N\) is large and \(P\) is closed. Then \(P\) is large.

**Proof.** Let \(P \leq T \in \Gamma_P\). Then \(C_T(P) \leq N_T(N) \in \Gamma_N\) and since \(N\) is large, \(C_T(P) \leq N_T(N) \cap C_G(N) \leq N \leq P\). Thus \(P\) is large. \(\Box\)

**Lemma 3.4** Let \(\Gamma\) be an NSS of rank 1 and \(P \in \Gamma\) with \(P \not\leq R(\Gamma)\).

1. \(P\) is contained in a unique maximal \(\Gamma\)-subgroup \(P^*\).

2. Suppose \(R(\Gamma)\) is large and \(x \in Q \in \Gamma\). If \(\langle P, P^x \rangle \in \Gamma\) then \(x \in P^*\).

**Proof.** (1) By Theorem 3.1, \(S \cap T = R(\Gamma)\) for all \(S \neq T \in R(\Gamma)\).

(2) By (1) \(P^* = \langle P, P^x \rangle^* = P^{xx} = P^{xx}\). Thus \(x \in N_Q(P^*)\). By Lemma 3.2(2) \(N_Q(P^*) \leq P^*\) and (2) holds. \(\Box\)

**Lemma 3.5** Let \(\Gamma\) be an NSS of rank 1 and \(S \in \Gamma^*\). Define \(\Pi = \bigcup_{g \in G} \Gamma S^g\). Then \(\Pi \leq \Gamma\), \(\Pi\) has rank at most one and \(\Pi^* = S^G \subseteq \Gamma^*\). If in addition \(R(\Gamma)\) is large then \(\Pi\) has rank 1 and \(R(\Pi) = R(\Gamma)\).
Proof. Clearly Π fulfils (Suba) and (Subc). Now let A, B ∈ Π with ⟨A, B⟩ ∈ Γ. If A ≤ R(Γ), then ⟨A, B⟩ ≤ B R(Γ) ∈ Π and so also ⟨A, B⟩ ∈ Π. So suppose A ∉ R(Γ) and B ∉ R(Γ). Then by Lemma 3.4(1)

\[ A^* = ⟨A, B⟩^* = B^*. \]

Thus ⟨A, B⟩ ≤ A* and ⟨A, B⟩ ∈ Π. Thus Π ≤ Γ. Clearly Π* = S G ⊆ Γ*.

Suppose first that |Π*| > 1. By Theorem 3.1, A ∩ B = R(Γ) for all A, B ∈ Π* and R(Π) = R(Γ). Hence by Theorem 3.1, Π has rank 1. So the lemma holds in this case.

Suppose next that |Π*| = 1. Then Π* = {S}, S is normal in G and Π has rank 0. So we may now assume that that R(Γ) is large. Since S is normal in G, Lemma 3.2 implies PS ∈ Γ for all P ∈ Γ. But then P ≤ S by maximality of S and Γ* = S, a contradiction to \( \text{rank}(Γ) = 1 \). □

**Lemma 3.6** Suppose that Γ has rank 1. Let K ≤ G with ⟨ΓK⟩ ∉ Γ and P ∈ ΓK with P ∉ R(Γ). Then ⟨P, Px⟩ ∉ Γ for some x ∈ K.

**Proof.** Since ⟨ΓK⟩ ∉ Γ, Proposition 2.3(6) implies Q ∉ P* for some Q ∈ ΓK. Let x ∈ Q\ P*. Then by Lemma 3.4(2), ⟨P, Px⟩ ∉ Γ. □

**Proposition 3.7** Let N ∈ Γ be closed of co-rank 1, (here the co-rank of N is the supremum of the lengths of chains of closed Γ-subgroup starting with N).

1. Let N ≤ S_1 ∩ S_2 with S_1 ≠ S_2 ∈ Γ*. Then N = S_1 ∩ S_2.
2. Γ_N has rank 1.
3. Let N < P ∈ Γ. Then P lies in a unique maximal Γ-subgroup P*. Moreover, N_G(P) ≤ N_G(P*).
4. Let P, S ∈ Γ with S ∈ Γ* and N < S ∩ P. Then P ⊆ S.

**Proof.** (1) By Proposition 2.11(2) N ≤ S_1 ∩ S_2 ≤ T < S for some closed T ∈ Γ and some S ∈ Γ*. Since N has co-rank 1 we conclude that N = T and so N = S_1 ∩ S_2.
(2) Let \( Q_1 \neq Q_2 \in \Gamma^*P \) and \( Q_i \leq S_i \in \Gamma^* \). Since \( \langle Q_1, Q_2 \rangle \notin \Gamma, S_1 \neq S_2 \).
So by (1) and Theorem 3.1, \( \Gamma_N \) has rank at most 1. Suppose that \( \Gamma_N \) has rank 0. Then since \( N \) is closed \( \{ N \} = \Gamma_N \). Let \( N \leq S \in \Gamma^* \). Then \( N \leq N_S(N) \in \Gamma_N \) and so \( N = N_S(N) \). Hence \( S = N \), a contradiction to \( N \notin \Gamma^* \).

(3) and (4) are easy consequences of (1) and (2). \( \square \)

**Theorem 3.8** If \( \Gamma \) is reduced of rank 2 then one of the following holds:

1. There are \( S \in \Gamma^* \) and closed \( P, Q \in \Gamma S \setminus \{ S, 1 \} \) such that \( \Gamma \langle \Gamma_P, \Gamma_Q \rangle \) is reduced.

2. There is an reduced NSS \( \Delta \) of \( G \) with \( \text{rank}(\Delta) = 1 \) and \( \Delta \leq \Gamma \).

**Proof.** Suppose first that there are \( S \in \Gamma^* \) and closed \( P, Q \in \Gamma S \setminus \{ 1, S \} \) with \( P \neq Q \). Let \( N := R(\Gamma(\Gamma_P, \Gamma_Q)) \). By Corollary 2.7, \( N \subseteq P \cap Q \) and \( N \) is closed. Since \( \text{rank}(\Gamma) = 2 \) we get \( N = 1 \) and (1) holds.

Suppose next that for all \( S \in \Gamma^* \) there is at most one closed \( P \in \Gamma S \) with \( 1 \neq P \neq S \). If such a \( P \) exists we denote it by \( P(S) \). Otherwise let \( P(S) = 1 \).

We will show that

\[(*) \quad P(S) = P(T) \neq 1 \text{ for all } S, T \in \Gamma^* \text{ with } S \cap T \neq 1.\]

If \( S \cap T \) is closed, \( P(S) = S \cap T = P(T) \). So we may assume that \( S \cap T \) is not closed. Then by Proposition 2.8(10) there exists a closed \( M \in \Gamma \) with \( S \cap T \subseteq M \) and \( \Gamma_{S \cap T} \subseteq \Gamma_M \). By Lemma 2.5(2) \( N_S(S \cap T)M \subseteq \Gamma \). So there exists \( \tilde{S} \in \Gamma^* \) with \( N_S(S \cap T)M \subseteq \tilde{S} \) and similarly choose \( \tilde{T} \). Then \( S \cap T \subseteq N_S(S \cap T) \subseteq S \cap \tilde{S}, S \cap T \subseteq M \subseteq \tilde{S} \cap \tilde{T} \) and \( S \cap T \cap N_T(S \cap T) \subseteq T \cap \tilde{T} \). So by downwards induction on \( S \cap T, P(S) = P(\tilde{S}) = P(\tilde{T}) = P(T) \neq 1 \). Thus \((*) \) holds.

Put \( \Delta = \bigcup \{ \Gamma P(S) \mid S \in \Gamma^* \} \). We claim that \( \Delta \leq \Gamma \). (Suba) and Sub(c) are obvious from the definition of \( \Delta \). Let \( A, B \in \Delta \) and \( S, T \in \Gamma^* \) with \( A \subseteq P(S) \) and \( B \subseteq P(T) \).

To show (Subc) we assume \( A \neq 1 \neq B \) and \( \langle A, B \rangle \leq \Gamma \). Pick \( Q \in \Gamma^* \) with \( \langle A, B \rangle \leq Q \). Then \( A \leq S \cap Q \) and \( B \leq Q \cap T \) and \((*) \) implies \( P(S) = P(Q) = P(T) \). Thus \( \langle A, B \rangle \leq P(Q) \) and \( \langle A, B \rangle \in \Delta \). Thus (Subb) holds. Thus \( \Delta \leq \Gamma \) and by Lemma 2.4, \( \Delta \) is an NSS.
Suppose that $|\Delta^*| > 1$. Let $A, B \in \Gamma^*$ with $A \cap B \neq 1$ and let $S, T$ be as above. Then by (*), $A = P(S) = P(T) = B$ and by Theorem 3.1, $\Delta$ is reduced of rank 1. Thus 2. holds in this case.

Suppose that $|\Delta^*| = 1$ and let $A$ be the unique member of $\Delta^*$. Assume that $A = 1$. Then $P(S) = 1$ for all $S \in \Gamma^*$ and so $\Delta$ has rank 1, a contradiction. Thus $A \neq 1$. Let $\Lambda = \Gamma \setminus \Gamma_A \cup \{1\}$. We claim that $\Lambda \leq \Gamma$. Let $P \leq Q \leq S$ with $1 \neq P \in \Gamma$, $Q \in \Lambda$ and $S \in \Gamma^*$. Since $\Gamma_A \leq \Gamma$ and $Q \notin \Gamma_A$, $S \notin \Lambda$. Suppose that $P \in \Gamma_A$. Then $PA \neq 1$ for all $S \in \Gamma^*$ and so $\Lambda$ has rank 1, a contradiction. Thus $A \neq 1$.

We proved $\Lambda \leq \Gamma$. Since $A \notin R(\Gamma)$, $\Lambda \neq \{1\}$. Suppose that $\Lambda$ has a unique maximal element $B$. Then $B \notin \Gamma_A$. Hence $\Lambda$ has a unique maximal element $B$. Then $B \in \Gamma_A$ and by (b) implies $A \leq S$. Thus $S \in \Gamma_A$, a contradiction. So $P \in \Lambda$ and we conclude that (Suba) holds for $\Lambda$. Clearly (Subb) and (Subc) hold.

We proved $\Lambda \leq \Gamma$. Since $A \neq R(\Gamma)$, $\Lambda \neq \{1\}$. Suppose that $\Lambda$ has a unique maximal element $B$. Then $B \in \Gamma_A$. Since both $A$ and $B$ are normal in $G$, $[A, B] = 1$. Thus $AB$ is nilpotent and $AB \in \Gamma$, a contradiction to $B \notin \Gamma_A$. Thus $|\Lambda^*| > 1$. By (*) $X \cap Y = 1$ for any two maximal members of $\Lambda$ and so Theorem 3.1 implies that $\Lambda$ is a reduced NSS of rank 1. Thus 2. holds for $\Lambda$ in place of $\Delta$. □

4 Minimal $\Gamma$-subgroups

In this section we continue to assume that a $G$ is group with an NSS $\Gamma$ and $1 \in \Gamma$. We consider elements $X \in \Gamma_*$. Recall that this just means that $X$ is a minimal non-trivial element of $\Gamma$. In particular for two different elements $X, Y \in \Gamma_*$ we have $X \cap Y = 1$.

**Proposition 4.1** Assume $P \in \Gamma$ and $X, Y \in \Gamma P_*$ with $X \neq Y$. If $N_X(Y) \neq 1$ or $[x, y] = 1$ for some $x \in X^\#$ and $y \in Y^\#$, then $\langle X, Y \rangle = X \times Y$.

**Proof.** If $[x, y] = 1$, then $y \in Y \cap Y^x$ and so $Y = Y^x$. So we may assume $N_X(Y) \neq 1$. Using Lemma 2.1 we get $X = N_X(Y)$. Since $XY \subseteq P$, $XY$ is nilpotent. As $Y$ is normal in $XY$, $C_Y(X) \neq 1$. Hence $N_Y(X) \neq 1$ and $Y = N_Y(X)$. So $[X, Y] \leq X \cap Y = 1$. □

**Proposition 4.2** Let $P \in \Gamma$ with $P = \langle X, Y \rangle$ where $X, Y \in \Gamma P_* \setminus \{P\}$. If $X' \neq 1$ then $X$ is a normal subgroup of $P$ (here $X' := [X, X]$ is the commutator subgroup of $X$).
Proof. Consider a counterexample with $P$ minimal. Then there is $y \in Y$ with $X^y \neq X$. Set $E := \langle X, X^y \rangle$. So by Proposition 2.3(6) and Proposition 1.6, $E \in \Gamma$ and $E < P$. Of course $X, X^y \neq E$ and by minimality of $P$, $X$ and $X^y$ are normal in $E$. Therefore by Proposition 4.1 $E = X \times X^y$. Let $Q := \langle Y^p \rangle$. Then $Q$ is a proper $\Gamma$-subgroup of $P$ by Proposition 2.3(6) and Proposition 1.6. Since $P = \langle X, Y \rangle$, $X \notin Q$ and $Q \cap X = 1$ as $X \in \Gamma$ and $Q \cap X \in \Gamma$ by (I). Now $[X, y] \leq E \cap Q$ and $E \cap Q$ is normal in $E$. Since $X \times X^y = X[X, y]$ we have $1 \neq [X^y, X^y] = [X^y, [X, y]] \leq X^y \cap Q$. Hence also $X \cap Q \neq 1$, a contradiction. 

Corollary 4.3 Let $P \in \Gamma$ and $\Delta := \{X \in \Gamma P, X' \neq 1\}$. Then $\Delta$ is finite and $\langle \Delta \rangle = X_1 \times \ldots \times X_n$, where $\Delta = \{X_1, \ldots, X_n\}$.

Proof. Let $X \neq Y \in \Delta$. Then by Proposition 4.2 and Proposition 4.1, $[X, Y] = 1$. Let $Z = \langle \Delta \setminus \{X\} \rangle$. Then $Z \in \Gamma$ and $[X, Z] = 1$. Thus $X \cap Z$ is a proper $\Gamma$-subgroup of $X$ and so $X \cap Z$. Thus the Corollary holds by the definition of the direct product. (Note also that $\Delta$ is finite by (MM)) \hfill \Box

Define an elementary abelian $p$-group to be an an abelian group so that all non-trivial elements have order $p$. Note that this makes sense for $p$ a prime or $p = \infty$. Indeed, an elementary abelian $\infty$-group is just a torsion free abelian group.

Proposition 4.4 Let $X, Y \in \Gamma$, $X \neq Y$, $H := \langle X, Y \rangle \in \Gamma$ and $[X, Y] \neq 1$. Then $X$ and $Y$ are both elementary abelian $p$-groups, $p = \infty$ or a prime.

Proof. Suppose first $Y$ is not elementary abelian. Let $M \in \Gamma$ maximal with respect to $X \leq M < H$. Then by Proposition 2.2, $M$ is normal in $H$. Also $Y \not\subseteq M$. Since $Y \cap M \in \Gamma$ and $Y \in \Gamma$, $Y \cap M = 1$. Let $1 \neq x \in X$. By Proposition 4.1, $N_X(Y) = 1$ and so $Y \neq Y^x$. Hence by Proposition 4.2, $Y$ is abelian. Since $\langle Y, Y^x \rangle \neq H$ we get by induction that $[Y, Y^x] = 1$. Let $D = YY^x \cap M$. Since $[Y, x] \subseteq D$, $YY^x = YD = Y^xD$. Let $E \in \Gamma, D$. Then $1 \neq Y \cap (EY^x) \in \Gamma$ and so $Y \subseteq EY^x$. Thus $E = D$. Note that $D$ is isomorphic to $Y$ and $\langle D, X \rangle \leq M$. In particular, $Y$ is not elementary abelian and so by induction $[D, X] = 1$. Since $[Y, x] \leq D$ we get $[Y, x] \leq Z(\langle Y, x \rangle)$. Let $y \in Y$ has order $p$, $p$ a prime. Then by Proposition 1.1, $[y, x^p] = [y^p, x] = 1$ and so by Proposition 4.1 $x^p = 1$. Hence for all $z \in Y, [z^p, x] = [z, x^p] = 1$ and so by Proposition 4.1 $z^p = 1$. 

19
Hence $Y$ is an elementary abelian $p$-group and by symmetry $X$ is an elementary abelian $q$-group. To show $p = q$ we may assume $p \neq \infty$. Then by Proposition 1.7 $[y, x^k] = 1$ for some positive integer $k$. So by Proposition 4.1, $x^k = 1$ and $q = p$.

**Proposition 4.5** Let $A_1$ be a $\Gamma$-subgroup of the decomposable abelian $\Gamma$-subgroup $A$. Then there is a decomposable $\Gamma$-subgroup $A_2$ of $A$ with $A = A_1 \times A_2$.

*Proof.* Let $K$ be a decomposable $\Gamma$-subgroup maximal with $A_1K = A_1 \times K$. If $A = A_1K$ we are done. So suppose $A_1K < A$. Since $A$ is decomposable, there exists $X \in \Gamma A$ with $X \not\leq A_1K$. Then $A_1K \cap X = 1$ and $A_1KX = (A_1 \times K) \times X = A_1 \times (K \times X)$. But $K < KX$ and we obtain a contradiction to the maximal choice of $K$. \[ \Box \]

**Proposition 4.6** $\Gamma$-subgroups of decomposable abelian $\Gamma$-subgroups are decomposable.

*Proof.* Let $A$ be a decomposable abelian $\Gamma$-subgroup and $B$ a $\Gamma$-subgroup of $A$. By Proposition 4.5 there exists $D \in \Gamma A$ with $A = \Omega(B) \times D$. By Proposition 1.8 $B = \Omega(B) \times (B \cap D)$. Also $\Omega(B \cap D) \leq \Omega(B) \cap D = 1$ and since $B \cap D \in \Gamma$, $B \cap D = 1$ and $B = \Omega(B)$. \[ \Box \]

**Proposition 4.7** Let $A, B \in \Gamma$ such that $A$ is decomposable abelian and $B$ is generated by abelian $\Gamma$-subgroups. If $\langle A, B \rangle \in \Gamma$, then $C_A(B)$ is a decomposable abelian $\Gamma$-subgroup.

*Proof.* Since $C_A(B) = \bigcap \{C_A(E) \mid E \in \Gamma B, E \text{ abelian} \}$ we may by (I) assume that $B$ is abelian. By Proposition 4.6 we only need to show $C_A(B) \in \Gamma$. By Proposition 2.3(1) we get $C_A(B) \leq D := \bigcap_{b \in B} A^b \in \Gamma$. Note that $C_A(B) = C_D(B)$ and that $B$ normalizes $D$. By Proposition 4.6 $D$ is decomposable.

If $D = C_D(B) = C_A(B)$ we are done. So suppose $[D, B] \neq 1$. Since $DB$ is nilpotent, there exists $d \in D$ with $1 \neq [d, B] \leq C_D(B)$. Then $B^d \leq C_D(B)B \leq C_G(B)$. Thus $BB^d$ is abelian and $BB^d \in \Gamma$. Thus

$$1 \neq [d, B] \leq BB^d \cap D \leq C_D(B).$$
Put \( E := BB^d \cap D \). Then \( E \) is a non-trivial \( \Gamma \) subgroup of \( C_D(B) \). By Proposition 4.5, \( D = E \times F \) for some decomposable \( \Gamma \) subgroup \( F \) of \( D \). Then \( C_D(B) = E \times C_F(B) \). Since \( F < A \), induction on \( A \) shows \( C_F(B) \in \Gamma \). Hence also \( C_D(B) \in \Gamma \) and the Proposition is proved. \( \square \)

**Proposition 4.8** Let \( A, B \in \Gamma \) such that \( A \) is decomposable, \( B \) an abelian \( \Gamma \)-subgroup and \( A \in \Gamma_B \). Then \([B, A]\) is a \( \Gamma \)-subgroup of \( G \).

**Proof.** Since \([B, A] = \langle [B, E] \mid E \in \Gamma_A \rangle\) we may by Proposition 2.3(6) assume that \( A \in \Gamma_A \). If \( A \leq B \), then since \( B \) is abelian \([A, B] = 1 \in \Gamma \). We therefore may assume \( A \not\subseteq B \) and so \( A \cap B = 1 \) by minimality of \( A \). Note that \( \langle A^B \rangle = A[B, A] \) and so \[
\langle A^B \rangle \cap B = [B, A](A \cap B) = [B, A].
\]

By Proposition 2.3(6) \( \langle A^B \rangle \in \Gamma \) and so by (I), \([B, A] \in \Gamma \). \( \square \)

## 5 Measure and the Thompson subgroup

\( G \) continues to be a group with an NSS \( \Gamma \) with \( 1 \in \Gamma \). We define a measure function and use it to state and prove a variant of the Thompson Replacement Theorem.

**Proposition 5.1** Let \( X, Y \in \Gamma \) with \( XY \in \Gamma \). Let

\[
X = X_0 < X_1 < \ldots < X_r = XY
\]

be any maximal chain of \( \Gamma \)-subgroups from \( X \) to \( XY \). Then

\[
X \cap Y = X_0 \cap Y < X_1 \cap Y < \ldots X_r \cap Y = Y
\]

is a maximal chain of \( \Gamma \)-subgroups from \( X \cap Y \) to \( Y \).

**Proof.** Let \( A \) be a \( \Gamma \) subgroup with \( X_i \cap Y \leq A \leq X_{i+1} \cap Y \). Since \( X \leq X_i \leq XY \), Proposition 1.8 implies \( X_i = X(X_i \cap Y) \). Thus

\[
X_i \leq AX \leq X_{i+1}.
\]
$X_i$ is a maximal $\Gamma$-subgroup of $X_{i+1}$ and so by Proposition 2.2 $X_i$ is normal in $X_{i+1}$. Thus $AX = AX_i$ is a subgroup of $X_{i+1}$. Since $X_{i+1}$ is nilpotent, Proposition 2.3(6) implies $AX \in \Gamma$. By the maximality of the $X_i$-chain, $AX = X_k$ for some $k \in \{i, i+1\}$. Thus $X_k \cap Y = AX \cap Y = A(X \cap Y) = A$. \hfill \box

**Proposition 5.2** Let $X \in \Gamma$. Then there exists a maximal chain of $\Gamma$-subgroups from 1 to $X$ and any two such chains have the same length. We denote this common length by $\mu(X)$.

*Proof.* The existence of a maximal $\Gamma$-chain from 1 to $X$ follows from (MM). Let $A$ and $B$ be maximal $\Gamma$-subgroups of $X$. By induction any maximal $\Gamma$-chain from 1 to $X$ through $A$ has unique length $\mu(A) + 1$. It remains to show that $\mu(A) = \mu(B)$. Without loss $A \neq B$. By maximality of $A$ and $B$, $A$ is normal in $X$, $AB \in \Gamma$ and $AB = X$. Note that $A \leq X$ is a maximal chain from $A$ to $X$ and so by Proposition 5.1, $A \cap B < B$ is a maximal chain from $A \cap B$ to $B$. Thus $\mu(B) = \mu(A \cap B) + 1 = \mu(A)$. \hfill \box

Abusing the term we call $\mu$ of Proposition 5.2 a *measure function* on $\Gamma$ and $\mu(A)$ is called the *measure* of $A$.

**Proposition 5.3** $\mu(P) = \mu(P^g)$ for all $P \in \Gamma$ and $g \in G$.

*Proof.* This follows from (C) and Proposition 5.2. \hfill \box

**Proposition 5.4** Assume $X, Y \in \Gamma$.

1. Suppose $X \leq Y$, then any maximal $\Gamma$-chain from $X$ to $Y$ has length $\mu(Y/X) := \mu(Y) - \mu(X)$.

2. Suppose $XY \in \Gamma$. Then $\mu(XY) = \mu(X) + \mu(Y) - \mu(X \cap Y)$.

*Proof.* (1) follows from Proposition 5.2.

(2) By (1) $\mu(XY/X) = \mu(XY) - \mu(X)$. By (1) and Proposition 5.1, $\mu(XY/X) = \mu(Y/X \cap Y)$. Again by (1) $\mu(Y/X \cap Y) = \mu(Y) - \mu(X \cap Y)$. Thus (2) holds. \hfill \box
Definition. For $P \in \Gamma$ let $\mathcal{A}(P)$ be the set of all decomposable abelian $\Gamma$-subgroups of $P$ with maximal measure. Let $J(P) := \langle \mathcal{A}(P) \rangle$, the Thompson-subgroup of $P$ (compare with the introduction of [5]).

Then $J(P)$ is a $\Gamma$-subgroup of $P$ by Proposition 2.3(6).

**Proposition 5.5** Let $V$ be a decomposable abelian $\Gamma$-subgroup of $G$ and $A \in \Gamma_V$ with $A \in \mathcal{A}(AV)$. Then $C_V(A) = V \cap A$ and $\mu(A/C_A(V)) \geq \mu(V/C_V(A))$

**Proof.** By Proposition 4.7 and (P), $C_V(A)A$ is an abelian decomposable $\Gamma$-subgroup of $P$. The maximality of $\mu(A)$ implies $C_V(A) \leq A$ and thus $C_V(A) = V \cap A$. Thus $V \cap C_A(V) = V \cap A = C_V(A)$ and by maximality of $A$, Proposition 4.7 and Proposition 5.4:

$$\mu(A) \geq \mu(VC_A(V)) = \mu(VC_A(V)/C_A(V)) + \mu(C_A(V)) = \mu(V/C_V(A)) + \mu(C_A(V)).$$

The next lemma is our version of the Thompson Replacement Theorem.

**Lemma 5.6** Let $A, V$ be decomposable abelian $\Gamma$-groups with $A \in \Gamma_V \cap \mathcal{A}(AV)$. Let $x \in N_V(N_V(A)A)$ and define

$$D = ((AA^x) \cap V)(A \cap A^x).$$

Then

1. $D \in \mathcal{A}(AV)$ and $\langle x \rangle N_V(A)A \subseteq N_G(D)$.
2. If $[V, A] \neq 1$, then $[V, D] \neq 1$.

**Proof.** (1) Let $P = N_V(A)A$. Since $x$ normalizes $P$, both $A$ and $A^x$ are normal in $P$. Thus $AA^x = \langle A, A^x \rangle$. Since $A$ is abelian, $A \cap A^x \subseteq Z(AA^x)$. By Proposition 4.6 both $AA^x \cap V$ and $A \cap A^x$ are decomposable $\Gamma$-groups and so $D$ is an abelian decomposable $\Gamma$-group. Also $[x, A] \subseteq V \cap (AA^x) \subseteq D$ and so $x \in N_G(D)$. Note that

$$\mu(AA^x) = \mu(A) + \mu(A^x) - \mu(A \cap A^x) = 2\mu(A) - \mu(A \cap A^x).$$

Also $AA^x \subseteq VA$ and so $AA^x = AA^x \cap VA = A(V \cap AA^x) = AD$. Moreover, $D \cap A = (V \cap A)(A \cap A^x)$ and $V \cap A \subseteq C_A(x) \subseteq A \cap A^x$. Thus
D \cap A = A \cap A^x. Hence \mu(AA^x) = \mu(DA) = \mu(D) + \mu(A) - \mu(A \cap A^x).
Comparing with (*) we obtain \mu(A) = \mu(D) and so D \in \mathcal{A}(AV).

(2) Suppose that [V, D] = 1. Then \( A \cap A^x \leq C_A(V) \) and so by Proposition 5.5, \( A \cap A^x = A \cap V \). Hence \( D \leq V \). Since \( D \in \mathcal{A}(AV) \) we get \( V = D \subseteq P \subseteq N_G(A) \). Thus \( A = A^x = A \cap A^x \leq D \) and \( [V, A] = 1 \). Thus (2) holds.

\[ \Box \]

**Proposition 5.7** Let \( V \) be a decomposable abelian \( \Gamma \)-subgroup of \( G \) and \( P \in \Gamma_V \) with \( V \leq P \) and \( J(P) \not\leq C_G(V) \). Then there exists \( A \in \mathcal{A}(P) \) such that \([ [V, A], A] = 1 \neq [V, A] \leq A \).

**Proof.** Since \( J(P) \not\leq C_G(V) \) there exists \( A \in \mathcal{A}(P) \) with \( 1 \neq [V, A] \). Choose such an \( A \) with \( N_V(A) \) maximal.

Suppose that \( V \) does not normalize \( A \). Then \( V \not\leq N_V(A)A \) and so by Proposition 1.6(2) there exists \( x \in N_V(N_V(A)A) \) with \( x \not\in N_V(A) \). Let \( D \) be defined as in Lemma 5.6. Then \( D \in \mathcal{A}(AV) \), \( [V, D] \neq 1 \) and \( (x)N_V(A) \leq N_V(D) \), contradiction to the maximal choice of \( N_V(A) \).

Thus \( V \) normalizes \( A \), \([ [V, A], A] \leq A \) and \([ [V, A], A] = 1 \).

\[ \Box \]

**Lemma 5.8** Let \( A, B \) be abelian \( \Gamma \)-subgroups with \([ A, B ] \leq A \cap B \). Let \( a \in A \). Suppose that \( B \) is decomposable and \( C_B(a) \in \Gamma \). Then \([ a, B ] \in \Gamma \) and \( \mu([a,B]) = \mu(B/C_B(a)) \).

**Proof.** By Proposition 4.5 there exists a \( \Gamma \)-subgroup \( D \) of \( B = C_B(a) \times D \). Then \([D,a] \leq A \cap B \leq C_A(D) \) and so by Proposition 1.2(2) \( DD^a = D[D,a] \in \Gamma \). Moreover, \( DD^a \cap A = (D \cap A)[D,a] \) and \( D \cap A \leq C_D(a) = 1 \). \([D,a] = DD^a \cap A \in \Gamma \). In particular, \( D \cap [D,a] = 1 \) and so by Proposition 1.2(3), \( D \cap D^a = C_D(a) = 1 \). Thus \( 2\mu(D) = \mu(DD^a) = \mu(D) + \mu([D,a]) \) and

\[ \mu([a,B]) = \mu([a,D]) = \mu(D) = \mu(B/C_B(a)). \]

\[ \Box \]
6 Glauberman’s Theorem, Part I

In this section we begin the proof of Theorem A stated in the introduction. Assume $G, V, S, A, \Gamma$ have the meaning and the properties mentioned there.

**Proposition 6.1** Set $\Pi := \bigcup_{g \in G} \Gamma S^g$. Let $T \in \Pi^*$.

1. $\Pi$ is an NSS of rank 1.
2. $\Pi^* = S^G \subseteq \Gamma^*$.
3. $R(\Pi) = R(\Gamma)$.
4. $[C_G(V), (\Pi)] \leq R(\Gamma)$.
5. Let $P \in \Gamma(TC_G(V))$, then $P \leq T$.
6. $R(\Gamma) = C_T(V)$.

**Proof.** By (a) (that is assumption (a) of Theorem A), $\Gamma$ has rank 1. By (e) $R(\Gamma)$ is large. So (1), (2) and (3) follow from Lemma 3.5.

(4) By (c) $[C_G(V), S] \leq R(\Gamma)$. Thus (4) follows by conjugation.

(5) By (4) $[P, T] \leq [TC_G(V), T] \leq T R(\Gamma) \leq T$. Thus $P \leq N_G(T)$. By (e), $R(\Gamma)$ is large and so by Lemma 3.2 $PT \in \Gamma$. Since $T \in \Pi^* = S^G \subseteq \Gamma^*$, $P \leq T$.

(6) Let $R \in \Pi^*$. By (4) $[T \cap C_G(V), R] \leq R(\Gamma) \leq R$. Thus $C_T(V) \leq E := \bigcap_{R \in \Pi^*} N_T(R)$. By (I) and (MM) $E \in \Gamma$ and by Proposition 2.3(8), $E \leq R(\Pi) = R(\Gamma)$.

**Lemma 6.2** There exists a non-trivial quadratic $\Gamma$-offender $E$ in $S$ on $V$ with $C_V(E) = V \cap E$.

**Proof.** By (d) there exists a non-trivial $\Gamma$-offender $A$ in $S$ on $V$. Since $V \leq R(\Gamma)$, $A \in \Gamma_V$. Let $B = C_A(V)V$ and $D = C_V(A)A$. By Proposition 4.7, $B \in \Gamma$. Since $A$ is an offender on $V$, $\mu(V/C_V(A)) \leq \mu(A/C_A(V))$. But this is equivalent to $\mu(B) \leq \mu(D)$. We will show that

(*) $J(AV) \not\leq C_{AV}(V)$. 

25
Since $AV = AB$, $C_{AV}(V) = C_A(V)B = B$. From $\mu(D) \geq \mu(B)$, we get $D \in A(AB)$ and $D \not\leq B$, since $1 \not\leq [A, V] \leq [B, D]$.

Thus (*) holds. The existence of $E$ now follows from Proposition 5.7 and Proposition 5.5.

\[\square\]

**Notation.**

\[\Delta := E^G,\] where $E$ is as in Lemma 6.2

$E, F \in \Delta$ such that $\mu([V, E][V, F])$ is minimal with respect to $\langle E, F \rangle \not\in \Gamma$.

\[W := [V, E][V, F].\]

\[H := \langle E, F \rangle R(\Gamma).\]

\[Z := [V, E] \cap [V, F].\]

\[\Lambda := \Delta H.\]

\[q := \mu(A/C_A(V)) \text{ for } A \in \Delta.\]

\[m := \mu([V, A]).\]

Notice that $C_A(V), [V, A], W$ and $Z$ are decomposable abelian subgroups by Proposition 4.5 - Proposition 4.8, (P) and (I). Hence the measure of these groups is defined. Note that by Proposition 5.4 and the choice of $E, F$, $\mu(Z)$ is maximal with respect to $\langle E, F \rangle \not\in \Gamma$. The existence of $F \in \Delta$ with $\langle E, F \rangle \not\in \Gamma$ is guaranteed by Lemma 3.6. In view of Lemma 3.4(1) we denote by $D^*$ the unique member of $\Gamma^*$ which contains $D$ provided $D \in \Gamma$ with $D \not\in R(\Gamma)$. Observe that by Proposition 6.1(6), $A \not\in R(\Gamma)$ for all $A \in \Delta$.

**Proposition 6.3** Let $A, B \in \Delta$

1. $\langle A, B \rangle \not\in \Gamma$ if and only if $A^* \not= B^*$.

2. If $A^* \not= B^*$, then $\langle A, A^b \rangle \not\in \Gamma$ for all $b \in B \setminus R(\Gamma)$.

3. If $A^* \not= B^*$, then $[V, A] \not= [V, B]$. 

**Proof.** (1) If $\langle A, B \rangle \in \Gamma$ then $A^* = \langle A, B \rangle^* = B^*$. If $A^* = B^*$ then $\langle A, B \rangle \in \Gamma$ by Proposition 2.3(6). Hence (1).

(2) Let $b \in B$ with $\langle A, A^b \rangle \in \Gamma$. Then by Lemma 3.4(2), $b \in A^* \cap B^*$. So by Theorem 3.1(2), $b \in R(\Gamma)$.

(3) Assume $[V, A] = [V, B]$. By (c), $A$ and $B$ are quadratic and so $[[V, A], B] = [[B, V], A] = 1$. Thus $[[A, B], V] = 1$ by Proposition 1.1 (5)
and \([A, B] \subseteq C_G(V)\). Let \(b \in B \setminus R(\Gamma)\). Then \(A^b \in AC_G(V) \leq A^*C_G(V)\) and so by Proposition 6.1(5), \(A^b \leq A^*\), a contradiction to (2).

\[\square\]

**Proposition 6.4** Let \(A \in \Lambda\) and \(a \in A \setminus R(\Gamma)\). Then

1. \([V, A] = Z \times [W, a] = C_W(a) = C_W(A)\).
2. \(\langle B, B^a \rangle \not\in \Gamma\) and \([V, B] \cap C_V(a) = [V, A] \cap [V, B] = Z\) for all \(B \in \Lambda\) with \(\langle A, B \rangle \not\in \Gamma\).

**Proof.** By Lemma 3.6 there is \(B \in \Lambda\) with \(\langle A, B \rangle / \not\in \Gamma\). So by Proposition 6.3 \(\langle B, B^a \rangle \not\in \Gamma\). Hence \(W = [V, A][V, B] = [V, B][V, B^a]\) by minimality of \(\mu(W)\). Put \(D := C_{[V, B]}(a)\). Then by Proposition 5.4, Proposition 1.2, and quadratic action,

\[C_W(a) = D \times [[V, B], a] = [V, A] = C_W(A)\]

Also \([W, a] = [[V, B], a]\) and

\[D = [V, B] \cap C_W(a) = [V, B] \cap [V, A] = C_W(B) \cap C_W(A) = C_W(\langle A, B \rangle)\]

Since \(H\) centralizes \(Z\), \(Z \leq D\). The maximality of \(\mu(Z)\) now implies \(Z = D\).

\[\square\]

**Proposition 6.5** Let \(A, B \in \Lambda\) with \(\langle A, B \rangle \not\in \Gamma\), \(w \in [V, B] \setminus Z\), and \(a \in A \setminus R(\Gamma)\). Then

1. \(W = [V, B] \times [w, A]\).
2. \(V = WC_V(A)\).
3. \(q = \mu(V/C_V(A)) = \mu(W/C_W(A)) = \mu(A/C_A(V)) = \mu([w, A]) = \mu([V, a])\).

**Proof.** (1) By Proposition 4.8, \(W \in \Gamma\). Since \(V\) is decomposable, also \(W\) is decomposable and by Proposition 4.7 \(C_W(A) \in \Gamma\). Hence also \(C_W(A)A \in \Gamma\) and we can assume \([V, A] = C_W(A) \subseteq A\). Then \(A \cap W = [V, A]\). By Proposition 6.4(2), \([w, a] \neq 1\) for all \(a \in A \setminus R(\Gamma)\) and so \(C_A(w) = A \cap R(\Gamma) \in \Gamma\). From Lemma 5.8 we conclude \([w, A] \in \Gamma\) and

27
\( \mu([w, A]) = \mu(A/C_A(V)) = q. \)

Note that \( \mu([V, B]) = m = \mu([V, A]) = \mu(C_W(A)) \) and so
\( (*) \quad \mu(V/C_V(A)) \geq \mu(W/C_W(A)) = \mu(W/[V, B]). \)

By \((*)\) and since \( A \) is an offender, \( \mu([w, A]) \geq \mu(W/[V, B]) \). By Proposition 6.4 \( [w, A] \cap [V, B] \leq [w, A] \cap Z = 1 \) and we conclude that\( \mu([w, A]) = \mu(W/[V, B]) \) and \( W = [V, B] \times [w, A] \). So (1) holds.

(2) We also conclude that the inequality in \( (**) \) actually is an equality. So \( \mu(V/C_V(A)) = \mu(W/C_W(A)) \). Hence (2) holds.

(3) By Proposition 6.4, \( C_W(a) = C_W(A) \in \Gamma \). So by Lemma 5.8, \( [W, a] \in \Gamma \) and \( \mu([W, a]) = \mu(W/C_W(A)) = q. \) By (2), \( [V, a] = [W, a] \) and all parts of (3) are proved.

\begin{proposition}
Let \( A, B \in \Lambda \) with \( \langle A, B \rangle \not\in \Gamma \) and \( \Sigma := [V, A]^H \). Then:
\begin{enumerate}
\item If \( B \in \Lambda \) with \( \langle A, B \rangle \not\in \Gamma \) then \( \Sigma = \{[V, A]\} \cup [V, B]^A \).
\item If \( M, N \in \Sigma \) with \( M \neq N \) then \( M \cap N = Z \).
\item \( W = \bigcup_{M \in \Sigma} M \).
\item For \( D \in \Delta \) put \( \hat{D} = DR(\Gamma) \). Let \( D \in \Lambda \) with \( D^* = A^* \), then \( \hat{D} = \hat{A} \).
\item Let \( \hat{\Lambda} = \{\hat{B} \mid B \in \Lambda\} \). Then \( \hat{\Lambda} = \{\hat{A}\} \cup \{\hat{B}^A\} \).
\item \( H = \langle C, D \rangle R(\Gamma) \) for all \( C, D \in \Delta H \) with \( \hat{C} \neq \hat{D} \).
\item \( V = C_V(H)W \) and \( V = \bigcup_{D \in \Lambda} C_V(D) \).
\end{enumerate}
\end{proposition}

\begin{proof}
Let \( A, B \in \Lambda \) with \( \langle A, B \rangle \not\in \Gamma \). Then \( W = [V, A][V, B] = [V, B] \times [w, A] \) for \( w \in [V, B] \setminus Z \) by Proposition 6.5 and \( [V, A] \cap [V, B] = Z \) by Proposition 6.4. Therefore \( w^A Z = w[V, A] \), which shows
\( (*) \quad W = [V, A] \cup \bigcup_{a \in A} [V, B]^a. \)

By Lemma 3.6 we can apply \( (*) \) to an element of \( A^H \) in the role of \( B \) and so (3) holds.

Also \( [V, B]^a_1 \cap [V, B]^a_2 = Z \) for \( a_1, a_2 \in A \) with \( a_1 a_2^{-1} \not\in R(\Gamma) \) by Proposition 6.4. Let \( C \in \Lambda \). Then there is \( D \in \{A\} \cup B^A \) with \( [V, C] \cap [V, D] \supset Z \).

28
Hence by maximality of $\mu(Z)$, $\langle C, D \rangle \in \Gamma$, $C^* = D^*$ and $\langle C, K \rangle \notin \Gamma$ for $K \in (\{A\} \cup B^A) \setminus \{C\}$. But then by (*)

$$[V, C] \setminus Z \subseteq W \setminus \bigcup \{[V, K] \mid D \neq K \in \{A\} \cup B^A\} \subseteq [V, D]$$

and $[V, C] = [V, D]$. Thus (1) and (2) hold.

(4) Let $d \in D$. By (1) $[V, B]^d_a = [V, B]$ for some $a \in A$. Thus $da \in N_G(B^*)$ and so $da \in R(\Gamma)$. Hence $d \in AR(\Gamma) = \hat{A}$. Thus (4) holds.

(5) Let $C \in \Lambda$ with $\hat{C} \neq \hat{A}$. By (4), $C^* \neq A^*$ and by Proposition 6.3(3), $[V, C] \neq [V, A]$. So by (1), $[V, C] = [V, B]^a$ for some $a \in A$. By Proposition 6.3(3), $C^* = B^a$ and so by (4) $\hat{C} = \hat{B}^a$.

(6) By (5), $H$ is doubly transitive on $\hat{\Lambda}$. Since $H = \langle \hat{E}, \hat{F} \rangle$, (6) holds.

(7) Since $H = \langle A, B \rangle R(\Gamma)$, we have $C_V(H) = C_V(A) \cap C_V(B)$. Since $\mu(V/C_V(A)) = q$ we get $\mu(V/C_V(H)) \leq 2q$. Since $\mu(W/C_W(H)) = 2q$, the first part of (7) holds.

Let $v \in V$. Then $v = cw$ with $c \in C_V(H)$ and $w \in W$. By (1), $w \in [V, C]$ for some $C \in \Lambda$. So $v \in C_V(H)[V, C] \leq C_V(C)$. □

**Lemma 6.7** Let $t \in G$ and $B \in \Delta$. Suppose that one of the following holds:

1. $t \in A \in \Delta$ and $[V, t] \cap C_V(B) \neq 1$.
2. $\mu(C_V(B)/(C_V(B) \cap C_V(B)^t)) < q$.

Then $\langle B, B^t \rangle \in \Gamma$.

**Proof.** Suppose that 1. holds. Then by Proposition 1.2(2) $C_V(B^t) \subseteq [V, t]C_V(B)$. By Proposition 6.5(3), $\mu([V, t]) = q$ and so 1. implies 2.

So we may assume that 2. holds. Then

$$\mu([V, B]/([V, B] \cap C_V(T))) < q.$$ 

Since $[V, B] \cap C_V(t) \leq [V, B] \cap [V, B]^t$ and $\mu([V, B]/Z) = q$, the maximality of $\mu(Z)$ implies $\langle B, B^t \rangle \in \Gamma$.

**Lemma 6.8** Let $A \in \Delta$. Then $A \subseteq BR(\Gamma) \subseteq H$ for some $B \in \Lambda$. 29
Proof. Let \( a \in A \setminus C_A(V) \). By Proposition 6.6(7) there exists \( B \in \Lambda \) with \([V, a] \cap C_V(B) \neq 1\). By Lemma 6.7, \((B, B^a) \in \Gamma\). Thus by Lemma 3.4(2), \( a \in B^*\). Hence \( A \subseteq B^* \) and \( A^* = B^* \). Since \( R(\Gamma) \leq C_G(V) \), Proposition 6.6(5) implies \([V, a] \subseteq C_V(B)\). By Lemma 6.7 \( a \in B^* \) and so \( A^* = B^* \). So by Proposition 6.6(5), \( B R(\Gamma) \) is independent from the choice of \( a \). Hence \([V, A] \subseteq C_V(B)\). Let \( A = B^g \) for \( g \in G \). Then \( A \in \Lambda^g \) and so by symmetry \([V, B] \leq C_V(A)\). Thus \([V, A][V, B] \subseteq C_V(AB)\).

Let \( D \in \Lambda \setminus \Lambda^B^* \).

Put \( T = \langle A, B, D \rangle, \ U = [V, T] = [V, A][V, B][V, D] \) and \( Y = [V, A][V, B] \cap C_V(D) \). Then \( Y \) is centralized by \( A, B \) and \( D \) and so \( Y \subseteq C_U(T) \). Since \( \mu(V/C_V(D)) = q = \mu([V, B]C_V(D)/C_V(D)), [V, A][V, B] = [V, B]Y \). Let \( a \in A \) and \( w \in [V, D] \setminus Z \). Note that \( Z \subseteq [V, D] \cap C_V(B) \leq Y \). By Proposition 6.5(1), \([w, B]Z = [V, B]\) and so \([V, A][V, B] = [w, B]Y\). Hence \([w, a]Y = [w, b]Y\) for some \( b \in B \). Let \( t = b^{-1}a \). Then \( w^t Y = wY \). Since \( wY \subseteq [V, D]Y \subseteq C_V(D), wY = wY \subseteq C_V(D^t) \). Hence \( Z < (w^t Z) \subseteq [V, D] \cap C_V(D^t) \) and so \( \mu([V, D]/([V, D] \cap C_V(D^t))) < q \).

Thus by Lemma 6.7, \( \langle D, D^t \rangle \in \Gamma \) and by Lemma 3.2(2), \( t \in D^* \). Hence \( t \in D^* \cap B^* = R(\Gamma) \). So \( a = bt \in B R(\Gamma) \subseteq H \). \( \square \)

**Theorem 6.9** \( \langle \Gamma \rangle = H \) and \( \Gamma^* = \{ A R(\Gamma) \mid A \in \Delta \} \).

**Proof.** Let \( P \in \Gamma^* \). By Lemma 6.8, \( H = \langle \Delta \rangle R(\Gamma) \) and so \( H \) is normal in \( G \). So \( P \) normalizes \( W = [V, H] \) and \( Z = C_W(H) \). As \( PV \) is nilpotent, \( P \) centralizes some \( 1 \neq wY \) in \( W/Z \). By Proposition 6.6(3), \( w \in [V, A] \) for some \( A \in \Lambda \). Thus \( P \subseteq N_G([V, A]) \). By Proposition 6.3(3) \( P \subseteq N_G(A^*) \) and so by Lemma 3.2, \( P = A^* \). By Proposition 6.6(5), \( A \) acts transitively on \( \hat{\Lambda} \setminus \hat{A} \), whence \( P = AN_P(B) \) for \( B \in \Lambda \) with \( \langle A, B \rangle \notin \Gamma \). But \( N_P(B) \leq N_P(B^*) = P \cap B^* = R(\Gamma) \) and so \( P = A R(\Gamma) \). \( \square \)

**7 Glauberman’s Theorem, Part II**

In this section we complete the proof of Theorem A. We continue to use the notations from the previous section. In addition we define:

\[ V_0 = W/Z, \] written additively.
\[ V_1 = [V, E]/Z \] and \( V_2 = [V, F]/Z \).
We view $V_0$ as a left module over the endomorphism ring $\text{End}(V_0)$. In particular if $\alpha, \delta \in \text{End}(V_0)$ and $v \in V_0$, then $(\alpha \delta)(v) = \alpha(\delta(v))$. For $h \in H$ define $\sigma_h \in \text{End}(V_0)$ by $\sigma_h(wZ) = w^hZ$ for $w \in W$. Note that $\sigma_{hh'} = \sigma_h \sigma_{h'}$.

From Proposition 6.6 we obtain:

(i) $V_1 = C_{V_0}(E) = [V_0, a]$ for all $a \in E \setminus C_E(V_0)$.

(ii) $V_2 = C_{V_0}(F) = [V_0, b]$ for all $b \in F \setminus C_F(V_0)$.

(iii) $V_0 = V_1 \oplus V_2$.

(iv) For $g \in H$ with $\sigma_g(V_1) \neq V_1$ there is $a \in E$ with $\sigma_{ga}(V_1) = V_2$.

Take $b \in F$ fixed such that $\sigma_b(V_1) \neq V_1$ and set $\beta := \sigma_b - 1 \in \text{End}(V_0)$. Similarly set $\chi_a = \sigma_a - 1$. Moreover for $i = 1, 2$ let $\pi_i$ be the projection from $V_0$ on $V_i$ according to the direct sum decomposition $V_0 = V_1 \oplus V_2$.

**Proposition 7.1** The following equations hold, where $a, c \in E$:

1. $\sigma_b = \pi_1 + \pi_2 + \beta$.
2. $\sigma_a = \pi_1 + \pi_2 + \chi_a$.
3. $\chi_a \pi_1 = \pi_2 \chi_a = \beta \pi_2 = \pi_1 \beta = 0$.
4. $\beta \pi_1 = \pi_2 \beta = \beta$ and $\pi_1 \chi_a = \chi_a \pi_2 = \chi_a$.
5. $\beta^2 = \chi_a \chi_c = 0$.
6. $\pi_1 \sigma_a = \pi_1 + \chi_a$.
7. $\chi_{ac} = \chi_a + \chi_c$ and $\chi_a^{-1} = -\chi_a$.

**Proof.** Straightforward. \hfill $\square$

**Proposition 7.2** There exists $a_1 \in E$ such that $(\chi_a, \beta) |_{V_1} = \text{id}_{V_1}$.

**Proof.** By (iv) there exists $a \in E$ such that $\sigma_{ba}(V_1) = V_2$. Now Proposition 7.1 affords

$$\pi_1 \sigma_{ba} = (\pi_1 \sigma_a) \sigma_b = (\pi_1 + \chi_a)(\pi_1 + \pi_2 + \beta) = \pi_1 + \chi_a + \chi_a \beta$$

and $0 = \pi_1 \sigma_{ba} |_{V_1} = \text{id}_{V_1} + (\chi_a \beta) |_{V_1}$. Let $a_1 = a^{-1}$. \hfill $\square$
Proposition 7.3 For every $a \in E \backslash C_E(V_0)$ there exists $\widehat{a} \in A$ such that $(\chi_{\widehat{a}}\beta) |_{V_1} = ((\chi_a\beta) |_{V_1})^{-1}$.

Proof. Let $g = b^{-1}ab$. A straightforward calculation shows

$$(\ast) \quad \sigma_g = (\pi_1 + \chi_a - \chi_a\beta) + (\pi_2 + \beta \chi_a - \beta \chi_a\beta).$$

By Proposition 6.6 $\sigma_g(V_1) \neq V_1$. Hence by (iv) there is $c \in E$ such that $\sigma_{gc}(V_1) = V_2$. Then $\pi_1 \sigma_{ge} = (\pi_1 \sigma_c) \sigma_g = (\pi_1 + \chi_c) \sigma_g$. Using (\ast) we compute

$$0 = \pi_1 \sigma_{ge} |_{V_1} = id_{V_1} - (\chi_a\beta) |_{V_1} - (\chi_c \beta \chi_a \beta) |_{V_1}.$$

Multiplying this equation with $((\chi_a\beta) |_{V_1})^{-1}$ from the right we obtain

$$(\chi_c \beta) |_{V_1} = ((\chi_a\beta) |_{V_1})^{-1} - id_{V_1}.$$

By Proposition 7.2 there exists $a_1 \in E$ such that $(\chi_{a_1}\beta) |_{V_1} = id$. Let $\widehat{a} = ca_1$. Then $\chi_{\widehat{a}} = \chi_c + \chi_{a_1}$ we compute $(\chi_{\widehat{a}}\beta) |_{V_1} = ((\chi_a\beta) |_{V_1})^{-1}$.

In Proposition 7.4 and Proposition 7.5 we pick a fixed $v_1 \in V_1$ with $v_1 \neq 0$.

Proposition 7.4 Let $a, a' \in E$. Define

$$\overline{\chi}_a = (\sigma_a - 1)\beta \in \text{End}(V_0) \quad \text{and} \quad x_a := \overline{\chi}_a (v_1).$$

There is a unique coset $a''C_E(V_0)$ with $\overline{\chi}_{a'}(x_a) = x_{a'}$. Define

$$x_a + x_{a'} := x_{aa'} \quad \text{and} \quad x_a \cdot x_{a'} := x_{aa''}.$$

Set $D := \{x_a \mid a \in E\}$. Then $(D, +, \cdot)$ is a Cayley-Dickson-Division-Algebra or a skew field with $(D, +) \simeq E/C_E(V_0)$.

Proof. For each $v \in V_1^\#$ we have $\overline{\chi}_E(v) := \{\overline{\chi}_a(v) \mid a \in E\} = V_1$ by Proposition 6.5. As elements of $\overline{\chi}_A$ are not singular we get $a^{-1}a' \in C_E(V_0)$ if $\chi_a(v) = \overline{\chi}_{a'}(v)$. Hence for $v, v' \in V_1 \backslash \{0\}$ there is a unique coset $a''C_E(V_0)$ with $\overline{\chi}_{a''}(v) = v'$. Thus the product $x_a \cdot x_{a'}$ for $a, a' \in E$ is well defined. Now the proof of Glauberman [5, (IX) on page 7 f] shows that $(D, +, \cdot)$ is an alternative division ring or a skew field. Thus Proposition 7.4 follows from [2].
Proposition 7.5 Let $D$ be as in Proposition 7.4. Then $\{V_1\} \cup V_2^E$ is a congruence partition of an affine plane over $D$.

Proof. By Proposition 6.6 $\{V_1\} \cup V_2^E$ is a congruence partition. Let $a_0 \in E$. Then $(\chi_{a_0} \chi_a)(v_1) = - (\beta \chi_a)(v_1) + (\sigma_{a_0} \beta \chi_a)(v_1)$ for $a \in E$. Hence $\sigma_{a_0}(V_2) = \{ (\chi_{a_0} \chi_a)(v_1) + (\beta \chi_a)(v_1) \mid a \in E \}$. Now $\chi_a(v_1) \mapsto aC(v_0) \mapsto (\beta \chi_a)(v_1)$ define bijective maps between $D \cong E/C_E(v_0)$, $V_1$ and $V_2$ which induce a bijective map between $V_0$ and $D \times D$. Then $\sigma_{a_0}(V_2)$ is mapped on $\{(\chi_{a_0} \chi_a, \chi_a) \mid a \in E \}$ and we get Proposition 7.5 (see [4, page 131 f]).

Proposition 7.6 By Proposition 7.5 we may view $V_0$ as an affine plane over $D$. Then $E$ induces the group of shears with axis $V_1$ on $V_0$ and $H = L$ induces the subgroup of a point-stabilizer of $V_0$ generated by all shears.

Proof. Since $E$ is transitive on all lines through 0 different from $V_1$ by Proposition 6.6, $E$ contains all shears by [4, page 122]. As $H$ is transitive on the lines through 0 we get Proposition 7.6.

Theorem A now follows from Proposition 7.6 and Theorem 6.9.

8 Strong NSS’s

We say that an NSS $\Gamma$ is strong provided that

(Z) $\Omega(Z(N)) \neq 1$ for all $1 \neq N \in \Gamma$.

Throughout this section we assume that $G$ is a group with a reduced strong NSS $\Gamma$ with $1 \in \Gamma$. In addition to our previous notations we let

$\Theta := \{ N \in \Gamma \setminus \Gamma^* \mid N$ is large in $\Gamma \}$.

Lemma 8.1 Let $\text{rank} \Gamma = 2$, $N \in \Theta$, $V = \Omega(Z(N))$, $P \in \Gamma^*_N$ and $Z \in \Gamma$ with $Z$ normal in $P$. Then:

(1) Let $1 \neq D \in \Gamma$ be normal in $N_G(N)$. Then $D^o = N$ and $N_G(D) = N_G(N)$

(2) $N_G(V) = N_G(N)$.
(3) If \( R(\Gamma_N \cap \Gamma_Z) \neq N \) then \((\Gamma_N \cap \Gamma_Z)^* = \{P\}\).

(4) If \( Z \subseteq V \) and \( R(\Gamma_N \cap \Gamma_Z) \neq N \), then \([C_G(V), P] \subseteq C_P(V) = N\).

**Proof.** (1) Note that \( \Gamma_N \leq \Gamma_D \). So by Corollary 2.7, \( D^\circ \) is closed and contained in \( N \). Since \( 1 \neq D \leq D^\circ \leq N < S \in \Gamma^* \) and \( \Gamma \) is reduced of rank 2, \( D^\circ = N \). So \( N_G(D) \subseteq N_G(N) \). By assumption \( N_G(N) \leq N_G(D) \) and \( (1) \) holds.

(2) follows from (1) applied to \( D = V \).

(3) Put \( T = R(\Gamma_N \cap \Gamma_Z) \) and suppose \( T \not\subseteq N \). Since \( N \leq T \) we get \( N < T \). Since \( \Gamma_N \) has rank 1, Proposition 3.7(3) implies \((\Gamma_N \cap \Gamma_Z)^* = \{P\}\).

(4) Since \( Z \subseteq V \), \( C_G(V) \subseteq N_G(N) \cap N_G(Z) \) and so \( C_G(V) \subseteq N_G(((\Gamma_N)Z)^*) = N_G(P) \). Thus

\[
[C_G(V), P] \subseteq C_P(V).
\]

Suppose that \( N_G(N) \leq N_G(P) \) and let \( Q \in \Gamma_N \). By definition of \( \Theta \), \( C_P(N) \leq N \) and thus \([C_P(N), (P, Q)] \leq N \). So by Lemma 2.9, \( QP \in \Gamma \). Thus \( Q \in \Gamma_P \) and \( \Gamma_N \leq \Gamma_P \). Corollary 2.7 implies \( P \leq P^o \leq N \), a contradiction. Thus \( N_G(N) \not\subseteq N_G(P) \).

Let \( g \in N_G(N) \setminus N_G(P) \). Then \( C_P(V) \subseteq N_P(N) \cap C_G(Z^g) \subseteq N_P(P^g) \), whence \( C_P(V)C_P(V)^g \subseteq N_P(P^g)N_P^g(P) \in \Gamma \). Pick \( Q \in \Gamma_N^* \) with

\[ N_P(P^g)N_P^g(P) \subseteq Q. \]

If \( Q \neq P \), then \( C_P(V) \subseteq P \cap Q \subseteq N \), by Proposition 3.7(1).

If \( Q = P \), then \( C_P(V)^g \subseteq P^g \cap P = N \), again by Proposition 3.7(1). Since \( N = N^g \) we get \( C_P(V) \leq N \).

**Theorem 8.2** Let \( G \) be a group with a reduced strong NSS \( \Gamma \) of rank 2. Let \( N \in \Theta \), \( S \in \Gamma_N^* \), \( V := \Omega(Z(N)) \) and \( Z := C_V(J(S)) \). Then \( 1 \neq Z \in \Gamma \). Moreover,

(1) If \( J(S) \leq N \), then \( J(S)^o = N \) and \( N_G(J(S)) = N_G(N) \).

(2) If \( J(S) \not\subseteq N \) and \( N = Z^o \), then \( N = \Omega(Z(P))^o \) for any \( P \in \Gamma^* \) with \( S \leq P \).

(3) If \( N \neq Z^o \), then \( V/C_V((\Gamma_N)) \) is a natural \( SL_2 \)-module for \( \Gamma_N \) and \( S = J(S)N \).

34
Proof. Since $V \neq 1$ and $VJ(S)$ is nilpotent, $Z \neq 1$. By Proposition 4.7 $Z \in \Gamma$.

(1) Follows from Lemma 8.1(1).

(2) Suppose that $(\Gamma C_G(Z))^* = \{T\}$ for some $T$. Then $T$ is normal in $N_G(Z)$. Since $N = Z^\circ$ is large, we get from Lemma 3.2 that $T \subseteq Q$ for all $Q \in \Gamma^*_Z$. Thus $T \subseteq Z^\circ = N$, a contradiction since $J(S) \leq T$ and $J(S) \neq N$.

Thus there exist $L, Q \in \Gamma C_G(Z)^*$ with $L \neq Q$. Then $(L, Q) \notin \Gamma$. Put $M = \Omega(Z(P))^\circ$. Note that $N \subseteq L \cap Q$, and both $LN_M(N)$ and $QN_M(N)$ are in $\Gamma$. Thus $NN_M(N) \subseteq LN_M(N) \cap QN_M(N) \subseteq N$, by Proposition 3.7(1). Thus $1 \neq M \subseteq N$. Since $M \leq P$, Proposition 2.8(12) implies $M$ is closed and as rank $G = 2$, $M = N$.

(3) From rank$(\Gamma) = 2$, Proposition 3.7(1) and Theorem 3.1 we get $P \cap Q = N$ and rank$(\Gamma_N) = 1$ for $P, Q \in \Gamma^*_N$ with $P \neq Q$. Suppose $N = R(\Gamma_N \cap \Gamma_Z)$. Then by Lemma 2.6 (applied with $\Lambda = \Gamma_N \cap \Gamma_Z$, $\Delta = \Gamma_Z$ and $P = N$), $Z^\circ = R(\Gamma_Z) \subseteq N$ and $Z^\circ$ is closed. Thus

$$1 \neq Z \subseteq Z^\circ \subseteq N \notin \Gamma^*.$$

Since rank$(\Gamma) = 2$ we get $N = Z^\circ$, a contradiction. Therefore $N \neq R((\Gamma_N \cap \Gamma_Z))$. In particular, $\Gamma_N \cap \Gamma_Z \neq \Gamma_N$ and so $\Gamma_N \notin \Gamma_Z$.

Moreover, by Lemma 8.1(4) $[C_G(V), S] \subseteq N = C_S(V)$.

Assume $J(S) \subseteq N$. Then $Z = V$ and $\Gamma_N \subseteq \Gamma_Z$, a contradiction. Thus $J(S) \notin N = C_S(V)$. Pick $A \in A(S)$ with $A \notin C_S(V)$. Then by Proposition 5.5 $A$ is a non-trivial $\Gamma$-offender on $V$. By Proposition 3.7(2) $\Gamma_N$ has rank 1. By definition of $\Theta$, $N$ is large in $\Gamma_N$.

We verified that all the the assumptions of Theorem A are satisfied for $N_G(N), \Gamma_N, S, A$ and $V$. Hence $V/C_V(\langle \Gamma_N \rangle)$ is a natural $SL_2$-module for $\Gamma_N$.

By Theorem 6.9, $S = AR(\Gamma_N) = AN$ and so $S = J(S)N$. \hfill $\square$

**Theorem 8.3** Suppose rank$(\Gamma) = 2$, $N \in \Theta$ and $S \in \Gamma^*_N$ with $N_G(S) \not\subseteq N_G(N)$. Put $V = \Omega(Z(N))$. Then $V/C_V(\langle \Gamma_N \rangle)$ is a natural $SL_2$-module for $\Gamma_N$.

Proof. Suppose that $J(S) \leq N$. Then using Theorem 8.2(1)

$$N_G(S) \leq N_G(J(S)) \leq N_G(J(S)^\circ) = N_G(N),$$

35
a contradiction to the assumptions.

Hence $J(S) \not\subseteq N$. Set $Z := C_V(J(S))$. Suppose that $Z^o = N$. By Proposition 3.7(3) $S$ lies in a unique maximal $\Gamma$-subgroup $P$. Then by Theorem 8.22, $N_G(S) \subseteq N_G(P) = N_G(\Omega(Z(P))^o = N_G(N)$, a contradiction.

Hence $Z^o \neq N$ and Theorem 8.3 follows from Theorem 8.2(3) \hfill \Box

\textbf{Theorem 8.4} Suppose $\text{rank}(\Gamma) = 2$, $S \in \Gamma^*$ and $|\Theta S| > 2$. Then there is $N \in \Theta S$ such that $V/C_V(\langle \Gamma_N \rangle)$ is a natural $SL_2$-module for $\Gamma_N$, where $V = \Omega(Z(N))$

\textbf{Proof.} Let $N \in \Theta S$. By Proposition 3.7(3), $S$ is the unique maximal $\Gamma$-subgroup containing $N_S(N)$. Hence $N_S(N) \in \Gamma_N$. If $N_G(N_S(N)) \not\subset N_G(N)$ we are done by Theorem 8.3.

So we may assume that $N_G(N_S(N)) \leq N_G(N)$ for all $N \in \Theta S$. In particular $N_S(N_S(N)) \leq N_S(N)$ and so $N_S(N) = S$. Thus $S \in \Gamma_N$ and $N_G(S) \leq N_G(N)$.

Since $|\Theta S| \geq 3$ there exists $N \in \Theta S$ with $N \neq J(S)^o$ and $N \neq \Omega Z(S)^o$. Thus by Theorem 8.2 $J(S) \not\subseteq N$ and $N \neq Z^o$. So Theorem 8.4 follows from Theorem 8.2(c) \hfill \Box

The following theorem deals with a situation which had been considered more detailed for finite groups in [3].

\textbf{Theorem 8.5} Let $\text{rank}(\Gamma) = 2$, $S \in \Gamma^*$ and $M, N \in \Theta S$ with $M \neq N$. Assume there is $P \in \Gamma_M \cap \Gamma_N$ with

\begin{equation}
(*) \quad Z \cap Z^g = 1 \text{ for all } g \in G \setminus N_G(P)
\end{equation}

where $Z := \Omega(Z(J(P)))$. Then $N$ is a natural $SL_2$-module for $\Gamma_N$. Moreover $P = MN$ and $P$ is of nilpotency class 2.

\textbf{Proof.} For $L \in \{M, N\}$ set $V_L := \Omega(Z(L))$. As $\langle M, N \rangle \subseteq P \cap S$ and $\text{rank}(\Gamma) = 2$ we get $P \subseteq S$ by Proposition 3.7(4).

Since $\text{rank}(\Gamma) = 2$, $\langle M, N \rangle \not\in \{M, N\}$. Thus by Lemma 3.2(2),

$$\Gamma_L \cap \Gamma N_G(P) = \Gamma P.$$ 

Suppose that $J(P) \subseteq L$. Then $V_L \subseteq Z$ and so by $(*)$ $N_G(L) \subseteq N_G(P)$. Thus $\Gamma_L \subseteq \Gamma_L \cap \Gamma N_G(P) = \Gamma P$ and $L = P$, a contradiction.

36
Thus $J(P) \not\subseteq L$. Let $X = \Omega(Z(P))$. Then

$$1 \neq X \leq Z \cap V_L \subseteq V_L.$$ 

By (*) $N_G(X) \subseteq N_G(P)$, and so $\Gamma_L \cap \Gamma_X \subseteq \Gamma_L \cap \Gamma N_G(P)$ and $R(\Gamma_L \cap \Gamma_X) = P$. Thus by Lemma 8.1(4), $C_S(V_L) \subseteq L$. So we can apply Theorem 8.2(c) and $V_L/C_{V_L}(\Gamma_L)$ is a natural $SL_2$-module for $\Gamma_L$.

Let $\{K, L\} = \{M, N\}$. By Theorem 6.9 $KL = P = AL$ and $L = C_P(V_L)$ for all $A \in \mathcal{A}(P)$ with $A \not\subseteq L$. Moreover $X = C_{V_L}(A) = V_L \cap V_K = C_{V_L}([\Gamma_L])[V_L, A]$. Since $X \cap X^g = 1$ for $g \in G \setminus N_G(P)$ we conclude $C_{V_L}([\Gamma_L]) = 1$. Thus by Proposition 6.5(3)

$$q := \mu(X) = \frac{1}{2} \mu(V_L) = \mu(X) = \mu(A/A \cap L).$$

In particular, $\mu(V_L) = \mu(A)$ and so $V_L(A \cap L) \in \mathcal{A}(L) \cap \mathcal{A}(P)$. Using this and symmetry in $K$ and $L$, $\mathcal{A}(K) \cup \mathcal{A}(L) \subseteq \mathcal{A}(P)$. Suppose that $\mathcal{A}(K) = \mathcal{A}(L)$, then $\Gamma_L \cup \Gamma_K \subseteq \Gamma_{J(K)}$. Thus by Corollary 2.7 $R(\Gamma_{J(K)})$ is closed and contained in $L \cap K$, a contradiction to $\text{rank}(\Gamma) = 2$. So $\mathcal{A}(K) \neq \mathcal{A}(L)$ and interchanging $K$ and $L$ if necessary we assume $\mathcal{A}(K) \not\subseteq \mathcal{A}(L)$.

So we can choose $A \in \mathcal{A}(K)$.

Suppose for a contradiction that $[V_K, V_L] = 1$. Then $V_KV_L \subseteq K \cap L$. As $AL = P \not\subseteq C_G(V_K)$ we get $[V_K, L] = X$. Let $W \in V_K^{[\Gamma_L]} \setminus \{V_K\}$. Then $[V_K, W] \subseteq (V_K \cap V_L) \cap (W \cap V_L) = 1$. Since $A$ normalizes $[A \cap W, L]$ and $[A \cap N, L] \leq W \cap V_L$ we get $[A \cap W, L] = 1$ and so $A \cap W = A \cap W \cap V_L = 1$. Now $\mu(W) = q = 2\mu(A/A \cap L)$ implies $\mu(W(A \cap L)) > \mu(A)$. Thus $[A \cap N, W] \neq 1$ and so by Proposition 6.5(3) applied to $(A \cap L)V_L$, $[A \cap N, W] = V_L \cap W$ and $\mu(A)/\mu(C_A(W)) = \mu(W) - 2\mu(X)$. Thus $C_A(W)W \in \mathcal{A}(L)$.

Let $a \in A \cap L$. Since $W$ centralizes $C_A(W)$, also $W^a$ centralizes $C_A(W)$. Since $[W, W^a] \subseteq V_L \cap W \cap W^a = 1$ we conclude that $C_A(W)W^a$ is a decomposable abelian $\Gamma$-subgroup. Since $C_A(W)W \in \mathcal{A}(P)$, $C_A(W)W = C_A(W)W^a$. Thus

$$V_L \cap W = [A \cap L, W] = [A \cap L, W^a] = V_L \cap W^a,$$

a contradiction to $V_L \cap W \cap W^a = 1$.

Therefore $[V_K, V_L] \neq 1$ and so $V_N \not\subseteq M$ and $V_M \not\subseteq N$.

Let $h \in \langle \Gamma_M \rangle \setminus N_G(P)$. Note that $M = V_M(N \cap N^h)$. Hence $\Omega Z(N \cap N^h) \leq \Omega Z(M) = V_M$. But $V_N$ centralizes $N$ and so

$$\Omega Z(N \cap N^h) \leq C_{V_M}(V_N) \cap C_{V_M}(A^h) = V_M \cap V_N \cap V_N^h = 1.$$
By the assumptions of this section, $\Gamma$ is strong and so $N \cap N^h = 1$. Thus $M = V_M, N = V_N$ and $P = V_MV_N = MN$. Now $P' = X = M \cap N$ and $P$ has class 2. 

**Theorem 8.6** Suppose that $\Pi$ is a $G$-invariant subset of $\Theta$ such that

(i) $\bigcap_{g \in G} A^g = 1$ for all $A \in \Pi$.

(ii) If $S \in \Gamma^*$ with $|\Theta S \cap \Pi| \geq 2$, then $|\Theta S| = 2$.

(iii) Whenever $X, Y \in \Pi$ with $X \in Y^G$ and $X \neq Y$ then $R(\Gamma \langle X, Y \rangle) \in \Pi$.

Let $\Pi_p$ be an arbitrary orbit for $G$ on $\Pi$ and define $\bar{\Pi}_p = \{ \text{R}(\Gamma \langle A, B \rangle) | A, B \in \Pi_p \text{ with } A \neq B \}$. Then

(1) $\bar{\Pi}_p = \Pi_p$.

(2) $\Pi_p$ is the set of points, $\bar{\Pi}_p$ is the set of lines of a projective Moufang plane $\pi$ and $\langle \Pi_p \rangle = \langle \bar{\Pi}_p \rangle$ induces the group generated by all the elations on $\pi$.

(3) $C_G(\pi) \leq C_G(\langle \Pi_p \rangle)$.

We remark that using knowledge of the automorphism group of a Moufang plane it should not be to difficult to show that $G$ only has two orbits on $\Pi$.

**Proof.** From (i) we get

(1.) $N \not\leq G$ for all $N \in \Pi$.

We say $X, Y \in \Pi$ are *incident* if $X \neq Y$ and $\langle X, Y \rangle \in \Gamma$. We show next

(2.) *If $X, Y$ are incident then $X \in \Gamma_Y$ and $Y \in \Gamma_X$.*

Indeed by (ii) $\Theta \langle X, Y \rangle = \{ X, Y \}$ and so $X$ and $Y$ are normal in $\langle X, Y \rangle$. For $X, Y \in \Pi$ with $X \neq Y$ write $\overrightarrow{XY} := R(\Gamma \langle X, Y \rangle)$.

(3.) $F \not\leq E$ for all $E, F \in \Pi$.

Otherwise let $g \in G \setminus N_G(E)$. Then $|\Theta E \overrightarrow{EE^g}| = 2$ and so by (iii) $F = \overrightarrow{EE^g} = F^g$. Since $\{ F \} = \Theta E \setminus \{ E \}, F^g = F$ for all $g \in G$. Thus $F \not\leq G$, a contradiction to (1.), proving (3.)
Let $A \in \Pi$. By (1.) there exists $B \in A^G$ with $A \neq B$. By (iii) $\overline{AB} \in \Theta$. Let $D = \overline{AB} \in \Gamma$.

Suppose that $A$ and $B$ are incident. Then $\langle A, B \rangle \in \Gamma$ and $D = \langle A, B \rangle$. By (ii), $|\Theta D| = 2$. Since $A \neq B$ we may assume $A = D$. Hence $B \leq A$. Since $A$ and $B$ are conjugate $\mu(A) = \mu(B)$ and we conclude that $A = B$, a contradiction.

We proved

(4.) No two distinct conjugate elements of $\Pi$ are incident.

Suppose $C \in \Pi$ is incident with $A$ and $B$, then $\langle A, B \rangle \leq \Gamma_C$ and so $C \leq D$. Since $\Theta AD = \{A, D\}$ and $A \neq D$, $C = D$. Thus

(5.) $\overline{AB}$ is the unique element of $\Pi$ incident with $A$ and $B$.

Let $\Sigma(A) = \Pi \cap \Gamma_A \setminus \{A\}$, the set of elements of $\Pi$ incident with $A$. Let $\Xi_A := \bigcup \{\Gamma AE \mid E \in \Sigma(A)\}$.

(6.) Let $A < X \in \Xi_A$. Then there exists a unique $X^* \in \Gamma^*$ with $X \leq X^*$ and a unique $E \in \Sigma(A)$ with $\Theta(X^*) = \{E, A\}$ and $X \leq AE$.

Pick $E \in \Sigma(A)$ with $X \leq EA$ and $P \in \Gamma^*$ with $EA \leq P$. Suppose there exists $Q \in \Gamma^*$ with $X \leq Q$ but $Q \neq P$. Choose such a $Q$ with $P \cap Q$ maximal. Then by Proposition 2.11(1), $P \cap Q$ is closed. Since $A \leq P \cap Q$, Lemma 3.3 implies that $P \cap Q$ is large. So $P \cap Q \in \Theta P$. But $\Theta P = \{A, E\}$ and we conclude that $E = P \cap Q$, but then $A < E$, a contradiction to (3.).

(7.) $\Xi_A \leq \Gamma_A$, $\Xi_A$ is an NSS of rank 1 for $N_G(A)$, $\Xi_A^* = \{AE \mid E \in \Sigma(A)\}$ and $R(\Xi_A) = A$.

Clearly $\Xi_A^* = \{AE \mid E \in \Sigma(A)\}$, and (Suba) and (Subc) are fulfilled. Let $X, Y \in \Xi_A$ with $\langle X, Y \rangle \in \Gamma$. We need to show that $\langle X, Y \rangle \in \Xi_A$. If $X \leq A$ or $Y \leq A$ this is obvious. We may assume $A < X$ and $A < Y$. Pick $Q \in \Gamma^*$ with $\langle X, Y \rangle \leq Q$. Let $E, F \in \Sigma(A)$ with $X \leq EA$ and $Y \leq FA$. By (6.), $\Theta(Q) = \{A, E\} = \{A, F\}$ and so $E = F$. Thus $\langle X, Y \rangle \leq EA$ and $\langle X, Y \rangle \in \Xi_A$.

By Theorem 3.1 it remains to show that $|\Xi_A^*| > 1$. Otherwise we conclude that $\Sigma_A = \{K\}$ for some $K$, and $K = \overline{AA^9}$ for all $g$ with $A \neq A^g$ and then $K \leq G$, a contradiction to (1.). This completes the proof of (7.).

(8.) Let $1 \neq X \in \Gamma A$. Then $N_{\Gamma}(X) \subseteq N_{\Gamma}(A)$.

Suppose not and pick $Q \in N_{\Gamma}(X)$ with $Q \not\subseteq N_{\Gamma}(A)$. Pick $g \in Q$ with $A \neq A^g$. Let $E \in \Sigma(A)$. 39
Suppose that $X \not\in L$ for some $L \in \Sigma(A)$.

If $A^g$ is incident with $L$, then both $A$ and $A^g$ are incident with $L$ and so $L = AA^g$. By (6) applied to $L$ in place of $A$, $X \leq A \cap A^g \leq L$.

Thus neither $A^g$ nor $A^{g^{-1}}$ are incident with $L$. In particular $L \neq L^g$. Note that $X$ normalizes $L$ and $L^g$ and so also $F := LL^g$. By Lemma 3.2(1) applied to $L$ in the place of $N$, we get $X LF \in \Gamma$. By $X \not\in L$ and (6), $XL$ lies in a unique maximal $\Gamma$-subgroup of $G$. Hence $\langle AL, XL \rangle \in \Gamma$ and (ii) implies $A = F$. Thus $L^g$ is incident with $A$ and so $A^{g^{-1}}$ is incident with $L$, a contradiction.

Thus $X \leq L$ for all $L \in \Sigma(A)$. Let $Y := \bigcap \Sigma(A)$. Then $X \leq Y$ and so $Y \neq 1$. Since $N_G(A) \leq N_G(Y)$ we have $N_G(Y) \not\leq N_G(L).$ The claim we just proved applied to $(Y, L)$ in place of $(X, A)$ yields $Y \leq K$ for all $K \in \Sigma(L)$ and all $L \in \Sigma(A)$. Thus $Y \leq A^g$ for all $g \in G$ and (i) implies $Y = 1$, a contradiction.

(9.) Let $E \in \Sigma(A)$ and $V_A = \Omega(Z(A))$. Then $[C_G(V_A), EA] \leq A$.

Let $F \in \Sigma(A)$ and put $X = \Omega(Z(AF))$. Since $A$ and $F$ are large, $X \leq A \cap F$. Since $\Gamma$ is strong, $X \neq 1$. By (8.) applied to $F$ in place of $A$, $N_G(X) \leq N_G(F)$. Since $X \leq V_A$ we conclude that $C_G(V_A) \leq N_G(F)$ for all $F \in \Sigma(A)$. So by (6.) $C_G(V_A) \leq N_G(P)$ for all $P \in \Xi^*_A$. Define $U := \bigcap \{N_{EA}(P) \mid P \in \Xi^*_A\}$. Then $U \in \Xi_A$ and by Proposition 2.3(8), $U = R(\Xi_A) = A$. But $[C_G(V_A), EA] = C_G(V_A) \cap EA \leq U$ and so (9.) holds.

(10.) Let $E \in \Sigma(A)$. Then $J(EA) \not\in E \cap A$.

If $J(EA) \in E \cap A$ then $J(E) = J(A)$. Then $N_G(E) \leq N_G(J(A))$ and so $N_G(J(A)) \not\leq N_G(A)$, a contradiction to (8.)

By (10.) and interchanging $A$ and $E$ if necessary

(11.) we can choose $A \in \Pi$ and $E \in \Sigma(A)$ with $J(EA) \not\in A$.

By (8.) and Proposition 5.5 there exists a non-trivial offender in $EA$ on $V_A$. Let $H_A =: \langle \Xi_A \rangle$. Note that $C_{V_A}(H_A) \leq C_{V_A}(E) \leq E$ and so by (8.) $C_{V_A}(H_A) = 1$. We conclude that the Hypothesis of Theorem A holds for $N_G(A), \Xi_A, V_A$ and $EA$. So $V_A$ is a natural $SL_2$-module for $\Xi_A$. In particular,

(12.) $E$ acts transitively on $\Sigma(A) \setminus \{E\}$, $N_G(A)$ acts transitively on $\Sigma(A)$ and $C_{EA}(V_A) = A$.

By (8.), $V_A \not\leq E$. Let $X \in A(AE)$. By Proposition 6.5(3), $(X \cap A)V_A \in A(AE)$ and we reestablish symmetry in $A$ and $E$. Let $R := \langle V_E, V_E^2 \rangle$ for
some \( h \in H_A \) with \( V_E \neq V_E^h \). Then \( A \cap E \cap E^h \leq C_A(R) \) and so by (8.), \( A \cap E \cap E^h = 1 \). It follows that \( N = V_N \) and so \( N \) is a natural \( SL_2 \)-module for \( \Xi_N \) for all \( N \in \Pi \).

It follows from (12.) that \( \widehat{A}^g \in E^G \) and \( \widehat{E}^g \in A^G \) for all \( g \in G \). Thus \( A^G \) and \( E^G \) form a projective plane.

By (12.) we have \((N,N)\)-transitivity. Then [4, page 130] shows that we have got a projective Moufang plane.

Let \( C_A \) be the kernel of the action of \( G \) on \( A^G \). Then clearly \( C_A \) also acts trivially on \( E^G \). Moreover \([C_A,A] \leq A \cap C_A \leq C_A(E) \leq A \cap E \) and (8.) implies \([C_A,A] = 1 \).

\[ \square \]

References


