A note on the cohomology of finitary modules

U. Meierfrankenfeld

May 17, 2002

In this note we prove the following three theorems on the cohomology of finitary modules in terms of the cohomology of a local system of subgroups:

Theorem 1 *Let G be a group, K a field, V a finitary KG-module and $\mathcal{L}$ a local system of subgroups of G. Suppose that, for all $H \in \mathcal{L}$, V is completely reducible as a KH-module. Then $[V, G]$ is completely reducible as a KG-module.*

Theorem 2 *Let G be a group, D a division ring, V a finitary DG-module, $\mathcal{L}$ a local system of subgroups of G and H an extension of V by G, (i.e. $H/V \cong G$). Suppose that the following holds for all L in $\mathcal{L}$:*

(i) *The extension of V by L in H splits.*

(ii) *$V/C_V(L)$ is finite dimensional.*

(iii) *$H^1(L, V)$ is finite dimensional.*

Then H splits over V.

Theorem 3 *Let G be a group, D a division ring, $\mathcal{L}$ a local system of subgroups of G, W a DG-module and V a DG-submodule of W such that $W = V + C_W(H)$ for all $H \in \mathcal{L}$. Then there exists a canonical DG-monomorphism from $W/C_W(G)$ to $[V^*, G]^*$, where Y^* denotes the dual of a module Y.*

We remark that conditons (ii) and (iii) in Theorem 2 are automatically fulfilled if all members of $\mathcal{L}$ are finite groups generated by elements whose order is coprime to the characteristic of D.

Proof of Theorem 1: Let $H \in \mathcal{L}$. Then $[V, H] = [V, H, H]$ and so $[V, G] = [V, G, G]$. Hence we may assume that $V = [V, G]$. Let W be the sum of all the irreducible KG-submodules in V, where $W = 0$ if G has no irreducible submodules in V. We need to show that $W = V$.

So suppose that $V \neq W$. Then $[V, G] \not\leq W$ and we may assume that $[V, H] \not\leq W$ for all $H \in \mathcal{L}$. Let $H \in \mathcal{L}$ and let I_H be the set of irreducible KH-submodules I in

$[V,H]$ with $I \not\leq W$. For $I \in I_H$ let $m(I)$ be supremum of all positive integers t such that I^t is isomorphic to a KH-submodule of V. Pick $h \in H$ with $[I,h] \neq 0$. Then $m(I) \cdot \deg_I(h) \leq \deg_V(h)$. In particular, $m(I)$ is finite. Note that there exists a unique KH-submodule $\hat{I}$ in V isomorphic to $I^{m(I)}$, namely $\hat{I}$ is the submodule generated by all the H submodules in V isomorphic to I. Let $K(I) = \mathrm{Hom}_{KH}(I,I)$ and $d(I) = \dim_K K(I)$. Since $\dim_K[I,h] = \dim_{K(I)}[I,h] \cdot \dim_K K(I)$, $d(I) \leq \deg_V(h)$ and so $d(I)$ is finite. Let m be the minimum of all $m(I), I \in I_H, H \in \mathcal{L}$ and d the minimum of all $d(I), I \in I_H, H \in \mathcal{L}, m(I) = m$.

Pick $H \in \mathcal{L}$ and $I \in I_H$ with $m(I) = m$ and $d(I) = d$. Without loss $H \leq F$ for all $F \in \mathcal{L}$. Let $F \in \mathcal{L}$. Since V is completely reducible as a KF-module, there exists $J \in I_F$ such that I is isomorphic to a KH submodule of J. Let a be a positive integer such that I^a is isomorphic to a KH-submodule of J. Then $I^{a \cdot m(J)}$ is isomorphic to a KH-submodule of V and so $a \cdot m(J) \leq m$. By minimal choice of m, $m \leq m(J)$. Thus $a = 1$ and $m(J) = m$. In particular, $\hat{I} \leq \hat{J}$ and there exists a unique KH-submodule U in J isomorphic to I. Hence $K(J)$ acts on U and restriction $K(J) \mid_U$ of $K(J)$ to U is contained in $K(U)$. Since $\dim_K K(U) = \dim_K K(I) = d \leq \dim_K K(J)$, we conclude that $K(J)|_U = K(U)$. It is now easy to see that every irreducible KH submodule of $\hat{I}$ lies in an irreducible KF-submodule of $\hat{J}$. Hence $\langle I^F \rangle$ is an irreducible KF-module for all $F \in \mathcal{L}$ and $\langle I^G \rangle$ is an irreducible KG-submodule in V not contained in W. This contradiction completes the proof of Theorem 1.

The following definition and lemma are used in the proof of Theorem 2.

Definition 4 (a) *Let R be a ring, A a set, M an R-module and for $a \in A$ let $\rho_a : A \to M$ be a bijection. Then A is called an affine R-module provided that for all a, b, c in A, $\rho_a(b) + \rho_b(c) = \rho_a(c)$.*

(b) *Let R be a ring, A and B affine R-modules and $\pi : A \to B$. Then π is called an affine R-homomorphism if for some a in A and b in B, $\rho_b \pi \rho_a^{-1}$ is a R-homomorphism of modules.*

(c) *Let R be a ring and A an affine R-module. A subset B of A is called an affine R-submodule if $\rho_a(B)$ is a R-submodule of M for some a in A.*

Remark: Let M be an R-module and define $\rho_x : M \to M, y \to y - x$. Then M is an affine R-module. Moreover, if A is any affine R-module with M as underlying module, then for all a in A, ρ_a is an isomorphism of affine R-modules. Finally if a, b are in A and C is a subset of A, then $\rho_a(C) = \rho_b(C) + \rho_a(b)$ and so C is an affine submodule if and only if $\rho_a(C)$ is the coset of a R-submodule in M.

Lemma 5 *Let G be a group, R a ring and V an RG-module. Let A_G be the set of complements to V in $V \rtimes G$. Then*

(a) *A_G is an affine R-module.*

(b) *Let $H \leq G$, then the canonical map from A_G to A_H is affine.*

(c) *Let $I_G = \{G^v \mid v \in V\}$. Then I_G is an affine RG submodule of A_G, $I_G \cong V/C_V(G)$ and $A_G/I_G \cong H^1(G,V)$.*

Proof of the Lemma: Identify V and G with their images in the semidirect product $V \rtimes G$. So $V \rtimes G = VG$.

(a) Let M_G the set of functions $f : VG/V \to V$ with $f(ab) = f(a)^{b^{-1}} + f(b)$ for all a, b in VG/V, i.e M_G is the set of derivations for G on V. Note that M_G is an R-module via $(r \cdot f)(a) = r \cdot f(a)$. For K, L in A_G define $\rho_K(L) \in M_G$ by $\rho_K(L)(Va) = v$, whenever $a \in K$ and $v \in V$ with $va \in L$. Then ρ_K is a bijection from A_G onto M_G (see for example [As, 17.1]).

Let K, L, N be in A_G and a in K. Put $b = \rho_K(L)(Va)a$ and $c = \rho_L(N)(Vb)b$. Then $Va = Vb = Vc$, $b \in L$, $c \in N$ and $c = \rho_L(N)(Va)\rho_K(L)(Va)a$. Thus $\rho_K(L) + \rho_L(N) = \rho_K(N)$. (Here we write the binary operation on V multiplicatively whenever V is regarded as a subgroup of $V \rtimes G$).

(b) For L in A_G let $\pi(L) = L \cap VH$. Then its is easy to check that $\rho_H \pi \rho_G^{-1}$ is just the restriction map $M_G \to M_H, \phi \to \phi_{VH/V}$. Thus π is affine.

(c) Define $\alpha : V \to M$ by $\alpha(v)(a) = v^a - v$. Then $\ker \alpha = C_V(G)$ and $\alpha(V) = \rho_G(I_G)$ is the set of inner derivations. In particular $H^1(G,V) = M/\alpha(V) \cong A_G/I_G$ and (c) holds.

Proof of Theorem 2: Let $L \in \mathcal{L}$. By (i) we may view $V \rtimes L$ as a subgroup of H and by part (a) of the Lemma , A_L is a affine D-module and by (ii),(iii) and Part (c) of the Lemma, A_L is finite dimensional. For L and K in $\mathcal{L}$ with $L \leq K$ let $\pi_{K,L}$ be the affine map defined in Part (b) of the Lemma . We claim that the inverse limit of $(\pi_{K,L})_{L \leq K}$ is not empty. Note that finite dimensional affine D-modules fulfill the descending chain condition on affine subspaces and so a set of affine subspaces whose intersection is empty has a finite subset whose intersection is empty. Moreover, images and inverse images of affine subspaces under affine maps are affine. Now the proof in [KW, 1K1] that inverse limits of non-empty finite sets are not empty carries over word for word, except that "subset" has to be replaced by "affine subspace". Let $(C_L)_{L \in \Lambda}$ be an element in the inverse limit. Then $\bigcup\{C_L | L \in \mathcal{L}\}$ is a complement to V in H and Theorem 2 is proved.

Proof of Theorem 3: For $X \leq V^*$ let $X^\perp = \{v \in V | x(v) = 0 \text{ for all } x \in X\}$. We will first prove that:

$$(*) \quad \text{For all } K \leq G,\ [V^*, K]^\perp = C_V(K).$$

Indeed, let $x \in V^*, k \in K$ and $v \in V$. Then

$$[x,k](v) = (x^k - x)(v) = x^k(v) - x(v) = x(v^{k^{-1}}) - x(v) = x([v, k^{-1}])$$

It follows that $v \in [V^*, K]^\perp$ if and only if $[v, K] \leq V^{*\perp} = 0$ and so if and only if $v \in C_V(K)$.

Let $H \in \mathcal{L}$. Define a map $a_H : W \to [V^*, H]^*$ by $a_H(w)(x) = x(u)$ where $x \in [V^*, H], w \in W$ and $u \in V$ with $w \in u + C_W(H)$. Note that by (*) this definition does not depend on the choice of u. If $K \leq H$ with $K \in \mathcal{L}$ then $C_W(H) \leq C_W(K)$ and so $w \in u + C_W(K)$ and $a_H(w)(x) = a_K(w)(x)$ for all $x \in [V^*, K]$. Define $a : W \to [V^*, G]^*$ by $a(w)(x) = a_H(w)(x)$ whenever $w \in W, x \in [V^*, G]$ and $H \in \mathcal{L}$ with $x \in [V^*, H]$. By the preceeding observation and since $\mathcal{L}$ is a local system this definition does not depend on the choice of H. Let $w \in W$ with $a(w) = 0$. Then $a_H(w) = 0$ for all $H \in \mathcal{L}$ and so $u \in [V^*, H]^\perp$, where u is as above. By (*), $u \in C_V(H)$ and so $w \in C_W(H)$ for all $H \in \mathcal{L}$. Thus $\ker a = C_W(G)$. It remains to show that a is a DG-homomorphism. Clearly a is D-linear. Let w, x, u and H be as above and $g \in G$. We may assume without loss that $g \in H$. Then $w^g \in u^g + C_W(H)$ and so

$$a(w^g)(x) = a_H(w^g)(x) = x(u^g) = x^{g^{-1}}(u) =$$

$$= a_H(w)(x^{g^{-1}}) = a(w)(x^{g^{-1}}) = a(w)^g(x).$$

Thus $a(w^g) = a(w)^g$ and a is a DG-homomorphism, completing the proof of Theorem 3.

References

[As] M. Aschbacher, *Finite Group Theory*, Cambridge Unversity Press, Cambridge (1986)

[KW] O. Kegel and B.A.F. Wehrfritz, *Locally Finite Groups*, North Holland (1973).

Department of Mathematics
Michigan State University
East Lansing. Michigan 48824
meier@math.msu.edu