The $\widetilde{P}!$-Theorem

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In this paper we prove a result that is used in the investigation of finite $\mathcal{K}_p$-groups of local characteristic p. It is part of an attempt to revise a major part of the classification of the finite simple groups. An overview of this revision can be found in [MSS].

More precisely, the $\widetilde{P}!$-Theorem proved in this paper together with the $P!$-Theorem in [PPS] show that under certain hypotheses there exist parabolic parabolic subgroups P and $\widetilde{P}$ (containing a common Sylow p-subgroup) in a $\mathcal{K}_p$-group H of local characteristic p that behave like the two minimal parabolic subgroups of a group of Lie type in characteristic p that correspond to the end node and its neighbor in the Dynkin diagram. They also establish that the remaining part of a hypothetical Dynkin diagram for H can be found in a single maximal p-local subgroup, which is called $\widetilde{C}$ further below.

Moreover, as it is outlined in [MSS, 2.4.9], these two theorems allow to restrict the structure of $\widetilde{C}$ and consequently that of the missing part of the diagram.

To get started we need some definitions. Let H be a finite group and p a fixed prime. Then H is of **characteristic p** if

$$C_H(O_p(H)) \leq O_p(H),$$

and H is of **local characteristic p** if every p-local subgroup of H is of characteristic p. Moreover, H is a **$\mathcal{K}_p$-group** if the simple sections of p-local subgroups are "known" simple groups[1].

[1]Which means, they are groups of prime order, groups of Lie type, alternating groups or one of the 26 sporadic groups.

For any p-local subgroup $L \le H$ let Y_L be the largest elementary abelian normal p-subgroup of L satisfying

$$O_p(L/C_L(Y_L)) = 1.$$

For elementary properties of this subgroup see [MSS] and [PPS].

By $\mathcal{L}_H(X)$ we denote the set of subgroups $L \le H$ containing a given subgroup X and satisfying $C_H(O_p(L)) \le O_p(L)$, and by $\mathcal{M}_H(X)$ the set of maximal p-local subgroups containing X.

We fix $S \in Syl_p(H)$ and $\widetilde{C} \in \mathcal{M}_H(N_H(\Omega_1 Z(S)))$ and put

$$Q := O_p(\widetilde{C}) \text{ and } X^\circ := \langle Q^h \mid h \in H \text{ with } Q^h \le X\rangle \text{ for } X \le H.$$

By $\mathcal{P}_H(S)$ we denote the set of subgroups $P \le H$ such that $P \not\le N_H(S)$, $O_p(P) \neq 1$ and S is contained in a unique maximal subgroup of P. The elements of $\mathcal{P}_H(S)$ are called the **minimal parabolic subgroups** of H containing S.

Let $K \cong SL_n(p^k)$ or $Sp_{2n}(p^k)$ and V be an irreducible $GF(p)K$-module. Set $F = End_K(V)$. Then V is a **natural** $SL_n(p^k)$**-module** for K if $K \cong SL_n(p^k)$ and $\dim_F V = n$, and V is a **natural** $Sp_{2n}(p^k)$**-module** for K, if $K \cong Sp_{2n}(p^k)$, $dim_F V = 2n$, and K leaves invariant a non-degenerate symplectic form s on the F-space V.

Note that a natural module for $SL_2(p^k)$ is unique up to isomorphism. For $SL_n(p^k)$, $n > 2$, and $Sp_4(2^k)$ there are two isomorphism classes of natural modules. The second class can be obtained from the first one by applying a graph automorphism.

By a ***k*-dimensional subspace** of V we mean an k-dimensional F-subspace of V.

For $K \cong Sp_{2n}(p^k)$ a k-dimensional subspace U of V is **singular** if the symplectic form s restricted to U is zero. Since s is unique (up to scalar multiplication by elements of F), the definition of a singular subspace does not depend on the choice of s.

For $K \cong SL_n(p^k)$ any subspace of V is called **singular**.

In this paper we investigate finite $\mathcal{K}_p$-groups of local characteristic p satisfying in addition:

$$C_H(x) \le \widetilde{C} \text{ for every } 1 \neq x \in C_H(Q) \quad (\textbf{\textit{Q}-uniqueness}).$$

We prove:

$\widetilde{P}$!-Theorem. *Let H be a $\mathcal{LK}_p$-group of local characteristic p that satisfies Q-uniqueness. Suppose that there exists $P \in \mathcal{P}_H(S)$ such that $P \not\leq \widetilde{C}$ and $Y_M \leq Q$ for every $M \in \mathcal{M}_H(P)$. Then one of the following holds:*

(a) *There exists at most one $\widetilde{P} \in \mathcal{P}_H(S)$ such that $\widetilde{P} \not\leq N_H(P^\circ)$ and $\langle P, \widetilde{P} \rangle \in \mathcal{L}_H(P)$. Moreover, if such $\widetilde{P}$ exists and $M_1 := \langle P, \widetilde{P} \rangle^\circ C_S(Y_P)$, then*

  (a_1) $M_1/C_{M_1}(Y_{M_1}) \cong SL_3(p^n)$, $Sp_4(p^n)$, *or* $Sp_4(2)'$ (*and* $p = 2$), *and*

  (a_2) $[Y_{M_1}, M_1]$ *is the corresponding natural module for* $M_1/C_{M_1}(Y_{M_1})$.

(b) *There exist at least two $\widetilde{P}_1, \widetilde{P}_2 \in \mathcal{P}_H(S)$ such that $\widetilde{P}_i \not\leq N_H(P^\circ)$ and $\langle P, \widetilde{P}_i \rangle \in \mathcal{L}_H(P)$, $i = 1, 2$. Moreover, for any such $\widetilde{P}_i$ and $M_i := \langle P, \widetilde{P}_i \rangle^\circ C_S(Y_P)$, $i = 1, 2$:*

  (b_1) $p = 3$ *or* 5 *and* $O^{p'}(M_1 \cap M_2) = P$,

  (b_2) $M_i/O_p(M_i) \cong SL_3(p)$,

  (b_3) $O_p(M_i)/Z(O_p(M_i))$ *and* $Z(O_p(M_i))$ *are natural* $SL_3(p)$*-modules for* $M_i/O_p(M_i)$ *dual to each other.*

The $\widetilde{P}$!-Theorem is a corollary of the following more general result on amalgams. By $3Sp_4(2)'$ we denote a non-split central extension of a group of order 3 by $Sp_4(2)'$, and by $3Sp_4(2)$ a group that has $3Sp_4(2)'$ as a subgroup of index 2 and $Sp_4(2)$ as a factor group.

Theorem 1. *Let G be group generated be two finite subgroups M_1 and M_2. Set $B := M_1 \cap M_2$ and $\overline{M}_i := M_i/C_{M_i}(Y_{M_i})$, and suppose that for $i = 1, 2$ the following hold:*

(1) $Syl_p(M_1) \cap Syl_p(M_2) = Syl_p(B)$, *and* M_1 *and* M_2 *are of characteristic* p.

(2) *No non-trivial normal subgroup of G is contained in B.*

(3) $\overline{M}_i \cong SL_3(q_i)$, $Sp_4(q_i)$, $q_i = p^{n_i}$, *or* $Sp_4(2)'$ (*and* $q_i = p = 2$).

(4) $[Y_{M_i}, O^p(M_i)]$ *is a natural module for* $\overline{M}_i$, *and* $Z(M_i) = 1$.

(5) *There exists a 2-dimensional singular subspace W in $[Y_{M_i}, O^p(M_i)]$ such that $O^{p'}(N_{\overline{M}_i}(W)) \leq \overline{B}$.*

(6) $C_{M_i}(Y_{M_i}) = O_p(M_i)$, *or* $q_i = 2$ *and*

$$M_i/O_2(M_i) \cong 3Sp_4(2) \text{ or } 3Sp_4(2)'.$$

Then one of the following holds for $i = 1, 2$*:*

(a) $p = 2$, $Y_{M_i} = O_2(M_i)$, $M_i/O_2(M_i) \cong Sp_4(2)'$ *or* $Sp_4(2)$, *and* $|Y_{M_i}| = 2^4$ *or* 2^5.

(b) $q := q_1 = q_2$, $p = 3$ *or* $q = 5$, $M_i/O_p(M_i) \cong SL_3(q)$, *and* $O_p(M_i)/Y_{M_i}$ *and* Y_{M_i} *are natural* $SL_3(q)$*-modules for* $M_i/O_p(M_i)$ *dual to each other.*

Since Theorem 1 does not depend on the hypothesis of the $\widetilde{P}$!-Theorem, it may also be useful in more general situations; for example, when the condition $Y_M \leq Q$ is not satisfied.

We also want to remark that Theorem 1 is in the same vein as the (much more general) main result of [ST]. Unfortunately, the hypotheses there are not compatible with the situation here, our Hypothesis (5) being the reason.

1 Elementary Properties

Throughout this section H is a finite group of local characteristic p satisfying Q-uniqueness (with the notation given in the introduction), and X is an arbitrary finite group.

1.1 *Let* X *be of characteristic* p, $S \in Syl_p(X)$ *and* $P \in \mathcal{P}_X(S)$. *Then the following hold:*

(a) $\Omega_1(Z(S)) \leq Y_X$.

(b) $X = N_X(S)\langle \mathcal{P}_X(S)\rangle$.

(c) *For every normal subgroup* N *of* P *either* $O^p(P) \leq N$ *or* $S \cap N \leq O_p(P)$.

(d) *For every normal subgroup* T *of* S *either* $O^p(P) = [O^p(P), T]$ *or* $T \leq O_p(P)$.

Proof. See (1.2) (c) and (1.3) (a), (b), (c) of [PPS]. □

1.2 *Let X be of characteristic p, $S \in Syl_p(X)$, $Y_X \leq N \trianglelefteq X$, and $V := \langle \Omega_1(Z(S))^X \rangle$. Then the following hold:*

(a) $Y_X \leq Y_N$.

(b) $Y_X = Y_N$ *if* $C_S(Y_X) \leq N$.

(c) $V = C_V(X)[V, O^p(X)]$ *and* $V \leq Y_X$.

Proof. Recall that Y_N is the unique maximal elementary abelian normal p-subgroup of N satisfying $O_p(N/C_N(Y_N)) = 1$. Hence, Y_N is characteristic in N and thus normal in X.

Set $\widetilde{X} := X/C_X(Y_X)$. Then $1 = O_p(\widetilde{N}) \cong O_p(N/C_N(Y_X))$, so $Y_X \leq Y_N$. This is (a).

Assume now that $C_S(Y_X) \leq N$, and let $D \leq S$ such that

$$DC_X(Y_N)/C_X(Y_N) = O_p(X/C_X(Y_N)).$$

Since $Y_X \leq Y_N$ we have $C_X(Y_N) \leq C_X(Y_X)$. Thus $DC_X(Y_X)/C_X(Y_X) \leq O_p(X/C_X(Y_X)) = 1$ and $D \leq C_S(Y_X) \leq N$. It follows that

$$DC_X(Y_N)/C_X(Y_N) \cong D/C_D(Y_N) \cong DC_N(Y_N)/C_N(Y_N) \leq O_p(N/C_N(Y_N)),$$

and thus $D \leq C_S(Y_N)$ and $O_p(X/C_X(Y_N)) = 1$. Hence $Y_N \leq Y_X$, and (a) implies (b).

A proof for the first part of (c) can be found in [CD], Lemma 2.5, the second part follows from 1.1 (a). □

1.3 *The following hold:*

(a) $Q = Q^h$ *and* $h \in \widetilde{C}$ *for every* $h \in H$ *with* $Q^h \leq \widetilde{C}$.

(b) *If T is a p-subgroup containing Q, then* $N_H(T) \leq \widetilde{C}$.

Proof. For (a) see (1.6) of [PPS]; (b) is a direct consequence of (a).

□

1.4 *Let $L \in \mathcal{L}_H(S)$. Then* $[C_L(Y_L), L^\circ] \leq O_p(L)$.

Proof. See 2.4.2 (dc) in [MSS]. □

1.5 *Let $L \in \mathcal{L}_H(S)$, $P \in \mathcal{P}_L(S)$ and $L_0 := L^\circ C_S(Y_L)$. Then one of the following holds:*

(*a*) $P \leq L_0 S$,

(*b*) $P \leq N_L(S \cap L_0) \cap N_L(L_0) \cap \widetilde{C}$,

(*c*) $[O^p(P), L^\circ] \leq O_p(L)$ *and* $P \leq \widetilde{C}$.

Proof. If $O^p(P) \leq C_L(Y_L)$, then 1.4 and 1.1 (a) give (c). Thus, we may assume that $O^p(P) \not\leq C_L(Y_L)$. Then 1.1 (c) yields $C_S(Y_L) \leq O_p(P)$, in particular $C_S(Y_L) = C_P(Y_L) \cap O_p(P)$. Hence, $C_S(Y_L)$ is normal in P, and $P \leq N_L(L_0)$. Now $S \cap L_0$ is normal in S, and 1.1 (d) shows that either $S \cap L_0 \leq O_p(P)$ or $O^p(P) = [O^p(P), S \cap L_0]$.

In the first case $S \cap L_0 = O_p(P) \cap L_0$, so $S \cap L_0$ is normal in P. As $Q \leq S \cap L_0$, 1.3 (b) implies $P \leq \widetilde{C}$, and (b) holds.

In the second case $O^p(P) \leq L_0$ since $P \leq N_L(L_0)$, and (a) holds. □

1.6 *Let* V *be an irreducible* $GF(p)X$*-module and* $F := End_X(V)$*. Then the direct sum* $V \oplus V$ *contains exactly* $|F| + 1$ $GF(p)X$*-submodules isomorphic to* V*.*

Proof. This follows from Theorem 3.5.6 in [Go]. □

Let $L = SL_2(q)$, $q = p^n$, and V be a natural $GF(p)L$-module. Set $F := End_L(V)$. Since $L \cong Sp_2(q)$ there exists an L-invariant non-degenerated symplectic form

$$s: \ V \times V \to F.$$

Let $\{v, w\}$ be a basis of the F-space V such that $s(v, w) = 1$, and set

$$\Omega_0 := \{(\mu(v), v) \mid \mu \in F\} \text{ and } \Omega_1 = \{(\mu(v), w) \mid \mu \in F^\sharp\}.$$

Note that $s(\mu(v), w) = \mu$, so $s(\Omega_1) \cup \{0\} = F$. Note further that $\Omega := \Omega_0 \cup \Omega_1 \cup \{(v, 0), (0, 0)\}$ is a set of representatives for the L-orbits on $V \times V$.

We consider F as a trivial $GF(p)L$-module.

1.7 *Let* C *be any trivial* $GF(p)L$*-module and* $\alpha: \ V \times V \to C$ *be* L*-invariant and* $GF(p)$*-bilinear. Then there exists a* $GF(p)L$*-module homomorphism* $\beta: \ F \to C$ *such that* $\beta(s(x, y)) = \alpha(x, y)$ *for all* $x, y \in V$*.*

Proof. For every $\mu \in F$ the pair $(\mu(v), v + w)$ is in the same L-orbit as $(\mu(v), w)$. It follows that

$$\alpha(\mu(v), w) = \alpha(\mu(v), v + w) = \alpha(\mu(v), v) + \alpha(\mu(v), w),$$

and $\alpha(\mu(v), v) = 0$. Moreover, the linearity of α gives $\alpha(v, 0) = \alpha(0, 0) = 0$.

We define

$$\beta: \ F = s(\Omega_1) \cup \{0\} \to C \text{ with } s(\mu(v), w) \mapsto \alpha(\mu(v), w) \text{ and } 0 \mapsto 0,$$

so $\beta s|_\Omega = \alpha|_\Omega$. As s and α are both L-invariant and Ω is a set of representatives for the L-orbits of $V \times V$, this shows that $\beta s = \alpha$. Moreover, it is easy to check that β is a $GF(p)L$-module homomorphism from F into C. □

2 Properties of $Sp_4(q)$ and $SL_3(q)$

In this section $G \in \{SL_3(q), Sp_4(q)\}$, where q is a power of the prime p, and V is a natural $GF(p)G$-module. We fix a 2-dimensional singular subspace W of V and we set $P := O^{p'}(N_G(W))$.

The next two Lemmata give properties of G we assume the reader to be familiar with:

2.1 *Let $G = SL_3(q)$ and $S \in Syl_p(P)$. Then $|P| = q^2|SL_2(q)|$, and the following hold:*

(a) *G is 2-transitive on the 1-dimensional and on the 2-dimensional subspaces of V.*

(b) *$[V, O_p(P)] = W$ and $C_G(W) = O_p(P)$.*

(c) *$P/O_p(P) \cong SL_2(q)$ and $Syl_p(P) \subseteq Syl_p(G)$.*

(d) *W and $O_p(P)$ are natural $SL_2(q)$-modules for $P/O_p(P)$.*

(e) *$|Z(S)| = |C_V(S)| = q$ and $Z(S) = C_G(W) \cap C_G(V/C_V(S))$.*

(f) *G is generated by three conjugates of $Z(S)$.*

(g) *If $A \leq S$ such that $|[V, A]| = q$ or $|V/C_V(A)| = q$, then A is conjugate to a subgroup of $Z(S)$.*

(h) *$N_G(W)$ is the unique maximal subgroup of G containing P.* □

2.2 *Let $G = Sp_4(q)$, $S \in Syl_p(P)$ and A be a minimal normal subgroup of $N_G(W)$. Then $|P| = q^3|SL_2(q)|$, and the following hold:*

(a) *$N_G(W)$ is a maximal subgroup of G.*

(b) $P/O_p(P) \cong SL_2(q)$, $Syl_p(P) \subseteq Syl_p(G)$, *and* $O_p(P)$ *is elementary abelian.*

(c) $[W, O_p(P)] = 0$, *and* $[v, O_p(P)] = W$ *for every* $v \in V \setminus W$.

(d) W *and* V/W *are natural* $SL_2(q)$*-modules for* $P/O_p(P)$.

(e) *Either*

(e_1) $A = O_p(P)$ *and* $p \neq 2$, *or*

(e_2) $p = 2$, $|A| = q$ *and* $O_2(P)/A$ *is a natural* $SL_2(q)$*-module for* $P/O_2(P)$, *or*

(e_3) $q = 2$, $|A| = 4$ *and* $[v, A] = W$ *for every* $v \in V \setminus W$.

(f) *There exists* $g \in G$ *with* $W \cap W^g = 0$. *Moreover,*

$$X \not\leq N_G(W^g) \text{ and } G = \langle P^g, X \rangle$$

for any such g *and for any* $1 \neq X \leq O_p(P)$.

(g) $A \not\leq O_p(C_G(x))$ *for every* $x \in W^\sharp$.

(h) $N_G(W)$ *is the unique maximal subgroup of* G *containing* P. □

2.3 *Let* $G = SL_3(q)$ *and* $g \in G \setminus N_G(W)$. *Then* $O_p(P) \cap O_p(P^g) = 1$, *and there exists* $S \in Syl_p(P)$ *such that* $O_p(P) \cap P^g = Z(S)$; *in particular* $|O_p(P)/O_p(P) \cap P^g| = q$.

Proof. Let $V_0 = W \cap W^g$. Then V_0 is 1-dimensional, and by 2.1 (a) and (e) there exists $S \in Syl_p(P)$ such that $V_0 = C_V(S)$. Hence 2.1 (b) and (e) give

$$Z(S) \leq O_p(P) \cap C_G(V/V_0) \leq O_p(P) \cap N_G(W^g),$$

so $Z(S) \leq O_p(P) \cap P^g$. On the other hand $O_p(P) \cap O_p(P^g) \leq C_G(WW^g) = 1$, so by 2.1 (c) and (e) $Z(S) = O_p(P) \cap P^g$. □

2.4 *Let* $G = Sp_4(q)$ *and* $B \leq O_p(P)$ *be such that* $|V/C_V(B)| = q$ *or* $|[V, B]| = q$. *Then the following hold:*

(a) $[V, A] = W$ *for every non-trivial normal subgroup* A *of* P *in* $O_p(P)$.

(b) $|V/C_V(B)| = |[V, B]| = q$ *and* $|B| \leq q$.

Proof. By 2.2 (d) P acts irreducible on W and by 2.2 (c) $[V, O_p(P)] = W$. This gives (a).

Note that $[V, B]$ is perpendicular to $C_V(B)$ with respect to the symplectic form on V. Hence, the dimension formula gives

$$dim\, V = dim\, C_V(B) + dim\, [V, B]$$

and the first part of (b). Then $|B/C_B(v)| \leq q$ for $v \in V \setminus C_V(B)$, and $V = C_V(B) + V_0$, where V_0 is the subspace generated by v. This gives $C_B(v) = 1$ and $|B| \leq q$. □

The next result follows from Theorem A in [CD]:

2.5 *Let Y be a faithful $GF(p)G$-module and $1 \neq A$ an elementary abelian p-subgroup of G. Suppose that $G = SL_3(q)$ and $|Y/C_Y(A)| \leq |A|$. Then $[Y, G]/C_{[Y,G]}(G)$ is a natural module or the direct sum of two isomorphic natural modules for G.* □

2.6 *Let M be a finite group of characteristic p with subgroups $Y \leq D \leq M$ such that D is a normal p-subgroup of M and $V_1 := \Phi(D) \leq \Omega_1 Z(O_p(M))$. Suppose that for $\widetilde{V} = D/Z(D)$ and $P_1 := O^{p'}(N_M(Y \cap V_1))$:*

(a) $M/O_p(M) = SL_3(q)$, and V_1 is a natural module for $M/O_p(M)$.

(b) $D = \langle Y^M \rangle$, $Y \not\leq Z(D)$ and $C_D(y) = C_D(Y)$ for all $y \in Y \setminus V_1$.

(c) $Y \cap V_1$ is 2-dimensional subspace of V_1, and $[Y, P_1] \leq Y \cap V_1$.

(d) $|\widetilde{V}/C_{\widetilde{V}}(T)| \leq |T/O_p(M)|$ where $T := O_p(C_M(z))$, $1 \neq z \in V_1$.

Then $\widetilde{V}$ is a natural $SL_3(q)$-module dual to V_1.

Proof. Set $\overline{M} := M/O_p(M)$, so $\overline{M} = SL_3(q)$ and by 2.1 (c)

$$\overline{P}_1/O_p(\overline{P}_1) \cong SL_2(q) \text{ and } Syl_p(P_1) \subseteq Syl_p(M).$$

By (b) and (c) $[D, O_p(M)] \leq V_1$ and $\widetilde{V} = [\widetilde{V}, \overline{M}]\widetilde{Y}$; in particular $\widetilde{V}$ is a $GF(p)\overline{M}$-module. As $\widetilde{Y}$ is centralized by a Sylow p-subgroup of $\overline{M}$, Gaschütz' Theorem implies that $\widetilde{V} = [\widetilde{V}, \overline{M}]C_{\widetilde{V}}(\overline{M})$.

Let $V_1 \leq U \leq D$ be such that U is M-invariant and $\widetilde{U} \neq 1$, and pick $y \in Y \setminus V_1$. By (c) $[y, U]$ is a P_1-submodule of V_1, so by (a) either $[y, U] = 1$ or

$[y,U] = Y \cap V_1$. In the first case (b) yields $U \leq C_D(Y)$ and then $U \leq Z(D)$, which contradicts $\widetilde{U} \neq 1$. Hence, we have

$$(*) \qquad [y,U] = Y \cap V_1;$$

in particular $C_{\widetilde{V}}(\overline{M}) = 1$ and $\widetilde{V} = [\widetilde{V}, \overline{M}]$.

Assume now, in addition, that $\widetilde{U}$ is an irreducible $\overline{M}$-module. We apply 2.5. Then $\widetilde{U}$ is a natural $SL_3(q)$-module for $\overline{M}$. Moreover, since $U/C_U(y)$ and $[y,U] = Y \cap V_1$ are isomorphic P_1-modules, the module $\widetilde{U}$ is dual to V_1. It remains to prove that $\widetilde{U} = \widetilde{D}$.

Assume that $\widetilde{D} \neq \widetilde{U}$. Then, again by 2.5, $\widetilde{V}$ is the direct sum of two natural modules isomorphic to $\widetilde{U}$. Thus $\widetilde{V}/C_{\widetilde{V}}(P_1)$ is the direct sum of two natural modules for $P_1/O_p(P_1)$. By 1.6 there are exactly $q+1$ $\overline{M}$-submodules $\widetilde{U}_0, \dots, \widetilde{U}_q$ isomorphic to $\widetilde{U}$ in $\widetilde{V}$ and also $q+1$ irreducible P_1-submodules in $\widetilde{V}/C_{\widetilde{V}}(P_1)$, so $\widetilde{U}_i C_{\widetilde{V}}(P_1)/C_{\widetilde{V}}(P_1)$, $i = 0, \dots, q$, are these P_1-submodules.

Let $D_y := C_D(y)$. As seen above D/D_y is a natural P_1-module. Hence, also $\widetilde{D}_y$ involves a natural P_1-module. Moreover, an application of the Three-Subgroups Lemma shows that $C_{\widetilde{V}}(P_1) \leq \widetilde{D}_y$. Thus, there exists $k \in \{0, \dots, q\}$ such that $\widetilde{D}_y = (\widetilde{U}_k \cap \widetilde{D}_y) C_{\widetilde{V}}(P_1)$. Then

$$\widetilde{U}_k = (\widetilde{U}_k \cap \widetilde{D}_y) C_{\widetilde{U}_k}(P_1) \leq \widetilde{D}_y,$$

which contradicts $(*)$. $\square$

2.7 *Let L be a finite group, p an odd prime, and D a normal p-subgroup of L such that $Z(D) = \Omega_1(Z(D)) =: V$, and let $\mathcal{A}$ be the set of all normal subgroups A of L satisfying $A \leq D$, $|A| = q^3$ and $C_D(A)' = A$. Suppose that*

(*i*) *$L/O_p(L) \cong SL_2(q)$, $q = p^n$,*

(*ii*) *V is a natural module and D/D' is the direct sum of two natural modules for $L/O_p(L)$,*

(*iii*) *$|\mathcal{A}| \geq 2$, $[D', L] \leq V$ and $|D'| = q^4$.*

Then one of the following holds:

(*a*) *$p \neq 3$ and $|\mathcal{A}| = 3$.*

(*b*) *$p = 3$ and $|\mathcal{A}| = q$.*

Proof. Set $R := D'$ and $\overline{D} := D/R$. Note that by (ii) any non-trivial proper L-invariant subgroup of $\overline{D}$ has order q^2. Also

$$(1) \qquad A = C_D(A)' \leq R \text{ for every} A \in \mathcal{A}.$$

By (ii) $[R, D, D] \leq [V, D] = 1$, so the Three Subgroups Lemma gives $[R, R] = 1$. Thus

$$(2) \qquad R \text{ is abelian.}$$

Let $1 \neq x \in A \setminus V$. By (1) and (iii) L normalizes $[x, D]$ and thus $[x, D] = V$ and $|D/C_D(x)| = q^2$. On the other hand, by (1) and (2)

$$R \leq C_D(A) \leq C_D(x) \text{ and } C_D(A) \neq R$$

since $A = C_D(A)'$. Now (ii) implies that

$$C_D(A) = C_D(x) \text{ for every } x \in A \setminus V.$$

According to (iii) there exists $B \in \mathcal{A}$ such that $A \neq B$. As $A = C_D(x)'$ for every $x \in A \setminus V$ and $B = C_D(x)'$ for every $x \in B \setminus V$, we get

$$(3) \qquad V = A \cap B,\ R = AB \text{ and } \overline{D} = \overline{C_D(A)} \times \overline{C_D(B)}.$$

Moreover $\overline{C_D(A)}$ and $\overline{C_D(B)}$ are natural $L/O_p(L)$-modules.

Let $K := End_L(V)$ and $\widetilde{R} := R/V$, so $\widetilde{R} = \widetilde{A} \times \widetilde{B}$. By (ii) $K \cong GF(q)$. From now on we will view V as a vector space over K; in particular addition replaces multiplication. Moreover, we write kv rather than $k(v)$ for $k \in K$ and $v \in V$.

There exist $GF(p)L$-isomorphisms

$$\phi_A:\ V \to \overline{C_D(A)} \text{ and } \phi_B:\ V \to \overline{C_D(B)}.$$

Let $r \in R$ and $x, y \in D$. Then $[\overline{x}, \overline{y}] := \widetilde{[x, y]}$ and $[\overline{x}, \widetilde{r}] := [x, r]$ are well-defined by (2). Let $z \in A$. Since $[\widetilde{z}, L] = 1$, the map

$$\mu_A^*(\widetilde{z}):\ V \to V \text{ with } v \mapsto [\phi_B(v), \widetilde{z}]$$

is L-invariant. Thus $\mu_A^*(\widetilde{z}) \in K$.

If $\mu_A^*(\widetilde{z}) = 0$, then $[z, C_D(B)] = 1$ and thus by (3) $z \leq Z(V)$, so $\widetilde{z} = 1$. Since

$$\mu_A^*:\ \widetilde{A} \to K, \widetilde{z} \mapsto \mu_A^*(\widetilde{z})$$

is a group homomorphism, we conclude that it is injective. Hence, as both $\widetilde{A}$ and K have order q, μ_A^* is an isomorphism. Let μ_A be its inverse. Similarly define μ_B.

Let $k, h \in K$ and $v, w \in V$. Then

$$kv = [\phi_B(v), \mu_A(k)] = [\phi_A(v), \mu_B(k)] \tag{4}$$

and

$$[\phi_A(v)\phi_B(w), \mu_A(k)\mu_B(h)] = kw + hv, \tag{5}$$

(recall that we write V additively).

For every $z \in R \setminus A$ there exist $k \in K$ and $h \in K^\sharp$ such that $\widetilde{z} = \mu_A(k)\mu_B(h)$. Let $z_{(h,k)} \in R$ be such that $\widetilde{z}_{(h,k)} = \mu_A(-kh^{-1})\mu_B(-1)$. Then (2) and (5) yield

$$C_D(z) = C_D(z_{(h,k)}).$$

Hence

$$\{C_D(z) \mid z \in R \setminus V\} = \{C_D(A)\} \cup \{C_D(\mu_A(t)\mu_B(-1)) \mid t \in K\}.$$

For $t \in K$ set $D_t := C_D(\mu_A(t)\mu_B(-1))$ and $B_t := Z(D_t)$. From (5) we obtain:

$$\overline{D}_t = \{\phi_A(tv)\phi_B(v) \mid v \in V\} \text{ and } \widetilde{B}_t = \{\mu_A(tk)\mu_B(-k) \mid k \in K\}. \tag{6}$$

In the following we determine for which $t \in K$ actually $B_t \in \mathcal{A}$. As $B_0 = B \in \mathcal{A}$ we can assume that $t \neq 0$.

We investigate the L-invariant $GF(p)$-bilinear form κ on $\overline{D}$ induced by the commutator mapping

$$\kappa : \overline{D} \times \overline{D} \to \widetilde{R} \text{ with } (\overline{x}, \overline{y}) \mapsto [\overline{x}, \overline{y}] \text{ for all } x, y \in D.$$

As $\widetilde{R} = \widetilde{A} \times \widetilde{B}$, the projection mappings

$$\pi_A : \widetilde{R} \to \widetilde{A} \text{ and } \pi_B : \widetilde{R} \to \widetilde{B}$$

give rise to L-invariant $GF(p)$-bilinear forms $\kappa\pi_A$ and $\kappa\pi_B$ satisfying

$$[\overline{x}, \overline{y}] = [\overline{x}, \overline{y}]\pi_A[\overline{x}, \overline{y}]\pi_B,$$

so $\kappa\pi_A\mu_A^*$ and $\kappa\pi_B\mu_B^*$ are $GF(p)$-bilinear forms with values in K.

We now restrict κ to $\overline{C_D(A)} \times \overline{C_D(A)}$, $\overline{C_D(B)} \times \overline{C_D(B)}$ and $\overline{C_D(A)} \times \overline{C_D(B)}$, respectively. Let s be an L-invariant non-degenerate symplectic form on V. According to 1.7 there exist $GF(p)L$-module homomorphisms

$$\lambda_A, \lambda_B, \rho_A, \rho_B : K \to K$$

such that for all $v, w \in V$

$$[\phi_A(v), \phi_A(w)] = \mu_A(\lambda_A(s(v,w))), \tag{7}$$

$$[\phi_B(v), \phi_B(w)] = \mu_B(\lambda_B(s(v,w))), \tag{8}$$

$$\begin{aligned}[\phi_A(v), \phi_B(w)] &= [\phi_A(v), \phi_B(w)]\pi_A[\phi_A(v), \phi_B(w)]\pi_B \\ &= \mu_A(\rho_A(s(v,w)))\mu_B(\rho_B(s(v,w))).\end{aligned} \tag{9}$$

It follows that

$$\begin{aligned}[\phi_B(v), \phi_A(w)] &= [\phi_A(w), \phi_B(v)]^{-1} \\ &= (\mu_A(\rho_A(s(w,v)))\mu_B(\rho_B(s(w,v))))^{-1} \\ &= (\mu_A(\rho_A(s(w,v))))^{-1}(\mu_B(\rho_B(s(w,v))))^{-1} \\ &= \mu_A(\rho_A(-s(w,v)))\mu_B(\rho_B(-s(w,v))) \\ &= \mu_A(\rho_A(s(v,w)))\mu_B(\rho_B(s(v,w))).\end{aligned} \tag{10}$$

Let $x, y, z \in D$. The Jacobi identity

$$[x, y^{-1}, z]^y [y, z^{-1}, x]^z [z, x^{-1}, y]^x = 1$$

yields

$$[\overline{y}, \overline{x}, \overline{z}] + [\overline{z}, \overline{y}, \overline{x}] + [\overline{x}, \overline{z}, \overline{y}] = 0 \tag{11}$$

since $V \leq Z(D)$ and $\widetilde{R} \leq Z(\widetilde{D})$.

Let $v, w \in V$ and $k \in K^\sharp$. We will compute all three triple commutators for $\overline{x} = \phi_A(v)$, $\overline{y} = \phi_A(kw)$ and $\overline{z} = \phi_B(w)$. From (9) we get

$$\begin{aligned}[\phi_A(kw), \phi_B(w)] &= \mu_A(\rho_A(s(kw,w)))\mu_B(\rho_B(s(kw,w))) \\ &= \mu_A(\rho_A(0))\mu_B(\rho_B(0)) = 1.\end{aligned}$$

Thus, (5), (7), (9) and (10) give

$$\begin{aligned}[\phi_A(v), \phi_A(kw), \phi_B(w)] &= [\mu_A(\lambda_A(s(v,kw))), \phi_B(w)] \\ &= -\lambda_A(s(v,kw))w = \lambda_A(s(kw,v))w, \\ [\phi_B(w), \phi_A(v), \phi_A(kw)] &= [\mu_A(\rho_A(s(w,v)))\mu_B(\rho_B(s(w,v))), \phi_A(kw)] \\ &= -\rho_B(s(w,v))kw, \\ [\phi_A(kw), \phi_B(w), \phi_A(v)] &= 0.\end{aligned}$$

Now (11) yields:

(12) $\lambda_A(s(kw,v))w = \rho_B(s(w,v))kw$ for every $v,w \in V$ and $k \in K$.

Choosing $v,w \in V$ such that $s(w,v) = k_2$ one gets from (12)

$$\lambda_A(k_1k_2) = \rho_B(k_2)k_1 \text{ for all } k_1,k_2 \in K.$$

A similar argument (with the roles of A and B reversed) gives

$$\lambda_B(k_1k_2) = \rho_A(k_2)k_1 \text{ for all } k_1,k_2 \in K.$$

In particular, with $\ell_A := \lambda_A(1)$ and $\ell_B := \lambda_B(1)$ the cases $k_1 = 1$ or $k_2 = 1$, respectively, yield:

(13) $\rho_B(k) = \lambda_A(k) = \ell_A k$ and $\rho_A(k) = \lambda_B(k) = \ell_B k$ for all $k \in K$.

We now determine for which t the subgroup B_t is in $\mathcal{A}$. Let $v,w \in V$ and $t \in K^\#$. Put $k := s(v,w)$ and recall that μ_A, μ_B are homomorphisms from the additive group of K into the multiplicative group $\widetilde{A}$ or $\widetilde{B}$, respectively. It follows from (7) – (10) and (13) that

$$\begin{aligned}
&[\phi_A(tv)\phi_B(v), \phi_A(tw)\phi_B(w)] = \\
&[\phi_A(tv), \phi_A(tw)][\phi_A(tv), \phi_B(w)][\phi_B(v), \phi_A(tw)][\phi_B(v), \phi_B(w)] = \\
&\mu_A(\lambda_A(s(tv,tw)))\mu_A(\rho_A(s(tv,w)))\mu_B(\rho_B(s(tv,w)))\mu_A(\rho_A((s(v,tw))) \\
&\mu_B(\rho_B(s(v,tw)))\mu_B(\lambda_B(s(v,w))) \\
&= \mu_A(\lambda_A(t^2k) + 2\rho_A(tk))\mu_B(2\rho_B(tk) + \lambda_B(k)) \\
&= \mu_A(k(t^2\ell_A + 2t\ell_B))\mu_B(k(2t\ell_A + \ell_B)).
\end{aligned}$$

For $\ell := \frac{\ell_B}{\ell_A}$ we get

(14) $$|\{[\phi_A(tv)\phi_B(v), \phi_A(tw)\phi_B(w)] \mid v,w \in V\}| = q \iff t \notin \{-2\ell, -\tfrac{1}{2}\ell\}.$$

By (6)

(15) $$D'_t \leq B_t \iff t^2\ell_A + 2t\ell_B = (-t)(\ell_B + 2t\ell_A).$$

The equation on the right hand is equivalent to

(16) $$3t = -3\ell,$$

since we are assuming that $t \neq 0$. Hence, according to (14)

$$D'_t = B_t \iff 3t = -3\ell \text{ and } t \notin \{-2\ell, -\frac{1}{2}\ell\}.$$

If $p \neq 3$, then (16) has a unique solution, namely $t = -\ell$, so $\mathcal{A} = \{A, B, B_{-\ell}\}$ and (a) holds.

Assume now that $p = 3$. Then every $t \in K^\sharp$ satisfies (16), and every $t \neq \ell$ satisfies $t \notin \{-2\ell, -\frac{1}{2}\ell\}$. Hence, there are $q-2$ elements in $\mathcal{A}$ different from A and B, and (b) holds. □

2.8 *Let L be as in 2.7. Suppose in addition that L is a subgroup of the finite group M, and*

(*i*) $M/O_p(M) \cong SL_3(q)$

(*ii*) *$Z(O_p(M))$ and $O_p(M)/Z(O_p(M))$ are natural $SL_3(q)$-modules dual to each other.*

Then $q = 5$ or $p = 3$.

Proof. We may assume that $p \neq 3$. Then by 2.7 $|\mathcal{A}| = 3$. A comparison of orders shows that L contains a Sylow p-subgroup of M and $D = O_p(L)$ (D as in 2.7). Hence there exists $L \leq R \leq M$ such that $L = O^{p'}(R)$ and $R/O_p(M)$ is a maximal parabolic subgroup of $M/O_p(M)$.

Let $Y = Z(O_p(M))$. By our hypothesis $O_p(M) = C_D(Y)$ and $O_p(M)' = Y$, so $Y \in \mathcal{A}$. Let A and B be the other two elements of $\mathcal{A}$, and let R_0 be the unique subgroup of index 2 in R with $L \leq R_0$, so $K := R_0/L$ is cyclic of order $\frac{q-1}{2}$. Since R normalizes Y and acts on $\mathcal{A}$, R_0 normalizes A and B.

Let $\widetilde{D} := D/Z(D)$. Then $\widetilde{Y}$, $\widetilde{A}$, and $\widetilde{B}$ are three different $GF(p)K$-submodules of order q in $\widetilde{D}'$, so the $GF(p)K$-modules $W_1 := \widetilde{Y}$ and $W_2 := \widetilde{D}'/\widetilde{Y}$ are isomorphic.

Let $F_i = End_K(W_i)$, $i = 1, 2$, and note that K is embedded in F_i since K is abelian. Then there exists a field isomorphism

$$\pi : F_2 \to F_1 \text{ with } \pi|_K = id.$$

On the other hand, by our hypothesis W_2 is a submodule of the module $O_p(M)/Y$, which is dual to Y. Thus, there exists a field isomorphism

$$\mu : F_1 \to F_2 \text{ with } k \mapsto k^{-1} \text{ for } k \in K.$$

Hence, $\sigma := \mu\pi$ is an automorphism of F_1 that inverts the elements of K.

As every automorphism of F_1 is a power of the Frobenius automorphism $x \mapsto x^p$, there exists $0 \leq m < n$ (where $q = p^n$) such that $f^\sigma = f^{p^m}$ for $f \in F_1$ and

$$k^\sigma = k^{-1} = k^{p^m} \text{ for } k \in K.$$

It follows that $k^{p^m+1} = 1$ and thus

$$|K| = \frac{q-1}{2} \mid p^m + 1.$$

This shows that

$$q - 1 = p^n - 1 \leq 2p^m + 2 \text{ and } p^m(p^{n-m} - 2) \leq 3,$$

so $m = 0$ and $q = 5$. $\square$

2.9 *Let G be a finite group, and let L, M_1, and M_2 be subgroups of G such that $L \leq M_1 \cap M_2$. Suppose that the following hold:*

(*a*) *L satisfies* 2.7.

(*b*) *M_1 and M_2 satisfy* 2.8 *in place of M.*

(*c*) *$O_p(M_1) \neq O_p(M_2)$, and $C_G(O_p(M_i)) \leq O_p(M_i)$, $i = 1, 2$.*

(*d*) *$C_G(L/D)/D$ is a p'-group (D as in* 2.7*).*

Then $q = 3$ or 5.

Proof. Let $T \in Syl_p(L)$ and set

$$U := N_G(L) \cap N_G(T),\ U_0 := U \cap L,\ U_i := U \cap M_i.$$

Moreover, let $\mathcal{A}$ and V be as in 2.7. We use a similar approach as in 2.8 getting $T \in Syl_p(M_i)$, $D = O_p(M_1)O_p(M_2)$, $Z(O_p(M_i)) \in \mathcal{A}$, and $Z(O_p(M_1))Z(O_p(M_2)) = D'$.

Observe that $U_i/T \cong C_{q-1} \times C_{q-1}$ and that U_1 and U_2 centralize U_0/T. Note further that $N_G(L)/C_G(L/D)$ is a subgroup of $Aut(L_2(q))$. Thus, the structure of $Aut(L_2(q))$ together with the hypothesis on $C_G(L/D)$ shows that $\langle U_1, U_2\rangle/T$ is a p'-group. Now the Theorem of Schur-Zassenhaus gives a complement X for T in $\langle U_1, U_2\rangle$.

According to 2.7 and 2.8 we may assume that $p = 3$ and $|\mathcal{A}| = q$. Set $\widetilde{D} = D/D'$. By 1.6 there are exactly $q + 1$ natural L/D-submodules in $\widetilde{D}$. On the other hand, for every $A \in \mathcal{A}$ the factor group $C_D(A)/D'$ is one of these submodules. Hence there exists a unique $D' \leq W_0 \leq D$ such that $\widetilde{W_0}$ is a natural L/D-module and $W_0' \notin \mathcal{A}$. In particular X normalizes W_0.

Assume first that $|X| = (q-1)^2$. Then $U_1 = U_2$ and, similar as in 2.8, the $GF(p)U_1$-modules $Z(O_p(M_1))/V$ and $D'/Z(O_p(M_1))$ are isomorphic and dual to each other with $C_{U_1}(Z(M_1)/V) = U_0$. As $|U_1/U_0| = q-1$, an argument as in 2.8 gives

$$q - 1 = p^n - 1 \mid p^k + 1 \text{ for some } 0 \leq k < n,$$

and $n = 1$, $k = 0$ and $q = 3$.

Assume now that $|X| > (q-1)^2$. Since $XL/C_{XL}(\widetilde{W_0}) \cong GL_2(q)$ we conclude that $C_X(\widetilde{W_0}) \neq 1$ and $X' \leq C_X(\widetilde{W_0})$. On the other hand

$$C_X(\widetilde{D}) = C_X(D) \leq C_X(O_p(M_1)) = 1$$

since X is a p'-group. It follows that also $W_1 := [D, C_X(\widetilde{W_0})]D' \neq D'$, so $\widetilde{W_1}$ is also a natural L/D-module. In particular $A := W_1' \in \mathcal{A}$.

Now with the same argument as above $C_X(\widetilde{W_1}) \neq 1$ and $X' \leq C_X(\widetilde{W_1})$, so

$$X' \leq C_X(\widetilde{W_0}) \cap C_X(\widetilde{W_1}) = C_X(\widetilde{D}) \leq C_X(O_p(M_1)) = 1,$$

and X is abelian.

If $A = Z(O_p(M_1))$, then $W_1 = O_p(M_1)$ and

$$C_X(\widetilde{W_1}) = C_X(O_p(M_1)) = 1,$$

a contradiction. With the same argument $A \neq Z(O_p(M_2))$, so there are three X-submodules of order q in D'/V, namely A/V, $Z(O_p(M_1))/V$, and $Z(O_p(M_2))/V$. Hence, these submodules are isomorphic.

If X normalizes $Z(O_p(M_2))$, then $Z(O_p(M_1))/V$ and $D'/Z(O_p(M_1))$ are isomorphic $GF(p)U_1$-modules and dual to each other, so as above $q = 3$. Thus we may assume that there exists $x \in X$ such that $Z(O_p(M_2))^x \neq Z(O_p(M_2))$. Now $Z(O_p(M_2))/V$ and $D'/Z(O_p(M_2))^x$ are isomorphic and dual to each other as $GF(p)U_2$-modules, and again $q = 3$. □

3 The Local $\widetilde{P}!$-Theorem

In this section we assume that H is a $\mathcal{K}_p$-group of local characteristic p, which satisfies Q-uniqueness. Moreover, we assume that there exists $P \in \mathcal{P}_H(S)$ such that $P \not\leq \widetilde{C}$ and $Y_M \leq Q$ for every $M \in \mathcal{M}_H(P)$.

Notation. For $L \in \mathcal{L}_H(S)$ set $L_0 := L^\circ C_S(Y_L)$.

We first collect two results from [PPS]:

3.1 *$P_0/O_p(P_0) \cong SL_2(p^m)$, and Y_P is a natural $SL_2(p^m)$-module for $P_0/O_p(P_0)$.*

Proof. This follows from the $P!$-Theorem in [PPS]. □

3.2 *Let $L \in \mathcal{L}_H(P)$ and $\overline{L} := L/C_L(Y_L)$. Then the following hold:*

(a) *$F^*(\overline{L}) = [\overline{L}_0, \overline{L}_0]$, $\overline{L}_0 \cong SL_n(p^m)$, $Sp_{2n}(p^m)$ or $Sp_4(2)'$ (and $p = 2$),*

(b) *$[Y_L, L_0]$ is the corresponding natural module, and $Z(L) = 1$.*

(c) *Y_P is a 2-dimensional (singular) subspace of $[Y_L, L_0]$.*

(d) *Either $C_{L_0}(Y_L) = O_p(L_0)$, or $p = 2$ and $L_0/O_2(L_0) \cong 3Sp_4(2)'$.*

(e) *$P_0 \leq L_0$ and $S \cap P_0 \in Syl_p(L_0)$, or $\overline{L}_0 \cong Sp_4(2)'$ and $\overline{L}_0\overline{C_S(Y_P)} \cong Sp_4(2)$.*

Proof. Claims (a) – (d) follow from the Corollary in [PPS]. For the proof of (e) set $P_1 := P_0 \cap L_0$ and $S_0 := S \cap L_0$. Since $P^\circ \leq P_1$

$$P^\circ C_S(Y_P) = P_0 = P_1 C_S(Y_P).$$

From (a), (b) and (c) we get that $|S_0/C_{S_0}(Y_P)| = p^m$. On the other hand, $C_{S_0}(Y_P) = C_S(Y_P) \cap P_1$ and thus

$$P_1/C_{S_0}(Y_P) \cong P_0/C_S(Y_P) \overset{3.1}{\cong} SL_2(p^m),$$

so $S_0 \leq P_1$.

If $C_S(Y_P) = C_{S_0}(Y_P)$, then the first possibility of (e) holds. Thus we may assume that there exists $t \in C_S(Y_P) \setminus S_0$; in particular $\overline{t} \notin \overline{L}_0$. This element induces an automorphism in $\overline{L}_0$ that centralizes a 2-dimensional singular subspace of $[Y_L, L_0]$.

An inspection of the automorphisms of $SL_3(p^m)$ and $Sp_4(p^m)$ having this property shows that either $\overline{t} \in \overline{L}_0$, which is not the case, or $\overline{L}_0 \cong Sp_4(2)'$ and the second possibilty of (e) holds. □

3.3 *$N_H(Y_P)^\circ = P^\circ$ and P° is normal in $N_H(Y_P)$.*

Proof. By 3.1, P is acts transitively on Y_P, so the claim follows from (2.4.2)(dd) in [MSS]. □

3.4 *Let $L \in \mathcal{L}_H(P)$ and $S_1 := L \cap S$. Then*

$$N_L(S_1) \le N_L(Y_P) \le N_L(P^\circ).$$

In particular, for every $\widetilde{P} \in \mathcal{P}_L(S)$ either $\widetilde{P} \le N_L(P^\circ)$ or $\widetilde{P} \le L_0 S$.

Proof. The structure of $\overline{L}_0$ and $[Y_L, L_0]$ described in 3.2 shows that Y_P is the unique S_1-invariant 2-dimensional singular subspace of $[Y_L, L_0]$. Together with 3.3 we get that $N_L(S_1) \le N_L(Y_P) \le N_L(P^\circ)$. Now 1.5 yields the second part of the claim. □

3.5 Local $\widetilde{P}$! -Theorem. *Let $L \in \mathcal{L}_H(P)$ be such that $P^\circ \ne L^\circ$. Then there exists a unique $\widetilde{P} \in \mathcal{P}_L(S)$ such that $\widetilde{P} \not\le N_L(P^\circ)$. Moreover, for $U := \langle P, \widetilde{P} \rangle$, $U^* := U^\circ C_S(Y_P)$ and $\overline{U}^* := U^*/C_{U^*}(Y_U)$ the following hold:*

(*a*) *$U = U^* S$ and $\overline{U}^* \cong SL_3(p^m)$, $Sp_4(p^m)$ or $Sp_4(2)'$ (and $p = 2$).*

(*b*) *$Z(U^*) = 1$, and $[Y_U, U^*]$ is a corresponding natural module for $\overline{U}^*$.*

(*c*) *Y_P is a 2-dimensional singular subspace of $[Y_U, U^*]$.*

(*d*) *$C_{U^*}(Y_U) = O_p(U^*)$ or $p = 2$ and $U^*/O_2(U^*) \cong 3Sp_4(2)'$ or $3Sp_4(2)$.*

(*e*) *$P_0 \le U^*$ and $S \cap P_0 \in Syl_p(U^*)$.*

Proof. Set $S_1 := S \cap L_0$. It is evident that $P^\circ \le L^\circ$. Thus, if $L_0 \le P$, then $L^\circ = P^\circ$, which contradicts the hypothesis. Hence, we have $L_0 \not\le P$, and the structure of $\overline{L}_0$ and $[Y_L, L_0]$ described in 3.2 shows:

($*$) There exists a unique minimal parabolic subgroup $R \in \mathcal{P}_{L_0}(S_1)$ such that $R \not\le N_{L_0}(Y_P)$.

Pick $\widetilde{P} \in \mathcal{P}_L(S)$ with $\widetilde{P} \not\le N_L(P^\circ)$. We first prove the uniqueness of $\widetilde{P}$. Let X be the unique maximal subgroup of $\widetilde{P}$ containing S, and set $\widetilde{P}_1 := \widetilde{P} \cap L_0$. By 3.4 and 1.1 (b) $\widetilde{P} = S\widetilde{P}_1$,

$$\widetilde{P}_1 = N_{\widetilde{P}_1}(S_1)\langle \hat{P} \mid \hat{P} \in \mathcal{P}_{\widetilde{P}_1}(S_1)\rangle \text{ and } N_{\widetilde{P}}(S_1) \le N_{\widetilde{P}}(P^\circ) \le X.$$

Hence, there exists $U \in \mathcal{P}_{\widetilde{P}_1}(S_1)$ such that $U \not\le X$, and ($*$) implies $U = R$, consequently $RS = \widetilde{P}$. This shows that $\widetilde{P}$ is uniquely determined.

We now apply 3.2 with U and $\overline{U} = U/C_U(Y_U)$ in place of L and $\overline{L}$. Note that by 3.2 (e) $C_S(Y_U) \le U_0 \le U^*$ and that U_0 and U^* are both normal in U. Hence 1.2 yields $Y_U = Y_{U_0} = Y_{U^*}$. Note further that by 3.2 (e) either $U_0 = U^*$ or $\overline{U}_0 \cong Sp_4(2)'$ and $\overline{U}^* \cong Sp_4(2)$. Then the claims (a) – (e) follow from the corresponding claims in 3.2 since $\overline{U}$ has Lie rank 2. □

3.6 *Suppose that there exist two different $\widetilde{P}_1, \widetilde{P}_2 \in \mathcal{P}_H(S)$ such that*

$$\widetilde{P}_i \not\leq N_H(P^\circ) \text{ and } O_p(\langle \widetilde{P}_i, P\rangle) \neq 1 \quad (i = 1, 2).$$

Set $M_i := \langle P, \widetilde{P}_i\rangle^\circ C_S(Y_P)$ and $G := \langle M_1, M_2\rangle$. Then G together with M_1 and M_2 satisfies the hypothesis of Theorem 1.

Proof. Note that 3.5 applies to M_i in place of U^*, for $i = 1$ and 2. The Hypotheses (1), (3), (4), and (6) of Theorem 1 follow from (2.5) (e), (a), (b), and (d), respectively.

Assume that there exists $1 \neq N \trianglelefteq G$ such that $N \leq B$. Since B has characteristic p, we get $O_p(N) \neq 1$ and thus $O_p(G) \neq 1$. Since S normalizes G we get that $GS \in \mathcal{L}_H(P)$, and 3.5 applies to GS.

In particular, there exists a unique $\widetilde{P} \in \mathcal{P}_{GS}(S)$ not normalizing P°. This contradicts $\widetilde{P}_1 \neq \widetilde{P}_2$, and this contradiction shows Hypothesis (2) of Theorem 1.

By 3.5 (e) $P_0 \leq M_1 \cap M_2$. Hence 3.5 (a), (c) and (e) give Hypothesis (5) of Theorem 1.

4 The Coset Graph

In this section we assume that the Hypothesis of Theorem 1 holds. We will apply the amalgam method to the group $G := \langle M_1, M_2\rangle$ and the pair of subgroups M_1 and M_2 using the standard notation (see for example [PPS] or [ST]). For the convenience of the reader we will repeat some of the notation.

Let Γ be the coset graph of G with respect to the subgroups M_1 and M_2; so the vertices are the right cosets of M_1 and M_2 in G, and two vertices are adjacent if and only if they are different and have non-empty intersection. Then G acts by right multiplication on Γ, and the vertex stabilizers in G are conjugate to M_1 or M_2, while the edge stabilizers are conjugate to $M_1 \cap M_2$. By $d(\, , \,)$ we denote the usual distance metric on Γ.

For a finite group L define $Z_L := \langle \Omega_1(Z(T)) \mid T \in Syl_p(L)\rangle$.

For every $\delta \in \Gamma$ define

$$\begin{array}{llll} \Delta(\delta) & := \{\lambda \mid d(\delta, \lambda) = 1\}, & G_\delta & := \{g \in G \mid \delta^g = \delta\}, \\ Y_\delta & := Y_{G_\delta}, & Z_\delta & := Z_{G_\delta}, \\ Q_\delta & := O_p(G_\delta), & V_\delta^{(i)} & := \langle Z_\lambda \mid d(\lambda, \delta) = i\rangle, \\ V_\delta & := V_\delta^{(1)}, & D_\delta & := C_{Q_\delta}(V_\delta), \\ \overline{G}_\delta & := G_\delta / C_{G_\delta}(Z_\delta), & & \end{array}$$

and

$$b := min\{d(\delta,\lambda) \mid \delta,\lambda \in \Gamma \text{ and } Z_\delta \not\leq Q_\lambda\}.$$

For adjacent vertices α and β define

$$G^*_{\alpha\beta} := O^{p'}(G_\alpha \cap G_\beta), \quad Q_{\alpha\beta} := O_p(G^*_{\alpha\beta}), \quad Z_{\alpha\beta} := Z_{G^*_{\alpha\beta}}.$$

A pair (α,α') of vertices is called **critical** if $Z_\alpha \not\leq Q_{\alpha'}$ and $d(\alpha,\alpha') = b$. If (α,α') is a critical pair, then we fix a path of length b from α to α' and denote its vertices by

$$(\alpha,\alpha+1,...,\alpha+b) \text{ or } (\alpha'-b,,...,\alpha'-1,\alpha'),$$

so $\alpha+i = \alpha'-(b-i)$.

The first lemma mostly phrases the hypothesis of Theorem 1 in terms of this new setup and notation.

4.1 *Let α and β be adjacent vertices in Γ. Then the following hold:*

(*a*) *G_α is of characteristic p, and $Syl_p(G^*_{\alpha\beta}) = Syl_p(G_\alpha) \cap Syl_p(G_\beta)$.*

(*b*) *No non-trivial subgroup of $G^*_{\alpha\beta}$ is normal in both G_α and G_β.*

(*c*) *$\overline{G}_\alpha \cong SL_3(q_\alpha)$, $Sp_4(q_\alpha)$ or $Sp_4(2)'$ (and $q_\alpha = 2$).*

(*d*) *$Z_\alpha = [Y_\alpha, O^p(G_\alpha)]$ is a natural module for $\overline{G}_\alpha$ and $Z(G_\alpha) = 1$.*

(*e*) *$Z_{\alpha\beta} = Z_\alpha \cap Z_\beta$, $|Z_{\alpha\beta}| = q_\alpha^2 = q_\beta^2$, and $O^{p'}(N_{\overline{G}_\alpha}(Z_{\alpha\beta})) = \overline{G}^*_{\alpha\beta}$.*

(*f*) *$q := q_\alpha = q_\beta$.*

(*g*) *$C_{G_\alpha}(Z_\alpha) = Q_\alpha$, or $p = q_\alpha = 2$ and $G_\alpha/Q_\alpha \cong 3Sp_4(2)$ or $3Sp_4(2)'$.*

(*h*) *$N_{G_\alpha}(Z_{\alpha\beta})$ is the unique maximal subgroup of G_α containing $G^*_{\alpha\beta}$.*

Proof. The pair $\{G_\alpha, G_\beta\}$ is conjugate to $\{M_1, M_2\}$, so the hypothesis of Theorem 1 applies to both G_α and G_β.

From 1.2 (c) we get that $Z_\alpha = [Z_\alpha, O^p(G_\alpha)]\Omega_1(Z(G_\alpha))$ and $Z_\alpha \leq Y_\alpha$. Now Hypothesis (4) of Theorem 1 yields $Z(G_\alpha) = 1$ and $Z_\alpha = [Y_\alpha, O^p(G_\alpha)]$. In particular, $C_{G_\alpha}(Y_\alpha) \leq C_{G_\alpha}(Z_\alpha)$, and the action of G_α on $[Y_\alpha, O^p(G_\alpha)]$ given in Hypothesis (4) of Theorem 1 shows that $C_{G_\alpha}(Y_\alpha) = C_{G_\alpha}(Z_\alpha)$.

With these remarks in mind the statements (a), (b), (c), (d) and (g) follow from the Hypotheses (1), (2), (3), (4) and (6) of Theorem 1.

By Hypothesis (5) of Theorem 1 there exists a 2-dimensional singular subspace $W_\alpha \le Z_\alpha$ with $O^{p'}(N_{\overline{G}_\alpha}(W_\alpha)) \le \overline{G_\alpha \cap G_\beta}$. By 2.1 (d) or 2.2 (d) $N_{G_\alpha}(W_\alpha)$ acts irreducibly on W_α, so

$$W_\alpha \le Z_{\alpha\beta} \le Z_\alpha \cap Z_\beta.$$

With a symmetric argument there also exists a 2-dimensional singular subspace $W_\beta \le Z_\beta$ such that $O^{p'}(N_{\overline{G}_\beta}(W_\beta)) \le \overline{G_\alpha \cap G_\beta}$ and

$$W_\beta \le Z_{\alpha\beta} \le Z_\alpha \cap Z_\beta.$$

Suppose that $W_\alpha \ne Z_\alpha \cap Z_\beta$. Let $T \in Syl_p(G^*_{\alpha\beta})$. If $\overline{G}_\alpha \cong SL_3(q_\alpha)$, then clearly $C_T(Z_\alpha \cap Z_\beta) = Q_\alpha$ since Z_α is 3-dimensional. If $G_\alpha \cong Sp_4(q_\alpha)$ or $Sp_4(2)'$, then $C_T(Z_\alpha \cap Z_\beta) = Q_\alpha$ follows from 2.2 (d). Thus $C_T(Z_\alpha \cap Z_\beta) = Q_\alpha$ holds in both cases. Now (b) implies that $C_T(Z_\alpha \cap Z_\beta) \ne Q_\beta$. Consequently, the above argument shows that $W_\beta = Z_\alpha \cap Z_\beta$. But now the irreducibility of W_α as a $G^*_{\alpha\beta}$-module yields $W_\alpha = W_\beta = Z_\alpha \cap Z_\beta$, a contradiction.

We have shown that $W_\alpha = Z_\alpha \cap Z_\beta$. Then with a symmetric argument also $W_\beta = Z_\alpha \cap Z_\beta$ and thus

$$W_\alpha = Z_{\alpha\beta} = W_\beta,$$

in particular $q_\alpha = q_\beta$ and $G^*_{\alpha\beta} \le N_{G_\alpha}(W_\alpha)$. Hence, (e), (f) and (h) hold. □

In the following we use the parameter q as defined in 4.1 (f). Observe that $Q_\alpha Q_\beta \le Q_{\alpha\beta}$. Thus, because of 4.1 (c), (d), (e), properties of the action of $Q_\alpha Q_\beta$ on Z_α and Z_β are given in 2.1 or 2.2, respectively. This fact will be used frequently.

4.2 *Let (α, α') be a critical pair. Then*

(*a*) $[Z_\alpha, Z_{\alpha'}] \le Z_\alpha \cap Z_{\alpha+1}$, *and*

(*b*) $[Z_\alpha, Z_{\alpha'}] \ne 1$*; in particular (α', α) is also a critical pair.*

Proof. As $Z_{\alpha'} \le Q_{\alpha+1}$, claim (a) follows from 4.1 and 2.1 (b) or 2.2 (c), respectively. Claim (b) is a consequence of 4.1 (g). □

4.3 *Let $\alpha, \beta \in \Gamma$ be adjacent. Then $Q_\alpha Q_\beta = Q_{\alpha\beta}$, or*

(∗) $q = 2$, $|Q_{\alpha\beta}/Q_\alpha Q_\beta| = 2$, *and $\overline{G}_\delta \cong Sp_4(2)$ for every $\delta \in \Gamma$.*

Proof. Set $A := Q_\alpha Q_\beta$. Since A is a normal p-subgroup of $G^*_{\alpha\beta}$ we have $A \le Q_{\alpha\beta}$. By 4.1 (b) $A \not\le Q_\alpha$ or $A \not\le Q_\beta$, and we choose our notation such that $A \not\le Q_\alpha$.

Assume that $A \ne Q_{\alpha\beta}$. Then 2.1 (d) shows that $\overline{G}_\alpha \not\cong SL_3(q)$, and 2.2 (e) implies that either

$$q = 2 \text{ and } |Q_{\alpha\beta}/A| = 2, \text{ or } p = 2 \text{ and } |Q_{\alpha\beta}/A| = q^2.$$

In the first case computing $|Q_{\alpha\beta}/A|$ in $\overline{G}_\beta$ shows $(*)$.

Hence, we may assume that we are in the second case. Then either

(I) $\overline{G}_\beta \cong Sp_4(q)$ and $|A/Q_\beta| = q$, or

(II) $\overline{G}_\beta \cong SL_3(q)$ and $Q_\alpha \le A = Q_\beta$.

From 2.2 (d) and $|Q_{\alpha\beta}/A| = q^2$ we get $J(A) \le Q_\alpha$. Similarly, (I) and (II) both imply $J(A) \le Q_\beta$. Thus $J(Q_\alpha) = J(A) = J(Q_\beta)$, a contradiction. □

4.4 *Let $\alpha, \beta \in \Gamma$ be adjacent. Suppose that $X \le G_\alpha$ such that $X \not\le N_{G_\alpha}(Z_{\alpha\beta})$ and $A \le Z_\beta$ such that $A \not\le Y_\alpha$. Then $[A, X] \not\le Y_\alpha$.*

Proof. Assume that $[A, X] \le Y_\alpha$. Then X normalizes AY_α and thus also $[AY_\alpha, Q_\alpha] = [A, Q_\alpha]$. Now 4.3, 2.1 (b) and 2.2 (c) and (e) show that $[A, Q_\alpha] = Z_{\alpha\beta}$ and $X \le N_{G_\alpha}(Z_{\alpha\beta})$, a contradiction. □

4.5 *Let $\alpha, \beta \in \Gamma$ be adjacent such that $\overline{G}_\alpha \cong \overline{G}_\beta \cong SL_3(q)$, and let $x \in Q_\alpha$ be such that $[x, G_\alpha] = Z_\alpha$. Suppose that $V_\beta \not\le Q_\alpha$. Then $x \in Z(Q_\alpha)$.*

Proof. Assume that $x \notin Z(Q_\alpha)$. We set $Y := [Q_\alpha, G^*_{\alpha\beta}]C_{Q_\alpha}(x)$ and $\widetilde{Q}_\alpha := Q_\alpha/C_{Q_\alpha}(x)$. Note that $C_{Q_\alpha}(x) = C_{Q_\alpha}(xZ_\alpha)$, so $C_{Q_\alpha}(x)$ is normal in G_α. The mapping

$$\widetilde{Q}_\alpha \to Z_\alpha \text{ such that } yC_{Q_\alpha}(x) \mapsto [y, x]$$

shows:

(1) Z_α and $\widetilde{Q}_\alpha$ are isomorphic $GF(p)\overline{G}_\alpha$-modules, in particular

(2) $\widetilde{Y}$ is contained in every non-trivial $G^*_{\alpha\beta}$-invariant subgroup of $\widetilde{Q}_\alpha$.

On the other hand, $\overline{G}_\beta \cong SL_3(q)$ and thus by 4.3 and 2.1 (d) $\overline{Q}_\alpha = \overline{Y}$. It follows that $Q_\alpha = (Q_\alpha \cap Q_\beta)Y$. On the other hand, by (1) $Q_\alpha \cap Q_\beta \not\leq C_{Q_\alpha}(x)$ since $|Q_\alpha / Q_\alpha \cap Q_\beta| \leq q^2$, so (2) implies that $Q_\alpha = (Q_\alpha \cap Q_\beta)C_{Q_\alpha}(x)$. In particular

(3) $[Q_\alpha, V_\beta] \leq [Q_\beta, V_\beta]C_{Q_\alpha}(x)$.

Again, as $\overline{G}_\beta \cong SL_3(q)$, we get that $Z_\beta / Z_{\alpha\beta}$ is a $G^*_{\alpha\beta}$-invariant subgroup of order q. But $Q_{\alpha\beta}/Q_\alpha$ is an irreducible $G^*_{\alpha\beta}$-module of order q^2, so $Z_\beta \leq Q_\alpha$, and (2) yields $Z_\beta \leq C_{Q_\alpha}(x)$.

In particular $Z_\alpha \leq Q_\beta$ and thus $V_\beta \leq Q_\beta$. By 2.1 (b) $[Z_\alpha, Q_\beta] \leq Z_{\alpha\beta} \leq Z_\beta$, so $[V_\beta, Q_\beta] \leq Z_\beta \leq C_{Q_\alpha}(x)$. Now (3) shows that V_β centralizes $\widetilde{Q}_\alpha$. Since $V_\beta \not\leq Q_\alpha$ we conclude that $\overline{G}_\alpha = \langle \overline{V}_\beta^{\overline{G}_\alpha} \rangle$, and $\widetilde{Q}_\alpha$ is a central $\overline{G}_\alpha$-module. This contradicts (1). □

4.6 *Suppose that $\delta \in \Gamma$ with $\overline{G}_\delta \cong SL_3(q)$. Then G_δ acts transitively on set of pairs (β, γ) with $\beta, \gamma \in \Delta(\delta)$ and $Z_{\beta\delta} \neq Z_{\gamma\delta}$.*

Proof. It is easy to see that $G^*_{\beta\delta}$ acts transitively on the 2-dimensional subspaces of Z_δ that are distinct from $Z_{\beta\delta}$. Now let $\gamma, \gamma' \in \Delta(\delta)$ be such that $Z_{\gamma\delta} = Z_{\gamma'\delta} \neq Z_{\beta\delta}$. By 4.1 (h) there exists $x \in N_{G_\delta}(Z_{\gamma\delta})$ such that $\gamma^x = \gamma'$. On the other hand

$$N_{G_\delta}(Z_{\gamma\delta}) = (G^*_{\beta\delta} \cap N_{G_\delta}(Z_{\gamma\delta}))G^*_{\gamma'\delta},$$

so this x can be chosen in $G^*_{\beta\delta}$, and the result follows. □

5 The Discussion of the Amalgam Problem.

In this section we adopt the hypothesis and notation of Section 4. Moreover, (α, α') is always a critical pair, and

$$\Lambda(\alpha, \alpha') \quad := \{\mu \in \Delta(\alpha) \mid Z_{\alpha'} \not\leq N_{G_\alpha}(Z_{\alpha\mu})\}.$$

5.1 *The following hold:*

(*a*) $\Lambda(\alpha, \alpha') \neq \emptyset$.

(*b*) $\langle Z_{\alpha'}, G^*_{\alpha\mu} \rangle = G_\alpha$ *for every* $\mu \in \Lambda(\alpha, \alpha')$.

Proof. By 4.2 (b) $Z_{\alpha'} \not\leq Q_\alpha$, so $Z_{\alpha'}$ does not stabilize every 2-dimensional subspace of Z_α. Hence, 4.1 (e) gives (a).

Claim (b) follows from 4.1 (h). □

5.2 *Suppose that $\alpha - 1 \in \Lambda(\alpha, \alpha')$ with $Z_{\alpha-1} \cap Q_{\alpha'-1} \neq Z_{\alpha-1\alpha}$. Then the following hold:*

(*a*) $\overline{G}_{\alpha'} \cong SL_3(q)$.

(*b*) $|[Z_\alpha, Z_{\alpha'}]| = q$.

(*c*) $Z_{\alpha'\alpha'-1} \not\leq Z_\alpha$ *and* $Z_{\alpha'\alpha'-1} \leq V_\alpha$.

(*d*) *If* $Z_{\alpha-1} \leq Q_{\alpha'-1}$, *then* $\overline{G}_{\alpha-1} \cong SL_3(q)$.

Proof. Set $A := Z_{\alpha-1} \cap Q_{\alpha'-1}$. Since $A \not\leq Z_{\alpha\alpha-1}$ and $Z_{\alpha'} \not\leq N_{G_\alpha}(Z_{\alpha\alpha-1})$, 4.4 implies $[A, Z_{\alpha'}] \not\leq Z_\alpha$, in particular $[Z_\alpha, Z_{\alpha'}] < [AZ_\alpha, Z_{\alpha'}]$. As $AZ_\alpha \leq Q_{\alpha'-1} \leq Q_{\alpha'-1\alpha'}$, we get from 2.1 (b) and 2.2 (c) that $[AZ_\alpha, Z_{\alpha'}] \leq Z_{\alpha'-1\alpha'}$. Since $Z_{\alpha'-1\alpha'}$ is a 2-dimensional subspace of $Z_{\alpha'}$ this implies:

$$Z_{\alpha'\alpha'-1} = [AZ_\alpha, Z_{\alpha'}],\ |[Z_\alpha, Z_{\alpha'}]| = q \text{ and } Z_{\alpha'\alpha'-1} \not\leq Z_\alpha.$$

This gives (b) and the first part of (c). As $AZ_\alpha \leq V_\alpha$ and V_α is normal in G_α, we get $Z_{\alpha'\alpha'-1} = [AZ_\alpha, Z_{\alpha'}] \leq V_\alpha$ and the second part of (c).

Suppose now that $\overline{G}_{\alpha'} \cong Sp_4(q)$ or $Sp_4(2)'$. Let $X := Z_{\alpha'} \cap Q_\alpha = C_{Z_{\alpha'}}(Z_\alpha)$. Since $|[Z_\alpha, Z_{\alpha'}]| = q$ we conclude from 2.4 that $|Z_{\alpha'}/X| = q$ and $|C_{AZ_\alpha}(X)Q_{\alpha'}/Q_{\alpha'}| \leq q$. As $|Z_\alpha Q_{\alpha'}/Q_{\alpha'}| = q$ and $A \not\leq Z_\alpha Q_{\alpha'}$, this shows that $[A, X] \neq 1$. Note that $[A, X] \leq Z_{\alpha-1\alpha} \leq Z_\alpha$.

If $[A, X] \not\leq [Z_\alpha, Z_{\alpha'}]$ then

$$Z_{\alpha'\alpha'-1} = [AZ_\alpha, Z_{\alpha'}] = [A, X][Z_\alpha, Z_{\alpha'}] \leq Z_\alpha,$$

a contradiction. Thus $[Z_\alpha, Z_{\alpha'}] = [A, X] \leq Z_{\alpha-1\alpha}$ and so $Z_{\alpha'}$ normalizes $Z_{\alpha-1\alpha}$, again a contradiction. This shows that $\overline{G}_{\alpha'} \cong SL_3(q)$, and (a) holds.

Set $Y := A \cap (Z_\alpha Q_{\alpha'})$. Then by 2.1 (d) $|A/Y| \leq q$. Since $[Y, Z_{\alpha'}] \leq Z_\alpha$, 4.4 implies $Y = Z_{\alpha-1\alpha}$ and thus $|A/Z_{\alpha-1\alpha}| \leq q$. This rules out the case $A = Z_{\alpha-1}$ and $\overline{G}_{\alpha-1} \cong Sp_4(q), Sp_4(2)'$, so (d) holds. □

5.3 *Suppose that $\overline{G}_\delta \cong Sp_4(q)$ or $Sp_4(2)'$ for some $\delta \in \Gamma$. Then $b = 1$.*

Proof. We say that G_δ is of symplectic type if $\overline{G}_\delta \cong Sp_4(q)$ or $Sp_4(2)'$. First we will show that there exists a critical pair (α, α') such that

(∗) G_α is of symplectic type, and either $G_{\alpha'}$ is of symplectic type or $|[Z_\alpha, Z_{\alpha'}]| = q^2$.

Let $\alpha - 1 \in \Lambda(\alpha, \alpha')$. Suppose first that b is even. Then G_α is conjugate to $G_{\alpha'}$, so $(*)$ holds if G_α is of symplectic type. Assume that $\overline{G}_\alpha \cong SL_3(q)$. Then by our hypothesis $G_{\alpha-1}$ is of symplectic type, and by 5.2 $(\alpha-1, \alpha'-1)$ is critical. Hence $(\alpha - 1, \alpha' - 1)$ satisfies $(*)$.

Suppose now that b is odd. Since also (α', α) is critical we may assume that $G_{\alpha'}$ is of symplectic type. Thus by 5.2 $Z_{\alpha-1} \cap Q_{\alpha'-1} = Z_{\alpha-1\alpha}$, in particular $(\alpha - 1, \alpha' - 1)$ is critical. Since b is odd, we get that $G_{\alpha-1}$ is conjugate to $G_{\alpha'}$, so $G_{\alpha-1}$ is of symplectic type and $|Z_{\alpha-1}/Z_{\alpha-1\alpha}| = q^2$. It follows that $|Z_{\alpha-1}Q_{\alpha'-1}/Q_{\alpha'-1}| = q^2$ and either $G_{\alpha'-1}$ is of symplectic type or $|[Z_{\alpha'-1}, Z_{\alpha-1}]| = q^2$, so $(\alpha - 1, \alpha' - 1)$ satisfies $(*)$.

Choose a critical pair (α, α') that satisfies $(*)$. Let D be a three dimensional subspace of Z_α containing $Z_\alpha \cap Q_{\alpha'}$, and let Σ be the set of 1-dimensional subspaces E of D with $E \not\leq Z_{\alpha\alpha+1}$. For $E \in \Sigma$ pick a 2-dimensional singular subspace W_E of Z_α with $W_E \cap D = E$. Note that $Z_\alpha = W_E \times Z_{\alpha\alpha+1}$.

By 2.2 (f) $Z_{\alpha'}$ does not normalize W_E. Pick $\mu_E \in \Delta(\alpha)$ with $Z_{\alpha\mu_E} = W_E$. Then $\mu_E \in \Lambda(\alpha, \alpha')$, so by $(*)$ and 5.2 $(\mu_E, \alpha' - 1)$ is critical.

Assume that $b > 1$. Then $[Z_{\mu_E}, Z_{\alpha'-1}] \leq Z_{\alpha'-1} \leq Q_{\alpha'}$, and

$$[Z_{\mu_E}, Z_{\alpha'-1}] \leq Z_{\alpha\mu_E} \cap Q_{\alpha'} \leq Z_{\alpha\mu_E} \cap D = E.$$

Hence $E = [Z_{\mu_E}, Z_{\alpha'-1}] \leq Z_{\alpha'-2\alpha'-1}$.

Since this is true for all $E \in \Sigma$ we get that $D = \langle \Sigma \rangle \leq Z_{\alpha'-2\alpha'-1}$. But D has order q^3 while $Z_{\alpha'-2\alpha'-1}$ has only order q^2. This contradiction shows that $b = 1$. □

5.4 $b \leq 2$.

Proof. Suppose $b \geq 3$. Then by 5.3 $\overline{G}_\delta \cong SL_3(q)$ for all $\delta \in \Gamma$. We will first show that there exists a critical pair (α, α') such that

$$(*) \qquad Z_{\alpha'-1\alpha'} \leq V_\alpha.$$

Let $\alpha - 1 \in \Lambda(\alpha, \alpha')$. If $(\alpha-1, \alpha'-1)$ is not critical, then 5.2 (c) implies $(*)$ for (α, α'). Suppose that $(\alpha - 1, \alpha' - 1)$ is critical. Since $[Z_{\alpha-1}, Z_{\alpha'-1}] \leq Z_{\alpha-1\alpha}$ and $Z_{\alpha'}$ does not normalize $Z_{\alpha-1\alpha}$ we have $[Z_\alpha, Z_{\alpha'}] \not\leq [Z_{\alpha-1}, Z_{\alpha'-1}]$. Thus $A := [Z_\alpha, Z_{\alpha'}][Z_{\alpha-1}, Z_{\alpha'-1}] \leq Z_\alpha$ has order at least q^2. Moreover $A \leq Z_{\alpha'-1}Z_{\alpha'} \leq V_{\alpha'}$, and $Z_{\alpha'}$ centralizes A since $b > 1$. Thus $A \leq C_{Z_\alpha}(Z_{\alpha'}) = Z_{\alpha\alpha+1}$, and since $Z_{\alpha\alpha+1}$ has order q^2 we get $Z_{\alpha\alpha+1} = A \leq V_{\alpha'}$, so $(*)$ holds for (α', α).

Now choose a critical pair (α, α') satisfying $(*)$. Suppose first that $b > 3$. Then $[V_\alpha^{(2)}, V_\alpha] = 1$, so $(*)$ implies $[V_\alpha^{(2)}, Z_{\alpha'\alpha'-1}] = 1$. Thus 4.4 with

$$(\alpha' - 2, \alpha' - 1, Z_{\alpha'\alpha'-1}, V_\alpha^{(2)}) \text{ in place of } (\alpha, \beta, A, X)$$

gives $Z_{\alpha'\alpha'-1} = Z_{\alpha'-1\alpha'-2}$ or $V_\alpha^{(2)} \leq N_{G_{\alpha'-2}}(Z_{\alpha'-1\alpha'-2})$. Evidently the first case implies the second one. Thus we have in both cases $V_\alpha^{(2)} \leq G_{\alpha'-1}$ by 4.1 (e). Hence $V_\alpha^{(2)} \leq C_{G_{\alpha'-1}}(Z_{\alpha'\alpha'-1}) = Q_{\alpha'\alpha'-1}$ and $[V_\alpha^{(2)}, Z_{\alpha'}] \leq Z_{\alpha'\alpha'-1} \leq V_\alpha$. This shows that $Z_{\alpha'}$ and $G^*_{\alpha\alpha-1}$ normalize $V_{\alpha-1}V_\alpha$. Since $\alpha - 1 \in \Lambda(\alpha, \alpha')$ we conclude from 5.1 (b) that $V_{\alpha-1}V_\alpha$ is normal in G_α. The transitivity of G_α on $\Delta(\alpha)$ gives $V_\alpha^{(2)} = V_\alpha V_\beta$ for all $\beta \in \Delta(\alpha)$. Conjugation to $G_{\alpha+2}$ yields $V_{\alpha+2}^{(2)} = V_{\alpha+2}V_{\alpha+3}$, in particular

$$Z_\alpha \leq V_{\alpha+2}V_{\alpha+3} \leq Q_{\alpha'},$$

a contradiction.

We have shown that $b = 3$. Let (μ, μ') be any critical pair. Suppose that $Z_{\mu+1\mu+2} = Z_{\mu'\mu+2}$. Then $V_\mu \leq C_{G_{\mu+2}}(Z_{\mu'\mu+2}) \leq Q_{\mu'\mu+2}$ and thus

$$[Z_{\mu'}, V_\mu] \leq Z_{\mu'\mu+2} = Z_{\mu+1\mu+2} \leq Z_{\mu+1}.$$

Let $\mu - 1 \in \Lambda(\mu, \mu')$. Then 4.4, with $(\mu, \mu - 1, Z_{\mu'}, Z_{\mu-1})$ in place of (α, β, X, A), implies that $[Z_{\mu-1}, Z_{\mu'}] \not\leq Z_\mu$. In particular $[V_\mu, Z_{\mu'}] \not\leq Z_\mu$ and thus $[Z_{\mu'}, V_\mu] > [Z_{\mu'}, Z_\mu]$. Since $Z_{\mu'}$ is a natural $SL_3(q)$-module for $\overline{G}_{\mu'}$ we get that

$$[Z_{\mu'}, V_\mu] = Z_{\mu'\mu+2} \text{ and } |[Z_{\mu'}, Z_\mu]| = q.$$

From 4.1 (e), applied to the vertices in $\Delta(\mu)$, we get $[V_\mu, Q_\mu] = Z_\mu$. Hence, as $[V_\mu, Z_{\mu'}] \not\leq Z_\mu$,

$$[V_\mu, Q_\mu Z_{\mu'}] = Z_\mu[V_\mu, Z_{\mu'}] = Z_\mu Z_{\mu+1}.$$

Moreover, $|[Z_{\mu'}, Z_\mu]| = q$ together with 2.4 shows that $\overline{Z}_\mu$ is a 1-dimensional subspace of $\overline{Q}_{\mu+1}$. As $\overline{G}^*_{\mu\mu+1}$ is transitive on these 1-dimensional subspaces and $N_{G_\mu}(Z_{\mu'}Q_\mu) \leq N_{G_\mu}(Z_{\mu\mu+1})$, we get with the Frattini argument

$$N_{G_\mu}(Z_{\mu\mu+1}) = G^*_{\mu\mu+1}N_{G_\mu}(Z_{\mu'}Q_\mu).$$

Since $[V_\mu, Q_\mu Z_{\mu'}]$ is normalized by $N_{G_\mu}(Z_{\mu'}Q_\mu)$ while $Z_\mu Z_{\mu+1}$ is normalized by $G^*_{\mu\mu+1}$, we conclude that $N_{G_\mu}(Z_{\mu\mu+1})$ normalizes $Z_\mu Z_{\mu+1}$. Since $Z_\mu Z_{\mu-1}$

is not normal in G_μ and $N_{G_\mu}(Z_{\mu\mu+1})$ is a maximal subgroup of G_μ, this shows that

$$N_{G_\mu}(Z_{\mu\mu+1}) = N_{G_\mu}(Z_\mu Z_{\mu+1}).$$

Hence, edge-transitivity also gives

$$N_{G_{\mu+2}}(Z_{\mu'}Z_{\mu+2}) = N_{G_{\mu+2}}(Z_{\mu'\mu+2}) = N_{G_{\mu+2}}(Z_{\mu+1\mu+2}) = N_{G_{\mu+2}}(Z_{\mu+1}Z_{\mu+2}),$$

and the transitivity of $G_{\mu+2}$ on $\Delta(\mu+2)$ yields

$$Z_{\mu'} \leq Z_{\mu'}Z_{\mu+2} = Z_{\mu+2}Z_{\mu+1} \leq Q_\mu,$$

a contradiction.

This contradiction shows that $Z_{\mu+1\mu+2} \neq Z_{\mu'\mu+2}$ for all critical pairs (μ, μ'); in particular

$$(**) \qquad Z_{\alpha+1\alpha+2} \neq Z_{\alpha'\alpha+2} \text{ and } Z_{\alpha\alpha+1} \neq Z_{\alpha+2\alpha+1}.$$

But then $Z_{\alpha+2} = Z_{\alpha+1\alpha+2}Z_{\alpha'\alpha+2}$, and $(*)$ implies that $Z_{\alpha+2} \leq V_\alpha$. On the other hand, by $(**)$ and 4.6 there exists an element in $G_{\alpha+1}$ that maps $(\alpha, \alpha+2)$ to $(\alpha+2, \alpha)$, so also $Z_\alpha \leq V_{\alpha+2}$. Since $b > 2$ we get that $V_{\alpha+2}$ is abelian and Z_α and $Z_{\alpha'}$ centralize each other. This contradicts 4.2. □

5.5 *Suppose that $b > 1$. Let $\delta \in \Gamma$ and $\gamma, \gamma' \in \Delta(\delta)$. Then the following hold:*

(a) $D_\delta = Z_\delta$.

(b) *(γ, γ') is critical if and only if $Z_{\gamma\delta} \neq Z_{\gamma'\delta}$.*

Proof. From 5.3 and 5.4 we get that $b = 2$ and $G_\delta/Q_\delta \cong SL_3(q)$ for every $\delta \in \Gamma$. Let $\beta = \alpha + 1$.

Since $G_\alpha = \langle Z_{\alpha'}^{G_\alpha}\rangle Q_\alpha$ we get from 4.4, with $(Z_\beta, O^p(G_\alpha))$ in place of (A, X), that $[Z_\beta, \langle Z_{\alpha'}^{G_\alpha}\rangle] \not\leq Y_\alpha$. Hence

(1) $[V_\alpha, Z_{\alpha'}] \not\leq Y_\alpha$.

Assume next that $Z_{\alpha\beta} = Z_{\alpha'\beta}$. Then $V_\alpha \leq C_{G_\beta}(Z_{\alpha'\beta}) \leq Q_{\alpha'\beta}$ and so

$$[V_\alpha, Z_{\alpha'}] \leq Z_{\alpha'\beta} = Z_{\alpha\beta} \leq Z_\alpha.$$

This contradicts (1). Thus

(2) $Z_{\alpha\beta} \neq Z_{\alpha'\beta}$.

From 4.6 we conclude that (b) holds for all $\delta \in \beta^G$.

Assume that $D_\alpha = Z_\alpha$. Then there exists $\alpha - 1 \in \Delta(\alpha)$ such that $(\alpha - 1, \beta)$ is a critical pair. By a symmetric argument (b) now also holds for all $\delta \in \alpha^G$ and thus for all $\delta \in \Gamma$. It remains to show that $D_\alpha = Z_\alpha$. So we assume:

(3) $D_\alpha \neq Z_\alpha$.

Note that $D_\alpha \leq Q_\beta \cap Q_\alpha \leq Q_{\alpha'\beta}$, so by 2.1 (d) and 4.1 (e) $\Phi(D_\alpha) \leq Q_{\alpha'}$. It follows that $[\Phi(D_\alpha), \langle Z_{\alpha'}^{G_\alpha} \rangle] = 1$. As $G_\alpha = \langle Z_{\alpha'}^{G_\alpha} \rangle Q_\alpha$ and by 4.1 (d) $Z(G_\alpha) = 1$, we get that $\Phi(D_\alpha) \cap Z(Q_\alpha) = 1$ and thus $\Phi(D_\alpha) = 1$; i.e.,

(4) D_α is elementary abelian.

Let N be the largest normal subgroup of G_α in D_α such that $N \leq Z_\alpha Q_{\alpha'}$. Then $[N, Z_{\alpha'}] \leq Z_\alpha$ and thus also $[N, O^p(G_\alpha)] = Z_\alpha$. Hence 4.5 shows that

$$N \leq \Omega_1(Z(Q_\alpha)) = Y_\alpha.$$

Since $|N/C_N(Z_{\alpha'})| = q$ we get from 2.1 (f) $N = Z_\alpha \times C_N(G_\alpha)$. Thus, 4.1 (d) implies that $N = Z_\alpha$. In particular, by (3) $D_\alpha \not\leq N$. Hence, 2.1 (b) gives $[Z_{\alpha'}, D_\alpha] = Z_{\alpha'\beta}$. From (2) we get

$$Z_\alpha[D_\alpha, Z_{\alpha'}] = Z_\alpha Z_{\alpha'\beta} = Z_\alpha Z_\beta \leq D_\alpha,$$

so

(5) $V_\alpha \leq D_\alpha$; in particular, V_α is abelian and $V_\alpha \leq Q_\beta$.

Let $\alpha - 1 \in \Lambda(\alpha, \alpha')$. Then 4.4, with $(\alpha, \alpha - 1, Z_{\alpha'}, Z_{\alpha-1})$ in place of (α, β, X, A), implies that $Z_{\alpha-1} \cap (Z_\alpha Q_{\alpha'}) = Z_{\alpha\alpha-1}$ and thus $V_\alpha Q_{\alpha'} = Q_\beta Q_{\alpha'}$. By symmetry in α and α',

(6) $V_\alpha Q_{\alpha'} = D_\alpha Q_{\alpha'} = Q_\beta Q_{\alpha'}$ and $Q_\beta Q_\alpha = D_{\alpha'} Q_\alpha = V_{\alpha'} Q_\alpha$.

Set $\widetilde{Q}_\alpha := Q_\alpha / D_\alpha$. We apply 2.1 (d). Then $Q_{\alpha\beta}/Q_\beta$ is a natural module for $G^*_{\alpha\beta}/Q_{\alpha\beta}$, so $\Phi(Q_\alpha) \leq Q_\beta$, and $\widetilde{Q}_\alpha$ is elementary abelian. Similarly $C_{Q_{\alpha\beta}/Q_\beta}(G^*_{\alpha\beta}) = 1$ and thus $C_{\widetilde{Q}_\alpha}(G_\alpha) = 1$. Moreover, by 2.3 $|Q_\alpha / Q_\alpha \cap G_{\alpha'}| = q$, so $[Q_\alpha \cap G_{\alpha'}, Z_{\alpha'}] \leq Z_\beta \leq D_\alpha$ yields $|\widetilde{Q}_\alpha / C_{\widetilde{Q}_\alpha}(Z_{\alpha'})| \leq q$. Hence 2.5 and 2.1 (f) imply:

(7) $\widetilde{Q}_\alpha$ is a G_α-module dual to Z_α.

From (7) we get $|Q_\alpha/D_\alpha| = q^3$ and $[x, Q_\beta]D_\alpha/D_\alpha = (Q_\alpha \cap Q_\beta)/D_\alpha$ for every $x \in Q_\alpha \setminus Q_\beta$. Hence (6) gives

$$Q_\alpha \cap Q_\beta = [Q_\alpha \cap G_{\alpha'}, D_{\alpha'}]D_\alpha = (D_{\alpha'} \cap Q_\alpha)D_\alpha \text{ and } Q_\beta = D_\alpha D_{\alpha'}.$$

In particular

$$\Phi(Q_\beta) \leq D_\alpha \cap D_{\alpha'} \overset{(4)}{\leq} \Omega_1(Z(Q_\beta)) \overset{1.2}{=} Y_\beta \leq D_\alpha \cap D_{\alpha'}.$$

Thus, we get

(8) $D_\alpha \cap D_{\alpha'} = Y_\beta$, $\Phi(Q_\beta) \leq Y_\beta$ and $|Q_\beta/Y_\beta| = q^6$.

Note that $|D_\alpha/D_\alpha \cap (Z_\alpha Q_{\alpha'})| \leq q$ and $[D_\alpha \cap (Z_\alpha Q_{\alpha'}), Z_{\alpha'}] \leq Z_\alpha$. Thus $Z_{\alpha'}$ centralizes a subgroup of index q in $W := D_\alpha/Z_\alpha$. From 2.1 (f) we get that $W/C_W(G_\alpha)$ is a natural $SL_3(q)$-module for $\overline{G}_\alpha$. Since $G^*_{\alpha\beta}$ normalizes $Z_\beta Z_\alpha/Z_\alpha$, this module is dual to Z_α.

Let U be the inverse image of $C_W(G_\alpha)$ in D_α. By 4.5 and (4)

$$U \leq \Omega_1(Z(Q_\alpha)) = Y_\alpha.$$

On the other hand, $Y_\alpha \leq D_\alpha$ since $b > 1$, so (1) and (6) imply that $Y_\alpha \leq Z_\alpha Q_{\alpha'}$. Since $Z(G_\alpha) = 1$, again 2.1 (f) yields $Y_\alpha = Z_\alpha$. We have shown:

(9) D_α/Z_α is a $\overline{G}_\alpha$-module dual to Z_α; in particular $Y_\beta = Z_\beta$ and $|Q_\alpha| = |Q_\beta| = q^9$.

By (9) $[D_\alpha, Q_\alpha] = Z_\alpha$. Pick $x \in Q_\alpha \cap Q_\beta$, so $[x, G^*_{\alpha\beta}] \leq D_\alpha$ by (7). Then $[Q_\alpha, x]Z_\alpha$ is $G^*_{\alpha\beta}$-invariant and $|[Q_\alpha, x]Z_\alpha/Z_\alpha| < q^3$. Hence (9) shows that $[Q_\alpha, x] \leq Z_\alpha Z_\beta$. It follows that $[x, Q_\alpha, G^*_{\alpha\beta}] \leq Z_\alpha$ and $[G^*_{\alpha\beta}, x, Q_\alpha] \leq Z_\alpha$. Thus, also $[Q_\alpha, G^*_{\alpha\beta}, x] \leq Z_\alpha$. Since $[Q_\alpha, G^*_{\alpha\beta}]D_\alpha = Q_\alpha$ by (7), we conclude that

(10) Q_α/Z_α is abelian.

From (6) and (10) we get $C_{Q_\alpha/Z_\alpha}(D_{\alpha'}) = C_{Q_\alpha/Z_\alpha}(Q_{\alpha\beta})$, so (7) and (9) imply $|C_{Q_\alpha/Z_\alpha}(Q_{\alpha\beta})| \leq q^2$. On the other hand, by (9) $|D_\alpha| = |D_{\alpha'}| = q^6$, so by (6) $|D_{\alpha'} \cap Q_\alpha| = q^4$ and $|(D_{\alpha'} \cap Q_\alpha)Z_\alpha/Z_\alpha| = q^2$. Hence (4) implies

(11) $C_{Q_\alpha/Z_\alpha}(Q_{\alpha\beta}) = C_{Q_\alpha/Z_\alpha}(D_{\alpha'}) = (D_{\alpha'} \cap Q_\alpha)Z_\alpha/Z_\alpha$.

Set $A := (D_{\alpha'} \cap Q_\alpha)Z_\alpha$. Recall from the above considerations that $|A| = q^5$ and $|(D_{\alpha'} \cap Q_\alpha)/Z_\beta| = q$. Let $\Sigma \subseteq \Delta(\beta)$ be such that $\alpha \in \Sigma$ and for each 2-dimensional subspace W of Z_β there exists a unique $\sigma \in \Sigma$ with $Z_{\sigma\beta} = W$.

Then $|\Sigma| = q^2 + q + 1$, and by (2) and 4.6 any distinct pair of elements in Σ is critical. Let $\sigma \in \Sigma$ be with $\sigma \neq \alpha$. Then by (11), applied to σ in place of α', we get

$$(D_\sigma \cap Q_\alpha) Z_\alpha = A.$$

We conclude that for all $\sigma \in \Sigma$, $D_\sigma \cap Q_\alpha$ is a subgroup of index q in A. On the other hand, by (8) and (9) the groups $(D_\sigma \cap Q_\alpha)/Z_\beta$ have pairwise trivial intersection, so $|A/Z_\beta| > q^2 + q + 1$, a contradiction as $|A/Z_\beta| = q^2$.
□

5.6 *Suppose that $b > 1$. Then $b = 2$, and for every $\delta \in \Gamma$ the following hold:*

(a) $\overline{G}_\delta \cong SL_3(q)$, *$p = 3$ or $q = 5$.*

(b) $Q_\delta = V_\delta$,

(c) V_δ / Z_δ *is a natural $SL_3(q)$-module dual to Z_δ.*

Proof. From 5.3, 5.4 and 5.5 we get the following information for every $\delta \in \Gamma$:

(1) $b = 2$.

(2) $G_\delta / Q_\delta \cong SL_3(q)$ and $Z_\delta = D_\delta$.

(3) Let $\gamma, \gamma' \in \Delta(\delta)$. Then (γ, γ') is critical if and only if $Z_{\gamma\delta} \neq Z_{\gamma'\delta}$.

We fix a critical pair (α, α') and set $\beta := \alpha + 1$. If $|[Z_\alpha, Z_{\alpha'}]| = q^2$, then $Z_{\alpha\beta} = [Z_\alpha, Z_{\alpha'}] = Z_{\alpha'\beta}$ which contradicts (3). Thus $|[Z_\alpha, Z_{\alpha'}]| = q$, and by 2.1 (g)

(4) $Z_{\alpha'} Q_\alpha / Q_\alpha$ is the center of a Sylow p-subgroup of G_α / Q_α for every critical pair (α, α').

According to (3) it suffices to show (a) – (c) for $\delta = \alpha'$.

By (4) and 2.3 there exists $\alpha - 1 \in \Delta(\alpha)$ such that $Z_{\alpha'} \leq G_{\alpha-1}$ but $Z_{\alpha'} \not\leq Q_{\alpha\alpha-1}$. In particular $R := [Z_{\alpha\alpha-1}, Z_{\alpha'}] \neq 1$, so $Z_{\alpha\alpha-1} \neq Z_{\alpha\beta}$, and by (3) $(\alpha - 1, \beta)$ is critical.

The action of $G^*_{\alpha\alpha-1}$ on $Z_{\alpha-1\alpha}$, see 2.1 (d), gives

$$R = Z_{\alpha\alpha-1} \cap Z_{\alpha\beta} = [Z_{\alpha-1}, Z_\beta].$$

Hence, since $|[Z_\alpha, Z_{\alpha'}]| = q$, we get

$$R = [Z_{\alpha-1}, Z_\beta] = [Z_\alpha, Z_{\alpha'}] \leq Z_\beta \cap Z_{\alpha'}$$

and thus

(5) $Z_{\alpha-1} \leq G_{\alpha'}$.

Again by 2.3 we have $Z_{\alpha-1} \not\leq Q_{\beta\alpha'}$ since $(\alpha-1,\beta)$ is critical. As $Z_\alpha Q_{\alpha'} = Z_{\alpha-1\alpha} Q_{\alpha'}$ we conclude that $|Z_{\alpha-1}Q_{\alpha'}/Q_{\alpha'}| = q^2$ and thus $|Z_{\alpha-1} \cap Q_{\alpha'}| = q$. We have shown:

(6) $|Z_{\alpha-1}Q_{\alpha'}/Q_{\alpha'}| = q^2$ and $Z_{\alpha-1} \cap Q_{\alpha'} = R \leq Z_{\alpha'}$.

Let $A := V_{\alpha'} \cap G_\alpha$. There are $q+1$ 2-dimensional subspaces of Z_α containing R. Since A fixes $Z_{\alpha\beta}$ we get that $|Z^A_{\alpha\alpha-1}| \leq q$, so $|A/A \cap G_{\alpha-1}| \leq q$. By 2.3, applied to $\overline{G}_\beta$, also $|V_{\alpha'}/V_{\alpha'} \cap G_\alpha| \leq q$, so we get

(7) $|V_{\alpha'}/V_{\alpha'} \cap G_{\alpha-1}| \leq q^2$.

On the other hand, (5) and (6) imply that $[Z_{\alpha-1}, V_{\alpha'} \cap G_{\alpha-1}] \leq Z_{\alpha-1} \cap Q_{\alpha'} \leq Z_{\alpha'}$. Thus (6) yields

$$|\widetilde{V}_{\alpha'}/C_{\widetilde{V}_{\alpha'}}(Z_{\alpha-1})| \leq q^2 = |Z_{\alpha-1}Q_{\alpha'}/Q_{\alpha'}|,$$

where $\widetilde{Q}_{\alpha'} := Q_{\alpha'}/Z_{\alpha'}$.

Clearly, $\Phi(Q_{\alpha'}) \leq D_{\alpha'} = Z_{\alpha'}$, so $V'_{\alpha'} \neq 1$ shows that $\Phi(Q_{\alpha'}) = \Phi(V_{\alpha'}) = Z_{\alpha'}$. Hence, we are able to apply 2.6 with $M := G_{\alpha'}$, $D := V_{\alpha'}$, $Y := Z_\beta$, and $V_1 := Z_{\alpha'}$, and get that $\widetilde{V}_{\alpha'}$ is a natural $G_{\alpha'}/Q_{\alpha'}$-module dual to $Z_{\alpha'}$. Thus (c) holds.

Note that G_α acts transitively on $(V_\alpha/Z_\alpha)^\#$. So every element in V_α has order p. Since V_α is not abelian we conclude that $p \neq 2$.

By (3) $Q_{\alpha'} = (Q_{\alpha'} \cap Q_\beta)V_{\alpha'}$ and

$$[Q_{\alpha'} \cap Q_\beta, Z_\alpha] \leq Z_\alpha \cap Z_\beta \leq Z_\beta,$$

so (c) gives $[\tilde{V}_{\alpha'}, Z_\alpha] \leq \widetilde{Z}_\beta$. It follows that $|[\widetilde{Q}_{\alpha'}, Z_\alpha]| = |\widetilde{Z}_\beta| = q$.

Let $\overline{U}$ be the parabolic subgroup of $\overline{G}_{\alpha'}$ fixing $\widetilde{Z}_\beta$. Then $\overline{Z}_\alpha \leq O_p(\overline{U})$, and since $p \neq 2$ there exists $\overline{K} \leq \overline{U}$ of order $q-1$ such that $[\widetilde{Z}_\beta, \overline{K}] = 1$ and $[\overline{Z}_\alpha, \overline{K}] = \overline{Z}_\alpha$. In addition, since $\widetilde{Q}_{\alpha'}/\widetilde{V}_{\alpha'}$ is central

$$\widetilde{Q}_{\alpha'} = \widetilde{V}_{\alpha'}\widetilde{C}, \text{ where } \widetilde{C} = C_{\widetilde{Q}_{\alpha'}}(\overline{K}).$$

It follows that

$$[\widetilde{C}, \overline{K}, \overline{Z}_\alpha] = 1 = [\overline{Z}_\alpha, \widetilde{C}, \overline{K}],$$

and thus also $[\overline{K}, \overline{Z}_\alpha, \widetilde{C}] = [\overline{Z}_\alpha, \widetilde{C}] = 1$. As $\widetilde{V}_{\alpha'}$ is a natural module, (4) implies that $|\widetilde{Q}_{\alpha'}/C_{\widetilde{Q}_{\alpha'}}(Z_\alpha)| = q$. Hence, 2.1 (f) yields $\widetilde{Q}_{\alpha'} = \widetilde{V}_{\alpha'}C_{\widetilde{Q}_{\alpha'}}(G_{\alpha'})$, and 4.5 gives $V_{\alpha'} = Q_{\alpha'}$. Thus, also (b) holds.

Now let $L := G^*_{\alpha'\beta}$, $D := Q_{\alpha'\beta}$, $V := Z_{\alpha'\beta}$ and $Y := Z_\beta Z_{\alpha'}$. From (b) and (c) we get that $Y = Q_\beta \cap Q_{\alpha'}$, $|Y| = q^4$ and $[Y, L] = V$. Moreover, $L/D \cong SL_2(q)$, and $D/Y = Q_\beta/Y \times Q_{\alpha'}/Y$ is the direct sum of the natural L/D-modules.

Let $\lambda \in \{\alpha', \beta\}$. Then $C_D(Z_\lambda) = Q_\lambda$ and so $C_D(Z_\lambda)' = Z_\lambda$. Thus $D' = Z_{\alpha'} Z_\beta = Y$. Hence, the hypotheses of 2.7 are satisfied, and (a) follows. □

5.7 *Suppose that $b = 1$. Then $G_\delta/Q_\delta \cong Sp_4(2)$ or $Sp_4(2)'$, $Q_\delta = Y_\delta$, and $|Y_\delta| = 2^4$ or 2^5 for all $\delta \in \Gamma$.*

Proof. As $\alpha' \in \Delta(\alpha)$ we get from 4.1 together with 2.1 (d) or 2.2 (b) that $Q_{\alpha\alpha'}/Q_{\alpha'}$ is elementary abelian and

$$[Z_\alpha, Z_{\alpha'}] = Z_\alpha \cap Q_{\alpha'} = Z_{\alpha\alpha'}.$$

In particular $\Phi(Q_\alpha) \leq Q_\alpha \cap Q_{\alpha'}$ since $b = 1$, so $\Phi(Q_\alpha) \leq D_\alpha$ and $[D_\alpha, Z_{\alpha'}] = 1$. Since $G_\alpha = \langle Z_{\alpha'}^{G_\alpha} \rangle Q_\alpha$, we conclude that

$$[Q_\alpha, O^p(G_\alpha)] \leq Z_\alpha \text{ and } [D_\alpha, O^p(G_\alpha)] = 1.$$

Now $Z(G_\alpha) = 1$ (4.1 (d)) implies $\Phi(Q_\alpha) = 1$ and thus $Q_\alpha = Y_\alpha$.

Suppose that $\overline{G}_\alpha \cong SL_3(q)$. Then by 4.1 (d) and (e) $[Q_\alpha, O^p(G^*_{\alpha\alpha'})] \leq Z_{\alpha\alpha'} \leq Q_{\alpha'}$ and so by 4.3 $[Q_{\alpha\alpha'}/Q_{\alpha'}, O^p(G^*_{\alpha\alpha'})] = 1$, which contradicts 2.1 (d).

By 4.2 (b) and a symmetric argument $Y_\delta = Q_\delta$ and $\overline{G}_\delta \cong Sp_4(q)$ or $Sp_4(2)'$ for all $\delta \in \Gamma$. Hence $Z_\alpha Q_{\alpha'}/Q_{\alpha'}$ is a normal subgroup of order q^2 in $G^*_{\alpha\alpha'}/Q_{\alpha'}$, so 2.2 (e) implies $q = 2$. In particular

$$O_2(O^2(G^*_{\alpha\alpha'})) \leq Q_\alpha Q_{\alpha'}. \quad (1)$$

It remains to prove that $|Q_\delta| = 2^4$ or 2^5.

From 2.2 (d) we get that $C_{Z_\alpha}(G^*_{\alpha\alpha'}) = 1$. In particular, by 4.1 (a), $C_{\Omega_1(Z(T))}(G^*_{\alpha\alpha'}) = 1$ for $T \in Syl_2(G^*_{\alpha\alpha'})$. It follows that also

$$C_{Q_\alpha}(O^2(G^*_{\alpha\alpha'})) = 1. \quad (2)$$

Let $D := Q_\alpha \cap Q_{\alpha'}$. Then $Q_\alpha Q_{\alpha'}$ centralizes D and

$$[D, O^2(G^*_{\alpha\alpha'})] \leq Z_\alpha \cap Q_{\alpha'} \leq Z_\alpha \cap Z_{\alpha'} = Z_{\alpha\alpha'}.$$

Moreover, by (1) $O^2(G^*_{\alpha\alpha'})$ acts as a cyclic group of order 3 on D. Thus

$$D = Z_{\alpha\alpha'} C_D(O^2(G^*_{\alpha\alpha'})) \overset{(2)}{=} Z_{\alpha\alpha'},$$

and $|Q_\delta/Z_{\alpha\alpha'}| \leq 2^3$ for $\delta = \alpha, \alpha'$. As $|Z_{\alpha\alpha'}| = 4$, the edge-transitivity of G on Γ gives $|Q_\delta| = 2^4$ or 2^5 for every $\delta \in \Gamma$. □

6 The Proof of Theorem 1 and the $\widetilde{P}$!-Theorem

Theorem 1 follows from 5.7 and 5.6.

The proof of the $\widetilde{P}$!-Theorem: Suppose that there exists $\widetilde{P}_1 \in \mathcal{P}_H(S)$ such that $\widetilde{P}_1 \not\leq N_H(P^\circ)$ and $\langle \widetilde{P}_1, P\rangle \in \mathcal{L}_H(P)$. Set $M_1 := \langle \widetilde{P}_1, P\rangle^\circ C_S(Y_P)$. Then we are allowed to apply the Local $\widetilde{P}$-Theorem 3.5 with M_1 in place of U^*. Observe (with the notation from 3.5) that $\overline{U}^\circ = \overline{U}_0$, and also $\overline{U}^\circ = \overline{U}^*$ or $\overline{U}^\circ \cong Sp_4(2)'$ and $\overline{U}^* \cong Sp_4(2)$. Hence 3.5 gives (a_1) and (a_2) for M_1.

Suppose now, in addition, that there exists $\widetilde{P}_2 \in \mathcal{P}(S)$ such that $\widetilde{P}_1 \neq \widetilde{P}_2$, $\widetilde{P}_2 \not\leq N_H(P^\circ)$ and $\langle \widetilde{P}_2, P\rangle \in \mathcal{L}_H(P)$. Set

$$M_i := \langle \widetilde{P}_i, P\rangle^\circ C_S(Y_P) \text{ and } V_i := [Y_{M_i}, O^p(M_i)],\ i = 1, 2$$

According to 3.6 $G = \langle M_1, M_2\rangle$ satisfies the hypothesis of Theorem 1 with respect to M_1 and M_2.

Assume first that we are in case (a) of Theorem 1. Then $p = 2$ and $Y_{M_2} \not\leq Y_{M_1}$, and $P \cap M_1$ stabilizes a 2-dimensional singular subspace of V_1. Moreover, by 2.4 Y_{M_2} does not centralize a hyperplane in V_1. It follows that

$$|V_2/V_1 \cap V_2| = |V_1 \cap V_2| = 4.$$

Pick $1 \neq x \in Z(S) \cap V_1 \cap V_2$. Then 2.2 (g) implies that $V_2 \not\leq O_2(C_{M_i}(x))$. On the other hand, Q-uniqueness gives $C_H(x) \leq \widetilde{C}$. Hence, also $V_2 \not\leq Q$, but this contradicts the hypothesis of the $\widetilde{P}$!-Theorem.

Assume now that case (b) of Theorem 1 holds. Let $L := P^\circ C_P(Y_P)$ and observe that $C_H(L/O_p(L))/O_p(L)$ is a p'-group since P contains a Sylow p-subgroup of H. Then L satisfies the hypothesis of 2.7, and H, L, M_1 and M_2 the hypothesis of 2.9. Hence, $q = 3$ or 5 follows.

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