Nearly Quadratic Modules

U. Meierfrankenfeld

B. Stellmacher

Department of Mathematics, Michigan State University, East Lansing MI 48840

meier@math.msu.edu

Mathematisches Seminar, Christian-Albrechts- Universität, D24098 Kiel

stellmacher@math.uni-kiel.de

1 Introduction

Let p a prime, G a finite group and $T \in \mathrm{Syl}_p(G)$. Then G has characteristic p if $C_G(O_p(G)) \le O_p(G)$; and G has local characteristic p if every p-local subgroup of G has characteristic p. This paper is part of a program to understand and classify the finite groups of local characteristic p, see [MSS1].

It was shown in [MS1] that in such groups there always exists a maximal p-local subgroup M containing T satisfying one of the following three cases (where Y_M is the largest elementary abelian normal p-subgroup of M with $O_p(M/C_M(Y_M)) = 1$):

1. M is the unique maximal p-local subgroup of G containing T.
2. There exists $A \le T$ such that $[Y_M, A] \ne 1$, $[\Phi(A), Y_M] = 1$, and
$$|Y_M/C_{Y_M}(A)| \le |A/C_A(Y_M)|.$$
3. There exists $A \le T$ such that $[Y_M, A] \ne 1$, $[\Phi(A), Y_M] = 1$, and
   - (i) $[Y_M, A]C_{Y_M}(A) = [v, A]C_{Y_M}(A)$ for every $v \in Y_M \setminus [Y_M, A]C_{Y_M}(A)$ and $[Y_M, A, A, A] = 1$,
   - (ii) $|Y_M/C_{Y_M}(A)| \le |A/C_A(Y_M)|^2$.

The nature of these properties give rise to questions about the embedding of M in G (in case (1)), and about the structure of $\overline{M} := M/C_M(Y_M)$ and the $\mathbb{F}_pM$-module Y_M (in case (2) and (3)).

In case (1) the Local $C(G,T)$-Theorem [BHS] gives the structure of all maximal p-local subgroups not containing a Sylow p-subgroup of G – and if there are none, then G has a strongly p-embedded subgroup.

In the other two cases one usually assumes that the composition factors of M are known simple groups. Then a forthcoming paper [MS2] describes the structure of Y_M and $\overline{M}$ in case (2). It generalizes known results about FF-modules.

In case (3) results of Guralnick, Lawther and Malle (see [GM1], [GM2] and [GLM]) use property (3:ii) to determine $\overline{M}$ and Y_M under the additional assumption that $F^*(\overline{M})$ is quasisimple and Y_M is a simple module. The purpose of our paper is to use property (3:i) to determine those $\overline{M}$ and Y_M that do not satisfy this additional assumption. In fact, we prove a stronger result giving more information about the action of M on Y_M, see Theorem 2. In addition, we do not need to assume that the composition factors of M are known simple groups.

We turn (3:i) into a definition:

Let $\mathbb{F}$ be a field, A a group and V an $\mathbb{F}A$-module. Then V is a *nearly quadratic* $\mathbb{F}A$-module (and A acts *nearly quadratically* on V) if $[V, A, A, A] = 0$ and

$$[V, A] + C_V(A) = [v\mathbb{F}, A] + C_V(A) \text{ for every } v \in V \setminus [V, A] + C_V(A).$$

Our main theorems:

Theorem 1 *Let $\mathbb{F}$ be field, H a group and V be a faithful semisimple $\mathbb{F}H$-module. Let $\mathcal{Q}$ be the set of nearly quadratic, but not quadratic subgroups of H. Suppose that $H = \langle \mathcal{Q} \rangle$. Then there exists a partition $(\mathcal{Q}_i)_{i\in I}$ of $\mathcal{Q}$ such that*

(a) $H = \bigtimes_{i\in I} H_i$, where $H_i = \langle \mathcal{Q}_i \rangle$.

(b) $V = C_V(H) \oplus \bigoplus_{i\in I}[V, H_i]$.

(c) For each $i \in I$, $[V, H_i]$ is a faithful simple $\mathbb{F}H_i$-module.

Theorem 2 *Let H be a finite group, and V a faithful simple $\mathbb{F}_pH$-module. Suppose that H is generated by subgroups that act nearly quadratically but not quadratically on V.*

Let W a Wedderburn-component for $\mathbb{F}_p\,\mathrm{F}^*(H)$ on V and $\mathbb{K} := \mathrm{Z}(\mathrm{End}_{\mathrm{F}^*(H)}(W))$. Then W is a simple $\mathbb{F}_p\mathrm{F}^*(H)$-module and one of the following holds:

(I) *$V = W$, $\mathbb{K} = \mathrm{End}_H(W)$, $\mathrm{F}^*(H) = \mathrm{Z}(H)K$, K a component of H and V is a simple $\mathbb{F}_pK$ module.*

(II) *$H, V, W, \mathbb{K}$ and (if $V = W$) $H/C_H(\mathbb{K})$ fulfil one the thirteen cases in Table 1. Moreover, in case 13. H is not generated by abelian nearly quadratic subgroups.*

Some notations used in the above table and through this paper:

All our actions are from the right. We write abc for $(ab)c$, $ab.cd$ for $(ab)(cd)$, $ab.cde.fg$ for $((ab)((cd)e))(fg)$ and so on.

By $C_n, \mathrm{Frob}_n, D_n$ and Q_n, respectively, we denote a cyclic, Frobenius, dihedral or quaternion group of order n, and $\mathbb{F}_q$ is a finite field of order q.

Let $\mathbb{K}$ be a field and V a $\mathbb{K}$-space. Then $\Gamma\mathrm{GL}_{\mathbb{K}}(V)$, $\mathrm{GL}_{\mathbb{K}}(V)$ and $\mathrm{SL}_{\mathbb{K}}(V)$, respectively, denotes the group of semilinear $\mathbb{K}$-isomorphisms, $\mathbb{K}$-isomorphisms, or $\mathbb{K}$-isomorphisms with determinant 1 of V.

Let $\mathbb{K}_0$ be the base field of $\mathbb{K}$ and V_0 a $\mathbb{K}_0$-subspace of V such that the map $\tau : V_0 \otimes_{\mathbb{K}_0} \mathbb{K} \to V$, $v_0 \otimes k \to vk$ is a $\mathbb{K}$-isomorphism. For $\sigma \in \mathrm{Aut}(\mathbb{K})$ let $\tilde{\sigma}$ be the semilinear $\mathbb{K}$-isomorphism of V with $(v_0 \otimes k)\tau\tilde{\sigma} = (v_0 \otimes k\sigma)\tau$. Let $\Gamma\mathrm{SL}_{\mathbb{K}}(V) = \{g\tilde{\sigma} \mid g \in \mathrm{SL}_{\mathbb{K}}(V), \sigma \in \mathrm{Aut}_{\mathbb{K}}(V)\}$. Note that $\Gamma\mathrm{SL}_{\mathbb{K}}(V)$ depends on the choice of V_0, but is unique up to conjugation under $\mathrm{GL}_{\mathbb{K}}(V)$.

$\mathcal{P}_{\mathbb{K}}(V)$ is the set of 1-dimensional $\mathbb{K}$-subspaces of V. For $\mathrm{X} = \mathrm{SL}, \mathrm{GL}$, $\Gamma\mathrm{GL}$ and $\Gamma\mathrm{SL}$ define $\mathrm{PX}_{\mathbb{K}}(V) = \mathrm{X}_{\mathbb{K}}(V)/Z$, where Z is the kernel of the action of $\mathrm{X}_{\mathbb{K}}(V)$ on $\mathcal{P}_{\mathbb{K}}(V)$, so $Z = \mathrm{Z}(\mathrm{GL}_{\mathbb{K}}(V) \cap \mathrm{X}_{\mathbb{K}}(V))$. If $\mathbb{K} = \mathbb{F}_q$ and $V = \mathbb{F}_q^n$ we write $\mathrm{X}_n(q)$ or $\mathrm{X}_n(\mathbb{F}_q)$ for $\mathrm{X}_{\mathbb{F}_q}(\mathbb{F}_q^n)$.

Table 1: The exceptional nearly quadratic modules

	H	V	W	$\mathbb{K}$	$H/C_H(\mathbb{K})$	conditions
1.	$(C_2 \wr \mathrm{Sym}(m))'$	$(\mathbb{F}_3)^m$	$\mathbb{F}_3$	$\mathbb{F}_3$	–	$m \geq 3, m \neq 4$
2.	$\mathrm{SL}_n(\mathbb{F}_2) \wr \mathrm{Sym}(m)$	$(\mathbb{F}_2^n)^m$	$\mathbb{F}_2^n$	$\mathbb{F}_2$	–	$m \geq 2, n \geq 3$
3.	$\mathrm{Wr}(\mathrm{SL}_2(\mathbb{F}_2), m)$	$(\mathbb{F}_2^n)^m$	$\mathbb{F}_2^n$	$\mathbb{F}_4$	–	$m \geq 2$
4.	Frob_{39}	$\mathbb{F}_{27}$	V	$\mathbb{F}_{27}$	C_3	
5.	$\Gamma\mathrm{GL_n}(\mathbb{F}_4)$	$\mathbb{F}_4^n$	V	$\mathbb{F}_4$	C_2	$n \geq 2$
6.	$\Gamma\mathrm{SL_n}(\mathbb{F}_4)$	$\mathbb{F}_4^n$	V	$\mathbb{F}_4$	C_2	$n \geq 2$
7.	$\mathrm{SL}_2(\mathbb{F}_2) \times \mathrm{SL}_n(\mathbb{F}_2)$	$\mathbb{F}_2^2 \otimes \mathbb{F}_2^n$	V	$\mathbb{F}_4$	C_2	$n \geq 3$
8.	$3^{\cdot}\,\mathrm{Sym}(6)$	$\mathbb{F}_4^3$	V	$\mathbb{F}_4$	C_2	
9.	$\mathrm{SL}_n(\mathbb{K}) \circ \mathrm{SL}_m(\mathbb{K})$	$\mathbb{K}^n \otimes \mathbb{K}^m$	V	any	1	$n, m \geq 3$
10.	$\mathrm{SL}_2(\mathbb{K}) \circ \mathrm{SL}_m(\mathbb{K})$	$\mathbb{K}^2 \otimes \mathbb{K}^m$	V	$\mathbb{K} \neq \mathbb{F}_2$	1	$m \geq 2$
11.	$\mathrm{SL}_n(\mathbb{F}_2) \wr C_2$	$\mathbb{F}_2^n \otimes \mathbb{F}_2^n$	V	$\mathbb{F}_2$	1	$n \geq 3$
12.	$(C_2 \wr \mathrm{Sym}(4))'$	$(\mathbb{F}_3)^4$	V	$\mathbb{F}_3$	1	
13.	$\mathrm{SU}_3(2)'$	$\mathbb{F}_4^3$	V	$\mathbb{F}_4$	1	

Let L be a group and m an integer with $m > 1$. Then $\mathrm{Wr}(L, m)$ denotes the augmented wreath-product of L with $\mathrm{Sym}(m)$. That is, $\mathrm{Wr}(L, m)$ is the normal closure of $\mathrm{Sym}(m)$ in $L \wr \mathrm{Sym}(m)$. An elementary argument shows that $L \wr \mathrm{Sym}(m)/\mathrm{Wr}(L, n) \cong L/L'$, so $\mathrm{Wr}(L, n) = L \wr \mathrm{Sym}(n)$ if L is perfect.

The proof of Theorem 1 is straight forward. It is entirely based on an elementary property of groups A that act nearly quadratically but not quadratically on a module V: If V is the direct sum of two A-submodules, then A acts trivial on one of them (see 2.9). This also indicates that non-quadratic nearly quadratic action has some properties stronger than quadratic action.

The proof of Theorem 2 uses two well-known facts: For every finite group H, $\mathrm{F}^*(H)$ is the central product of subgroups $N_1, \ldots, N_r$, where N_i is either a component of H or $N_i = O_q(H)$, q a prime divisor of $\mathrm{F}(H)$, and for every finite dimensional simple $\mathrm{F}^*(H)$-module V, V can be written as a tensor product of N_i-modules V_i.

If in addition V is also an $\mathbb{F}H$-module, the action of H on V can be described explicitly by means of this tensor decomposition. It turns out that the action of a nearly quadratic subgroup on such a tensor decomposition is very restricted. This is then used to determine the exceptions given in Theorem 2.

Similar arguments also give the following theorem, which is a generalization of a result of Chermak [Ch].

Theorem 3 *Let G be a finite group, K a component of G and V a faithful $\mathbb{F}G$-module. Suppose that there exists a p-subgroup $A \leq G$ with $|A/C_A(K)| > 2$ acting nearly quadratically on V. Then $|A/\mathrm{N}_A(K)| \leq 2$ and either $A \leq \mathrm{N}_G(K)$, or $p = 2$ and $K/O_2(K) \cong \mathrm{SL}_n(2)$ or $\mathrm{SL}_2(2^m)$.*

We would like to remark all the results in this paper are proved without using the classification of finite simple groups. In fact, apart from text book results, the proofs are selfcontained.

2 Cubic and Nearly Quadratic Action

In this section A is a group, $\mathbb{F}$ is a field and V an $\mathbb{F}A$-module.

Definition 2.1 *V is a*

(a) quadratic *$\mathbb{F}A$-module if $[V, A, A] = 0$,*

(b) cubic *$\mathbb{F}A$-module if $[V, A, A, A] = 0$,*

(c) nearly quadratic *$\mathbb{F}A$-module if V is a cubic $\mathbb{F}A$-module such that*

$$[V, A] + C_V(A) = [v\mathbb{F}, A] + C_V(A) \text{ for every } v \in V \setminus [V, A] + C_V(A).$$

In the corresponding cases we also say that A acts quadratically, cubically and nearly quadratically on V.

Definition 2.2 *$Q_V(A)$ is the sum of all quadratic $\mathbb{F}A$-submodules of V (and so the largest quadratic $\mathbb{F}A$-submodule of V).*

Definition 2.3 *A system of imprimitivity for A in V is a set Δ of $\mathbb{F}$-subspaces of V such that*

(i) $|\Delta| > 1$ and $\Delta^A = \Delta$, and

(ii) $V = \bigoplus \Delta$ $(= \bigoplus_{W \in \Delta} W)$.

Definition 2.4 *Let $\mathbb{K}$ be a field extension of $\mathbb{F}$ such that V is also a $\mathbb{K}$-vector space, and let $\sigma : A \to \mathrm{Aut}(\mathbb{K})$ be a homomorphism. Then V is a* semi-linear *$\mathbb{K}A$-module with respect to σ provided that $vka = va.k\sigma$ for every $k \in K$, $a \in A$ and $v \in V$. Set $A_{\mathbb{K}} := \ker \sigma$ and $\mathbb{K}_A := C_{\mathbb{K}}(A\sigma)$.*

Lemma 2.5 *Let V be a quadratic $\mathbb{F}A$-module. Then V is a nearly quadratic $\mathbb{F}A$-module.*

Proof: Since A is quadratic, $[V, A] \leq C_V(A) \leq [v\mathbb{F}, A] + C_V(A)$ for all $v \in V$. □

Lemma 2.6 *Let V be a nearly quadratic $\mathbb{F}A$-module and W be an $\mathbb{F}A$-submodule of V. Then the following hold:*

(a) Either $W \leq [V, A] + C_V(A)$ or $[V, A] \leq [W, A] + C_V(A)$.

(b) Either $Q_V(A) = V$ or $Q_V(A) = [V, A] + C_V(A)$.

(c) W and V/W are nearly quadratic $\mathbb{F}A$-modules.

(d) A is quadratic on W or on V/W.

Proof: (a) If $W \not\leq [V,A]+C_V(A)$ the definition of nearly quadratic implies $[V,A] \leq [W,A] + C_V(A)$.

(b) Since A is cubic, $[V,A]+C_V(A) \leq Q_V(A)$. Since A is quadratic on $Q_V(A)$, $[Q_V(A),A] \leq C_V(A)$. By (a) with $W := Q_V(A)$ either $Q_V(A) \leq [V,A]+C_V(A)$ or $[V,A] \leq [Q_V(A),A]+C_V(A)$. In the first case $[V,A]+C_V(A) = Q_V(A)$. In the second case $[V,A] \leq C_V(A)$, so A acts quadratically on V and $V = Q_V(A)$.

(c) and (d) We first show that $\overline{V} := V/W$ is a nearly quadratic $\mathbb{F}A$-module. Let $v \in V$ with $\overline{v} \notin [\overline{V},A]+C_{\overline{V}}(A)$. Since $\overline{[V,A]+C_V(A)} \leq [\overline{V},A]+C_{\overline{V}}(A)$ we get that $v \notin [V,A]+C_V(A)$. Thus $[V,A] \leq [v\mathbb{F},A]+C_V(A)$ and $[\overline{V},A] \leq [\overline{v}\mathbb{F},A]+C_{\overline{V}}(A)$. Hence $\overline{V}$ is nearly quadratic for A.

To show that A is nearly quadratic on W and is quadratic on W or V/W we may assume that A is not quadratic on W, so $W \not\leq Q_V(A)$. Then by (a) and (b) $[V,A] \leq [W,A]+C_V(A)$. It follows that $[\overline{V},A] \leq \overline{C_V(A)}$ and A is quadratic on $\overline{V}$. Hence (d) holds.

Moreover, $Q_V(A) = [V,A]+C_V(A) = [W,A]+C_V(A)$ and so $Q_W(A) = W \cap Q_V(A) = [W,A] + C_W(A)$. Hence if $w \in W$ with $w \notin [W,A]+C_W(A)$, then $w \notin Q_V(A) = [V,A]+C_V(A)$ and so $[V,A] \leq [\mathbb{F}w,A]+C_V(A)$. Thus $[W,A] \leq [V,A] \cap W \leq [w\mathbb{F},A]+C_W(A)$, and W is nearly quadratic. So also (c) is proved. $\square$

Lemma 2.7 *Let V be a cubic $\mathbb{F}A$-module and put $A_0 = C_A(Q_V(A))$. Then the following hold:*

(a) A_0 acts quadratically on V.

(b) For $z \in \mathbb{Z}$, $a \in A$ and $v \in V$ with $[v,a,a]=0$,

$$[v,a]z = [v,a^z] = [vz,a].$$

(c) If A acts quadratically on V, then $A/C_A(V)$ is an elementary abelian $\operatorname{char}\mathbb{F}$*-group.* [1]

(d) $A_0 \trianglelefteq A$, and A/A_0 and $A_0/C_A(V)$ are elementary abelian $\operatorname{char}\mathbb{F}$*-groups.*

(e) If $\operatorname{char}\mathbb{F}=0$*, then all non-trivial elements in $A/C_A(V)$ have infinite order. If* $\operatorname{char}\mathbb{F}$ *is a prime, then $A/C_A(V)$ is a* $\operatorname{char}\mathbb{F}$*-group.*

Proof: (a): Since A is cubic, $[V,A_0] \leq [V,A] \leq Q_V(A) \leq C_V(A_0)$.

(b): Note that $[v,a]$ is centralized by a and v. So (b) follows from [Gor, II2.2(i)]

(c): Since $[V,A,A]=0=[A,V,A]$ the Three Subgroup Lemma gives $[A,A,V]=0$. So $A/C_A(V)$ is abelian. Now let $a \in A$ and $v \in V$ with $[v,a] \neq 0$. Let i be a positive integer and put $p := \operatorname{char}\mathbb{F}$. By (b) $[v,a^i] = [v,a]i$. If $p>0$ we conclude that $[v,a^p]=0$ and if $p=0$, then $[v,a^i] \neq 0$. So $aC_A(V)$ has order p in $A/C_A(V)$ if $p>0$ and infinite order if $p=0$.

(d): This follows from (c), since A acts quadratically on $Q_V(A)$ and A_0 acts quadratically on V.

(e) follows immediately from (d). $\square$

Lemma 2.8 *Let $a \in A$ and $v \in V$. Suppose that* $\operatorname{char}\mathbb{F}=2$*. Then $[v,a^2] = [v,a,a]$ and $[V,a^2] = [V,a,a]$.*

Proof: This follows for example since $(a-1)^2 = a^2-1$ in $\operatorname{End}_{\mathbb{F}}(V)$. $\square$

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[1] Here an elementary abelian m-group is an abelian group all of its non-trivial elements have order m, if m is a prime, and infinite order if $m=0$.

Lemma 2.9 *Let V be a nearly quadratic, but not quadratic $\mathbb{F}A$-module and X and Y be $\mathbb{F}A$-submodules of V such that $V = X \oplus Y$. Then at least one of the submodules X and Y is centralized by A.*

Proof: Since $V = X \oplus Y$ and V is not quadratic at least one of the summands, say X, is not a quadratic $\mathbb{F}A$-module. Then by 2.6(a),(b)

$$[V, A] + C_V(A) = [X, A] + C_V(A) = Q_V(A).$$

In particular, $[V, A, A] = [X, A, A] \leq X$ and $[Y, A, A] \leq Y \cap X = 0$. Hence $Y \leq Q_V(A) = [X, A] + C_V(A)$ and $[Y, A] \leq Y \cap [X, A, A] \leq Y \cap X = 0$. □

Lemma 2.10 *Suppose that Δ is a system of imprimitivity for A in V. Let Δ_1 be an orbit for A on Δ and $\Delta_0 \subseteq \Delta_1$. Then each of following conditions implies that $\Delta_0 = \Delta_1$.*

1. $\bigoplus \Delta_0 \cap C_V(A) \neq 0$.
2. *A is cubic and* $\bigoplus \Delta_0 \cap [V, A, A] \neq 0$.
3. *A is quadratic and* $\bigoplus \Delta_0 \cap [V, A] \neq 0$.

Proof: Put $U = \bigoplus \Delta_0$. Observe that each of the conditions (2) and (3) imply (1), so we may assume that $C_U(A) \neq 0$. For $X \in \Delta$ let π_X be the projection of V onto X. Set

$$\Delta_2 := \{X \in \Delta_1 \mid C_U(A)\pi_X \neq 0\}.$$

Since $U\pi_X = 0$ for all $X \in \Delta_1 \setminus \Delta_0$ we have $\Delta_2 \subseteq \Delta_0$. Since $\Delta_2 \neq \emptyset$ and Δ_2 is A-invariant we conclude that $\Delta_2 = \Delta_1$ and so also $\Delta_0 = \Delta_1$. □

Lemma 2.11 *Let V be a quadratic $\mathbb{F}A$-module, Δ a system of imprimitivity for A in V, Δ_1 a non-trivial orbit for A on Δ, and $W \in \Delta_1$. Then $|\Delta_1| = \operatorname{char} \mathbb{F} = |A/C_A(\bigoplus \Delta_1)| = 2$.*

Proof: Let $W \in \Delta_1$ and $B = N_A(W)$. Then $\{W\} \neq \Delta_1$ and so by 2.10(3), $W \cap [V, A] = 0$. In particular, $[W, B] = 0$.

Let $a \in A \setminus B$. Then $0 \neq [W, a] \leq W + W^a$ and so by 2.10(3), $\Delta_1 = \{W, W^a\}$ and $|A/B| = 2$. In particular, $B \trianglelefteq A$ and $B = C_A(\bigoplus \Delta_1)$, so

$$|\Delta_1| = |A/C_A(\bigoplus \Delta_1)| = 2.$$

Moreover, 2.7(e) gives $\operatorname{char} \mathbb{F} = 2$. □

Lemma 2.12 *Let V be a cubic $\mathbb{F}A$-module, Δ a system of imprimitivity for A in V, Δ_1 a non-trivial orbit for A on Δ, and $W \in \Delta_1$. Then*

(a) $A/C_A(\bigoplus \Delta_1)$ is an elementary abelian p-group for some prime p.

(b) $p = \operatorname{char} \mathbb{F} \in \{2, 3\}$.

(c) One of the following holds:

1. $|A/C_A(\bigoplus \Delta_1)| = |\Delta_1| \le 4$ *and* $N_A(W) = C_A(\bigoplus \Delta_1) = C_A(\Delta_1)$
2. $p = |\Delta_1| = 2$ *and* $N_A(W) = C_A(\Delta_1)$ *acts quadratically on* $\bigoplus \Delta_1$.

Proof: We may assume without loss that $V = \bigoplus \Delta_1$ and V is a faithful $\mathbb{F}A$-module. If A is quadratic on V, then the lemma follows from 2.11. Hence we may assume

1° *A is not quadratic on V.*

Next we prove

2° *Suppose* $\operatorname{char} \mathbb{F} = 2$. *Then A is an elementary abelian 2-group.*

Let $a \in A$ and suppose that $a^2 \neq 1$. Then there exists $W \in \Delta_1$ with $[W, a^2] \neq 0$. Put $\Delta_0 = \{W, W^{a^2}\}$. By 2.8 $[W, a, a] = [W, a^2] \le \bigoplus \Delta_0$. Hence 2.10(2) implies that $\Delta_1 = \Delta_0$. Thus $\Delta_1 = \{W, W^a\}$ and so a^2 acts trivially on Δ_1. But then $W = W^{a^2}$ and $\Delta_1 = \Delta_0 = \{W\}$, a contradiction.

This shows that $a^2 = 1$ for all $a \in A$, and (1°) holds.

3° *Let $W \in \Delta_1$ and $a, b \in V$ with $[W, b, a] \neq 0$. Then $\Delta_1 = \{W, W^b, W^a, W^{ba}\}$.*

Note that $[W, b] \le W + W^b$ and so also $[W, b, a] \le W + W^b + W^a + W^{ba}$. Hence 2.10(2) implies that $\Delta_1 = \{W, W^b, W^a, W^{ba}\}$.

4° $|\Delta_1| \le 4$

By (1°) there exist $a, b \in A$ with $[V, b, a] \neq 0$. Since $V = \bigoplus \Delta_1$ there exists $W \in \Delta_1$ with $[W, b, a] \neq 0$ and so (4°) follows from (3°).

Case 1 *Suppose that $[W, N_A(W)] \neq 0$ for some $W \in \Delta_1$.*

Pick $b \in B := N_A(W)$ with $[W, b] \neq 0$ and $a \in A \setminus B$. Since $[W, b] \le W$ we get $[W, b, a] \neq 0$, so (3°) yields $\Delta_1 = \{W, W^a\}$ and $|A/B| = 2$, in particular $B \trianglelefteq A$. Since $\Delta_1 \neq \{W\}$, 2.10(2) gives $[W, B, B] = 0$, so B acts quadratically on $W \oplus W^a = V$.

Since A is cubic on V, A is quadratic on $[V, A]$ and thus also on $[W, B] \oplus [W^a, B]$. Hence 2.11, applied to A and $[W, B] \oplus [W^a, B]$, shows that $p = 2$. Thus (c:2) holds. By (2°), A is elementary abelian and so the lemma holds in this case.

Case 2 *Suppose that $[W, N_A(W)] = 0$ for all $W \in \Delta_1$.*

Note that $C_A(\Delta_1) = 1$ in this case. Hence by (4°) A is isomorphic to a subgroup of $\mathrm{Sym}(4)$. Put $p = \operatorname{char} \mathbb{F}$. By 2.7(e), $p > 0$ and A is a p-group. Hence (b) holds. Moreover, if $|\Delta_1| \le 3$ we conclude that $|\Delta_1| = p = |A|$ and the lemma holds. In the other case (4°) shows that $|\Delta_1| = 4$. Hence $p = 2$ and by (2°), A is elementary abelian. Since A acts transitively and faithfully on Δ_1, this implies $|A| = 4$ and $N_A(W) = C_A(\Delta_1) = 1$ for $W \in \Delta_1$. Again the lemma holds. □

Lemma 2.13 *Let V be a nearly quadratic $\mathbb{F}A$-module, and let Δ be a system of imprimitivity for A in V. Then one of the following holds:*

1. *A acts trivially on Δ and there exists at most one $W \in \Delta$ with $[W, A] \neq 0$.*

2. *A acts trivially on Δ and quadratically on V.*

3. *A acts quadratically on V, $\operatorname{char}\mathbb{F}=2$, and $|A/C_A(W)|\leq 2$ for every $W\in\Delta\setminus C_\Delta(A)$.*

4. *A does not act quadratically on V, $A/C_A(V)$ is elementary abelian and there exists a unique A-orbit $W^A\subseteq\Delta$ with $[W,A]\neq 0$. Moreover, $B:=N_A(W)$ acts quadratically on V, $B=C_A(\Delta)$ and one of the following holds:*

   1. *$\operatorname{char}\mathbb{F}=2$, $|W^A|=4$, $\dim_{\mathbb{F}} W=1$, $B=C_A(V)$, and $A/C_A(V)\cong C_2\times C_2$.*
   2. *$\operatorname{char}\mathbb{F}=3$, $|W^A|=3$, $\dim_{\mathbb{F}} W=1$, $B=C_A(V)$, and $A/C_A(V)\cong C_3$.*
   3. *$\operatorname{char}\mathbb{F}=2$, $|W^A|=2$, and $C_A(W)=C_A(V)$. Moreover, $\dim_{\mathbb{F}} W/C_W(B)=1$ and $C_W(B)=[W,B]$.*

Proof: Suppose first that A acts quadratically on V. Then 2.11 shows that (2) or (3) holds.

Suppose next that A does not act quadratically on V. Pick $W\in\Delta$ with $[W,A]\neq 0$ and set

$$B:=N_A(W),\ U:=\bigoplus W^A \text{ and } U_0:=\bigoplus C_\Delta(A).$$

By 2.9 $[E,A]=0$ for all $E\in\Delta\setminus W^A$. It follows that $V=U\oplus U_0$ and $[U_0,A]=0$. Thus after replacing V by U, we may assume that $\Delta=W^A$.

If $W^A=W$ then (1°) holds. Thus we may assume that $|W^A|\geq 2$. We prove next that

1° *W is not the direct sum of two proper $\mathbb{F}B$-submodules.*

Suppose that $W=W_1\oplus W_2$ for some proper $\mathbb{F}B$-submodules W_1 and W_2. Then

$$V=\bigoplus W_1^A\oplus\bigoplus W_2^A,$$

and A acts non-trivially on both direct summands. But this contradicts 2.9.

Note that we can apply 2.12. In particular, $\operatorname{char}\mathbb{F}$ is a prime $p\in\{2,3\}$, and $A/C_A(V)$ is an elementary abelian p-group. We now discuss the two cases given in 2.12(c) separately.

Suppose that 2.12(c:1) holds. Then $[W,B]=0$, and (1°) gives $\dim_{\mathbb{F}} W=1$. In addition $|A/C_A(V)|>2$ since A is not quadratic on V. Thus (4:1) or (4:2) holds in this case.

Suppose that 2.12(c:2) holds. Then $|A/B|=2$ and B is quadratic on W, so $[W,B]\leq C_W(B)$. Moreover, as above $[W,B]\neq 0$ since A is not quadratic on V. Pick an $\mathbb{F}$-subspace $W_1\leq W$ with $W_1\cap C_W(B)=[W,B]$ and $W_1+C_W(B)=W$. Also pick an $\mathbb{F}$-subspace $W_2\leq C_W(B)$ with $C_W(B)=W_2\oplus[W,B]$. Then $W=W_1\oplus W_2$ and W_1 and W_2 are $\mathbb{F}B$-submodules of W. Thus by (1°), $W_2=0$ and so $C_W(B)=[W,B]$.

Let $a\in A\setminus B$ and $w\in W\setminus C_W(B)$. Put $W_0=w\mathbb{F}+C_W(B)$ and $V_0=\langle W_0^A\rangle$. Then $V=W+W^a$ and $V_0=W_0+W_0^a$.

By 2.6(b), $Q_V(A)=[V,A]+C_V(A)$. Since $[W_0,B]\neq 0$ we have $[V_0,B,A]\neq 0$. Thus $V_0\not\leq Q_V(A)=[V,A]+C_V(A)$ and so by 2.6(a), $[V,A]\leq V_0=W_0+W_0^a$. This gives

$$V=W\oplus W^a=W+[V,A]=W+W_0+W_0^a=W+W_0^a$$

Hence $W^a=W_0^a$, $W=W_0$, and $C_W(B)=[W,B]$ is an $\mathbb{F}$-hyperplane in W. Thus (4:3) holds. □

Lemma 2.14 *Let $\mathbb{K}$ be a field, $1\neq A\leq\operatorname{Aut}(\mathbb{K})$, and $\mathbb{E}:=C_{\mathbb{K}}(A)$. Suppose that $\mathbb{K}$ is a cubic $\mathbb{E}A$-module. Then* *(a)* $p := \operatorname{char} \mathbb{K} \in \{2, 3\}$.

(b) *A is an elementary abelian p-group and $A = \operatorname{Aut}_{\mathbb{E}}(\mathbb{K})$.*

(c) $\dim_{\mathbb{E}} \mathbb{K} = |A|$ *and* $\mathbb{K} \cong \mathbb{E}A$ *as an* $\mathbb{E}A$*-module.*

(d) *One of the following holds:*

1. $|A| = 2$, *A acts quadratically on $\mathbb{K}$ and* $[\mathbb{K}, A] = \mathbb{E}$.
2. $|A| = 3$, *A does not act quadratically on V and* $[\mathbb{K}, A, A] = \mathbb{E}$.
3. $A \cong C_2 \times C_2$, *A does not act quadratically on V,* $[\mathbb{K}, A, A] = \mathbb{E}$ *and $\mathbb{K}$ is infinite.*

Proof: We consider first the case where A is cyclic. Then $A = \langle \sigma \rangle$ for some $\sigma \in A$ and $C_{\mathbb{K}}(\sigma) = \mathbb{E}$, so $C_{\mathbb{K}}(\sigma)$ is 1-dimensional over $\mathbb{E}$. Since σ acts cubically on V we have $(\sigma - 1)^3 = 0$. So σ is unipotent with Jordan blocks of size at most 3. As $\dim C_{\mathbb{K}}(\sigma) = 1$, σ has most one Jordan block, so $2 \le \dim_{\mathbb{E}} \mathbb{K} \le 3$.

Note that $\mathbb{K}$ over $\mathbb{E}$ is a finite Galois extension since the fixed field of the Galois group $\operatorname{Aut}_{\mathbb{E}}(\mathbb{K})$ is $\mathbb{E}$, so $|A| = |\operatorname{Aut}_{\mathbb{E}}(\mathbb{K})| = \dim_{\mathbb{E}} \mathbb{K} \in \{2, 3\}$. Since A is cubic we conclude from 2.7(e) that $\operatorname{char} \mathbb{E} = |A|$. Let $k \in \mathbb{K} \setminus [\mathbb{K}, A]$. Then it is easy to see that k^A is an $\mathbb{E}$-basis for $\mathbb{K}$ and so $\mathbb{K} \cong \mathbb{E}A$ as an $\mathbb{E}A$-module.

We now consider the general case and use the cyclic case we have treated already. Let $1 \ne \sigma \in A$ and put $\mathbb{L} = C_{\mathbb{K}}(\sigma)$. Then by the cyclic case $p = \operatorname{char} \mathbb{K} = |\sigma| = \dim_{\mathbb{L}} \mathbb{K}$.

Suppose that $p = 2$ and A acts quadratically on $\mathbb{K}$ or that $p = 3$. If $p = 2$, then $\mathbb{L} = [\mathbb{K}, \sigma]$ and if $p = 3$, then $\mathbb{L} = [\mathbb{K}, \sigma, \sigma]$. So in any case $\mathbb{L} \le C_{\mathbb{K}}(A) = \mathbb{E}$. Thus $\dim_{\mathbb{E}} \mathbb{K}$ =p and also $|\operatorname{Aut}_{\mathbb{L}}(\mathbb{K})| = p$. Thus $A = \langle \sigma \rangle$ and the lemma holds.

Suppose that $p = 2$ and A is not quadratic on $\mathbb{K}$. Then $A \ne \langle \sigma \rangle$ and there exists $\mu \in A$ with $\mu \notin \langle \sigma \rangle$. On the other hand the cyclic case implies $\mathbb{L} = [\mathbb{K}, \sigma]$. Hence

$$[\mathbb{L}, \mu] \le [\mathbb{K}, A, A] \le C_{\mathbb{K}}(A) = \mathbb{E} \le \mathbb{L},$$

so $\mathbb{L}^\mu = \mathbb{L}$. Since $\operatorname{Aut}_{\mathbb{L}}(\mathbb{K}) = \langle \sigma \rangle$, $[\mathbb{L}, \mu] \ne 0$. The cyclic case applied to $\mathbb{L}$ in place of $\mathbb{K}$ shows that $\dim_{[\mathbb{L}, \mu]} \mathbb{L} = 2$. We conclude that $[\mathbb{L}, \mu] = \mathbb{E}$ and so $\dim_{\mathbb{E}} \mathbb{K} = 4$. It follows that $A = \operatorname{Aut}_{\mathbb{E}}(\mathbb{K})$ and $|A| = 4$. Since $\sigma^2 = 1$ for all $\sigma \in A$, A is elementary abelian. Pick $k \in \mathbb{K} \setminus [\mathbb{K}, A]$, then k^A is a $\mathbb{E}$-basis for $\mathbb{K}$ and so $\mathbb{K} \cong \mathbb{E}A$ as an $\mathbb{E}A$-module. Since the automorphism group of a finite field is cyclic, $\mathbb{K}$ is infinite. □

Lemma 2.15 *Let V be a semi-linear, cubic $\mathbb{K}A$-module. Put $\mathbb{E} = \mathbb{K}_A$ and suppose that $\mathbb{E} \ne \mathbb{K}$. Then one of the following holds:*

1. $|A/C_A(V)| = \operatorname{char} \mathbb{E} = \dim_{\mathbb{E}} \mathbb{K} = 2$, $A_{\mathbb{K}} = C_A(V)$, *A is quadratic on V, and as an $\mathbb{E}A$-module V is the direct sum of $\mathbb{E}A$-submodules isomorphic to $\mathbb{K}$.*

2. $|A/C_A(V)| = \operatorname{char} \mathbb{E} = \dim_{\mathbb{E}} \mathbb{K} = 3$, $A_{\mathbb{K}} = C_A(V)$, *A is not quadratic on V, and as an $\mathbb{E}A$-module, V is the direct sum of $\mathbb{E}A$-submodules isomorphic to $\mathbb{K}$.*

3. $A/C_A(V) \cong C_2 \times C_2$, $A_{\mathbb{K}} = C_A(V)$, $\operatorname{char} \mathbb{E} = 2$, $\dim_{\mathbb{E}} \mathbb{K} = 4$, *$\mathbb{K}$ is infinite, A is not quadratic on V, and as an $\mathbb{E}A$-module, V is the direct sum of $\mathbb{E}A$-submodules isomorphic to $\mathbb{K}$.*

4. $|A/A_{\mathbb{K}}| = \operatorname{char} \mathbb{E} = \dim_{\mathbb{E}} \mathbb{K} = 2$, *$A$ is not quadratic on V, $A/C_A(V)$ is elementary abelian, and there exists an $\mathbb{E}A$ submodule W of V such that $V \cong W \otimes_{\mathbb{E}} \mathbb{K}$ as an $\mathbb{E}A$-module,* $A = C_A(W)A_{\mathbb{K}}$, *and $A_{\mathbb{K}}$ acts quadratically on V and W.*

Proof: We may assume that A acts faithfully on V and $V \neq 0$.

Case 1 *Suppose that $A_{\mathbb{K}} = 1$.*

By Zorn's Lemma there exists a subset $\mathcal{B} \subseteq C_V(A)$ that is maximal with respect to being linearly independent over $\mathbb{K}$. Since A is cubic, $C_V(A) \neq 0$ and so $\mathcal{B} \neq \emptyset$.

Let U be the $\mathbb{K}$-span of $\mathcal{B}$ and $b \in \mathcal{B}$. Then $b\mathbb{K}$ is isomorphic to $\mathbb{K}$ as an $\mathbb{E}A$-module. So A acts cubicly on $\mathbb{K}$. Since $A_{\mathbb{K}} = 1$, A acts faithfully on $\mathbb{K}$ and we can apply 2.14. It follows that $|A| \leq 4$, A is elementary abelian and, if $|A| = 4$, $\mathbb{K}$ is infinite. Moreover, either

$$\operatorname{char} \mathbb{E} = |A| = 2,\ [\mathbb{K}, A] = \mathbb{E} \text{ and } [\mathbb{K}, A, A] = 0, \quad \text{or } |A| > 2 \text{ and } [\mathbb{K}, A, A] = \mathbb{E}.$$

Suppose that $U \neq V$. Then V/U has an $\mathbb{E}A$-submodule isomorphic to $\mathbb{K}$. Hence $[V/U, A] \neq 0$, and, if $|A| > 2$, $[V/U, A, A] \neq 0$. So if $|A| = 2$ we can choose $v \in [V, A]$ with $v \notin U$, and if $|A| > 2$ we can choose $v \in [V, A, A]$ with $v \notin U$.

If $|A| = 2$, then $\operatorname{char} \mathbb{E} = 2$ and A acts quadratically on V. So in any case $v \in C_V(A)$. Since U is a $\mathbb{K}$-subspace, $\mathcal{B} \cup \{v\}$ is linearly independent over $\mathbb{K}$, a contradiction to the maximality of $\mathcal{B}$.

Thus $U = V$ and so $V = \bigoplus_{b \in \mathcal{B}} b\mathbb{K}$ is a direct sum of copies of $\mathbb{K}$ as an $\mathbb{E}A$-module. Now 2.14 shows that one of (1), (2) and (3) holds.

Case 2 *Suppose $A_{\mathbb{K}} \neq 1$.*

Note that $[V, A_{\mathbb{K}}, A_{\mathbb{K}}]$ is a $\mathbb{K}$-subspace of V centralized by A. Since A does not act $\mathbb{K}$-linearly on V, we get $[V, A_{\mathbb{K}}, A_{\mathbb{K}}] = 0$. Moreover, $A/A_{\mathbb{K}}$ acts quadratically and faithfully on the non-trivial $\mathbb{K}$-space $[V, A_{\mathbb{K}}]$ and so (Case 1) shows that $|A/A_{\mathbb{K}}| = 2$ and $\dim_{\mathbb{E}} \mathbb{K} = 2$.

Let $a \in A$. By 2.8 $[V, a^2] = [V, a, a]$. Since $a^2 \in A_{\mathbb{K}}$ we conclude that $[V, a^2]$ is a $\mathbb{K}$-subspace centralized by A. Thus $[V, a^2] = 0$ and A is elementary abelian. Let $a \in A \setminus A_{\mathbb{K}}$ and put $W := C_V(a)$. Then W is an $\mathbb{E}$-subspace of V. By (Case 1) applied to $\langle a \rangle$, $W = [V, a]$. Hence $[W, A] = [V, a, A] \leq C_V(A) \leq W$. Hence W is an quadratic $\mathbb{E}A$-submodule of V. Since $a \in C_A(W)$, $A = C_A(W)A_{\mathbb{K}}$.

By the universal property of the tensor product, there exists an $\mathbb{E}A$-homomorphism $\rho : W \otimes_{\mathbb{E}} \mathbb{K} \to V$ with $(w \otimes k)\rho = wk$ for all $w \in W$ and $k \in \mathbb{K}$. By (Case 1) applied to $\langle a \rangle$, ρ is a bijection. Thus (4) holds in this case. □

3 Tensor Decomposition

Lemma 3.1 *Let $\mathbb{K}$ be a field, V a $\mathbb{K}$-space of dimension at least 2 and $\mathbb{F}$ a subfield of $\mathbb{K}$, and let $\alpha \in \mathrm{GL}_{\mathbb{F}}(V)$ with $v\mathbb{K}\alpha = v\mathbb{K}$ for all $v \in V$. Then there exists $k \in \mathbb{K}$ with $v\alpha = vk$ for all $v \in V$.*

Proof: Let $0 \neq v \in V$. Then by assumption $v\alpha = vk_v$ for a unique $k_v \in \mathbb{K}$. Let $0 \neq w \in V$. It suffices to show that $k_v = k_w$.

Suppose first that $v\mathbb{K} \neq w\mathbb{K}$. We have

$$vk_v + wk_w = v\alpha + w\alpha = (v + w)\alpha = (v + w)k_{v+w} = vk_{v+w} + wk_{v+w}.$$

Since v and w are linearly independent over $\mathbb{K}$ we conclude that $k_v = k_{v+w} = k_w$.

Suppose next that $v\mathbb{K} = w\mathbb{K}$. Since V is at least two dimensional over $\mathbb{K}$ there exists $u \in V \setminus v\mathbb{K}$. Thus by the preceding case $k_v = k_u = k_w$. □

Definition 3.2 *Let $\mathbb{K}$ be a field, G group and V a quadratic $\mathbb{K}G$-module.*

(a) We say that G acts $\mathbb{K}$-commutator dependently on V if $[v,a]\mathbb{K} = [v,b]\mathbb{K}$ for all $a,b \in G \setminus C_G(V)$ and $v \in V$.

(b) Let $\lambda : G \to (\mathbb{K},+)$ be a homomorphism. We say that G acts λ-dependently on V if there exists $\alpha \in \mathrm{End}_{\mathbb{K}}(V)$ with $\alpha^2 = 0$ and $[v,a] = v\alpha.a\lambda$ for all $a \in G$ and $v \in V$.

Lemma 3.3 *Let $\mathbb{K}$ be a field, G a group, and V a quadratic $\mathbb{K}G$-module. Then G acts $\mathbb{K}$-commutator dependently on V iff G acts λ-dependently on V for some homomorphism $\lambda : G \to (\mathbb{K},+)$.*

Proof: If G acts λ-dependently on V, then clearly G acts $\mathbb{K}$-commutator dependently on V.

Suppose now that G is $\mathbb{K}$-commutator dependent on V and fix $a \in G \setminus C_G(V)$. Define

$$\alpha : V \to V, v \mapsto [v,a].$$

Since G is quadratic on V, $\alpha^2 = 0$.

Let $b \in G \setminus C_G(V)$. Then by assumption $[v,a]\mathbb{K} = [v,b]\mathbb{K}$ for all $v \in V$. Thus $C_V(a) = C_V(b)$ and $[V,a] = [V,b]$. Hence we obtain $\mathbb{K}$-isomorphisms

$$\beta : V/C_V(a) \to [V,a],\ v + C_V(a) \mapsto [v,a] \text{ and } \gamma : V/C_V(a) \to [V,a],\ v + C_V(a) \mapsto [v,b].$$

Put $\delta = \gamma\beta^{-1}$. From $[v,a]\mathbb{K} = [v,b]\mathbb{K}$ for all $v \in V$ we conclude that $u\mathbb{K}\delta = u\mathbb{K}$ for all $u \in V/C_V(a)$. Thus by 3.1 there exists $k_b \in \mathbb{K}$ with $u\delta = uk_b$ for all $u \in V/C_V(a)$. Hence $u\gamma = uk_b\beta = u\beta k_b$ for all $u \in V/C_V(a)$ and so $[v,b] = [v,a]k_b = v\alpha k_b$ for all $v \in V$.

For $b \in C_G(V)$ put $k_b = 0$. Define $\lambda : G \to \mathbb{K}, b \mapsto k_b$. Then for all $v \in V$ and $b \in G$, $[v,b] = v\alpha.a\lambda$. Let $b,c \in G$. Using the quadratic action, $[v,bc] = [v,b] + [v,c]$ and so $bc\lambda = b\lambda + c\lambda$. Thus λ is a homomorphism and G acts λ-dependently on V. $\square$

Lemma 3.4 *Let $\mathbb{K}$ be a field , G a group, $\lambda : G \to (\mathbb{K},+)$ a homomorphism, and V a λ-dependent $\mathbb{K}G$-module. Let W_λ be the $\mathbb{K}G$-module with $W_\lambda = \mathbb{K}^2$ as $\mathbb{K}$-space and $(k,l)a = (k, l + k.a\lambda)$ for $a \in G$. Then $V = W \oplus C$, where W and C are $\mathbb{K}G$ submodules of V such that G centralizes C and W is the direct sum of $\mathbb{K}G$-submodules isomorphic to W_λ.*

Proof: By the definition of λ-dependent there exists $\alpha \in \mathrm{End}_{\mathbb{K}}(V)$ with $\alpha^2 = 0$ and $[v,a] = v\alpha.a\lambda$ for all $v \in V, a \in G$. Choose $\mathcal{V} \subseteq V$ such that $(v + C_V(G))_{v \in \mathcal{V}}$ is a $\mathbb{K}$-basis for $V/C_V(A)$. For $v \in V$ put $W_v = \langle v, v\alpha \rangle_{\mathbb{K}}$. Let C be a $\mathbb{K}$-subspace of $C_V(G)$ with $C_V(G) = [V,G] \oplus C$. Then it is readily verified that W_v is a $\mathbb{K}$-subspace of V isomorphic to W_λ and $V = C \oplus \bigoplus_{v \in \mathcal{V}} W_v$. $\square$

Definition 3.5 *Let $\mathbb{F}$ be a field and V an $\mathbb{F}$-space.*

(a) A tensor decomposition *$\mathcal{V}$ of V is a tuple $(\Phi, \mathbb{K}, (V_i, i \in I))$ where $\mathbb{F} \leq \mathbb{K}$ is a field extension, $(V_i, i \in I)$ is a finite family of pairwise disjoint $\mathbb{K}$-spaces, and $\Phi : \bigotimes_{i \in I}^{\mathbb{K}} V_i \to V$ is an $\mathbb{F}$-isomorphism.*

(b) A tensor decomposition $\mathcal{V}$ (as in (a)) is called proper *if $|I| > 1$ and $\dim_{\mathbb{K}} V_i \geq 2$ for all $i \in I$.*

Notation 3.6 *Let $\mathbb{K}$ be a field and $(V_i, i \in I)$ be a finite family of pairwise disjoint $\mathbb{K}$-spaces. For $J \subseteq I$ let*

$$V_J := \bigotimes_{j\in J}^{\mathbb{K}} V_j \text{ and } V^J := V_{I\setminus J},$$

and for 1-element sets we write V^i rather than $V^{\{i\}}$.

Let $(u_i, i \in I)$ be a tuple of elements such that there exists $\pi \in \mathrm{Sym}(I)$ with $u_{i\pi} \in V_i$. Then $\otimes_{i\in I}u_i$ (or just $\otimes u_i$) denotes the element $\otimes_{i\in I}u_{i\pi}$ in $V_I = \otimes_{i\in I}^{\mathbb{K}}V_i$. (Note here that π and thus $\otimes_{i\in I}u_i$ is uniquely determined by the elements $(u_i, i \in I)$ since the spaces V_i are pairwise disjoint.) In the same spirit we identify $V_J \otimes V^J$ with V_I.

Definition 3.7 *Let G be a group, $\mathbb{F}$ a field and V an $\mathbb{F}G$-module.*

(a) A G-invariant tensor decomposition $\mathcal{T} = (\Phi, \mathbb{K}, (V_i, i \in I), \sigma, (g_i; g \in G, i \in I))$ of V is a tuple consisting of

  (i) a tensor decomposition $(\Phi, \mathbb{K}, (V_i, i \in I))$ of V;

  (ii) an action $I \times G \to I, (i, g) \mapsto ig$ of G on I;

  (iii) a homomorphism $\sigma : G \to \mathrm{Aut}_{\mathbb{F}}(\mathbb{K})$; and

  (iv) a family $(g_i; g \in G, i \in I)$ of maps such that for each $g \in G$ and $i \in I$, $g_i : V_i \to V_{ig}$ is a $g\sigma$-linear map from V_i to V_{ig}. such that

    (i) $(\otimes_{i\in I}v_i)\Phi g = (\otimes_{i\in I}v_ig_i)\Phi$ for all $g \in G$, and

    (ii) for each $g, h \in G$ and $i \in I$ there exists an element $\lambda_{i,g,h} \in \mathbb{K}^\sharp$ with

$$g_ih_{ig} = (gh)_i\lambda_{i,g,h}.$$

(b) A G-invariant tensor decomposition as in (a) is called strict if $\lambda_{i,g,h} = 1$ for all i, g, h, that is if

$$g_ih_{ig} = (gh)_i$$

for all $g, h \in G$ and $i \in I$.

(c) A G-invariant tensor decomposition is called regular if the action of G on I is trivial.

(d) A G-invariant tensor decomposition is called $\mathbb{K}$-linear if $G\sigma = 1$, that is if G acts $\mathbb{K}$-linearly on V.

(e) A G-invariant tensor decomposition is ordinary if its $\mathbb{K}$-linear, regular and strict.

Abusing notation we will often say that $\Phi : \bigotimes_{i\in I}^{\mathbb{K}} V_i \to V$ is a G-invariant tensor decomposition of V, assuming that the remaining parts of a tensor decomposition are just as in 3.7(a).

Definition 3.8 *Let G be a group, $\mathbb{K}$ a field, and $\sigma : G \to \mathrm{Aut}(\mathbb{K})$ a homomorphism. A projective σ-linear $\mathbb{K}G$–module is a $\mathbb{K}$-space V together with a map $V \times G \to V, (v, g) \mapsto vg$, such that the following hold:*

(i) For each $g \in G$, the map $V \to V, v \mapsto vg$, is $g\sigma$-linear.

(ii) For each $g, h \in G$ there exists $\lambda_{g,h} \in \mathbb{K}^\sharp$ with

$$v.gh = vgh\lambda_{g,h}$$

for all $v \in V$.

In the case $\sigma = 1$ (that is $g\sigma = \mathrm{id}_\mathbb{K}$ for all $g \in G$) a σ-linear projective $\mathbb{K}G$-module is called projective $\mathbb{K}G$-module.

Lemma 3.9 *Let G be a group, $\mathbb{F}$ a field, V an $\mathbb{F}G$-module, and*

$$\mathcal{T} = (\Phi, \mathbb{K}, (V_i, i \in I), \sigma, (g_i; g \in G, i \in I))$$

a G-invariant tensor decomposition. Then the following hold:

(a) G acts on $\bigcup_{i \in I} \mathcal{P}_\mathbb{K}(V_i)$ via $v_i\mathbb{K}g = v_i g_i \mathbb{K}$ for all $v_i \in V_i^\sharp$ and $g \in G$.

(b) For each $i \in I$, V_i is a projective σ-linear $\mathbb{K}C_G(i)$-module.

(c) For each $g \in G$ and $i \in I$, $g_i : V_i \to V_{ig}$ is a σg-linear isomorphism.

Proof: This follows immediately from the definition of a G-invariant tensor decomposition. □

4 Strict Tensor Decompositions

Throughout this section we assume the following hypothesis:

Hypothesis 4.1 *Let G be a group, $\mathbb{F}$ a field, V an $\mathbb{F}G$-module, and*

$$\mathcal{T} = (\Phi, \mathbb{K}, (V_i, i \in I), \sigma, (g_i; g \in G, i \in I))$$

a strict G-invariant tensor decomposition of V with $\Phi = \mathrm{id}_V$, that is $V = \otimes_{i \in I}^{\mathbb{K}} V_i$ and $\Phi = \mathrm{id}_{\otimes V_i}$.

Lemma 4.2 *Assume Hypothesis 4.1.*

(a) G acts on $\bigcup_{i \in I} V_i$ via $vg := vg_i$ for all $g \in G$ and $v \in \bigcup_{i \in I} V_i$, where i is the unique element in I with $v \in V_i$.

(b) Let $i \in I$. Then $C_G(i)$ acts σ-semilinear on V_i.

Proof: This follows immediately from the definition of a strict tensor decomposition. □

Notation 4.3 *By 4.2(a) G acts on $\bigcup_{i \in I} V_i$ and we can use the usual notation $C_W(B)$ and $C_B(W)$ where $W \subseteq \bigcup_{i \in I} V_i$ and $B \subseteq G$. Note that if B fixes $v \in V_i$, then B also fixes i. (This is even true for $v = 0$, since we assumed the V_i's to be disjoint and so have distinct zero vectors.)*

By 4.2(b), V_j is a $\mathbb{F}C_G(i)$-module, and we can use the usual notation $[V_j, B]$ for $B \subseteq C_G(i)$.

Lemma 4.4 *Assume Hypothesis 4.1. For $i \in I$ let U_i be a non-trivial $\mathbb{F}$-subspace of V_i. Suppose that there exist $r \in I$ and $B \leq G$ such that*

$$B \nleq C_G(r) \text{ and } \{\otimes u_i \mid u_i \in U_i, i \in I\} \subseteq C_V(B).$$

Then $\dim_\mathbb{K}\langle U_r\rangle_\mathbb{K} = 1$.

Proof: Pick $0 \neq u_i \in U_i$, $i \in I$ and $a \in B$ with $ra \neq r$. Then

$$\otimes u_i = (\otimes u_i)a = \otimes u_i a_i$$

and so $u_r a_r = u_{ra} k$ for some $k \in \mathbb{K}^\sharp$. Fixing u_{ra} and allowing u_r to run through the elements of $U_r^\sharp$ shows that $U_r a_r \leq u_{ra}\mathbb{K}$. Thus $\langle U_r a_r \rangle_{\mathbb{K}} = u_{ra}\mathbb{K}$ is 1-dimensional $\mathbb{K}$-space. Since a_r is a $\mathbb{K}$-semilinear isomorphism, also $\langle U_r \rangle_{\mathbb{K}}$ is a 1-dimensional $\mathbb{K}$-space. □

Lemma 4.5 *Assume Hypothesis 4.1. Suppose that* $\operatorname{char}\mathbb{K} = p > 0$ *and that* G *is a finite* p*-group. Let* $j \in I$ *with* $\dim_{\mathbb{K}} V_j \geq 2$*. Then* $C_G(V) \leq C_G(V_j)$*.*

Proof: Pick $h \in C_G(V)$ and for $i \in I$ pick $0 \neq v_i \in V_i$. Then

$$\otimes v_i = (\otimes v_i)h = \otimes v_i h_i,$$

so $v_j h_j \in v_{jh}\mathbb{K}$.

If $j \neq jh$, then $V_j h_j \leq v_{jh}\mathbb{K}$, and since h_j is a $\mathbb{K}$-semilinear isomorphism, $\dim_{\mathbb{K}} V_j = 1$, a contradiction. Hence $j = jh$ and by 3.1 h acts via scalar multiplication by a fixed scalar $\lambda \in \mathbb{K}$ on V_j. On the other hand, since $\operatorname{char}\mathbb{K} = p$ and G is a finite p-group, $C_{V_i}(G) \neq 0$ and so $\lambda = 1$. □

Lemma 4.6 *Assume Hypothesis 4.1. Suppose* $\mathcal{T}$ *is ordinary and that* $|I| \geq 2$*. Suppose that there exists* $r \in I$ *such that* G *acts non-trivially on* $\mathcal{P}_{\mathbb{K}}(V_r)$*. Then*

$$\dim_{\mathbb{K}} \bigotimes\nolimits^{\mathbb{K}}_{r \neq i \in I} V_i \leq \dim_{\mathbb{K}} V/C_V(G).$$

Proof: Put $W := V^r = \bigotimes^{\mathbb{K}}_{r \neq i \in I} V_i$. Then V and $V_r \otimes_{\mathbb{K}} W$ are isomorphic $\mathbb{K}G$-modules.

Since G acts non-trivially on $\mathcal{P}_{\mathbb{K}}(V_r)$ there exist $a \in G$ and $v \in V_r$ with $v\mathbb{K} \neq va\mathbb{K}$. Hence

$$(v \otimes W)a \cap v \otimes W = va_r \otimes W \cap v \otimes W = 0.$$

In particular, $v \otimes W \cap C_{V_r \otimes W}(a) = 0$ and

$$\dim V/C_V(G) \geq \dim V/C_V(a) \geq \dim v \otimes W = \dim W.$$

Lemma 4.7 *Assume Hypothesis 4.1. Suppose* G *is transitive on* I *and* $|I| \geq 2$*. Fix* $r \in I$ *and let* X_r *be a proper* $C_G(r)$*-invariant* $\mathbb{K}$*-subspace of* V_r*. For* $h \in G$ *put* $X_{rh} := X_r h$ *and*

$$X := \bigotimes\nolimits^{\mathbb{K}}_I X_i, \quad U_i := V_i \otimes \bigotimes\nolimits_{j \neq i} X_j, \quad U := \sum_{i \in I} U_i, \quad \Delta := \{U_i/X \mid i \in I\}.$$

Then

(a) U*,* X *and* U/X *are semi-linear* $\mathbb{K}G$*-modules,*

(b) Δ *is a system of imprimitivity for* G *in* U/X*,*

(c) G *acts transitively on* Δ*.*

Proof: Observe that $X_{rh} = X_{rg}$ for $g \in G_r h$ since X_r is $C_G(r)$-invariant. So X_{rh} is well-defined. For $h \in G$

$$Xh = (\bigotimes X_i)h = \bigotimes X_{ih} = \bigotimes X_i = X$$

and similarly

$$U_i h = V_i h_i \otimes \bigotimes\nolimits_{j \neq i} X_j h_j = V_{ih} \otimes \bigotimes\nolimits_{j \neq i} X_{jh} = V_{ih} \otimes \bigotimes\nolimits_{k \neq ih} X_k = U_{ih}.$$

This shows that X and U are G-invariant, so (a) holds, and that G acts on Δ, so (c) holds.

Observe that $\sum_{j \neq i} U_j \leq X_i \otimes V^i$ and that $(X_i \otimes V^i) \cap U_i = X$. Thus also $U_i \cap \sum_{j \neq i} U_j = X$, and (b) holds. $\square$

We also need the dual version of the preceding lemma:

Lemma 4.8 *Assume Hypothesis 4.1. Suppose G is transitive on I and $|I| \geq 2$. Fix $r \in I$ and let X_r be a proper $C_G(r)$-invariant $\mathbb{K}$-subspace of V_r. For $h \in G$ put $X_{rh} := X_r h$ and*

$$\tilde{U} := \sum_{i \neq j \in I} V^{\{i,j\}} \otimes X_i \otimes X_j, \quad \tilde{U}_i = (V^i \otimes X_i) + \tilde{U}, \quad \tilde{X} = \sum_{i \in I} U_i, \quad \tilde{\Delta} := \{\tilde{U}_i/\tilde{X} \mid i \in I\}.$$

Then

(a) $\tilde{U}$, $\tilde{X}$ and $\tilde{X}/\tilde{U}$ are semi-linear $\mathbb{K}G$-modules,

(b) $\tilde{\Delta}$ is a system of imprimitivity for G in $\tilde{X}/\tilde{U}$,

(c) G acts transitively on $\tilde{\Delta}$.

Proof: This can be proved similarly to 4.7 or by applying 4.7 to the dual of V. $\square$

Lemma 4.9 *Assume Hypothesis 4.1 and in addition:*

(i) $|I| = 2$.

(ii) $\mathcal{T}$ is ordinary.

(iii) G acts quadratically on V and $V_i \neq C_{V_i}(G) \neq 0$ for every $i \in I$.

Then the following hold:

(a) For all $i \in I$, G acts quadratically on V_i and $C_G(V) = C_G(V_i)$.

(b) There exists a homomorphism $\lambda : G \to \mathbb{K}$ such that G acts λ-dependently on each $V_i, i \in I$.

Proof: Let $I = \{i, j\}$. Note that

$$[V_i, G] \otimes C_{V_j}(G) = [V_i \otimes C_{V_j}(G), G] \leq C_V(G),$$

so $[V_i, G, G] \otimes C_{V_i}(G) \leq [V, G, G] = 0$ and thus $[V_i, G, G] = 0$. Hence (a) holds.

Let $a, b \in G \setminus C_G(V_j)$ and $x \in V_i$. Then $V_j \neq C_{V_j}(a) \cup C_{V_j}(b)$ and so there exists $y \in V_j$ with $[y, a] \neq 0$ and $[y, b] \neq 0$. Then

(1) $$[x \otimes y, a] = [x, a] \otimes [y, a] + x \otimes [y, a] + [x, a] \otimes y.$$

Taking commutators with b and using that G acts quadratically on V_i, V_j and V we get

$$0 = [x \otimes y, a, b] = [x, b] \otimes [y, a] + [x, a] \otimes [y, b]. \tag{2}$$

By the choice of y, $[y, a] \neq 0 \neq [y, b]$ and so (2) implies $C_{V_i}(a) = C_{V_i}(b) = C_{V_i}(G)$ and that $[x, a]\mathbb{K} = [x, b]\mathbb{K}$. Now 3.3 implies that G acts λ_i-dependently with respect to α_i on V_i for some homomorphism $\lambda_i : G \to (\mathbb{K}, +)$ and some $\alpha_i \in \mathrm{End}_{\mathbb{K}}(V_i)$ with $\alpha_i^2 = 0$. By symmetry the same holds for j in place of i.

Recall that $k_j := a\lambda_j \neq 0$ since $[y, a] \neq 0$. Thus, after substituting α_j by $k_j\alpha_j$ and λ_j by $k_j^{-1}\lambda_j : g \mapsto k_j^{-1} g\lambda_j$, we may assume that $a\lambda_j = 1$ and with a similar argument that $a\lambda_i = 1$.

Substitution into (2) yields, $b\lambda_j = -b\lambda_i$. In the case $a = b$ we have $1 = -1$ and so $\mathrm{char}\,\mathbb{K} = 2$. It follows that $\lambda_j = \lambda_i$ and the lemma is proved.

□

Lemma 4.10 *Assume Hypothesis 4.1 and in addition:*

(i) $\mathcal{T}$ *is proper and* $\mathbb{K}$*-linear.*

(ii) G *acts transitively on* I*.*

(iii) $|G| > 2$ *and* V *is a faithful quadratic* $\mathbb{K}G$*-module.*

Then $\mathrm{char}\,\mathbb{F} = 2$*,* $|I| = 2$ *and for* $i \in I$*,* $\dim_{\mathbb{K}} V_i = 2$*, and* $[V_i, C_G(I)] = C_{V_i}(C_G(I))$ *is a* 1*-dimensional* $\mathbb{K}$*-subspace of* V_i*.*

Proof: Recall from 2.7(c) that G is an elementary abelian $\mathrm{char}\,\mathbb{F}$-group. Put $B := C_G(I)$ and fix $r \in I$. Then $B = C_G(r)$ since G is abelian and transitive on I.

Let X_r be an 1-dimensional $\mathbb{K}B$-subspace of V_r. We apply 4.7 with the notation given there. Then Δ is a system of imprimitivity for G in U/X, so we can apply 2.13. Since G acts transitively on I and quadratically on U/X, we are in case (3) of 2.13, so $\mathrm{char}\,\mathbb{F} = 2$, $|I| = 2$, say $I = \{1, 2\}$, and $[U_i, B] \leq X$. As $|G| > 2$ we also get that $B \neq 1$. Since the $\mathbb{K}B$-modules U_j/X and V_j/X_j are isomorphic, $[V_j, B] = X_j$.

Pick $1 \neq b \in B$ and $a \in G \setminus B$, and put $C_1 := C_{V_1}(b)$. Then by the quadratic action of G,

$$[C_1 \otimes V_2, b] = C_1 \otimes X_2 \leq C_V(a).$$

Hence 4.4 shows that $\dim_{\mathbb{K}} C_1 = \dim_{\mathbb{K}} X_2 = 1$, so also $\dim_{\mathbb{K}} X_1 = 1$. The quadratic action of b on V_1 gives $C_1 = X_1$ and $\dim_{\mathbb{K}} V_1 = 2$. □

5 Tensor Decompositions of Homogeneous Modules

Lemma 5.1 *Let* $\mathbb{F}$ *be finite field,* H *a group and* V *a finite dimensional simple* $\mathbb{F}H$*-module. Recall that* V *is called absolutely simple if* $V \otimes_{\mathbb{F}} \mathbb{E}$ *is an simple* $\mathbb{E}H$ *module for all field extensions* $\mathbb{F} \leq \mathbb{E}$*.*

(a) V *is absolutely simple iff* $\mathbb{F} = \mathrm{End}_{\mathbb{F}H}(V)$*.*

(b) Put $\mathbb{K} := \mathrm{End}_{\mathbb{F}H}(V)$*. Then* $\mathbb{K}$ *is a field and* V *is an absolutely simple* $\mathbb{K}H$*-module.*

Proof: (a) This is [As, 25.8].

(b) By Schur's Lemma $\mathbb{K}$ is a division ring. As $\dim_{\mathbb{F}} V$ is finite, also $\dim_{\mathbb{F}} \mathbb{K}$ is finite. Now the finiteness of $\mathbb{F}$ shows that $\mathbb{K}$ is finite, so by Wedderburn's Theorem $\mathbb{K}$ is a finite field.

Since $\mathbb{F}$ and $\mathbb{K}$ are commutative we have $\mathbb{F} \leq \mathbb{K}$ and $\mathbb{K} \leq \mathrm{End}_{\mathbb{K}H}(V)$, so $\mathbb{K} \leq \mathrm{End}_{\mathbb{K}H}(V) \leq \mathrm{End}_{\mathbb{F}H}(V) = \mathbb{K}$. Hence by (a), V is absolutely simple. $\square$

Lemma 5.2 *Let $\mathbb{F}$ be a finite field, G a group, and V a finite dimensional $\mathbb{F}G$-module. Let D and E be subgroups of G such that $[D,E] = 1$. Suppose that V is a homogeneous $\mathbb{F}D$-module and X is a simple $\mathbb{F}D$-submodule of V. Then the following hold, where $Y := \mathrm{Hom}_{\mathbb{F}D}(X,V)$, $\mathbb{K} := \mathrm{End}_{\mathbb{F}D}(X)$ and $\mathbb{E} := \mathrm{Z}(\mathrm{End}_{\mathbb{F}D}(V))$:*

(a) $\mathbb{K}$ is a finite field and Y is a $\mathbb{K}E$-module via

$$\alpha k : x \mapsto xk\alpha \ \text{ and } \alpha e : x \mapsto x\alpha e \quad (x \in X,\, \alpha \in Y,\, k \in \mathbb{K},\, e \in E).$$

(b) X is an absolutely simple $\mathbb{K}D$-module.

(c) There exists an $\mathbb{F}(D \times E)$-module isomorphism $\Phi : X \otimes_{\mathbb{K}} Y \to V$ with $(x \otimes \alpha)\Phi = x\alpha$ for all $x \in X$ and $\alpha \in Y$.

(d) For $\mathbb{K}$-subspaces $Z \leq Y$ the map $Z \mapsto (X \otimes Z)\Phi$ is an E-invariant bijection between the $\mathbb{K}$-subspaces Z of Y and the $\mathbb{F}D$-submodules of V with inverse $U \mapsto \mathrm{Hom}_{\mathbb{F}D}(X,U)$, U an $\mathbb{F}D$-subspace of V.

(e) V is a simple $\mathbb{F}DE$-module iff Y is a simple $\mathbb{K}E$-module.

(f) $\mathrm{End}_{\mathbb{F}D}(V) \cong \mathrm{End}_{\mathbb{K}}(Y)$

(g) $\mathbb{K} \cong \mathbb{E}$, X is $\mathbb{E}$-invariant and $\mathbb{K} = \{e\,|_X \mid e \in \mathbb{E}\}$. In particular, X is an absolutely simple $\mathbb{E}D$-module.

Proof: By 5.1(b), $\mathbb{K}$ is a field and X is an absolutely simple $\mathbb{K}F$-module. Statements (a)-(d) now follows from [As, 27.14].

(e): This is a direct consequence of (d) since a $\mathbb{K}$-subspace of Y is E-invariant if and only if the corresponding $\mathbb{F}D$-submodule of V is E-invariant.

(f): Let $x \in X$, $\alpha \in Y$ and $\beta \in \mathrm{End}_{\mathbb{F}D}(V)$. Then the map

$$\alpha\beta : x \mapsto x\alpha\beta$$

is in Y, and Y is an $\mathrm{End}_{\mathbb{F}D}(V)$-module. Moreover, for $k \in \mathbb{K}$

$$x.\alpha\beta k = xk.\alpha\beta = xk\alpha\beta = x.\alpha k.\beta = x.\alpha k\beta,$$

so $\alpha\beta k = \alpha k\beta$ and $\mathrm{End}_{\mathbb{F}D}(V)$ acts $\mathbb{K}$-linearly on Y. Hence we obtain a ring homomorphism $\tau : \mathrm{End}_{\mathbb{F}D}(V) \to \mathrm{End}_{\mathbb{K}} Y$.

Observe that $X \otimes_{\mathbb{K}} Y$ is an $\mathrm{End}_{\mathbb{K}}(Y)$-module via

$$(x \otimes \alpha)\delta = x \otimes \alpha\delta \quad (x \in X,\, \alpha \in Y,\, \delta \in \mathrm{End}_{\mathbb{K}}(Y)).$$

So by (c) V is an $\mathrm{End}_{\mathbb{K}}(Y)$-module with $(x \otimes \alpha)\Phi\delta := (x \otimes \alpha)\delta\Phi$. Since Φ is an $\mathbb{F}(D \times E)$-module homomorphism this action of $\mathrm{End}_{\mathbb{K}}(Y)$ on V is $\mathbb{F}D$-linear. Hence, we obtain a ring homomorphism $\mathrm{End}_{\mathbb{K}}(Y) \to \mathrm{End}_{\mathbb{F}D}(V)$ which is inverse to τ. Thus (f) holds.

(g) Let $e \in \mathbb{E}$. By (f) $\mathbb{E}\tau = \mathrm{Z}(\mathrm{End}_{\mathbb{K}}(Y)) = \{\mathrm{id}_Y k \mid k \in \mathbb{K}\}$. So there exists $k \in \mathbb{K}$ with $\alpha.e\tau = \alpha k$ for all $\alpha \in Y$. Hence for all $x \in X$

$$x\alpha k = x.\alpha k = x(\alpha.e\tau) = x\alpha e.$$

For $\alpha = \mathrm{id}_X$ this gives $xk = xe$. Together with (b) we have (g).

□

Lemma 5.3 *Let $\mathbb{F}$ be a finite field, G a group, and V a finite dimensional simple $\mathbb{F}G$-module. Let X a simple $\mathbb{F}_pG$-submodule of V and put $\mathbb{D} := \mathrm{End}_{\mathbb{F}G}(V)$ and $\mathbb{E} =: \mathrm{Z}(\mathrm{End}_{\mathbb{F}_pG}(V))$. Then $\mathbb{E} \le \mathbb{D}$ is a field extension and $V \cong X \otimes_{\mathbb{E}} \mathbb{D}$ as an $\mathbb{D}G$-module.*

Proof: Since V is a simple $\mathbb{F}G$-module, $V = \sum_{f\in\mathbb{F}} Wf$. Thus V is homogeneous and by 5.2(g) $\mathbb{E}$ is a field and X is an absolutely simple $\mathbb{E}G$-module. Note that $\mathbb{E} \subseteq \mathrm{End}_{\mathbb{F}_pG}(V)$ and so $\mathbb{E} \subseteq \mathrm{End}_{\mathbb{F}G}(V) = \mathbb{D}$. Thus by the definition of absolutely simple, $X \otimes_{\mathbb{E}} \mathbb{D}$ is a simple $\mathbb{D}G$-module.

By the universal property of the tensor product there exists a unique $\mathbb{E}$-linear map $\alpha : X \otimes_{\mathbb{E}} \mathbb{D} \to V$ with $(x \otimes d)\alpha = xd$ for all $x \in X, d \in \mathbb{D}$. Clearly this map is a $\mathbb{D}G$-homomorphism. Since both $X \otimes_{\mathbb{E}} \mathbb{D}$ and V are simple $\mathbb{D}G$-modules, α is an isomorphism. □

Lemma 5.4 *Let $\mathbb{F}$ be a finite field, G a group, and V a finite dimensional simple $\mathbb{F}G$-module. Let $(D_i,\, i \in I)$ be a finite family of subgroups of G with $G = \langle D_i \mid i \in I\rangle$ and $[D_i, D_j] = 1$ for all $i \neq j \in I$. Put $\mathbb{K} := \mathrm{End}_{\mathbb{F}G}(V)$. Then the following hold:*

(a) For each $i \in I$ there exists an absolutely simple $\mathbb{K}D_i$-module V_i isomorphic to a $\mathbb{K}D_i$-submodule of V, and there exists a $\mathbb{K}(\bigtimes_{i\in I} D_i)$-isomorphism $\Phi : \otimes^I_{\mathbb{K}} V_i \to V$ (where D_i acts trivially on V_j for $i \neq j$).

(b) For $0 \neq v \in V$ the following two statements are equivalent:

1. *For all $i \in I$, $v\mathbb{K}D_i$ is a simple $\mathbb{K}D_i$-submodule of V.*
2. *There exist $0 \neq v_i \in V_i$ with $v = (\otimes v_i)\Phi$.*

Proof: By 5.1 V is an absolutely simple $\mathbb{K}G$-module and $\mathbb{K} = \mathrm{End}_{\mathbb{K}}(V)$. By induction on $|I|$ we may assume that $|I| = 2$, say $I = \{1, 2\}$. Let V_1 be a simple $\mathbb{K}D_1$-submodule of V and put $V_2 := \mathrm{Hom}_{\mathbb{K}D_1}(V_1, V)$.

(a): Put $\mathbb{E} := \mathrm{End}_{\mathbb{K}D_1}(V_1)$ and note that $\mathbb{K}$ embeds into $\mathbb{E}$. By 5.2 there exists a $\mathbb{K}(D_1 \times D_2)$-isomorphism

$$\Phi :\, V_1 \otimes_{\mathbb{E}} V_2 \to V \text{ with } (w \otimes \alpha)\Phi = w\alpha \quad (w \in V_1,\, \alpha \in V_2).$$

It follows that $\mathrm{End}_{\mathbb{K}(D_1\times D_2)}(V_1 \otimes_{\mathbb{E}} V_2) \cong \mathrm{End}_{\mathbb{K}G}(V) = \mathbb{K}$. Since $\mathbb{E}$ embeds into $\mathrm{End}_{\mathbb{K}(D_1\times D_2)}(V_1 \otimes_{\mathbb{E}} V_2)$, we get conclude that $\mathbb{K}$ is isomorphic to $\mathbb{E}$ and V_1 is an absolutely simple $\mathbb{K}\mathbb{D}_1$-module. By symmetry any simple $\mathbb{K}D_2$ submodule of V is absolutely simple.

Let $0 \neq v_1 \in V_1$. Then, again by 5.2, V_2 is isomorphic to the $\mathbb{K}D_2$-submodule $(v_1 \otimes V_2)\Phi$ of V, and V_2 is a simple $\mathbb{K}D_2$-module since V is a simple $\mathbb{K}G$-module. It follows that $(v_1 \otimes V_2)\Phi$ and so also V_2 is an absolutely simple $\mathbb{K}D_2$-module.

(b): Let $0 \neq v \in V$. Suppose first that (b:1) holds. Since V is a homogeneous $\mathbb{K}D_1$-module, there exists a $\mathbb{K}D_1$-isomorphism $\alpha \in V_2$ such that $V_1\alpha = v\mathbb{K}D_1$. Put $v_1 = v\alpha^{-1}$. Then $(v_1 \otimes \alpha)\Phi = v$.

Suppose next that (b:2) holds. Then $v\mathbb{K}D_1 = (v_1 \otimes v_2)\Phi\mathbb{K}D_1 = (V_1 \otimes v_2)\Phi \cong V_1$ as $\mathbb{K}D_1$-module. By symmetry $v\mathbb{K}D_2 \cong V_2$ as $\mathbb{K}D_2$-module. Thus (b:1) holds. □

Proposition 5.5 *Let $\mathbb{F}$ be a finite field of characteristic p, G a finite group, V a finite dimensional $\mathbb{F}G$-module, and I a finite G-set. Further let T be a p-subgroup of G and $(D_i,\, i \in I)$ be a family of subgroups of G. Put $D := \langle D_i \mid i \in I\rangle$. Suppose that*

(i) $D_i^h = D_{ih}$ and $[D_i, D_j] = 1$ for all $i \neq j \in I, h \in H$, and

(ii) V is homogeneous as an $\mathbb{F}D$-module.

Put $\mathbb{K} := \mathrm{Z}(\mathrm{End}_{\mathbb{F}D}(V))$, $J := I$ if V is a simple $\mathbb{F}D$-module and otherwise $J := I \uplus \{0\}$, where J is viewed as a G-set with G fixing 0. Then there exist $\mathbb{K}D_i$-modules V_i, $i \in I$, a finite dimensional $\mathbb{K}$-space V_0 and a G-invariant tensor decomposition $\mathcal{T} = (\Phi, \mathbb{K}, (V_j, j \in J), \sigma, (g_j, j \in J, g \in G))$ of V such that the following hold:

(a) V_j is an absolutely simple $\mathbb{K}D_j$-module for $j \neq 0$, and V_0 is a trivial $\mathbb{K}D$-module. Moreover, every simple $\mathbb{K}D_j$-submodule of V is isomorphic to V_j as a $\mathbb{K}D_j$-module.

(b) $\Phi : \otimes_{\mathbb{K}}^J V_j \to V$ is a $\mathbb{K}(\bigtimes_{i\in I} D_i)$-module isomorphism (where D_i acts trivially on V_j for $j \neq i$).

(c) $\mathcal{T}$ restricted to T is strict.

Proof: To simplify notation we assume without loss that V is a faithful $\mathbb{F}G$-module and that G is subgroup of $\mathrm{GL}_{\mathbb{F}}(V)$. Let $D_0 := \mathrm{GL}_{\mathbb{F}D}(V)$, and for $j \in J$ let R_j be the subring of $\mathrm{End}_{\mathbb{F}}(V)$ spanned by $\mathbb{K}$ and D_j. By 5.2(f),(e) (with D_0 in place of E) V is a simple $\mathbb{F}DD_0$-module and so V is an absolutely simple $\mathbb{K}DD_0$-module. Thus 5.4 implies:

1° *There exist absolutely simple $\mathbb{K}D_j$-modules V_j and a $\mathbb{K}(\bigtimes_{i\in J} D_j)$-isomorphism*

$$\Phi : \otimes_{\mathbb{K}}^J V_j \to V.$$

Let α_j be canonical ring homomorphism from $\mathbb{K}D_j$ onto R_j. From $(\otimes v_j d_j)\Phi = (\otimes v_j)\Phi.\prod d_j$ for all $v_j \in V_j, d_j \in D_j$ we conclude that $(\otimes v_j a_j)\Phi = (\otimes v_j)\Phi.\prod a_j\alpha_j$ for all $v_j \in V_j, a_j \in \mathbb{K}D_j$. This implies $\ker \alpha_j = \mathrm{Ann}_{\mathbb{K}D_j}(V_j)$ and we conclude that

2° *V_j can be viewed as simple R_j-module such that $(\otimes v_j r_j)\Phi = (\otimes v_j)\Phi.\prod r_j$ for all $v_j \in V_j, r_j \in R_j$.*

Fix $0 \neq w_j \in V_j, j \in J$. Let $g \in G$ and $j \in J$. By 5.4(b), wR_j is a simple R_j-module. Since g normalizes $\mathbb{K}$ and $D_j^g = D_{jg}$ we have $R_j^g = R_{jg}$ and so wR_jg is a simple R_{jg}-module. Also $wR_jg = wgR_{jg}$ and so for $j \in J$, wgR_{jg} is a simple R_{jg}-module. Thus by 5.4(b), there exist $0 \neq u_j \in V_j$, $j \in J$, with $wg = (\otimes u_j)\Phi$. The number of elements of the form $(\otimes v_j)\Phi$, $0 \neq v_j \in V_j$, is not divisible by p and so we can and do choose the w_j's such that $w := (\otimes w_j)\Phi$ is centralized by T. Hence we may also choose $u_j = w_j$ if $g \in T$.

Let $v_j \in V_j$. Since V_j is a simple R_j-module there exists $r_j \in R_j$ with $v_j = w_j r_j$. Next we show:

3° *Let $i \in J$ and $r_i, s_i \in R_i$ with $w_i r_i = w_i s_i$. Then $u_{ig} r_i^g = u_{ig} s_i^g$.*

Put $t_i = r_i - s_i$. Then $w_i t_i = 0$ and by (2°) $wt_i = 0$. Thus $(\otimes u_j)\Phi t_i^g = wgt_i^g = wt_i g = 0$. Since $t_i^g \in R_{ig}$ we conclude from (2°) that $u_{ig} t_i^g = 0$ and so $u_{ig} r_i^g = u_{ig} s_i^g$.

We now define

$$g_j : V_j \to V_{jg} \text{ with } v_j \to u_{jg} r_j^g, \text{ where } r_j \in R_j \text{ and } v_j = w_j r_j.$$

Using (3°) we get

4° g_j *is independent from the choice of* r_j.

Clearly g_j is a homomorphism between the additive group V_j and V_{jg}. Next we define a homomorphism $\sigma : G \to \mathrm{Aut}(\mathbb{K})$. Observe that for fixed $g \in G$,

$$g\sigma : \mathbb{K} \to \mathbb{K} \text{ with } k \mapsto k^g$$

is an element of $\mathrm{Aut}_{\mathbb{F}}(\mathbb{K})$ and so $g \mapsto g\sigma$ defines the desired homomorphism σ.

Let $k \in \mathbb{K}$. Since $v_j k = w_j r_j k = w_j . r_j k$ and $r_j k \in R_j$, the definition of g_j shows that

$$v_j k g_j = u_{jg}(r_j k)^g = u_{jg} r_j^g k^g = v_j g_j (k.g\sigma).$$

Hence g_j is $g\sigma$-linear.

To verify 3.7(a) we compute

$$(*) \qquad \begin{aligned} (\otimes v_j g_j)\Phi &= (\otimes u_{jg} r_j^g)\Phi = (\otimes u_j)\Phi \prod r_j^g = wg(\prod r_j)^g = w(\prod r_j)g \\ &= (\otimes w_j)\Phi(\prod r_j)g = (\otimes w_j r_j)\Phi g = (\otimes v_j)\Phi g. \end{aligned}$$

This is 3.7(a).

Let $g, h \in G$ and put $v := \otimes v_j$. Note that $v\Phi.gh = v\Phi gh$ and so using $(*)$ three times

$$(\otimes v_j (gh)_j)\Phi = v\Phi.gh = v\Phi gh = (\otimes v_j g_j)\Phi h = (\otimes v_j g_j h_{jg})\Phi.$$

Since Φ is bijective, this implies:

$$\otimes v_j (gh)_j = \otimes v_j g_j h_{jg}.$$

Thus $v_j(gh)_j \mathbb{K} = v_j g_j h_{jg} \mathbb{K}$. Fix j, g and h and put $\delta = g_j h_{jg} (gh)_j^{-1}$. Then $\delta : V_j \to V_j$ is $\mathbb{K}$-linear and $v_j \delta \mathbb{K} = v_j \mathbb{K}$ for all $v_j \in V_j$. If $\dim_{\mathbb{K}} V_j \geq 2$ we conclude from 3.1 δ acts as a scalar μ on V_j. Obviously the same is true if $\dim_{\mathbb{K}} V_j = 1$. Thus $g_j h_{jg} = \mu (gh)_j = (gh)_j \lambda$ where $\lambda = \mu.gh\sigma$. Hence 3.7(b) holds.

Therefore $\mathcal{T} = (\Phi, \mathbb{K}, (V_j, j \in J), \sigma, (g_j; g \in G, j \in J))$ is a G-invariant tensor decomposition of V.

Let $g \in T$. Recall that we chose $w_j = u_j$ for such g. Hence for $v_j = w_j$ we can choose $r_j = 1$ and so $w_j g_j = w_{jg}$. For $a, b \in T$ we conclude

$$w_j a_j b_{ja} = w_{ja} b_{ja} = w_{jab} = w_j (ab)_j,$$

and $\lambda_{j,a,b} = 1$. Thus (c) holds. □

We remark that 5.5(c) maybe false if $\mathbb{K}$ is infinite. Indeed, let $\mathbb{F}$ be a finite field of characteristic 2 and $\mathbb{E} = \mathbb{F}(t)$ with t transcendental over $\mathbb{E}$. Put $\mathbb{K} = \mathbb{E}(t^2)$, $V = \mathbb{E} \otimes_{\mathbb{K}} \mathbb{E}$ and let $\alpha \in \mathrm{GL}_{\mathbb{K}}(V)$ with $(k \otimes l)\alpha = kt \otimes lt^{-1}$ for all $k, l \in \mathbb{E}$. Then $\alpha^2 = 1$ and so $\langle \alpha \rangle$ has order two. Moreover, it is easy to verify that the tensor decomposition $\mathbb{E} \otimes_{\mathbb{K}} \mathbb{E}$ is not strict for $\langle \alpha \rangle$.

6 Tensor Products and Nearly Quadratic Modules

The following hypothesis will be used throughout this section (except in 6.3).

Hypothesis 6.1 *Let p be a prime, $\mathbb{F}$ a field of characteristic p, A a finite p-group, J a finite A-set, V an $\mathbb{F}A$-module, and $\mathcal{T} = (\Phi, \mathbb{K}, (V_j, j \in J), \sigma, (g_j; g \in A, j \in J))$ a strict A-invariant tensor decomposition of V with $\Phi = \mathrm{id}_V$. The fixed field of A in $\mathbb{K}$ is denoted by $\mathbb{K}_A$.*

Lemma 6.2 *Suppose Hypothesis 6.1 holds, $\mathcal{T}$ is proper and ordinary, $J = \{1, 2\}$, and V is a nearly quadratic $\mathbb{F}A$-module but not a quadratic $\mathbb{F}A$-module. Then the following hold for $j \in J$:*

(a) A acts quadratically and non-trivially on V_j. In particular, A is elementary abelian.

(b) $[V_j, A] = C_{V_j}(A)$ is a $\mathbb{K}$-hyperplane of V_j.

(c) $[z\mathbb{F}, A] = [V_j, A]$ for all $z \in V_j \setminus [V_j, A]$.

(d) $Q_V(A) = [V_1, A] \otimes V_2 + V_1 \otimes [V_2, A]$ is a $\mathbb{K}$-hyperplane of V.

(e) One of the following holds:

1. *$C_V(A) = [V_1, A] \otimes [V_2, A]$ and if $\mathbb{F} = \mathbb{F}_p$ then $A = C_A(V_1)C_A(V_2)$.*
2. *$\dim_{\mathbb{K}} C_V(A) = 2$, $\mathrm{char}\,\mathbb{F} \neq 2$, $\dim_{\mathbb{K}} V_1 = 2 = \dim_{\mathbb{K}} V_2$, and V_1 and V_2 are isomorphic as $\mathbb{K}A$-modules.*

(f) If $\mathbb{F} = \mathbb{F}_p$ then A induces $C_{\mathrm{SL}_{\mathbb{K}}(V_j)}([V_j, A])$ on V_j.

Proof: As A is not quadratic on V, by 2.6(b) $[V, A] + C_V(A)$ is the largest quadratic $\mathbb{F}A$-submodule $Q_V(A)$ of V. Observe that $Q_V(A)$ is $\mathbb{K}$-subspace of V. Let $J = \{i, j\}$ and put $C_i = C_{V_i}(A)$.

1° *As a $\mathbb{K}A$-module, $V_i \otimes C_j$ is the direct sum of $\dim_{\mathbb{K}} C_j$ $\mathbb{K}A$-submodules isomorphic to V_i.*

Let $\mathcal{B}$ be a $\mathbb{K}$-basis for C_j. Then $V_i \otimes C_j = \bigoplus_{b \in \mathcal{B}} V_i \otimes b$ and each $V_i \otimes b$ is isomorphic to V_i as an $\mathbb{K}A$-module.

2° *A acts non-trivially on V_i.*

Suppose A centralizes V_i. Then $C_i = V_i$ and since $\dim_{\mathbb{K}} V_i \geq 2$ we conclude that from (1°) (with the roles of i and j reversed) and 2.9, that A centralizes V_j and V, a contradiction.

3° *V_i is not the direct sum of two proper $\mathbb{K}A$-submodules.*

Suppose $V_i = X \oplus Y$ with X and Y proper $\mathbb{K}A$ submodules of V_i. Then $V = X \otimes V_j \oplus X \otimes V_j$ and so by 2.9, A centralizes one of the summands, say $X \otimes V_j$. So by 4.5, A centralizes V_j, a contradiction to (2°).

4° *A acts quadratically on V_i, so (a) holds.*

Since $\dim_{\mathbb{K}} V_j \geq 2$ and A is a p-group, there exists a proper $\mathbb{K}A$-submodule $X \neq 0$ of V_j. By 2.6(d), A acts quadratically on $V/(V_i \otimes X)$ or on $V_i \otimes X$. Since $V/(V_i \otimes X) \cong V_i \otimes (V_j/X)$ we conclude that G acts quadratically on $V_i \otimes Y$ where $Y = X$ and $Y = V_j/X$, respectively. If $\dim_{\mathbb{K}} Y = 1$, then $V_i \cong V_i \otimes Y$ and (4°) holds. If $\dim Y \geq 2$, then 4.9(a) shows that G acts quadratically on V_i.

5° $C_i = [V_i, A]$.

By (4°), $[V_i, A] \leq C_i$. Let $C_i = X \oplus [V_i, A]$ and $V_i/[V_i, A] = C_i/[V_i, A] \oplus Y/[V_i, A]$ for some $\mathbb{K}$-subspaces X and Y of V_i with $[V_i, A] \leq Y$. Then $V_i = X \oplus Y$, $Y \neq 0$ and both X and Y are $\mathbb{K}A$-submodules of V_i. So by (3°), $X = 0$ and $C_i = [V_i, A]$.

6° $\quad C_1 \otimes V_2 + V_1 \otimes C_2 \leq Q_V(A)$.

By (4°), A is quadratic on V_i and so by (1°) also on $V_i \otimes C_j$. Thus $V_i \otimes C_j \leq Q_V(A)$.

7° $\quad$ *Let $v_1 \in V_1$, $v_2 \in V_2$ and $a \in A$. Then*

$$[v_1 \otimes v_2, a] = [v_1, a] \otimes [v_2, a] + v_1 \otimes [v_2, a] + [v_1, a] \otimes v_2.$$

This is readily verified.

Since $V \neq Q_V(A)$ there exists $x_i \in V_i$ with $x_1 \otimes x_2 \notin Q_V(A)$. By (6°) we have $x_i \notin C_i$.

8° $\quad Q_V(A) = x_1 \otimes C_2 + C_1 \otimes x_2 + C_V(A)$

By (6°) the right-hand-side is contained in the left-hand-side of (8°), and by the definition of nearly quadratic, $Q_V(A) = [(x_1 \otimes x_2)\mathbb{F}, A] + C_V(A)$. Since $[V_i, A] \leq C_i$ we conclude from (7°) that the left-hand-side is also contained in the right-hand-side.

9° $\quad$ *Let $t_1 \in V_1$ with $t_1 \otimes x_2 \in Q_V(A)$. Then $t_1 \in C_1$.*

By (8°) there exist $c \in C_V(A)$, $c_1 \in C_1$ and $c_2 \in C_2$ such that

$$t_1 \otimes x_2 = x_1 \otimes c_2 + c_1 \otimes x_2 + c.$$

Taking commutators with a on both sides and using (7°) we conclude

$$[t_1, a] \otimes [x_2, a] + t_1 \otimes [x_2, a] + [t_1, a] \otimes x_2 = [x_1, a] \otimes c_2 + c_1 \otimes [x_2, a].$$

Hence (4°) gives $[t_1, a] \otimes x_2 \in V_1 \otimes C_2$. Since $x_2 \notin C_2$ we get $[t_1, a] = 0$ and thus $t_1 \in C_{V_1}(A) = C_1$.

10° $\quad$ *C_i is a $\mathbb{K}$-hyperplane of V_i.*

Since $[V, A, A] \neq 0$, there exists a $\mathbb{K}$-hyperplane H_i of C_i with $[V, A, A] \not\leq H_i \otimes V_j$. Put $\overline{V_i} = V_i/H_i$. Hence by 2.6(c), $V/(H_i \otimes V_j) \cong \overline{V_i} \otimes V_j$ is a nearly quadratic, but not quadratic $\mathbb{F}A$-module. Thus by (5°),

$$\overline{C}_i = \overline{[V_i, A]} = [\overline{V}_i, A] = C_{\overline{V}_i}(A),$$

so we may replace V_i by $\overline{V}_i$ and assume that $\dim_{\mathbb{K}} C_i = 1$ for $i = 1, 2$. Thus we need to show that $\dim_{\mathbb{K}} V_i = 2$.

Assume for a contradiction that $\dim_{\mathbb{K}} V_i \geq 3$. By (3°), $V_i \otimes C_j \leq Q_V(A)$ and by (8°), $\dim_{\mathbb{K}} Q_V(A)/(C_i \otimes x_j + C_V(A)) \leq 1$. Hence $R := V_i \otimes C_j \cap (C_i \otimes x_j + C_V(A))$ contains a hyperplane of $V_i \otimes C_j$. Moreover, $C_i \otimes C_j \leq R$. Let $0 \neq c_j \in C_j$. Since C_j is 1-dimensional, the map $V_i \to V_i \otimes C_j$ with $v_i \to v_i \otimes c_j$ is a $\mathbb{K}$-isomorphism. Since $\dim V_i \geq 3$ we have $R \neq C_1 \otimes C_2$ and so there exists $t_i \in V_i$ with $t_i \otimes c_j \in R$ and $t_i \notin C_i$. As C_j is 1-dimensional, also $t_i \otimes C_j \leq R$.

From (9°) we get that $t_i \otimes x_j \notin Q_V(A)$, so (8°) applies with t_i in place of x_i; i.e., $Q_V(A) = t_i \otimes C_j + C_i \otimes x_j + C_V(A)$. Since $t_i \otimes C_j \leq R \leq C_i \otimes x_j + C_V(A)$ this gives $Q_V(A) = C_i \otimes x_j + C_V(A)$. Hence $C_V(A)$ is a hyperplane of $Q_V(A)$ and $(V_i \otimes C_j) \cap C_V(A)$ is a hyperplane of $V_i \otimes C_j$ containing $C_i \otimes C_j$. It follows that $C_{V_i}(A) = C_i$ is a hyperplane of V_i contradicting $\dim_{\mathbb{K}} C_i = 1$ and $\dim_K V_i \geq 3$.

11° *(b) and (d) hold.*

Claim (b) follows from (10°) and (5°). In particular, $C_1 \otimes V_2 + V_1 \otimes C_2$ is a $\mathbb{K}$-hyperplane of V. So (6°) and (b) imply (d).

12° *Suppose that $C_V(A) = C_1 \otimes C_2$. Then (c), (e:1) and (f) hold.*

To prove (c) let $z_1 \in V_1 \setminus [V_1, A]$. By (b) $C_1 = [V_1, A]$ and thus by (9°) $z_1 \otimes x_2 \notin Q_V(A)$. Hence, we may assume that $z_1 = x_1$.

From (6°) and the nearly quadratic action of A we get

$$C_1 \otimes x_2 \leq Q_V(A) = [(x_1 \otimes x_2)\mathbb{F}, A] + C_1 \otimes C_2.$$

Since by (b) $[V_2, A] = C_2$, (7°) implies that

$$[(x_1 \otimes x_2)\mathbb{F}, A] + C_1 \otimes C_2 \leq [x_1\mathbb{F}, A] \otimes x_2 + V_1 \otimes C_2.$$

Thus we have

$$[x_1\mathbb{F}, A] \otimes x_2 \leq C_1 \otimes x_2 \leq [x_1\mathbb{F}, A] \otimes x_2 + V_1 \otimes C_2.$$

Since $x_2 \notin C_2$ this implies $C_1 \otimes x_2 = [x_1\mathbb{F}, A] \otimes x_2$. Hence $C_1 = [x_1\mathbb{F}, A]$, and (c) follows.

The first part of (e:1) is true by assumption, so for the proof of (e:1) and (f) we can assume that $\mathbb{F} = \mathbb{F}_p$. Since $\mathbb{F} = \mathbb{F}_p$ and A is quadratic on $V/C_V(A)$ we have $[(x_1 \otimes x_2)\mathbb{F}, A] + C_V(A) = \{[x_1 \otimes x_1, a] \mid a \in A\} + C_V(A)$ and so

$$C_1 \otimes x_2 + x_1 \otimes C_2 \leq [(x_1 \otimes x_2)\mathbb{F}, A] + C_1 \otimes C_2 = \{[x_1, a] \otimes x_2 + x_1 \otimes [x_2, a] \mid a \in A\} + C_1 \otimes C_2.$$

Hence, for every $c_1 \in C_1$ and $c_2 \in C_2$, there exists $a \in A$ with $[x_1, a] = c_1$ and $[x_2, a] = c_2$. The particular case when $c_1 = 0$ (or $c_2 = 0$) gives (e:1). Moreover, (f) follows.

13° *Suppose that $C_V(A) \neq C_1 \otimes C_2$. Then (c), (e:2) and (f) hold.*

By (b) and (d),

$$Q_V(A) = C_1 \otimes C_2 + C_1 \otimes x_2 + x_1 \otimes C_2.$$

Since $C_1 \otimes C_2 < C_V(A)$ there exist $c_1 \in C_1$ and $c_2 \in C_2$ with $0 \neq c_1 \otimes x_2 - x_1 \otimes c_2 \in C_V(A)$. Hence (7°) implies that

$$(*) \qquad c_1 \otimes [x_2, a] = [x_1, a] \otimes c_2 \quad \text{for all } a \in A.$$

Suppose that $c_1 = 0$. By the choice of x_1, there exists $a \in A$ with $[x_1, a] \neq 0$, so $c_2 = 0$, which contradicts $c_1 \otimes x_2 - x_1 \otimes c_2 \neq 0$. Hence $c_1 \neq 0$ and similarly also $c_2 \neq 0$. Then $(*)$ implies that $[x_1, a] \in c_1\mathbb{K}$ for all $a \in A$. So by (b) $\dim_{\mathbb{K}} C_1 = 1$ and $\dim_{\mathbb{K}} V_1 = 2$. By symmetry $\dim_{\mathbb{K}} C_2 = 1$ and $\dim_{\mathbb{K}} V_2 = 2$.

Define $\lambda_i : A \to \mathbb{K}$ by $[x_i, a] = c_i.a\lambda_i$ for all $a \in A$. Then by $(*)$, $a\lambda_1 = a\lambda_2$ and $\lambda_1 = \lambda_2 =: \lambda$. Hence V_1 and V_2 are isomorphic as $\mathbb{F}A$-modules. Moreover,

$$C_V(A) = C_1 \otimes C_2 + (c_1 \otimes x_2 - x_1 \otimes c_2)\mathbb{K}.$$

Let L be the $\mathbb{F}$-subspace of $\mathbb{K}$ spanned by $A\lambda = \{a\lambda \mid a \in A\}$. Then

$$Q_V(A) = [(x_1 \otimes x_2)\mathbb{F}, A] + C_V(A) = (c_1 \otimes x_2 + x_1 \otimes c_2)L + (c_1 \otimes x_2 - x_1 \otimes c_2)\mathbb{K} + C_1 \otimes C_2.$$

Let $k \in \mathbb{K}$. Then $x_1 k \otimes c_2 \in Q_V(A)$, so there exists $\ell \in L$ and $s \in \mathbb{K}$ with

$$x_1 k \otimes c_2 \in (c_1 \otimes x_2 + x_1 \otimes c_2)\ell + (c_1 \otimes x_2 - x_1 \otimes c_2)s + C_1 \otimes C_2 = c_1(\ell + s) \otimes x_2 + x_1 \otimes c_2(\ell - s) + C_1 \otimes C_2.$$

This implies $s = -\ell$ and $k = 2\ell$. Since $k \in \mathbb{K}$ was arbitrary we conclude that $\operatorname{char} \mathbb{F} \neq 2$ and $L = \mathbb{K}$. Thus (e:2) and (c) hold.

If $\mathbb{F} = \mathbb{F}_p$, then $\mathbb{K} = L = A\lambda$ and so also (f) is proved. □

Lemma 6.3 *Let V be a semi-linear but not linear $\mathbb{K}A$-module. Suppose that there exists a subfield $\mathbb{F} \leq \mathbb{K}$ such that V is a nearly quadratic $\mathbb{F}A$-module. Then $A/C_A(V)$ is elementary abelian and one of the following holds:*

1. $[V, A, A] = 0$, $[V, A_{\mathbb{K}}] = 0$, *and* $\operatorname{char} \mathbb{K} = 2 = |A/A_{\mathbb{K}}|$.
2. $[V, A, A] \neq 0$, $[V, A_{\mathbb{K}}] = C_V(A_{\mathbb{K}})$, $\dim_{\mathbb{K}} V/C_V(A_{\mathbb{K}}) = 1$, $\mathbb{F} = \mathbb{K}_A$, *and* $\operatorname{char} \mathbb{K} = 2 = |A/A_{\mathbb{K}}| = \dim_{\mathbb{F}} \mathbb{K}$.
3. $[V, A, A] \neq 0$, $[V, A_{\mathbb{K}}] = 0$, $\mathbb{F} = \mathbb{K}_A$, $\dim_{\mathbb{K}} V = 1$, *and* $\operatorname{char} \mathbb{F} = 3 = |A/A_{\mathbb{K}}| = \dim_{\mathbb{F}} \mathbb{K}$.
4. $[V, A, A] \neq 0$, $[V, A_{\mathbb{K}}] = 0$, $\mathbb{F} = \mathbb{K}_A$, $\dim_{\mathbb{K}} V = 1$, $\operatorname{char} \mathbb{F} = 2$, $A/A_{\mathbb{K}} \cong C_2 \times C_2$, $\dim_{\mathbb{F}} \mathbb{K} = 4$, *and* $\mathbb{F}$ *is infinite.*

Proof: Without loss V is a faithful $\mathbb{F}A$-module. Put $\mathbb{E} = \mathbb{K}_A$. Since A is cubic on V we can apply 2.15. If A is quadratic on V, then 2.15(1) applies and gives (1). So we may assume that A is not quadratic.

Suppose first that $[V, A_{\mathbb{K}}] = 0$. Then 2.15(2) or (3) applies, and V is as an $\mathbb{E}A$-module the direct sum of $\mathbb{E}A$-submodules isomorphic to $\mathbb{K}$. Hence by 2.9, $V \cong \mathbb{K}$ as an $\mathbb{F}A$-module. Thus by 2.14(c), $V \cong \mathbb{E}A$ as an $\mathbb{E}A$-module. As an $\mathbb{F}A$-module, $\mathbb{E}A$ is a direct sum of $\dim_{\mathbb{F}} \mathbb{E}$-copies of $\mathbb{F}A$ and so 2.9 gives $\mathbb{F} = \mathbb{E}$. Now 2.15 implies (3) or (4).

Suppose next that $[V, A_{\mathbb{K}}] \neq 0$. Then 2.15(4) applies, so $p = |A/A_{\mathbb{K}}| = \dim_{\mathbb{E}} \mathbb{K} = 2$, and there exists an $\mathbb{E}A$-module W such that $V \cong W \otimes_{\mathbb{E}} \mathbb{K}$ as an $\mathbb{E}A$-module and $A = A_{\mathbb{K}} C_A(W)$. Hence we can apply 6.2 (with $\mathbb{E}$ in place of $\mathbb{K}$). Note that $[\mathbb{K}, A] = \mathbb{E}$. By 6.2(c), $[\mathbb{K}, A] = [z\mathbb{F}, A]$ for some $z \in \mathbb{K}$. Since $|A/A_{\mathbb{K}}| = 2$ this implies that $[\mathbb{K}, A]$ is 1-dimensional over $\mathbb{F}$. Hence $\mathbb{E} = \mathbb{F}$. Also by 6.2(b) $[W, A] = C_W(A)$ is an $\mathbb{E}$-hyperplane of W. Since $A = A_{\mathbb{K}} C_A(W)$, $[W, A_{\mathbb{K}}] = C_W(A_{\mathbb{K}})$ is a $\mathbb{E}$-hyperplane of W. Since $V \cong W \otimes_{\mathbb{E}} \mathbb{K}$ we conclude that $[V, A_{\mathbb{K}}] = C_V(A_{\mathbb{K}})$ is an $\mathbb{K}$-hyperplane of V. Thus (2) holds. □

Lemma 6.4 *Suppose Hypothesis 6.1 holds, $\mathcal{T}$ is proper and $\mathbb{K}$-linear, $[V_j, A] \neq 0$ for all $j \in J$, and $\operatorname{char} \mathbb{F} = 2$. If A acts cubically on V, then $A/C_A(V)$ is elementary abelian.*

Proof: We may assume that $C_A(V) = 1$ and that A is not elementary abelian. By 2.7(e) A is a 2-group. Hence there exists $a \in A$ with $a^2 \neq 1$. Since $\operatorname{char} \mathbb{F} = 2$, 2.8 gives $[V, a^2] = [V, a, a]$ and since A is cubic we conclude that

$$(*) \qquad [V, a^2] \leq C_V(A).$$

If A does not act transitively on J, let I be an orbit for A on J and $K := J \setminus I$. If A acts transitively on J, let A_0 be a maximal subgroup of A containing a point stabilizer and I and K

be the two orbits of A_0 on J. In both cases we obtain a strict A-invariant tensor decomposition $V_I \otimes_{\mathbb{K}} V_K \to V$. So we may assume that $|J| = 2$, say $J = \{1, 2\}$. Then a^2 acts trivially on J. Without loss $[V_1, a^2] \neq 0$. We have

$$[V_1 \otimes C_{V_2}(a^2), a^2] = [V_1, a^2] \otimes C_{V_2}(a^2),$$

and so by $(*)$

$$(**) \qquad [V_1, a^2] \otimes C_{V_2}(a^2) \leq C_V(A).$$

Suppose that a acts trivially on J. Then $(**)$ and 4.5 imply $C_{V_2}(a^2) = C_{V_2}(a)$. Put $Q_V(a) := Q_V(\langle a \rangle)$. By 2.8 $C_{V_2}(a^2) = Q_{V_2}(a)$, so

$$C_{V_2}(a) = C_{V_2}(a^2) = Q_{V_2}(a).$$

Thus a centralizes V_2. Hence $(**)$ and again 4.5 imply that also $C_A(J)$ centralizes V_2. It follows that $A \neq C_A(J)$ and since $C_A(J) \trianglelefteq A$, $C_A(J)$ centralizes V_1 and V. Thus $|A| = 2$, a contradiction.

We have shown that a acts non-trivial on J. Recall that by 4.2(a), A acts on $V_1 \cup V_2$ via $v_i a = v_i a_i$ for all $v_i \in V_i$. So $(v_1 \otimes v_2)a = v_2 a \otimes v_1 a$. Let $x_1 \in V_1$ with $[x_1, a^2] \neq 0$ and $[x_1, a^2, a^2] = 0$. Put $x_1 a =: x_2 \in V_2$. Then

$$(***) \qquad \begin{aligned} [x_1 \otimes x_2, a] &= (x_1 \otimes x_2)a - x_1 \otimes x_2 = x_2 a \otimes x_1 a - x_1 \otimes x_2 \\ &= x_1 a^2 \otimes x_2 - x_1 \otimes x_2 = [x_1, a^2] \otimes x_2. \end{aligned}$$

Since A acts quadratically on $[V_1 \otimes V_2, A]$, a^2 centralizes $[V_1 \otimes V_2, A]$, and since a^2 also centralizes $[x_1, a^2]$ we conclude from $(***)$ that a^2 centralizes x_2. But then a^2 also centralizes $x_1 = x_2 a^{-1}$, a contradiction. □

Proposition 6.5 *Suppose Hypothesis 6.1 holds,*

(i) $|A| > 2$, $\mathcal{T}$ *is proper, and*

(ii) V *is a faithful nearly quadratic* $\mathbb{F}A$*-module.*

Then A *acts* $\mathbb{K}$*-linearly on* V*,* A *is elementary abelian* $\operatorname{char}\mathbb{F}$*-group and one of the following holds, where* $B := C_A(J)$*:*

1. *A is quadratic on V, and there exists $j \in J$ such that A centralizes V_i for all $i \in J \setminus \{j\}$.*
2. *$\operatorname{char}\mathbb{F} = 2$, A is quadratic on V, and there exists an A-invariant subset J_0 in J with $|J_0| = 2$ such that A centralizes V_i for all $i \in J \setminus J_0$. Moreover, one of the following holds:*
    1. *A acts trivially on J_0 and there exists a homomorphism $\lambda : G \to (\mathbb{K}, +)$ such that V_j is a λ-dependent $\mathbb{K}A$-module for all $j \in J_0$.*
    2. *A acts non-trivially on J_0, $\dim_{\mathbb{K}} V_j = 2$ and $C_B(V_j) = C_B(V)$ for all $j \in J_0$.*
3. *$|J| = 2$, A is not quadratic on V and for $j \in J$, $[V_j, B] = C_{V_j}(B)$ is a $\mathbb{K}$-hyperplane of V_j. Moreover, one of the following holds:*
    1. *A acts trivially on J, and $[V_j, A] = [v_j\mathbb{F}, A]$ for all $v_j \in V_j \setminus [V_j, A]$ and $j \in J$.*

2. *A acts non-trivially on J, $\operatorname{char}\mathbb{F} = 2$, $\mathbb{F} = \mathbb{K}$, and $C_B(V_j) = C_B(V)$ for all $j \in J$.*

Proof: The proof is by induction on $|A|$ and $|J|$. Note that $\dim_{\mathbb{K}} V \geq 4$ since $|J| \geq 2$ and $\dim_{\mathbb{K}} V_j \geq 2$. First we show:

1° *A acts $\mathbb{K}$-linearly on V.*

Assume that $A \neq A_{\mathbb{K}}$. Then we can apply 6.3. Since $|A| > 2$ and $\dim_{\mathbb{K}} V \neq 1$ we are in case 6.3(2), so $p = 2$, $A_{\mathbb{K}} \neq 1$, $[V, A_{\mathbb{K}}] = C_V(A_{\mathbb{K}})$, and $\dim_{\mathbb{K}} V/C_V(A_{\mathbb{K}}) = 1$. If $|A_{\mathbb{K}}| = 2$, then $\dim_{\mathbb{K}}[V, A_{\mathbb{K}}] = 1$ and so $\dim_{\mathbb{K}} V = 2$, which contradicts $\dim_{\mathbb{K}} V \geq 4$. If $|A_{\mathbb{K}}| > 2$, then we can apply induction with $A_{\mathbb{K}}$ in place of A. Since $A_{\mathbb{K}}$ acts quadratically on V, one of the cases (1) or (2) holds for $A_{\mathbb{K}}$. In both cases $|A_{\mathbb{K}}/A_{\mathbb{K}} \cap B| \leq 2$, so $[V_r, A_{\mathbb{K}} \cap B] \neq 0$ for some $r \in J$. Hence 4.6 applied to $A_{\mathbb{K}} \cap B$ shows that $C_V(A_{\mathbb{K}} \cap B)$ is not a $\mathbb{K}$-hyperplane of V. This contradiction shows that A acts $\mathbb{K}$-linearly on V.

Case 1 *A is not transitive on J.*

Let L be an orbit of A on J. We choose L in such a way that $|L|$ is minimal and that A centralizes V_j for $j \in L$ if this is possible. Put $I := J \setminus L$. Then $V_L \otimes_{\mathbb{K}} V_I \cong V$ is an ordinary A-invariant tensor decomposition of V. By (1°) A induces $\mathbb{K}$-linear transformations on V_L and V_I.

2° *A acts quadratically on V_L and V_I.*

If A acts quadratically on V, then by 4.9(a) A also acts quadratically on V_L and V_I. If A is not quadratic on V, then 6.2(a) shows that A acts quadratically on V_L and V_I.

3° *Suppose that A centralizes V_L. Then (1) or (2) of the proposition holds.*

Note that by our choice of L and 4.5, $|L| = 1$. Moreover, the faithful action of A on V shows that A acts faithfully on V_I.

If $|I| = 1$, then (1) holds. If $|I| > 1$, then by induction on $|J|$ we see that (1) or (2) holds for V_I since A is quadratic on V_I. But then the same case also holds for V.

4° *Suppose that A acts non-trivially on V_L. Then (2:1) or (3:1) holds.*

By our choice of L and 4.5, A does not centralize any V_i for $i \in J$ and A acts non-trivially on V_I.

Assume first that A acts quadratically on V. Then A is elementary abelian and by 4.9 there exists a homomorphism $\lambda : G \to (\mathbb{K}, +)$ such that V_L and V_I are λ-dependent as $\mathbb{K}A$-modules. Suppose for a contradiction that $|I| \geq 2$. Then by induction $|I| = 2$, and (2:1) or (2:2) holds for V_I. Let $I = \{i, k\}$.

Suppose that (2:1) holds. Then there exists a homomorphism $\mu : G \to (\mathbb{K}, +)$ such that V_L and V_I are μ-dependent as $\mathbb{K}A$-module. Let $1 \neq a, b \in A$. As in the proof of 4.9 we can choose μ such that $a\mu = 1$. Put $\xi = b\mu$. For $j \in I$ let $x_j \in V_j \setminus C_{V_j}(A)$ and put $y_j := [x_j, a]$. Then

$$[x_i \otimes x_k, a] = x_i \otimes y_k + y_i \otimes x_k + y_i \otimes y_k$$

and

$$\begin{aligned} [x_i \otimes x_k, b] &= x_i \otimes y_k\xi + y_i \otimes x_k\xi + y_i\xi \otimes y_k\xi \\ &= (x_i \otimes y_k)\xi + (y_i \otimes x_k)\xi + (y_i \otimes y_k)\xi^2. \end{aligned}$$

Since A acts λ-dependently on V_I, we also get $[x_i \otimes x_k, b] = [x_i \otimes x_k, a]\ell$ for some $\ell \in \mathbb{K}$. A comparison of coefficients gives $\xi = \xi^2$. So $\xi = 1$ and $a = b$. Thus $|A| = 2$, a contradiction to the assumptions.

Suppose next that (2:2) holds. Let $a \in A \setminus C_A(I)$. Since $|A| > 2$ there also exists $1 \neq b \in C_A(i)$. Since $|I| = 2$, a interchanges i and k while b fixes i and k. Let $v_i \in V_i \setminus C_{V_i}(b)$ and $c_i := [v_i, b]$. Put $c_k := c_i a_i = c_i a$ and $v_k := v_i a_i = v_i a$. By 4.2(a) and since A is abelian, $[v_k, b] = c_k$. Since $\operatorname{char} \mathbb{K} = 2$ we have $a^2 = 1$, $v_k a = v_i$ and $c_k a = c_i$. Thus

$$[v_i \otimes c_k, b] = c_i \otimes c_k$$

and

$$[v_i \otimes c_k, a] = (c_k a_k \otimes v_i a_i) - (v_i \otimes c_k) = c_i \otimes v_k - v_i \otimes c_k.$$

Since c_j and v_j are $\mathbb{K}$-linearly independent we conclude that $[v_i \otimes c_k, b]\mathbb{K} \neq [v_i \otimes c_k, a]\mathbb{K}$, a contradiction to 3.3 since A acts λ-dependently on V_J.

Thus $|I| = 1$ and the minimal choice of $|L|$ gives $|L| = 1$. Now (2:1) holds.

Assume now that A is not quadratic on V. Then by 6.2(b) (with $V_1 = V_L$ and $V_2 = V_I$), $C_{V_I}(A) = [V_I, A]$ is a $\mathbb{K}$-hyperplane of V_I. Suppose for a contradiction that $|I| \geq 2$. If $[C_A(I), V_I] \neq 1$, then by 4.6 (applied to V_I and $C_A(I)$), $\dim_{\mathbb{K}} V_I / C_{V_I}(C_A(I)) > 1$, a contradiction. Thus $C_A(V_I) = C_A(I)$ and so $|A/C_A(V_I)| = 2$. But then $C_{V_I}(A) = [V_I, A]$ is 1-dimensional and $\dim_{\mathbb{K}} V_I = 2$, a contradiction to $|I| \geq 2$. Hence $|I| = 1$, and thus by our choice of L also $|L| = 1$. Now 6.2(c) gives (3:1).

Case 2 *A is transitive on J.*

Fix $1 \in J$ and put $B_1 := C_A(1)$. Since A is a finite p-group, there exists a 1-dimensional $\mathbb{K}B_1$-submodule X_1 of V_1. We apply 4.7 and 4.8 (with A in place of G) and use the notation introduced there. So we get systems Δ and $\tilde{\Delta}$ of imprimitivity for A in U/X and $\tilde{X}/\tilde{U}$, respectively, on which A acts transitively. Moreover, by 2.6 A is nearly quadratic on U/X and $\tilde{U}/\tilde{X}$. Thus we can apply 2.13 to U/X, Δ and A (and $\tilde{X}/\tilde{U}$, $\tilde{\Delta}$ and A).

5° *Either $|J| \geq 3$ and 2.13 (4:1) or (4:2) holds for Δ and A, or $|J| = 2$ and 2.13 (3) or (4:3) holds for Δ and A. In particular $[V_1, B, B] \leq X_1$.*

This follows from 2.13 using the transitivity of A on Δ.

6° *$|J| = 2 = \operatorname{char} \mathbb{K}$ and $|A/B| = 2$, in particular $B_1 = B \neq 1$.*

Suppose that $|J| \geq 3$. Then by (5°) and 2.13 A is not quadratic on U/X and not quadratic on $\tilde{X}/\tilde{U}$. Since $|J| \geq 3$ we have $U \leq \tilde{U}$. Hence A is neither quadratic on U nor on V/U, which contradicts 2.6.

Thus $|J| = 2$. It follows that $|A/B| = |J| = 2$ and $B_1 = B$. Moreover, $B \neq 1$ since $|A| \geq 3$.

According to (6°) we may assume $J = \{1, 2\}$.

7° *A is elementary abelian and $C_B(V_i) = C_B(V)$.*

By (6°) $\operatorname{char} \mathbb{F} = 2$. So by 6.4, A is elementary abelian. The transitive action of A on J and 4.2(a) also give $C_B(V_i) \leq C_B(V)$. By 4.5 $C_B(V) \leq C_B(V_i)$ and (7°) is proved.

8° *Suppose that A is quadratic on V. Then (2:2) holds.*

This follows from (1°), (7°) and 4.10.

9° *Suppose that $C_{V_1}(B) \neq X_1$. Then (2:2) or (3:2) holds.*

There exists a 1-dimensional $\mathbb{K}$-subspace X_1' of $C_{V_1}(B)$ different from X_1. Hence by (5°) $[V_j, B, B] \leq X_1 \cap X_1' = 0$, so B acts quadratically on V_1. By (6°) $|A/B| = 2$ and $B \neq 1$.

Assume first that $C_{V_1}(B) \neq [V_1, B]$. Then there exists a non-zero a $\mathbb{K}B$-submodule $Z_1 \leq C_{V_1}(B)$ with $C_{V_1}(B) = Z_1 \oplus [V_1, B]$. Hence, there also exists a $\mathbb{K}B$-submodule $Y_1 \leq V_1$ with $C_{V_1}(B) \cap Y_1 = [V_1, B]$ and $V_1 = Y_1 \oplus Z_1$. Pick $a \in A \setminus B$ and put

$$Z_2 := Z_1 a_1,\ Y_2 := Y_1 a_1,\ Y := Z_1 \otimes Y_2 + Y_1 \otimes Z_2,\ D := Y_1 \otimes Y_2.$$

By 4.2(a), Y, D and $Z_1 \otimes Z_2$ are $\mathbb{K}A$-submodules of V. Note that

$$V = (Z_1 \oplus Y_1) \otimes (Z_2 \oplus Y_2) = (Z_1 \otimes Z_2) \oplus (Z_1 \otimes Y_2) \oplus (Y_1 \otimes Z_2) \oplus (Y_1 \otimes Y_2) = (Z_1 \otimes Z_2) \oplus Y \oplus D.$$

4.5 implies that A neither centralizes Y nor D. Hence, 2.9 shows that A is quadratic on V, and (8°) implies (9°).

Assume now that $C_{V_1}(B) = [V_1, B]$. Then $X_1 \leq [V_1, B]$. We apply (5°). If 2.13 (4:3) holds for Δ then $[U_1/X, B]$ is a $\mathbb{F}$-hyperplane of U_1/X. Hence $\mathbb{K} = \mathbb{F}$, $[V_1, B]$ is a $\mathbb{K}$-hyperplane of V_1 and (3:2) follows from (7°). If 2.13 (3) holds for Δ then $[V_1, B] = X_1$ and so $C_{V_1}(B) = X_1$, a contradiction.

10° *Suppose that $C_{V_1}(B) = X_1$. Then (2:2) or (3:2) holds.*

If $\dim_{\mathbb{K}} V_1 = 2$, then (7°) implies (2:2) or (3:2). Hence we may assume that $\dim_{\mathbb{K}} V_1 \geq 3$. Let Y be a 3-dimensional $\mathbb{K}B$-submodule of V_1.

Since $\operatorname{char} \mathbb{K} = 2$, the elementary abelian 2-subgroups of $\mathrm{GL}_3(\mathbb{K})$ are quadratic. Hence $[Y, B, B] = 0$ and since $C_{V_1}(B) = X_1$ we conclude that $[Y, B] = X_1$. Fix $1 \neq b \in B$ with $[Y, b] \neq 0$ and $a \in A \setminus B$. Then $\dim_{\mathbb{K}} Y/C_Y(b) = \dim_{\mathbb{K}} X_1 = 1$, so there exists $x, y, z \in Y$ such that

$$Y = \langle x, y, z\rangle_{\mathbb{K}},\ X_1 = x\mathbb{K},\ [z, b] = 0,\ [y, b] = x.$$

By (7°) A is abelian and so by 4.2 the map

$$V_1 \to V_2 \text{ with } v_1 \mapsto v_1 a =: v_1'$$

is a $\mathbb{K}B$-module isomorphism. It is easy to calculate that

$$x \otimes x' \in C_V(A),\ y \otimes y' \in C_V(a) \text{ and } [y \otimes z', a] = y \otimes z' + z \otimes y'.$$

This shows that

$$[y \otimes z', a, b] = [y \otimes z' + z \otimes y', b] = x \otimes z' + z \otimes x' \neq 0,$$

Thus $y \otimes z' \notin Q_V(A)$. Since V is a nearly quadratic $\mathbb{F}A$-module we get for $Y' := Y a_1$

$$(*) \qquad \begin{aligned} Q_V(A) &= [y \otimes z', A]\mathbb{F} + C_V(A) = [y \otimes z', a]\mathbb{F} + [y \otimes z', B]\mathbb{F} + C_V(A) \\ &\leq (y \otimes z' + z \otimes y')\mathbb{F} + x \otimes Y' + Y \otimes x' + C_V(A). \end{aligned}$$

If $y \otimes y' \notin Q_V(A)$ then $Q_V(A) = [y \otimes y', A]\mathbb{F} + C_V(A) \leq C_V(a)$, since $y \otimes y' \in C_V(a)$. But then $[V, A, a] = 0$. Since A is abelian, we get $[V, a, A] = 0$, which contradicts $[y \otimes z', a, b] \neq 0$. Thus we have $y \otimes y' \in Q_V(A)$, so $(*)$ shows that there exist $u, w \in Y$, $t \in \mathbb{F}$ and $c \in C_V(A)$ such that

$$y \otimes y' = (y \otimes z' + z \otimes y')t + x \otimes u' + w \otimes x' + c.$$

Taking the commutator with b on both sides gives

$$x \otimes x' + x \otimes y' + y \otimes x' = (x \otimes z' + z \otimes x')t + x \otimes [u', b] + [w, b] \otimes x'$$

and

$$x \otimes y' + y \otimes x' \equiv (x \otimes z' + z \otimes x)t + (x \otimes x')k$$

for some $k \in \mathbb{K}$. But then x, y, z are not linearly independent in V_1 which contradicts $\dim_{\mathbb{K}} Y = 3$. This contradiction shows (10°) and completes the proof of 6.5. $\square$

7 The Nearly Quadratic Subgroup Theorem

Definition 7.1 *Let H be a group, $\mathbb{F}$ a field and V an $\mathbb{F}H$-module. We say that H acts nilpotently on V if there exists a finite ascending series*

$$0 = V_0 \leq V_1 \leq V_2 \leq \ldots \leq V_{d-1} \leq V_d = V$$

of $\mathbb{F}H$-submodules such that $[V_i, H] \leq V_{i-1}$ for all $1 \leq i \leq d$.

We say that V is H-reduced if $[V, N] = 0$ whenever N is a normal subgroup of H acting nilpotently on V.

Lemma 7.2 *Let $\mathbb{F}$ be a field, V a finite dimensional $\mathbb{F}$-space, and $G \leq \mathrm{GL}_{\mathbb{F}}(V)$. For $U \leq V$ let $L(U)$ be the largest subgroup of $\mathrm{SL}_{\mathbb{F}}(V)$ with $[U, L(U)] = 0$ and $[V, L(U)] \leq U$. Suppose that V is G-reduced and there exists 1-dimensional $\mathbb{F}$-subspace U of V with $L(U) \leq G$. Then $\langle L(U)^G \rangle = \mathrm{SL}_{\mathbb{F}}(V)$.*

Proof: Put $M = \langle L(U)^G \rangle$. We may assume that $\dim_{\mathbb{F}} V > 1$ since otherwise $M = \mathrm{SL}_{\mathbb{F}}(V) = 1$. Let $\mathcal{P} = \mathcal{P}_{\mathbb{K}}(V)$ be the set of 1-dimensional subspaces of V, and let

$$\mathcal{P}(M) := \{X \in \mathcal{P} \mid L(X) \leq M\}.$$

As $\mathrm{SL}_{\mathbb{F}}(V) = \langle L(X) \mid X \in \mathcal{P} \rangle$, it suffices to show that $\mathcal{P}(M) = \mathcal{P}$.

Since $[V, L(U)] = U$ we get $[V, M] = \sum_{U \in \mathcal{P}(M)} U$. If $[V, M] \neq V$, then $1 \neq C_{L(U)}([V, M]) \leq C_M([V, M])) \cap C_M(V/[V, M])$, a contradiction since the latter group is normal in M and acts nilpotently on V. Thus $V = [V, M] = \sum_{U \in \mathcal{P}(M)} U$.

Let $U_1, U_2 \in \mathcal{P}(M)$. Then $L(U_1)$ acts transitively on the 1-dimensional subspaces of $U_1 + U_2$ unequal to U_1. Hence $\mathcal{P}(M)$ contains all the 1-dimensional subspaces of $U_1 + U_2$. Since $V = \sum_{U \in \mathcal{P}(M)} U$ we conclude that $V = \sum_{U \in \mathcal{P}(M)} U = \bigcup_{U \in \mathcal{P}(M)} U$, and $\mathcal{P}(M)$ contains all the 1-dimensional subspaces of V. $\square$

Remark 7.3 *Let $\mathbb{F}$ be a field, H a group and V be a finite dimensional $\mathbb{F}H$-module. Then H acts on the dual module $V^* := \mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F})$ via*

$$v.w^*h := vh^{-1}.w^* \quad (h \in H,\, v \in V,\, w^* \in V^*).$$

Put

$$U^\perp := \{w^* \in V^* \mid Uw^* = 0\} \text{ and } U^{*\perp} := \{v \in V \mid vU^* = 0\},$$

where U is an $\mathbb{F}$-subspace of V and U^* an $\mathbb{F}$-subspace of V^*. Elementary linear algebra shows that

$$\dim_{\mathbb{F}} U + \dim_{\mathbb{F}} U^{\perp} = \dim_{\mathbb{F}} U^* + \dim_{\mathbb{F}} U^{*\perp} = \dim_{\mathbb{F}} V,$$

$$[V,U]^{\perp} = C_{V^*}(U) \text{ and } [V^*,U]^{\perp} = C_V(U) \text{ for } U \leq H.$$

In particular, if U is a hyperplane of V, $U^{\perp}$ is a 1-dimensional subspace of V and $L(U) = L(U^{\perp})$. Hence passing to the dual space V^* transforms 7.2 into a statement about reduced subgroups $G \leq \mathrm{GL}_{\mathbb{F}}(V)$ with $L(U) \leq G$ for some hyperplane $U \leq V$. We will refer to this version as the "dual version of 7.2".

Also in 7.5 below we will "dualize" in this way certain steps in the proof.

Proof of Theorem 1: Since V is semisimple, there exist simple $\mathbb{F}H$-submodules $V_j, j \in J$, such that $V = \bigoplus_{j\in J} V_j$. Let $I = \{i \in J \mid [V_i, H] \neq 0\}$. Then clearly

$$V = C_V(H) \oplus \bigoplus_{i\in I} V_i.$$

For $i \in I$ let $\mathcal{Q}_i = \{A \in \mathcal{Q} \mid [V_i, A] \neq 0\}$. Then by 2.9 each $A \in \mathcal{Q}$ is contained in a unique $\mathcal{Q}_i$ and so $(Q_i)_{i\in I}$ is a partition of $\mathcal{Q}$. Observe that H_i centralizes V_j for all $i \in I$ and $j \in J$ with $i \neq j$. In particular, V_i is a faithful H_i-module and $H_i \cap \langle H_j \mid i \neq j \in I\rangle = 1$. Thus (a) holds. Since V_i is a simple $\mathbb{F}H$ module, we conclude that V_i is a simple $\mathbb{F}H_i$. Since $0 \neq [V, H_i] \leq V_i$ we get $V_i = [V, H_i]$ and so (b) and (c) hold. □

Lemma 7.4 *Let $\mathbb{K}$ be a finite field, $\mathbb{F} \leq \mathbb{K}$ a subfield, V a $\mathbb{K}$-space, $L \leq \mathrm{GL}_{\mathbb{K}}(V)$ such that $L \cong \mathrm{SL}_2(\mathbb{F})$, $V = [V, L]$, $C_V(L) \neq 0$ and $V/C_V(L) \cong W_0 \otimes_{\mathbb{F}} \mathbb{K}$, where W_0 is a natural $\mathbb{F}\mathrm{SL}_2(\mathbb{F})$-module for L. Let $A \in \mathrm{Syl}_2(L)$. Put $H = \mathrm{N}_{\mathrm{GL}_{\mathbb{K}}(V)}(A)$, $B = C_{\mathrm{SL}_{\mathbb{K}}(V)}(C_V(A))$, $Z = \mathrm{Z}(\mathrm{GL}_{\mathbb{K}}(V))$, $V_1 = [C_V(A), H \cap L]$ and $V_2 = C_V(L)$. Then the following hold:*

(a) $\mathrm{char}\,\mathbb{F} = 2$, $|\mathbb{F}| \geq 4$ and $V \cong W \otimes_{\mathbb{F}} \mathbb{K}$, where W is a natural $\mathbb{F}\Omega_3(\mathbb{F})$-module for L.

(b) $C_V(A) = V_1 \oplus V_2$ and $C_V(A) \leq [V, B] = [V, A]$ for all $B \leq A$ with $|B| \geq 4$.

(c) If $|\mathbb{F}| > 4$, then V_1 and V_2 are H-invariant and $H = (H \cap L)ZB = C_H(V_2)Z$.

(d) If $|\mathbb{F}| = 4$, then $V_1^H = \{V_1, V_2\}$ and $N_H(V_1) = (H \cap L)ZB = C_H(V_2)Z$.

Proof: If $p := \mathrm{char}\,\mathbb{F} \neq 2$ or $|\mathbb{F}| = 2$, then $\mathrm{F}(L)$ is a non-trivial p'-group, $V = C_V(\mathrm{F}(L)) \oplus [V, \mathrm{F}(L)]$, $C_V(L) = C_V(\mathrm{F}(L))$ and $V = [V, L] = [V, \mathrm{F}(L)]$, a contradiction to $C_V(L) \neq 0$.

Thus $\mathrm{char}\,\mathbb{F} = 2$ and $q := |\mathbb{F}| \geq 4$. Let $E \leq A$ with $|E| = 4$ and pick $1 \neq e \in E$. Then $C_V(L)[V, e]$ is a $\mathbb{K}$-hyperplane of V. By Dickson's List [Hu, II.8.27] of maximal subgroups of $\mathrm{SL}_2(\mathbb{F})$, there exists a maximal subgroup $D \leq L$ with $D \cong D_{2(q+1)}$. As $q + 1$ is odd, $D = \langle e, e^g\rangle$ for some $g \in D$ and $E \not\leq D$, in particular $L = \langle E, e^g\rangle$. Since $[V, e, e] = 0$ we get $C_V(L) + [V, e] = C_V(e)$ and $[V, e]$ is 1-dimensional over $\mathbb{K}$. Thus $[V, E]$ is at most 2-dimensional and $V = [V, L] = [V, E] + [V, e^g]$ is at most 3-dimensional. Since $C_V(L) \neq 0$, $\dim_{\mathbb{K}} V \geq 3$. Thus $\dim_{\mathbb{K}} V = 3$, $\dim_{\mathbb{K}} C_V(L) = 1$ and $\dim_{\mathbb{K}}[V, E] = 2$. We have $\dim_{\mathbb{K}}[V/C_V(L), E] = 1$ and so $C_V(L) \leq [V, E]$.

Since $\dim_{\mathbb{K}} C_V(L) = 1$ for any such V, we conclude that V is unique up to $\mathbb{K}L$-isomorphism (see [As, 17.12]). Let W be a natural $\mathbb{F}\Omega_3(\mathbb{F})$ module. The $W = [W, L]$, $C_W(L)$ is 1-dimensional over $\mathbb{F}$ and $W/C_W(L) \cong W_0$. Thus $W \otimes_{\mathbb{F}} \mathbb{K}$ fulfills the assumption on V and so $V \cong W \otimes_{\mathbb{F}} \mathbb{K}$. By coprime action $C_V(A) = V_1 \oplus V_2$. Hence (a) and (b) hold.

Let q be the L-invariant quadratic form on W and s the corresponding bilinear form. We fix an $\mathbb{F}$-basis (w_1, w_2, w_3) satisfying

$$w_2 \in C_W(L),\ (w_1, w_3)s = 1,\ w_1q = w_3q = 0,\ w_2q = 1,\ w_1 \in C_V(A).$$

Let $a \in A$ and $w_3a = w_1f_1 + w_2f_2 + w_3f_3$, $f_i \in \mathbb{F}$. Then

$$1 = (w_1, w_3)s = (w_1a, w_3a)s = (w_1, w_3a)s = (w_1, w_3f_3) = f_3,$$

and a similar calculation using $0 = (w_3a)q$ yields $f_2^2 = f_1$, so $w_3a = w_1\lambda^2 + w_2\lambda + w_3$ for some $\lambda \in \mathbb{F}$. We denote this element of A by a_λ.

Let v_i be the image of $w_i \otimes 1$ in V under the isomorphism from $W \otimes_{\mathbb{F}} \mathbb{K}$ to V. Then (v_1, v_2, v_3) is an $\mathbb{K}$-basis for V and the matrix of a_λ with respect to this basis is

$$a_\lambda \leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \lambda^2 & \lambda & 1 \end{pmatrix}.$$

Note that B is abelian and $A \leq B$. Thus $ZB \leq H$. Since ZB acts transitively on $V \setminus C_V(A)$ we have $H = C_H(v_3)ZB$. Let $h \in C_H(v_3)$. Since H normalizes $C_V(A) = v_1\mathbb{K} + v_2\mathbb{K}$:

$$h \leftrightarrow \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for some $a, b, c, d \in \mathbb{K}$ with $ad - cb \neq 0$. Let $\lambda \in \mathbb{F}$. Since $h \in H$, h normalizes A and so $a_\lambda h = ha_\mu$ for some $\mu \in \mathbb{F}$. We have

$$a_\lambda h \leftrightarrow \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ \lambda^2 a + \lambda c & \lambda^2 b + \lambda d & 1 \end{pmatrix} \text{ and } ha_\mu \leftrightarrow \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ \mu^2 & \mu & 1 \end{pmatrix}.$$

Hence

$$\lambda^2 b + \lambda d = \mu \text{ and } \lambda^2 a + \lambda c = \mu^2 = \lambda^4 b^2 + \lambda^2 d^2.$$

Thus

$$\lambda c + \lambda^2(a + d^2) + \lambda^4 b^2 = 0 \text{ for all } \lambda \in \mathbb{F}.$$

Suppose that $|\mathbb{F}| > 4$ and consider the polynomial $f = cx + (a + d^2)x^2 + b^2x^4$. Then each $\lambda \in \mathbb{F}$ is a root of f. Since $\deg f \leq 4 < |\mathbb{F}|$ we conclude that f is the zero polynomial. Hence $c = 0, b = 0$ and $a = d^2$. From $\mu = \lambda^2 b + \lambda d = \lambda d$ we conclude that $d \in \mathbb{F}$. Moreover,

$$h \leftrightarrow \begin{pmatrix} d^2 & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} d & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & d^{-1} \end{pmatrix} \begin{pmatrix} d & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & d \end{pmatrix}$$

and so $h \in (H \cap L)Z$. Thus $H = (H \cap L)BZ$. Since $(H \cap L)B \le C_H(V_2)$ we see that (c) holds.

Suppose next that $|\mathbb{F}| = 4$. Then $\lambda^4 = \lambda$ for all λ in $\mathbb{F}$. Thus $\lambda(c + b^2) + \lambda^2(a + d^2) = 0$. Hence $f = (c + b^2)x + (a + d^2)x^2$ is the zero polynomial and so $c = b^2$ and $a = d^2$. From $\lambda^2 b + \lambda d = \mu \in \mathbb{F}$ for all $\lambda \in \mathbb{F}$ we conclude that $b, d \in \mathbb{F}$. Moreover, $0 \neq ad - bc = a^3 - b^3$. Since $u^3 = 1$ for all $0 \neq u \in \mathbb{F}$ we have $a = 0$ or $b = 0$. If $a = 0$, then $v_1 h = v_2 b \in V_2$ and if $b = 0$ then $v_1 h = v_1 a \in V_1$. Thus $V_1^H = \{V_1, V_2\}$. Also if $b = 0$ then as above $h \in (L \cap H)BZ \le C_H(V_2)Z$ and so (d) holds. $\square$

Proposition 7.5 *Let H be a finite group, $\mathbb{K}$ a finite field and V faithful finite dimensional $\mathbb{K}H$-module. Put*

$$\mathcal{H} = \{A \le H \mid C_V(A) = [V, A] \text{ and } \dim V/C_V(A) = 1\}.$$

Suppose that $H = \langle \mathcal{H} \rangle$. Then $C_H(V/C_V(H)) = O_p(H)$ and $C_V(H)$ is the unique maximal $\mathbb{K}H$-submodule of V. Moreover, put $\overline{V} = V/C_V(H)$, $\widetilde{H} = H/C_H(\overline{V})$, $p = \operatorname{char} \mathbb{K}$, and $n = \dim_{\mathbb{K}}(\overline{V})$. Then one of the following holds:

1. *$p = 2$, $n = 2$, and $\widetilde{H} \cong D_{2m}$ for some odd integer m with $m > 3$.*
2. *$p = 3$, $n = 2$, $\mathbb{F}_9 \le \mathbb{K}$, and $\widetilde{H} \cong \mathrm{SL}_2(5)$.*
3. *$p = 2$, $n = 3$, $\mathbb{F}_4 \le \mathbb{K}$ and $\widetilde{H} \cong 3.\mathrm{Alt}(6)$.*
4. *$\widetilde{H} \cong \mathrm{SL}_n(\mathbb{F})$ for some subfield $\mathbb{F}$ of $\mathbb{K}$. Moreover, $\overline{V} \cong W \otimes_{\mathbb{F}} \mathbb{K}$ for a natural $\mathbb{F}\tilde{H}$-module W.*

Proof: Let

$$\mathcal{H}^* = \{A \le H \mid C_V(A) = [V, A] \text{ and } \dim_{\mathbb{K}} C_V(A) = 1\},$$

and let $\mathcal{A}$ and $\mathcal{A}^*$ be the set of maximal elements of $\mathcal{H}$ and $\mathcal{H}^*$, respectively. By $\mathcal{H}_L$, $\mathcal{A}_L$,$\mathcal{H}^*_L$ and $\mathcal{A}^*_L$ we denote the set of elements of $\mathcal{H}$, $\mathcal{A}$, $\mathcal{H}^*$ and $\mathcal{A}^*$ contained in $L \le H$.

We now proceed by induction on n. First we show:

1° *Let $B \in \mathcal{H}$ and T be a p-subgroup of H with $[B, T] = 1$. Then $C_V(B) \le C_V(T)$, $BT \in \mathcal{H}$, and if $B \in \mathcal{A}$, then $T \le B$.*

Let $1 \neq b \in B$. Then $[V, b]$ is a 1-dimensional $\mathbb{K}$-space normalized by T and so $[V, b, T] = 0$. Since $C_V(B) = [V, B]$ we get $C_V(B) \le C_V(T)$. Hence $C_V(B) = C_V(BT) = [V, BT]$ and $BT \in \mathcal{H}$. If $B \in \mathcal{A}$ this gives $BT = B$.

2° *Let $L \le H$ such that $L = \langle \mathcal{H}_L \rangle$. Then $C_V(L)$ is the unique maximal $\mathbb{K}L$-submodule of V.*

Let U be a $\mathbb{K}L$-submodule of V with $U \not\le C_V(L)$. Then there exists $A \in \mathcal{H}_L$ with $[U, A] \neq 0$. Since $\dim_{\mathbb{K}} V/C_V(A) = 1$ we have $V = U + C_V(A)$. Thus $C_V(A) = [V, A] = [U, A] \le U$ and so $V = U$.

3° *Let $L \le H$ such that $L = \langle \mathcal{H}_L \rangle$. Then $V/C_V(L)$ is an absolutely simple $\mathbb{K}L$-module and $O_p(L) = C_L(V/C_V(L))$.*

Put $\hat{V} := V/C_V(L)$. It follows from (2°) that $\hat{V}$ is simple $\mathbb{K}L$-module. Let $A \in \mathcal{H}_L$ and put $\mathbb{D} = \mathrm{End}_{\mathbb{K}L}(\hat{V})$. Then

$$|\mathbb{D}| \le |\hat{V}/[\hat{V}, A]| = |V/[V, A]| = |\mathbb{K}|$$

and so $\mathbb{D} = \mathbb{K}$. Thus by 5.1(a), $\widetilde{V}$ is absolutely simple. Moreover, $O_p(L) \le C_L(\hat{V})$. Since $C_L(\hat{V})$ centralizes $C_V(L)$ and $V/C_V(L)$ we get that $C_L(\hat{V})$ is a p-group and so (3°) holds.

4° *$H = \langle \mathcal{A} \rangle$, each A in $\mathcal{A}$ is weakly closed in H, and H acts transitively on $\mathcal{A}$.*

Since each $B \in \mathcal{H}$ is contained in some $A \in \mathcal{A}$, $H = \langle \mathcal{A} \rangle$. Let $A, B \in \mathcal{A}$ such that B normalizes A. By (2°) applied to AB in place of H, $C_V(A) \leq C_V(AB)$. Thus $C_V(A) = C_V(B)$ and so $AB \in \mathcal{H}$. By maximality of A and B we get $A = AB = B$. Thus A is weakly closed in H. In particular, any Sylow p-subgroup of H contains a unique member of $\mathcal{A}$. So (4°) holds.

If $n \leq 1$, then (4) holds with $\mathbb{K} = \mathbb{F}$. If $n = 2$ the Proposition follows from Dickson's List of subgroups of $\mathrm{SL}_2(\mathbb{K})$ [Hu, II.8.27]. Replacing V be $\overline{V}$ and H by $\tilde{H}$ we may assume from now on:

5° *V is a faithful simple $\mathbb{K}H$ module and $n \geq 3$.*

Let V^* be the $\mathbb{K}H$-module dual to V. If X is an $\mathbb{K}H$-submodule of V^*, then $X^\perp$ is a $\mathbb{K}H$-submodule of V. So by (5°) V^* is a simple $\mathbb{K}H$-module. For $A \in \mathcal{H}^*$, observe that $\dim_{\mathbb{K}} C_{V^*}(A) = n - 1$ and $C_{V^*}(A) = [V^*, A]$. Hence, the elements of $\mathcal{H}^*$ act on V^* as the elements of $\mathcal{H}$ act on V. In particular any statement proved for $\mathcal{H}$ and subgroups generated by elements of $\mathcal{H}$ also gives rise to a dual statement with $\mathcal{H}$ and V replaced by $\mathcal{H}^*$ and V^*.

Since $C_V(H) = 0$ and $\dim V/C_V(A) = 1$ for all $A \in \mathcal{A}$, there exists $L \leq H$ with $L = \langle \mathcal{A}_L \rangle$ and $\dim C_V(L) = 1$. Let $\mathcal{L}$ be the set of such subgroups of H. Similarly let $\mathcal{L}^*$ be the set of all subgroups L^* such that $L^* = \langle \mathcal{A}_L^* \rangle$ and $\dim_{\mathbb{K}} C_{V^*}(L^*) = 1$.

6° *Suppose $L \in \mathcal{L}$ with $O_p(L) = 1$. Then $n = 3$, $p = 2$, $L \cong \mathrm{SL}_2(\mathbb{F})$ for some subfield $\mathbb{F}$ of $\mathbb{K}$ with $4 \leq |\mathbb{F}|$, $\mathcal{A}_L = \mathrm{Syl}_p(L)$ and $V \cong W \otimes_{\mathbb{F}} \mathbb{K}$ for a natural $\mathbb{F}\Omega_3(\mathbb{F})$-module W for L.*

By induction the theorem holds for L in place of H. By (2°) V is indecomposable as a $\mathbb{K}L$-module. In particular, $O_{p'}(L) = 1$. We conclude that Case (4) of 7.5 holds for L, so $L \cong \mathrm{SL}_{n-1}(\mathbb{F})$ for some subfield $\mathbb{F} \leq \mathbb{K}$.

Let $A \in \mathcal{A}_L$ and put $P^* = N_L(C_V(A))$ and $P = C_{P^*}(V/C_V(A))$. Note that P^* acts simply on $O_p(P^*)$. Since A is weakly closed in H and $A \leq O_p(P^*)$ we conclude that $A \trianglelefteq P^*$ and $A = O_p(P^*)$. If $n = 3$, then 7.4(a) implies that (6°) holds.

Suppose that $n > 3$. Then $A = O_p(P^*)$ is natural module for $P/O_p(P^*) \cong \mathrm{SL}_{n-2}(\mathbb{F})$. Let $x \in V \setminus C_V(A)$. Then $[x, A] \cong A/C_A(x) \cong A$ as an $\mathbb{F}_pP$-module. Thus $[x, A]$ is a nontrivial simple module for P. Hence $[V, A] = \sum_{v \in V} [v, A]$ is a sum of non-trivial simple $\mathbb{F}_pP$-modules and so $C_{[V,A]}(P) = 0$. But this contradicts $C_V(L) \leq C_V(A) = [V, A]$ and $\dim_{\mathbb{K}} C_V(L) = 1$.

7° *Suppose $L \in \mathcal{L}$ with $O_p(L) = 1$. Fix $A \in \mathcal{A}_L$.*

(a) If T is a p-subgroup of H with $L \leq N_H(T)$, then $T = 1$.

(b) Let $L \leq R \in \mathcal{L}$. Then $L = R$.

(c) Let $\mathcal{L}_-(A) = \{R \in \mathcal{L} \mid A \leq R, O_p(R) = 1\}$. If $|A| = 4$, then $|\mathcal{L}_-(A)| \leq 2$ and if $|A| > 4$, then $\mathcal{L}_-(A) = \{L\}$.

(d) There exists $R \in \mathcal{L}$ with $O_p(R) \neq 1$.

Since $A \cap T \leq O_p(L) = 1$ and by (4°) A is weakly closed in L, $[A, T] = 1$. Thus by (1°), $T \leq A$ and so $T \leq A \cap T = 1$. Hence (a) hold.

In particular, if $L \leq R \in \mathcal{L}$, then $O_p(R) = 1$. Hence by (6°), both L and R are isomorphic to $SL_2(|A|)$ and so $L = R$. Thus (b) holds.

Put $V_1 := [C_V(A), N_L(A)]$ and $V_2 := C_V(L)$. Let $R \in \mathcal{L}_-(A)$. By (6°) $N_R(A) = O^2(N_R(A))$, so (6°) and 7.4 imply that $N_R(A)$ normalizes V_1 and V_2. Since there are only two proper $\mathbb{K}N_R(A)$-submodules of $C_V(A)$, we conclude that $C_V(R) = V_1$ or $C_V(R) = V_2$. In the second case $\langle R, L\rangle \in \mathcal{L}$ and so by (b), $R = L$. So suppose $C_V(R) = V_1$, then $V_2 = [C_V(A), N_R(A)]$. Put $P = N_H(A) \cap N_H(V_2)$. From the action of $R \cong \mathrm{SL}_2(|A|)$ on V we conclude that $|P/C_P(V_2)| \geq |A| - 1$. On the other hand since $P \leq \mathrm{SL}_{\mathbb{K}}(V)$ and $V_2 = C_V(L)$, 7.4 implies that $P \leq \mathrm{Z}(\mathrm{SL}_{\mathbb{K}}(V))C_P(V_2)$ and so $|P/C_P(V_2)| \leq 3$. Thus $|A| = 4$. Moreover by (b), R is unique in $\mathcal{L}$ with $C_V(R) = V_1$ and so (c) holds.

Suppose that $O_p(R) = 1$ for all $R \in \mathcal{L}$. Let $B \in \mathcal{A}$ with $B \not\leq L$. Then $B \neq C$ for every $C \in \mathcal{A}_L$. Hence $C_V(C) \neq C_V(B))$. From (6°) we get $n = 3$, so $\langle C, B\rangle \in \mathcal{L}$. Again by (6°) $\langle C, B\rangle \cong \mathrm{SL}_2(\mathbb{F})$, $\langle C, B\rangle \in \mathcal{L}_-(B)$ and $\mathcal{A}_{L\cap\langle C,B\rangle} = \{C\}$. Thus $\langle C, B\rangle \neq \langle D, B\rangle$ for all $B \neq D \in \mathcal{A}_L$. Thus $|\mathcal{L}_-(B)| \geq |\mathcal{A}_L| > 2$, a contradiction to (c).

8° *Suppose $L \in \mathcal{L}$ with $O_p(L) \neq 1$. Then there exists a simple $\mathbb{F}_pL$-module W and a subfield $\mathbb{F} \leq \mathbb{K}$ with $\mathbb{F} \cong \mathrm{End}_L(W)$ such that the following hold:*

(a) $B \in \mathcal{H}^*$ for every $1 \neq B \trianglelefteq L$ with $B \leq O_p(L)$.

(b) $O_p(L) \in \mathcal{A}^*$ and $|O_p(L)| = |A|$ for $A \in \mathcal{A}$.

(c) $O_p(L)$ is a minimal normal subgroup of L.

(d) $|A| = |\mathbb{F}|^{n-1}$ and $|A \cap O_p(L)| = |\mathbb{F}|$ for $A \in \mathcal{A}_L$.

(e) $L/O_p(L) \cong \mathrm{SL}_{n-1}(\mathbb{F})$, and $O_p(L)$ is a natural module for $\mathbb{F}_p\mathrm{SL}_{n-1}(\mathbb{F})$.

(f) $V/C_V(L) \cong Y \otimes_{\mathbb{F}} \mathbb{K}$ as an $\mathbb{F}_pL$-module, and Y is a natural $\mathbb{F}_p\mathrm{SL}_{n-1}(\mathbb{F})$-module for L dual to $O_p(L)$.

(g) $H = \langle \mathcal{H}^*\rangle$.

By (2°) $V/C_V(L)$ is a simple $\mathbb{K}L$-module. Hence $\dim_{\mathbb{K}} C_V(L) = 1$ implies that $[V, B] = C_V(L) = C_V(O_p(L)) = C_V(B)$ for every non-trivial normal subgroup B of L contained in $O_p(L)$, in particular, $B \in \mathcal{H}^*$. This is (a).

Now let B be a minimal normal subgroup of L in $O_p(L)$. Then B is a simple $\mathbb{F}_pL$-module. There exists an $\mathbb{F}_pL$-submodule $W \leq V$ and a maximal $\mathbb{F}_pL$-submodule U of $C_V(L)$ such that $C_V(L) \leq W$, $W/C_V(L)$ is a simple $\mathbb{F}_pL$-module, and $[W, B] \not\leq U$. Then $B^* := W/C_V(L)$ is as an $\mathbb{F}_pL$-module dual to B. It follows that $O_p(L) = BB_0$, where $B_0 := C_{O_p(L)}(W/U)$ and $B \cap B_0 = 1$.

Put $\mathbb{F} := \mathrm{End}_L(B^*)$. Then B and B^* are also $\mathbb{F}L$-modules. By (3°), $V/C_V(L)$ is an absolutely simple $\mathbb{K}L$-module and so by 5.3 $V/C_V(L) \cong B^* \otimes_{\mathbb{F}} \mathbb{K}$ as an $\mathbb{F}_pL$-module. Let $A \in \mathcal{A}_L$. Since $[V/C_V(L), A] = C_{V/C_V(L)}(A)$ is a $\mathbb{K}$-hyperplane of $V/C_V(L)$, $[B^*, A] = C_{B^*}(A)$ is an $\mathbb{F}$-hyperplane of B^*. Hence duality shows that $\dim_{\mathbb{F}}[B, A] = C_B(A)$ is 1-dimensional over $\mathbb{F}$; in particular, $|A/C_A(B)| \leq |B/C_B(A)|$. Moreover, $[B, A] \leq A$ since A is weakly closed in H by (4°), so

$$[B, A] = C_B(A) = B \cap A.$$

By the dual version of (1°), $[V, C_A(B)] \leq [V, B] = C_V(L)$ and so $A \leq O_p(L)$. This gives $C_A(B) = A \cap O_p(L)$ and

$$\begin{aligned} |A| = |A/C_A(B)||C_A(B)| &\leq |B/C_B(A)||A \cap O_p(L)| = |B/B \cap A||A \cap O_p(L)| \\ &\leq |O_p(L)/O_p(L) \cap A||A \cap O_p(L)| = |O_p(L)|. \end{aligned}$$

By the dual version of (4°) all elements of $\mathcal{A}^*$ are conjugate and so have the same order. Together with (4°) we conclude

$$|A| \leq |O_p(L)| \overset{(a)}{\leq} |A^*| \text{ for every } A \in \mathcal{A} \text{ and every } A^* \in \mathcal{A}^*.$$

The dual version of (7°)(d) shows that there exists $L^* \in \mathcal{L}^*$ such that $O_p(L^*) \neq 1$. This leads to a dual version of the above chain of inequalities. Thus also $|A^*| \leq |A|$, and $|A| = |O_p(L)|$; in particular $O_p(L) \in \mathcal{A}^*$. This is (b).

Moreover, $|O_p(L)/A \cap O_p(L)| = |B/A \cap B|$ and so $O_p(L) = B(A \cap O_p(L))$; in particular, $[O_p(L), A] \leq B$. Since this holds for all $A \in \mathcal{A}_L$, $[O_p(L), L] = B$. Now the above factorization $O_p(L) = BB_0$ yields $[B_0, L] = 1$ and then with the Three Subgroups Lemma $[V, B_0] = 0$ and $B_0 = 1$. This is (c).

By induction the theorem holds for $V/C_V(L)$ and L. If we are not in case (4) we conclude that $|AB/B| < |B/C_B(A)|$, a contradiction. Thus (4) holds. So $L/O_p(L) \cong \mathrm{SL}_{n-1}(\mathbb{F})$ and $B = O_p(L)$ is a natural $\mathbb{F}_p\mathrm{SL}_{n-1}(\mathbb{F})$-module. In particular $|O_p(L)| = |\mathbb{F}|^{n-1}$, so by (c) also $|A| = |\mathbb{F}|^{n-1}$. Hence (e) and (d) are proved.

We have shown already that $V/C_V(L) \cong B^* \otimes_{\mathbb{F}} \mathbb{K}$. Hence (f) follows from (e).

By the dual version of (b) $O_p(L^*) \in \mathcal{A}$, where as above $L^* \in \mathcal{L}^*$ with $O_p(L^*) \neq 1$. Hence (4°) shows that

$$H = \langle O_p(L^*)^H \rangle \leq \langle L^{*H} \rangle = \langle \mathcal{H}^* \rangle,$$

and (g) follows.

9° *Let $A \in \mathcal{A}$ and put $L_A^* := \langle \mathcal{A}^*_{N_H(A)} \rangle$. Then $L_A^* \in \mathcal{L}^*$ and $A = O_p(L_A^*)$.*

By (7°)(d) there exists $R^* \in \mathcal{L}^*$ with $O_p(R^*) \neq 1$. By (8°)(b), $O_p(R^*) \in \mathcal{A}$ and so by (4°) we may assume that $O_p(R^*) = A$. Thus $A \leq R^* \leq L_A^*$. Note that $A \leq O_p(L_A^*)$. Since L_A^* normalizes $[V, A]$ we have $[V, L_A^*] = [V, A]$ and so $L_A^* \in \mathcal{L}^*$. Thus by (8°)(e), $|R^*| = |L_A^*|$ and so $R^* = L_A^*$ and $A = O_p(L_A^*)$.

10° *Let $A, B \in \mathcal{A}$ with $A \neq B$. Then exactly one of the following holds.*

1. *There exists $D \in \mathcal{A}^*$ with $D \leq N_H(A)$ and $B \leq N_H(D)$.*
2. *$\langle A, B \rangle \in \mathcal{L}$ and $O_p(\langle A, B \rangle) = 1$.*

Pick $L \in \mathcal{L}$ with $\langle A, B \rangle \leq L$. Suppose that $D := O_p(L) \neq 1$. Then by (8°)(b), $D \in \mathcal{A}^*$. Clearly $B \leq N_H(D)$ and since AD is a p-group and A is weakly closed, $D \leq N_H(A)$. Thus (1) holds.

Suppose that $O_p(L) = 1$. Then by (6°), $L = \langle A, B \rangle$ and so (2) holds.

Suppose for a contradiction that (1) and (2) hold. Since D is weakly closed by (4°), $A \leq N_H(D)$ and so $L \leq N_H(D)$, a contradiction to (7°)(a).

We now divide the proof into three cases.

Case 1 *Suppose that $O_p(L) = 1$ for some $L \in \mathcal{L}$. Then (3) holds.*

By (6°) $n = 3$, $p = 2$ and $L \cong \mathrm{SL}_2(\mathbb{F})$ for a subfield $\mathbb{F}$ of $\mathbb{K}$ with $|\mathbb{F}| = |A|$. Let $A \in \mathcal{A}_L$. By (7°) there exists $R \in \mathcal{L}$ with $O_p(L) \neq 1$ and by (4°) we can choose R such that $A \leq R$. Suppose that $|A \cap O_p(R)| > 2$. Then by 7.4(b) $[V, A \cap O_p(R] = [V, A]$. But this is a contradiction since by (8°)(b) $[V, O_p(R)]$ is 1-dimensional while $[V, A]$ is a hyperplane. Thus $|A \cap O_p(R)| = 2$. By (8°)(e),

$R/O_p(R) \cong \mathrm{SL}_2(\mathbb{E})$ for some subfield $\mathbb{E}$ of $\mathbb{K}$. Moreover, $|A| = |\mathbb{E}|^2$ and $|A \cap O_p(R)| = |\mathbb{E}|$. Thus $|\mathbb{E}| = 2$ and $|\mathbb{F}| = |A| = 4$.

By (9°), $N_H(A) \neq N_H(C_V(L))$. So $N_H(A)$ does not normalizes L. Thus (7°)(c) implies that $|\mathcal{L}_-(A)| = 2$. Each $T \in \mathcal{L}_-(A)$ is isomorphic to $\mathrm{SL}_2(4)$ and so contains four elements of $\mathcal{A}$ other than A. So there exist eight elements of $\mathcal{A}$ that satisfy (10°)(2) together with A.

By (9°) and (8°)(e), $L_A^* \cong \mathrm{Sym}(4)$ and so there exist exactly three $D \in \mathcal{A}^*$ with $D \leq N_H(A)$. Similarly there exist exactly three elements $B \in \mathcal{A}$ with $B \leq N_H(D)$, two of which are different from A. If $B \neq A$, then $\langle A, B\rangle = L_D$ and so $D = O_p(\langle A, B\rangle)$ is uniquely determined by A and B. Thus there are $6 = 2 \cdot 3$ elements of $\mathcal{A}$ that satisfy (10°)(1) together with A. This shows that $|\mathcal{A}| = 1 + 6 + 8 = 15$.

Since $N_L(A) \cong \mathrm{Alt}(4)$ and $L_A^* \cong \mathrm{Sym}(4)$ we have $N_L^*(A) \not\leq Z(H)N_L(A)$. Thus 7.4 implies $|N_H(A)/Z(H)N_L(A)| = 2$. Since $|N_L(A)| = 12$ and $N_L(A) \cap Z(H) = 1$, $|N_H(A)/\mathrm{Z}(H)| = 2 \cdot 12 = 24$ and $|H/\mathrm{Z}(H)| = 24 \cdot 15 = 360$. Since $|L\,\mathrm{Z}(H)/\mathrm{Z}(H)| = |L| = 60$ we have $|H/L\,\mathrm{Z}(H)| = 6$ and so $H/\mathrm{Z}(H) \cong \mathrm{Alt}(6)$. The elements of order three in L_A^* act fixed-point freely on $C_V(A)$, but the elements of order three in $N_L(A)$ do not. Thus $N_L(A) \not\leq L_A^*$ and $\mathrm{Z}(H) \neq 1$, so $|\mathrm{Z}(H)| = 3$. Therefore $H \sim 3.\mathrm{Alt}(6)$, $\mathbb{F}_4 \leq \mathbb{K}$ and (3) holds in this case.

Case 2 *Suppose that $O_p(L) = 1$ for some $L \in \mathcal{L}^*$. Then (3) holds.*

By duality the above argument also applies to $\mathcal{L}^*$. Thus, also in this case (3) holds.

Case 3 *Suppose that $O_p(L) \neq 1$ for all $L \in \mathcal{L} \cup \mathcal{L}^*$. Then (4) holds.*

Let $\mathbb{F}$ be as in (8°). By (4°) and (8°)(d), $|\mathbb{F}|$ and so also $\mathbb{F}$ is independent of the choice of $L \in \mathcal{L}$. Let

$$\mathcal{W} = \{x \in V \setminus \{0\} \mid x \in C_V(A^*) \text{ for some } A^* \in \mathcal{A}^*\} \text{ and } \mathcal{W}_0 := \mathcal{W} \cup \{0\}.$$

For $x \in \mathcal{W}$ let $A_x^* \in \mathcal{A}^*$ with $x \in C_V(A_x^*)$ and observe that A_x^* is uniquely determined by x since $C_V(A_x^*) = x\mathbb{K}$. Define the relation '$\sim$' on $\mathcal{W}$ by

$$x \sim y \;:\Longleftrightarrow\; x\mathbb{F} = y\mathbb{F} \text{ or } y \in [x, A_y^*].$$

Since $x\mathbb{F} = x\mathbb{F}$, $\sim$ is reflexive.

11° *Let $x, y \in \mathcal{W}$ with $x \sim y$. Then $y \sim x$ and $x\mathbb{F} + y\mathbb{F} \subseteq \mathcal{W}_0$. Moreover, if in addition $x\mathbb{F} \neq y\mathbb{F}$, then $\langle A_x^*, A_y^*\rangle / C_{\langle A_x^*, A_y^*\rangle}(x\mathbb{F} + y\mathbb{F}) \cong \mathrm{SL}_2(\mathbb{F})$ and $x\mathbb{F} + y\mathbb{F}$ is a natural $\mathbb{F}\mathrm{SL}_2(\mathbb{F})$-module for $\langle A_x^*, A_y^*\rangle$, in particular $y\mathbb{F} = [x, A_y^*]$.*

If $x\mathbb{F} = y\mathbb{F}$, then clearly $y \sim x$ and $x\mathbb{F} + y\mathbb{F} \subseteq \mathcal{W}_0$. So we may assume that $x\mathbb{F} \neq y\mathbb{F}$ and $y \in [x, A_y^*]$. Put $R^* := \langle A_x^*, A_y^*\rangle$ and $V_1 := x\mathbb{K} + y\mathbb{K}$, and pick $L^* \in \mathcal{L}^*$ with $R^* \leq L^*$. Let $z \in \{x, y\}$. Observe that R^* normalizes V_1. From (8°) applied to L^* we conclude that $R^*/C_R^*(V_1) \cong \mathrm{SL}_2(\mathbb{F})$ and $V_1 \cong W_1 \otimes_{\mathbb{F}} \mathbb{K}$ for some natural $\mathbb{F}\mathrm{SL}_2(\mathbb{F})$-module W_1 of R^*; in particular, $C_{W_1 \otimes_{\mathbb{F}} \mathbb{K}}(A_z^*) = C_{W_1}(A_z^*) \otimes_{\mathbb{F}} \mathbb{K}$.

Since $z \in C_{V_1}(A_z^*)$, $z \leftrightarrow w_z \otimes \ell_z$ for some $w_z \in W_1$ and $\ell_z \in \mathbb{K}$. On the other hand $W_1 \otimes \ell_x$ is R^*-invariant and $y \in [x, A_y^*]$, so $w_y \otimes \ell_y \in W_1 \otimes \ell_x$. Thus $x\mathbb{F} + y\mathbb{F} \leftrightarrow W \otimes \ell_x$, in particular $x\mathbb{F} + y\mathbb{F}$ is invariant under R^*. Hence $x\mathbb{F} + y\mathbb{F}$ is natural $\mathrm{SL}_2(\mathbb{F})$-module for R^* and R^* acts transitively on $(x\mathbb{F} + y\mathbb{F})^\sharp$. It follows that $x\mathbb{F} + y\mathbb{F} \subseteq \mathcal{W}_0$, $[x, A_y^*] = y\mathbb{F}$ and $x \in [y, A_x]$. So (11°) holds.

12° *$\sim$ is an equivalence relation on $\mathcal{W}$.*

We already have proved that $\sim$ is reflexive and symmetric. To show that $\sim$ is transitive, let $x, y, z \in \mathcal{W}^\sharp$ with $x \sim y$ and $y \sim z$. If $x\mathbb{F} = y\mathbb{F}$ and $y\mathbb{F} = z\mathbb{F}$, then $x\mathbb{F} = y\mathbb{F}$. If $x\mathbb{F} = y\mathbb{F}$ and $y\mathbb{F} \neq z\mathbb{F}$, then $x \in y\mathbb{F} \leq y\mathbb{F} + z\mathbb{F}$ and by (11°), $[x, A^*_z] = z\mathbb{F}$ and so $x \sim z$. So we may assume that $x\mathbb{F} \neq y\mathbb{F}$ and similarly that $y\mathbb{F} \neq z\mathbb{F}$.

Put $V_1 := x\mathbb{K} + y\mathbb{K}$, $R^* := \langle A^*_x, A^*_y \rangle$. and $N = \mathrm{N}_H(V_1)$. Then $\langle \mathcal{A}^*_N \rangle \leq L^*$ for some $L^* \in \mathcal{L}^*$ and (8°) gives that $\langle \mathcal{A}^*_N \rangle = R^*$. By (11°) $\langle y^{R^*} \rangle = x\mathbb{F} + y\mathbb{F}$.

Suppose first that $z \in V_1$. Then $V_1 = z\mathbb{K} + y\mathbb{K}$ and by symmetry $R = \langle \mathcal{A}^*_N \rangle = \langle A^*_z, A^*_x \rangle$ and $\langle y^{R^*} \rangle = z\mathbb{F} + y\mathbb{F}$. Hence by (11°), $x \sim z$.

Suppose next that $z \notin V_1$. Note that $C_V(C_{A^*_y}(x)) = V_1$. Thus $C_V(C_{A^*_y}(x)) \neq C_V(C_{A^*_y}(z))$. Since both $C_V(C_{A^*_y}(x))$ and $C_V(C_{A^*_y}(z))$ are $\mathbb{F}$-hyperplanes of A^*_y we conclude that $A^*_y = C_{A^*_y}(x)C_{A^*_y}(z)$. Thus

$$[z + V_1, C_{A^*_y}(V_1)] = [z, C_{A^*_y}(x)] = [z, A^*_y] = y\mathbb{F}.$$

By (11°) there exists $g \in R^*$ with $y^g = x$. Since $[V, R^*] \leq V_1$ we have $(z + V_1)^g = z + V_1$. So conjugating the preceding line by g gives

$$[z + V_1, C_{A^*_x}(V_1)] = x\mathbb{F}.$$

Hence $x \in [z, A^*_x]$ and $x \sim z$. Thus (12°) holds.

13° *Let W_0 be an equivalence class of $\sim$. Then $W := W_0 \cup \{0\}$ is an $\mathbb{F}$-subspace of V, H normalizes W, H induces $\mathrm{SL}_{\mathbb{F}}(W)$ on V, and $V \cong W \otimes_{\mathbb{F}} \mathbb{K}$ as an $\mathbb{K}H$-module.*

It follows from (11°) that W is an $\mathbb{F}$-subspace of V. Let $x \in W$, $A^* \in \mathcal{A}^*$ with $[x, A^*] \neq 0$, and $y \in [x, A^*]^\sharp$. Then $A^*_y = A^*$ and $x \sim y$. So $y \in W$ and $\langle \mathcal{A}^* \rangle$ normalizes W. Now (8°)(g) implies that H normalizes W. Note that A^*_y induces $C_{\mathrm{SL}_{\mathbb{F}}(W)}(W/x\mathbb{F})$ on W. Hence by 7.2, H induces $\mathrm{SL}_{\mathbb{F}}(W)$ on W. In particular, $\mathbb{F} = \mathrm{End}_{\mathbb{F}G}(W)$. By (3°), V is absolutely simple and so by 5.3 $V \cong W \otimes_{\mathbb{F}} \mathbb{K}$. Hence (13°) holds.

(13°) completes the proof for (Case 3) and for the Proposition. □

Proof of Theorem 2: Let $\mathcal{Q}$ be the set of subgroups that act nearly quadratically but not quadratically on V. To simplify notation we view H as a subgroup of $\mathrm{GL}_{\mathbb{F}_p}(V)$. By assumption $H = \langle \mathcal{Q} \rangle$.

Put $F := \mathrm{F}^*(H)$ and $\mathbb{F} := \mathbb{F}_p$. Let Δ be the set of Wedderburn components of F on V, and for $A \in \mathcal{Q}$ let

$$\Delta_A := \{W \in \Delta \mid [W, A] \neq 0\}.$$

Since V is a simple H-module, Clifford Theory ensures that $V = \bigoplus \Delta$. Observe that Δ is a system of imprimitivity if $|\Delta| > 1$.

Let $W \in \Delta$ and $A \in \mathcal{Q}$. Put

$$N := \mathrm{N}_G(W),\ \widetilde{N} := N/C_G(W),\ A_W := \mathrm{N}_A(W),\ E := \langle A^N_W \rangle.$$

By Clifford Theory W is a simple $\mathbb{F}N$-module. By 5.2(f) $\mathbb{K}$ is a finite field. We now divide the proof into several cases.

Case 1 *The case $V \neq W$.*

Since H acts transitively on Δ and $H = \langle \mathcal{Q} \rangle$ there exists $A \in \mathcal{Q}$ with $A \neq A_W$. Hence $|\Delta_A| > 1$, and 2.13 can be applied. Since A is not quadratic on V, we are in case 2.13(4).

Clearly $|W| > 2$ since $C_V(F) = 0$. Hence 2.13(4) shows that A is elementary abelian and either

(i) $p = 3$, $A_W = 1$ and $|W| = |A| = |\Delta_A| = 3$; or

(ii) $p = 2$, $|A/A_W| = |\Delta_A| = 2$, $C_W(A_W) = [W, A_W]$, and $|W/C_W(A_W)| = 2$.

Put $m := |\Delta|$ and let $a \in A \setminus A_W$. Suppose that (i) holds. Then a acts as 3-cycle on Δ. It follows that $H/C_H(\Delta)$ is a transitive subgroup of $\mathrm{Sym}(\Delta)$ generated by 3-cycles. Hence $H/C_H(\Delta) \cong \mathrm{Alt}(\Delta)$ and $m \geq 3$.

Put $D := \mathrm{N}_{\mathrm{GL}_{\mathbb{F}}(V)}(\Delta)$. Then $D \cong C_2 \wr \mathrm{Sym}(m)$ and $O^{3'}(D) = D' \sim 2^{m-1}\,\mathrm{Alt}(m)$. Put $X := C_{D'}(\Delta)$. Then $F \leq X$ and $D' = XH$. Moreover, X is an elementary abelian 2-group, $X = [X, D']$, and $X/C_X(D)$ is isomorphic to the simple constituent of the $\mathbb{F}\,\mathrm{Alt}(m)$-permutation module. Since $F \not\leq Z(H)$ we conclude that $F = X$ and $H = D'$. If $m = 4$, the $O_2(D') \not\leq X$. So $m \neq 4$ and the first case of Theorem 2 holds.

Suppose that (ii) holds. Then $H/C_H(\Delta)$ is a transitive subgroup of $\mathrm{Sym}(\Delta)$ generated by 2-cycles and so $H/C_H(\Delta) \cong \mathrm{Sym}(\Delta)$. Also $|A_W| = |[w, A_W]| = |[W, A_W]|$ for $w \in W \setminus C_W(A_W)$ and thus $|A_W| = |C_W(A_W)|$. Since W is a simple $\mathbb{F}N$-module, the dual version of 7.2 implies that $\widetilde{E} \cong \mathrm{SL}_{\mathbb{F}}(W)$. Since $C_V(F) = 0$, $\widetilde{F} \neq 1$, and since $\widetilde{E}$ normalizes $\widetilde{F}$ we conclude that either $\dim_{\mathbb{F}} W > 2$ and $\widetilde{F} = \widetilde{E}$ or $\dim_{\mathbb{F}} W = 2$ and $\widetilde{F} = \widetilde{E}' \cong C_3$.

Assume first that $n := \dim_{\mathbb{F}} W > 2$. Then there exists a component K of H such that $F = KC_F(W)$ and $C_F(W) = C_F(K)$. Suppose that K is A-invariant, then KA_W also induces $\mathrm{SL}_{\mathbb{F}}(W^a)$ on W^a. Moreover, $C_F(W) = C_F(K) = C_F(W^a)$, and W and W^a are isomorphic F-modules since A_W centralizes a hyperplane in both. This contradicts the fact that W and W^a are distinct Wedderburn components. Thus, we have that $K \neq K^a$. It follows that $F \cong K^m \cong \mathrm{SL}_n(\mathbb{F})^m$ and the second case of theorem 2 holds.

Assume now that $n = 2$. Put $D := \mathrm{N}_{\mathrm{GL}_{\mathbb{F}}(V)}(\Delta)$. Then $D \cong \mathrm{SL}_2(2) \wr \mathrm{Sym}(m)$. Let D_1 be the image of $\mathrm{Wr}(\mathrm{SL}_2(2), m)$ in D under this isomorphism. Put $B = C_D(\Delta) \cong \mathrm{SL}_2(2)^m$ and $B_1 = B \cap D_1$. Since A centralizes each $W_0 \in \Delta \setminus \Delta_A$ and A is elementary abelian we have that $A \leq D_1$. Each 2-subgroup of B acts quadratically on V and so each member of $\mathcal{Q}$ acts non-trivially on Δ. Thus $H \leq D_1$ and $D_1 = B_1 H$. It follows that $B_1 = B' \langle A_W^H \rangle$ and $D_1 = B'H$. Note that $1 \neq F \leq B'$. We claim that $F = B'$. If $m > 2$, then D and so also H acts simply on B'. Hence $B' = F$. If $m = 2$, then $|B'| = 3^2$. But C_3 has a unique non-trivial simple module over $\mathbb{F}$ and W is a Wedderburn component for F, so also $|F| > 3$ and again $F = B'$.

From $F = B'$ we conclude that $H = D_1$ and so the third case of Theorem 2 holds.

Case 2 *The case $V = W$, and H not $\mathbb{K}$-linear on V.*

Since $H = \langle \mathcal{Q} \rangle$, there exists $A \in \mathcal{Q}$ such that A is not $\mathbb{K}$-linear on V. Hence we can apply 6.3. Since A is not quadratic on V and $\mathbb{K}$ is finite, we are either in case 6.3(2) or (3). If 6.3(3) holds, then $|\mathbb{K}| = 27$, and it is easy to see that case 4 of Theorem 2 holds.

Assume now that 6.3(2) holds. Then A is elementary abelian and

$$(*) \qquad |\mathbb{K}| = 4,\ [V, A_{\mathbb{K}}] = C_V(A_{\mathbb{K}}),\ |A/A_{\mathbb{K}}| = 2,\ \text{and } |V/C_V(A_{\mathbb{K}})| = 4.$$

In particular $A_{\mathbb{K}} \neq 1$, $\dim_{\mathbb{K}} V > 1$, and $A_{\mathbb{K}}$ acts quadratically on V. Let $E_1 = \langle A_{\mathbb{K}}^H \rangle$. We apply 7.5. If 7.5(1) holds, then H is a subgroup of $\Gamma\mathrm{GL}_2(4)$, $|A| = 4$ and $E_1 \cong D_6$ or D_{10}. Since A is elementary abelian we get $A \cap O_2(H) \neq 1$, a contradiction. Thus, we get from 7.5 that either $E_1 \cong \mathrm{SL}_{\mathbb{K}}(V)$,

$E_1 \cong \mathrm{SL}_{\mathbb{F}}(U)$ where U is an $\mathbb{F}$-space with $V \cong U \otimes_{\mathbb{F}} \mathbb{K}$ as an $\mathbb{F}E_1$-module, or $\dim_{\mathbb{K}} V = 3$ and $E_1 \cong 3^{\cdot}\,\mathrm{Alt}(6)$.

In the first case $\Gamma\mathrm{GL}_{\mathbb{K}}(V)/E_1 \cong \mathrm{Sym}(3)$ and so either $H = \Gamma\mathrm{GL}_{\mathbb{K}}(V)$, or $|H/E_1| = 2$ and $H = \Gamma\mathrm{SL}_{\mathbb{K}}(V)$. In the second case $\mathrm{N}_{\mathrm{GL}_{\mathbb{F}}(V)}(E_1) \cong \mathrm{SL}_2(\mathbb{F}) \times \mathrm{SL}_{\mathbb{F}}(U)$ and so, since H acts simply on V and is generated by 2-groups, $H \cong \mathrm{SL}_2(\mathbb{F}) \times \mathrm{SL}_{\mathbb{F}}(U)$. In the last case $\mathrm{N}_{\Gamma\,\mathrm{GL}_{\mathbb{K}}(V)}(E_1) \cong 3^{\cdot}\,\mathrm{Sym}(6)$ and so $H \cong 3^{\cdot}\,\mathrm{Sym}(6)$. It follows that Case 5, 6, 7 or 8 of Theorem 2 holds.

We may assume from now on that $V = W$ and H acts $\mathbb{K}$-linearly on V. In particular, $\mathrm{Z}(F) = \mathrm{Z}(H)$. Let $\mathcal{I}$ be the set of all $X \trianglelefteq F$ such that either X is a component of H or $X = O_q(H)$ with $X' \neq 1$, q a prime. Put $D := \langle \mathcal{I} \rangle$ and $\mathbb{D} := \mathrm{Z}(\mathrm{End}_{\mathbb{F}D}(V))$. Next we show:

1° $\mathrm{Z}(F) = \mathrm{Z}(H) = F \cap \mathbb{K}$, $F = \mathrm{Z}(H)D$ *and* $\mathbb{D} \subseteq \mathbb{K}$. *In particular, H acts $\mathbb{D}$-linearly on V.*

The definition of $\mathcal{I}$ and $Z(H) = Z(F)$ imply that $F = \mathrm{Z}(F)\langle \mathcal{I} \rangle = Z(H)D$ and $\mathrm{Z}(H) = \mathrm{Z}(F) = F \cap \mathbb{K}$. Since $D \leq F$, $\mathrm{End}_{\mathbb{F}F}(V) \subseteq \mathrm{End}_{\mathbb{F}D}(V)$. Since $\mathrm{Z}(F) \subseteq \mathrm{End}_{\mathbb{F}D}(V)$ and $F = \mathrm{Z}(F)D$, $\mathbb{D} \subseteq \mathrm{End}_{\mathbb{F}F}(V)$ and so $\mathbb{D} \subseteq \mathrm{Z}(\mathrm{End}_{\mathbb{F}F}(V)) = \mathbb{K}$. Thus all parts of (1°) are proved.

2° *Let $A \in \mathcal{Q}$. Then $[D, A] \neq 1$.*

Otherwise (1°) implies that $[F, A] = 1$ and so $A \leq \mathrm{Z}(F)$ and $A \leq O_p(F) = 1$, a contradiction.

3° *V is a simple $\mathbb{F}D$-module and $\mathbb{K} = \mathbb{D} = \mathrm{End}_{\mathbb{F}D}(V)$.*

Suppose for a contradiction that V is not a simple $\mathbb{F}D$-module. Since V is a homogeneous $\mathbb{F}F$-module and $F = Z(F)D$, V is a homogeneous $\mathbb{F}D$-module. We apply 5.5 with $I = \{1\}$, $T := A \in \mathcal{Q}$, $D_1 = D$, and $\mathbb{D}$ in place of $\mathbb{K}$. Hence there exists an H-invariant tensor decomposition $\mathcal{T}$ with $\Phi : \otimes_{j \in J} V_j \to V$ where $J = \{0, 1\}$, H acts trivially on J, V_1 is a simple $\mathbb{D}D$-module, V_0 is a trivial $\mathbb{D}D$-module and $\Phi : V_0 \otimes_{\mathbb{D}} V_1 \to V$ is a $\mathbb{D}D$-isomorphism. Moreover, $\mathcal{T}$ is strict when restricted to A.

Since V is a simple $\mathbb{F}H$-module we conclude that V_0 and V_1 are simple projective $\mathbb{D}H$-modules. By (2°), $D \neq 1$ and so D is non-abelian. Thus $\dim_{\mathbb{D}} V_1 \geq 2$ and if $\dim_{\mathbb{D}} V_1 = 2$, then $|\mathbb{D}| > 2$. Since V is not a simple $\mathbb{F}D$-module, $\dim_{\mathbb{D}} V_0 \geq 2$. Thus $\mathcal{T}$ is proper, regular and $\mathbb{D}$-linear, and $\mathcal{T}$ is proper and ordinary when restricted to A. In particular 6.2 applies with $G := A$, J in place of $\{1, 2\}$ (and $\mathbb{D}$ in place of $\mathbb{K}$).

It follows from 6.2(b) and (f) that $[V_j, A]$ is a $\mathbb{D}$-hyperplane of V_j and A induces $C_{\mathrm{SL}_{\mathbb{D}}(V_i)}([V_j, A])$ on V_j for $j = 0, 1$. Thus the dual version of 7.2 shows that $\langle A^H \rangle$ induces $\mathrm{PSL}_{\mathbb{D}}(V_j)$ on $\mathcal{P}_{\mathbb{D}}(V_j)$. Since D is normal in H and acts faithfully on V_1 we conclude that $D \cong \mathrm{SL}_{\mathbb{D}}(V_1)'$. Put $Z := C_H(\mathcal{P}_{\mathbb{D}}(V_1))$. Using (1°), $[Z, F] = [Z, D] \leq Z \cap D \leq \mathrm{Z}(D) \leq \mathrm{Z}(F) \leq \mathrm{Z}(H)$. But $C_H(F/\mathrm{Z}(H)) \leq F$ and so $Z = \mathrm{Z}(F) = \mathrm{Z}(H)$. Hence $H/\mathrm{Z}(H)$ is isomorphic to a subgroup of $\mathrm{P\Gamma GL}_{\mathbb{D}}(V_1)$ containing $\mathrm{PSL}_{\mathbb{D}}(V_1)'$. Since F acts trivially on $\mathcal{P}_{\mathbb{D}}(V_0)$ we see that $H/C_H(\mathcal{P}_{\mathbb{D}}(V_0))$ is isomorphic to a section of $\mathrm{P\Gamma GL}_{\mathbb{D}}(V_1)/\mathrm{PSL}_{\mathbb{D}}(V_1)'$. Therefore $H/C_H(\mathcal{P}_{\mathbb{D}}(V_0))$ is solvable. On the other hand $H/C_H(\mathcal{P}_{\mathbb{D}}(V_0))$ contains a subgroup isomorphic to $\mathrm{PSL}_{\mathbb{D}}(V_0)$. Hence $p = 2$ or 3, $\mathbb{D} = \mathbb{F}$ and $\dim_{\mathbb{D}} V_0 = 2$. In particular, $\mathrm{P\Gamma GL}_{\mathbb{D}}(V_1) = \mathrm{PGL}_{\mathbb{D}}(V_1)$. Since $\mathrm{PSL}_{\mathbb{D}}(V_0)$ is not abelian, we conclude that $\mathrm{PGL}_{\mathbb{D}}(V_1)/\mathrm{PSL}_{\mathbb{D}}(V_1)'$ is not abelian. Hence $\mathrm{SL}_{\mathbb{D}}(V_1) \neq \mathrm{SL}_{\mathbb{D}}(V_1)'$, $\dim_{\mathbb{D}} V_1 = 2$, $\mathbb{D} = \mathbb{F}$, and $\mathrm{PGL}_{\mathbb{D}}(V_1)/\mathrm{PSL}_{\mathbb{D}}(V_1)' \cong \mathrm{Sym}(3)$. Thus $H/C_H(\mathcal{P}_{\mathbb{D}}(V_0))$ has order at most 6. But $\mathrm{PSL}_{\mathbb{D}}(V_0) \cong \mathrm{PSL}_2(3)$ has order 12, a contradiction.

Therefore V is a simple $\mathbb{F}D$-module. It follows that $\mathrm{End}_{\mathbb{F}D}(V)$ is a field, $\mathbb{D} = \mathrm{End}_{\mathbb{F}D}(V)$, $\mathbb{K} \subseteq \mathbb{D}$ and $\mathbb{K} = \mathbb{D}$.

Case 3 *The case H $\mathbb{K}$-linear on V, $|\mathcal{I}| \geq 2$, and V a simple $\mathbb{F}D$-module.*

Let $D_1, \ldots, D_r$ be the distinct elements of the set $\mathcal{I}$ and put $I := \{1, \ldots, r\}$. Since H acts on $\mathcal{I}$ by conjugation, I is an H-set with respect to the induced action; i.e., $D_{ig} := D_i^g$ for $g \in H$. Hence, the module V, I and $\mathcal{I}$ satisfy the conditions (i) and (ii) of 5.5 (with $T := A \in \mathcal{Q}$ and H in place of G) for the case that V is a simple $\mathbb{K}D$-module. Thus there exists an H-invariant tensor decomposition $\mathcal{T}$ with $\Phi : \bigotimes_I V_i \to V$, where V_i is a simple $\mathbb{K}D_i$-module and trivial $\mathbb{K}D_j$-module for $i, j \in I$ with $i \neq j$. Moreover, Φ is a $\mathbb{K}(D_1 \times \ldots \times D_r)$-module isomorphism and the decomposition restricted to A is strict. Since each D_i is non-abelian, $\dim_{\mathbb{K}} V_i \geq 2$ for each $i \in I$. By assumption of the current case, $|I| \geq 2$. So $\mathcal{T}$ is proper. Since A is not quadratic, $|A| > 2$, and we are allowed to apply 6.5. Again since A is not quadratic, one of the following holds:

(Case 3)(a) $|I| = 2$, I is a trivial A-set, and $[V_i, A] = C_{V_i}(A) = [v_i, A]$ is a $\mathbb{K}$-hyperplane of V_i for $v_i \in V_i \setminus [V_i, A]$ and $i \in I$.

(Case 3)(b) $|I| = 2$, I is a non-trivial A-set, $\mathbb{K} = \mathbb{F}$, and $[V_i, B] = C_{V_i}(B)$ is a $\mathbb{K}$-hyperplane of V_i for $i \in I$, where $B := C_A(I)$.

Put $H_0 := C_H(I)$ and $B := C_A(I)$ and let $I = \{i, j\}$. Since V_i is a simple $\mathbb{K}D_i$-module, V_i is a simple projective $\mathbb{K}H_0$-module.

Thus the dual version of 7.2 shows $\langle B^H \rangle$ induces $\mathrm{PSL}_{\mathbb{K}}(V_i)$ on $\mathcal{P}_{\mathbb{K}}(V_i)$. Since H_0 normalizes D_i and D_i acts faithfully on V_i we conclude that $D_i \cong \mathrm{SL}_{\mathbb{K}}(V_i)'$. If $\mathrm{SL}_{\mathbb{K}}(V_i)$ is perfect for $i = 1$ and 2 we see that one of the Cases 9, 10 and 11 of Theorem 2 holds. So suppose $\mathrm{SL}_{\mathbb{K}}(V_i)$ is not perfect for some $i \in I$. Since D_i is not abelian, $\mathbb{K} = \mathbb{F} = \mathbb{F}_3$ and $\dim_{\mathbb{K}} V_i = 2$. Thus $D_i = O_3(H)$. Hence $D_j \neq O_3(H)$ and so $\dim_{\mathbb{K}} V_j \geq 3$ and $D_j \cong \mathrm{SL}_{\mathbb{K}}(V_j)$. It follows that Case 10 of Theorem 2 holds.

Case 4 *The case H $\mathbb{K}$-linear on V, V a simple $\mathbb{F}D$-module, $|\mathcal{I}| = 1$, and D not solvable.*

Since $|\mathcal{I}| = 1$ and D is not solvable, D is a component of G. By (1°) $F = \mathrm{Z}(H)D$ and so Theorem 2 holds.

Case 5 *The case H $\mathbb{K}$-linear on V, V a simple $\mathbb{F}D$-module, $|\mathcal{I}| = 1$ and D solvable.*

Since $|\mathcal{I}| = 1$ and D is solvable there exists a prime r such that $D = \mathrm{O}_r(G)$. Put $K = [D, A]$. By coprime action, $D = C_D(A)K$ and since $F = \mathrm{Z}(H)D$, $F = C_F(A)K$. So $K = [K, A]$, $KA = \langle A^K \rangle = \langle A^F \rangle$ and $K \trianglelefteq FA$. We choose a normal subgroup R of FA contained in D that is minimal with $[R, A] \neq 1$.

4° *$R = [R, KA] \leq K$ and either R is elementary abelian, or $C_R(A) = \mathrm{Z}(R)$ and A acts simply on $R/\mathrm{Z}(R)$.*

Note that $[R, \langle A^K \rangle]$ is a normal subgroup of FA contained in R. Since $[R, A, A] = [R, A] \neq 1$, the minimality of R gives $R = [R, \langle A^K \rangle] = [R, KA] \leq K$.

As $[R, F] = [R, D] < R$ the minimality of R gives $[R, F, A] = 1$. So $[R, F] \leq C_R(A)$ and $C_R(A)$ is a normal subgroup of FA. Hence F centralizes $R/C_R(A)$ and the minimality of R implies that A acts simply on $R/C_R(A)$. Since FA normalizes $C_R(A)$, $\langle A^F \rangle = AK$ centralizes $C_R(A)$ and so $C_R(A) = C_R(KA) \leq \mathrm{Z}(R)$. If R is non-abelian, the minimality of R yields $C_R(A) = \mathrm{Z}(R)$. If R is abelian, then $R = [R, A] \times C_R(A)$. Hence $[\Omega_1(R), A] \neq 1$ and $\Omega_1(R) = R$. So R is elementary abelian.

5° $1 \neq C_R(F) \subseteq \mathbb{K}$*; in particular* $|\mathbb{K}| > 2$.

Since R is a normal subgroup of $D = O_r(F)$, $C_R(F) = C_R(D) \neq 1$. By the definition of $\mathbb{K}$, $C_R(F) \subseteq \mathbb{K}$ so $|\mathbb{K}| > 2$.

6° *V is a simple $\mathbb{F}KA$-module.*

Let V_1 be a simple $\mathbb{F}KA$-submodule of V. Since V is simple $\mathbb{F}F$-module and $F = C_F(A)K$ we get $V = \langle V_1^{C_F(A)} \rangle$. So V is a direct sum of A-submodules isomorphic to V_1. Since A is nearly quadratic but not quadratic, 2.9 shows that $V = V_1$.

7° *Suppose that V is not a homogeneous $\mathbb{F}R$-module. Then Case 10 or 12 of Theorem 2 holds.*

Let Δ be the set of Wedderburn components for $\mathbb{F}R$ on V. Then Δ is a system of imprimitivity for FA on V. Since V is a simple $\mathbb{F}D$-module, D acts transitively on Δ. Thus $|\Delta|$ is a power of r. We apply 2.13.

Let $U \in \Delta$. By (6°) V is a simple $\mathbb{F}KA$-module. Hence KA acts transitively on Δ and $\mathrm{N}_{KA}(U)$ acts simply on U. Also $[U, R] \neq 0$ since $C_V(R) = 0$. This excludes the case 2.13(4:1). The transitivity on Δ and the non-quadratic action of A on V excludes the cases 2.13(1), 2.13(2), and 2.13(3). Moreover, if 2.13(4:3) holds, then $|U/\,\mathrm{C}_U(\mathrm{N}_A(U))| = 2$ and so $|\mathbb{K}| = 2$, which contradicts (5°).

Thus we are left with case 2.13(4:2). In this case $p = 3 = |U| = |A|$, $\mathbb{K} = \mathbb{F}$ and A acts as a 3-cycle on Δ. On the other hand, $KA = \langle A^K \rangle$ acts transitively on Δ, so $KA/\,\mathrm{C}_{KA}(\Delta) \cong \mathrm{Alt}(\Delta)$. The solvability of KA gives $|\Delta| = 3$ or 4, and since $|\Delta|$ is a power of r, $r = 2$ and $|\Delta| = 4$. Thus FA is a subgroup of $\mathrm{N}_{\mathrm{GL}_{\mathbb{K}}(V)}(\Delta) \cong C_2 \wr \mathrm{Sym}(4)$. Since $R \leq \mathrm{C}_H(\Delta)$ and by (4°) $R = [R, KA]$ we conclude that $R = [R, KA] \cong C_2^3$ and $KA \cong 2^3 : \mathrm{Alt}(4)$. This shows that $K \cong Q_8 \circ Q_8$, $F \leq C_H(\Delta)K$ and $F \cap \mathrm{SL}_{\mathbb{K}}(V) = K$. Thus $K \trianglelefteq H$ and $H \leq \mathrm{N}_{\mathrm{GL}_{\mathbb{K}}(V)}(K) \sim (\mathrm{GL}_2(3) \circ \mathrm{GL}_2(3)).2$. Since $H = \langle \mathcal{Q} \rangle = O^{3'}(H)$ we conclude that either $H = KA$ and Case 12 of Theorem 2 holds or $H \cong \mathrm{SL}_2(3) \circ \mathrm{SL}_2(3)$ and Case 10 holds.

We may assume from now on that V is a homogeneous $\mathbb{F}R$-module. By 5.2 $\mathbb{E} := \mathrm{Z}(\mathrm{End}_R(V))$ is a field and so FA acts $\mathbb{E}$-semi-linearly on V.

8° *A acts $\mathbb{E}$-linearly on V, and R is not abelian.*

Suppose A does not act $\mathbb{E}$-linearly on V. Since A is not quadratic on V, 6.3(2) or (3) holds. In both cases $\dim_{\mathbb{F}} \mathbb{E} = p$ and so $|\,\mathrm{Aut}(\mathbb{E})| = p$. Since D is an r-group, $\mathbb{E} \subseteq \mathrm{End}_{\mathbb{F}D}(V)$. So by (3°), $\mathbb{E} \subseteq \mathbb{K}$. But A acts $\mathbb{K}$-linearly on V. Thus A is $\mathbb{E}$-linear.

If R is abelian, then $R \subseteq \mathrm{End}_R(V) \cap R \subseteq \mathbb{E}$ and $[R, A] = 1$, a contradiction.

9° *Suppose V is a homogeneous but not simple $\mathbb{F}R$-module. Then Case 10 or 12 of Theorem 2 hold.*

By 5.5 there exists a KA-invariant tensor decomposition $\Phi : V_0 \otimes_{\mathbb{E}} V_1 \to V$ such that $V \cong V_0 \otimes_{\mathbb{E}} V_1$ as an $\mathbb{E}R$-module, V_1 is a simple $\mathbb{E}R$-module and V_0 is a trivial $\mathbb{E}R$-module. Since R is not abelian, $\dim_{\mathbb{E}} V_1 > 1$, and since V is not a simple R-module, $\dim_{\mathbb{E}} V_0 > 1$. Moreover, by (8°) A acts $\mathbb{E}$-linearly on $V_0 \otimes_{\mathbb{E}} V_1$. Hence one of the two cases in 6.5(3) hold.

In both cases the dual version of 7.2 shows $KA = \langle A^K \rangle$ induces $\mathrm{PSL}_{\mathbb{E}}(V_i)$ on $\mathcal{P}_{\mathbb{E}}(V_i)$. Since R is solvable, $p = 2$ or 3, $\mathbb{E} = \mathbb{F}$ and $\dim_{\mathbb{E}} V_i = 2$. Since R is non-abelian, $p = 3$ and $R \cong Q_8$. Hence $FA \leq \mathrm{N}_{\mathrm{GL}_{\mathbb{K}}(V)}(R) \cong \mathrm{GL}_2(3) \circ \mathrm{GL}_2(3)$. Since A is not quadratic on V, A is not contained in any of the normal $\mathrm{GL}_2(3)$'s. Since A normalizes F we get that $F \leq O_2\big(\mathrm{GL}_2(3) \circ \mathrm{GL}_2(3)\big) \cong Q_8 \circ Q_8$ and then since F acts simply on V, $F \cong Q_8 \circ Q_8$. Thus $H \leq \mathrm{N}_{\mathrm{GL}_{\mathbb{K}}(V)}(F) \cong (\mathrm{GL}_2(3) \circ \mathrm{GL}_2(3)).2$. It follows that Case 10 or 12 of the Theorem holds.

10° *Suppose V is a simple $\mathbb{F}R$-module. Then Case 13 of the Theorem 2 holds.*

By (8°) R is non-abelian, and by (4°) $\overline{R} := R/\operatorname{Z}(R)$ is a simple A-module. Let X be maximal abelian normal subgroup of A. Put

$$\mathcal{Y} = \{C_X(U) \mid U \text{ a simple } X\text{-submodule of } \overline{R}\}.$$

For $Y \in \mathcal{Y}$ let

$$\overline{R}_Y = \langle U \mid U \text{ a simple } X\text{-submodule of } \overline{R} \text{ with } C_X(U) = Y\rangle.$$

Then $\overline{R}_Y$ is a sum of Wedderburn components for X on $\overline{R}$ and so

$$\overline{R} = \bigoplus_{Y\in\mathcal{Y}} \overline{R}_Y.$$

Suppose for a contradiction that $|\mathcal{Y}| \geq 2$. For $Y \in \mathcal{Y}$ let R_Y be the inverse image of $\overline{R}_Y$ in R. Let $Y, Z \in \mathcal{Y}$ with $Y \neq Z$. We claim that $[R_Y, R_Z] = 1$. Without loss $Y \not\leq Z$. Hence $\overline{R}_Z = [\overline{R}_Z, Y]$ and thus $R_Z \leq \operatorname{Z}(R)[R, Y]$. Since $R' \leq \operatorname{Z}(R) \leq C_R(Y)$, $[R, C_R(Y), Y] = 1$. Also $[C_R(Y), Y, R] = 1$ and so the Three Subgroups Lemma implies $[[Y, R], C_Y(R)] = 1$. Since $R_Y \leq C_R(Y))$ and $R_Z \leq [R, Y]\operatorname{Z}(R)$ the claim is proved.

Note that $R = \langle R_Y \mid Y \in \mathcal{Y}\rangle$. If R_Y is abelian we would conclude $R_Y \leq \operatorname{Z}(R)$, a contradiction. So R_Y is not abelian. Each R_Y is normal in R and so RA acts on $\mathcal{Y}$ and on $\{R_Y \mid Y \in \mathcal{Y}\}$ by conjugation. Since V is a simple $\mathbb{E}R$-module, 5.5 yields an RA-invariant tensor decomposition $\bigotimes^{\mathbb{E}}_{Y\in\mathcal{Y}} V_Y \cong V$, which is strict for A and where each V_Y is a faithful simple $\mathbb{E}R_Y$-module. Since R_Y is non-abelian $\dim_{\mathbb{E}} V_Y \geq 2$. 6.5 now implies that A is abelian. Thus $X = A$, $\overline{R}$ is a simple X-module and $|\mathcal{Y}| = 1$, a contradiction.

We have proved that $|\mathcal{Y}| = 1$. Let $\mathcal{Y} = \{Y\}$. Then $[\overline{R}, Y] = 1$. Since $[\operatorname{Z}(R), A] = 1$ coprime action gives $C_A(\overline{R}) \leq C_A(R) \subseteq A \cap \mathbb{E} \leq O_p(RA) = 1$. Thus $Y = 1$ and X is cyclic. Let $|X| = p^n$ with $n \in \mathbb{N}$.

Suppose $p^n \geq 4$. Since X acts cubically on V, Hall-Higman's Theorem B [Gor, 11.1.1] shows that there exists $n_0 \leq n$ with

$$p^{n-n_0}(p^{n_0} - 1) \leq 3$$

and $p^{n_0} - 1 = r^k$ for some positive integer k. Since $r^k \geq r \geq 2$, we get $p^{n-n_0} = 1$, $n = n_0$, $p^n - 1 \leq 3$ and $p^n \leq 4$.

It follows that $|X| = p^n \leq 4$ and if $|X| = 4$, then $r = 3$.

Suppose for a contradiction that $X = A$. Since X is cubic but not quadratic on V the Jordan Canonical Form for A on V shows that $V \cong V_0 \otimes_{\mathbb{F}} \mathbb{K}$ and $V_0 = V_1 \oplus V_2$, where V_1, V_2 are $\mathbb{F}A$-submodules of V_0 with $\dim_{\mathbb{F}} V_1 = 3$ and $[V_1, A, A] \neq 0$. Since $\mathbb{K}$ is a direct sum of 1-dimensional $\mathbb{F}$-subspaces, V is as an $\mathbb{F}A$-module the direct sum of copies of V_0. Thus by 2.9, $\mathbb{F} = \mathbb{K}$ and so $1 \neq \operatorname{Z}(R) \subset \mathbb{F}^\sharp$. Hence $p \neq 2$, $|X| = 3 = p$ and $r \neq 3$. Thus $|\mathbb{F}| = 3$, $|\operatorname{Z}(R)| = 2$ and $r = 2$. Since A has order 3 and acts simply on $R/\operatorname{Z}(R)$ we have $|R/\operatorname{Z}(R)| = 4$ and $R \cong Q_8$. Now Q_8 has a unique faithful simple module over $\mathbb{F}_3$ and this module has dimension 2. Thus $\dim_{\mathbb{F}} V = 2$ and A acts quadratically on V, a contradiction.

Thus $A \neq X$. Since $C_A(X) \leq X$ we conclude that $|X| = 4$, $r = 3$ and $A \cong D_8$ or Q_8. In the first case A has a non-cyclic maximal abelian normal subgroup, a contradiction. Therefore $A \cong Q_8$. Let $v \in V \setminus Q_V(A)$. Then $[V, A] \leq [v, A] + C_V(A)$ by the nearly quadratic action of A. Since A is quadratic on $\overline{V} := V/C_V(A)$ we have $A' \leq C_A(\overline{V})$ and

$$[\overline{V}, A] = \{[\overline{v}, a] \mid a \in A\} \leq |A/C_A(\overline{v})| \leq |A/A'| = 4.$$

As $[\overline{V}, A]$ is a $\mathbb{K}$-space we get $|\mathbb{K}| \leq 4$. By (5°) $|\mathbb{K}| > 2$ and so $|\mathbb{K}| = 4$ and $\dim_{\mathbb{K}}[\overline{V}, A] = 1$. In particular, $|A/C_A(\overline{v})| = 4$ and $C_A(\overline{v}) = A'$. So for every $a \in A \backslash A'$ and every $v \in V \backslash Q_V(A)$, $[\overline{v}, a] \neq 0$ and thus $C_{\overline{V}}(a) = [\overline{V}, A] = [\overline{V}, a]$. Since $[\overline{V}, a]$ is 1-dimensional we conclude that $\overline{V}/C_{\overline{V}}(a)$ is 1-dimensional, and so $C_{\overline{V}}(a) = [\overline{V}, a]$ gives $\dim_{\mathbb{K}} \overline{V} = 2$. This implies that $Q_V(A)$ is a $\mathbb{K}$-hyperplane of V and $\dim_{\mathbb{K}} V/C_V(A) = 2$. Since A' centralizes $Q_V(A)$ this show that $C_V(A') = Q_V(A)$.

Put $Q := [R, A]$ and let $g \in Q \setminus Z(Q)$. Then $QA = \langle A, A'^g \rangle$. Since $C_V(Q) = 0$ we get $C_V(A) \cap C_V(A'^g) = 0$ and so $\dim_{\mathbb{K}} V = 3$. Since the Sylow 3-subgroup of $\mathrm{SL}_3(4)$ is extra special of order 27 we conclude that $Q = D = O_3(H) \sim 3^{1+2}$ and $H = \mathrm{N}_{\mathrm{SL}_{\mathbb{K}}(V)}(Q) \sim 3^{1+2}.Q_8$. Hence Case 13 of Theorem 2 holds. $\square$

Proof of Theorem 3: We may assume without loss that $A \not\leq N_G(K)$. Let W be composition factor for $H := \langle K, A \rangle$ on V with $[W, K] \neq 0$. By 2.6(c), W is a nearly quadratic A-module. Note also that $H = \langle A^K \rangle$.

If A is not quadratic on W we conclude from Theorem 2 that p=2, $K/C_K(W) \cong \mathrm{SL}_n(2)$, $W \neq [W, K]$ and and $[W, K]$ is a Wedderburn component for $\mathrm{F}^*(H)$-module on V Thus by (i) and (ii) in Case 1 of the proof of Theorem 2 we conclude that $|A/N_A([W, K])| = 2$ and so also $|A/N_A(K)| = 2$. Hence Theorem 3 holds in this case.

Suppose that A is quadratic on W. Let $L = \langle K^A \rangle$. Then $H = LA$. Let U be a Wedderburn component for L on W. Then $W = \langle U^A \rangle$. From $|A/C_A(K)| > 2$ we have $|A/C_A(W)| > 2$, and 2.11 implies that $U = W$. So W is a homogeneous $\mathbb{F}_pL$-module. For example by 5.2(d), the number of simple $\mathbb{F}_pL$-modules in W is not divisible by p, so one of them is normalized by A. Since $H = LA$ acts simply on W, W is a simple $\mathbb{F}_pL$-module. Let $K^A = \{K_1, K_2, \dots K_r\}$ and $I = \{1, \dots, r\}$. Then $|I| \geq 2$ and A acts transitively on I via $K_{ia} = K_i^a$. By 5.5 there exists a H-invariant tensor decomposition $V \cong \bigotimes_{i \in I}^{\mathbb{K}} V_i$, where V_i is a simple $\mathbb{K}K_i$-module and a trivial $\mathbb{K}K_j$ for $i, j \in I$ with $i \neq j$. Thus by 4.10, $p = 2$, $|I| = 2$, $\dim_{\mathbb{K}} V_i = 2$ and $[V_i, C_A(I)] \neq 1$. Now 7.5 shows that $K/C_K(W) \cong \mathrm{SL}_2(2^m)$.

Thus $K/C_K(W) \cong \mathrm{SL}_n(2)$ or $\mathrm{SL}_2(2^m)$. Since this holds for all non-trivial composition factors of K on V, $K/O_2(K) \cong \mathrm{SL}_n(2)$ or $\mathrm{SL}_2(2^m)$. $\square$

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