# The Local Structure Theorem For Finite Groups With A Large $p$-Subgroup

U. Meierfrankenfeld

B. Stellmacher

G. Stroth

Author address:

Department of Mathematics, Michigan State University, East Lansing MI 48824
*E-mail address*: meier@math.msu.edu

Gneisenaustrasse 6, D24105 Kiel
*E-mail address*: stellmacher@math.uni-kiel.de

Martin-Luther-Universität Halle-Wittenberg, Institut für Mathematik, D06099 Halle
*E-mail address*: gernot.stroth@mathematik.uni-halle.de

# Contents


Contents i

Introduction vii

Chapter 1. Definitions and Preliminary Results 1
1.1. Elementary Properties of Finite Groups 1
1.2. The Largest $\boldsymbol{p}$-Reduced Elementary Abelian Normal Subgroup 6
1.3. $\boldsymbol{p}$-Irreducible Groups 10
1.4. $\boldsymbol{Y}$-Minimal Groups 12
1.5. Weakly Closed Subgroups 17
1.6. Large Subgroups 20

Chapter 2. The Case Subdivision and Preliminary Results 27
2.1. Notation and Elementary Properties 27
2.2. Asymmetry 29
2.3. Symmetric Pairs 37
2.4. Tall Natural Symplectic Modules 42

Chapter 3. The Orthogonal Groups 45
3.1. Notation and Elementary Properties 45
3.2. The Proof of Theorem C 50

Chapter 4. The Symmetric Case 65
4.1. The Proof of Theorem D 66

Chapter 5. The Short Asymmetric Case 81

Chapter 6. The Tall $\boldsymbol{char\,p}$-Short Asymmetric Case 97

Chapter 7. The $\boldsymbol{char\,p}$-Tall $\boldsymbol{Q}$-Short Asymmetric Case 99
7.1. The Proof of Theorem G 100

Chapter 8. The $\boldsymbol{Q}$-Tall Asymmetric Case I 119
8.1. Notation and Preliminary Results 121
8.2. The Case $\boldsymbol{I \leqslant A}$ 124
8.3. The Case $\boldsymbol{I \nleqslant A}$ and $\boldsymbol{\Omega_1 Z(A) \nleqslant Z(L)}$ 145
8.4. The Case $\boldsymbol{I \nleqslant A}$ and $\boldsymbol{\Omega_1 Z(A) \leqslant Z(L)}$ 145
8.5. The Proof of Theorem H 148

Chapter 9. The $\boldsymbol{Q}$-tall Asymmetric Case II 149
9.1. Notation and Preliminary Results 150
9.2. The Proof of Theorem I 156
9.3. The Proof of Corollary 9.1 169
9.4. The Proof of Corollary 9.2 171

Chapter 10. Proof of the Local Structure Theorem 175
10.1. Proof of Theorem J 176
10.2. Proof of the Local Structure Theorem 177

10.3. Proof of the Corollary to the Local Structure Theorem 183

Appendix A. Module theoretic Definitions and Results 185
A.1. Module-theoretic Definitions 185
A.2. Naming Modules 186
A.3. p-Reduced Modules 188
A.4. Wreath Product Modules 190
A.5. Offenders 194
A.6. Nearly Quadratic Modules 198
A.7. $\boldsymbol{Q!}$-Modules 199
A.8. Genuine Groups of Lie Type 202

Appendix B. Classical Spaces and Classical Groups 205
B.1. Elementary Properties 206
B.2. The Classification of Classical Spaces 211
B.3. The Clifford Algebra 212
B.4. Normalizers of Singular Subspaces 215
B.5. Point-Stabilizers 216
B.6. Simplicity of the Natural Module 220
B.7. Normalizers of Classical Groups 221
B.8. $\boldsymbol{Q}$-Uniqueness in Classical Groups 224

Appendix C. FF-Module Theorems and Related Results 229
C.1. FF-Module Theorems 229
C.2. $\boldsymbol{H^1}$- and $\boldsymbol{H^2}$-Results 236
C.3. $\boldsymbol{Q!}$-Module Theorems 239
C.4. The Asymmetric Module Theorems 242

Appendix D. The Fitting Submodule 245
D.1. The Definition of the Fitting Submodule and Results from [**MS2**] 245
D.2. The Fitting Submodule and Large Subgroups 246
D.3. The Nearly Quadratic $Q!$ -Module Theorem 247

Appendix E. The Amalgam Method 255
E.1. The Coset Graph 255
E.2. Elementary Properties 255
E.3. Critical Pairs 257
E.4. The Case $|I| = 2$ 258
E.5. An Application of the Amalgam Method 260

Appendix. Bibliography 263

Appendix. List of Symbols 265

Appendix. List of Definitions 267

# Abstract

Let $p$ be a prime, $G$ a finite $\mathcal{K}_p$-group $S$ a Sylow $p$-subgroup of $G$ and $Q$ a large subgroup of $G$ in $S$ (i.e., $C_G(Q) \leqslant Q$ and $N_G(U) \leqslant N_G(Q)$ for $1 \neq U \leqslant C_G(Q)$). Let $L$ be any subgroup of $G$ with $S \leqslant L$, $O_p(L) \neq 1$ and $Q \ntrianglelefteq L$. In this paper we determine the action of $L$ on the largest elementary abelian normal $p$-reduced $p$-subgroup $Y_L$ of $L$.



2010 *Mathematics Subject Classification.* Primary 20D05.

*Key words and phrases.* local characteristic $p$, large subgroup, finite simple groups.

partially supported by Humboldt foundation.

partially supported by DFG.

# Introduction

Historical Background. One of the great achievements of 20th century mathematics is the classification of the finite simple groups. At least from hindsight, the quest for this classification began with a talk of R. Brauer at the ICM in Amsterdam 1952, where he demonstrated a method, the *centralizer method*, that makes it possible to characterize finite simple groups by means of the centralizer of an involution, and of course with the celebrated Odd Order Theorem of Feit-Thompson 1963, [**FT**], which shows that every finite (non-abelian) simple group possesses involutions. It was quite natural from these beginnings that the prime 2 played an overwhelmingly important role in the classification.

On the other hand, apart from the alternating groups, the classes *Lie*($p$) of finite simple groups of Lie type in characteristic $p$, $p$ a prime, provide the generic examples for finite simple groups. So for these examples there exists a *distinguished* prime $p$ associated to these groups. Moreover, in 1974 J. Tits, [**T**], presented the theory of buildings of spherical type, which makes it possible to understand and characterize the groups in *Lie*($p$) by means of a geometry that reflects properties of their parabolic subgroups and focuses on that *distinguished* prime $p$. So one might wonder if there is also a way to classify the finite simple groups more prominently based on this *geometric approach*.

Both of theses approaches, Brauer's *centralizer approach* based on the prime 2 and used in the classification and Tits' *geometric approach*, can only be applied successfully to a general finite simple group, if one is able to set the stage properly. More precisely, one has to get a satisfactory answer to the following fundamental questions:

• In case of the centralizer approach: What does the centralizer of a (properly chosen) involution look like in a general simple group $G$?

• In case of the geometric approach: How can one detect a *distinguished* prime $p$ (if there is any) in a general simple group $G$? And how does this then lead to a geometry that characterizes $G$?

The answer of the first question can be read from the classification. The Standard Component Theorem of Aschbacher 1975, [**As1**], shows that either

$$(char\,2) \qquad C_G(O_2(M)) \leqslant O_2(M) \text{ for all 2-local subgroups } M \text{ of } G,$$ [1]

or that there exists an involution $t$ whose centralizer $C_G(t)$ is *classical* or of *standard form*. The latter case can be treated nicely using Brauer's centralizer method; either by the Classical Involution Theorem of Aschbacher 1977, [**As2**], or by solving various standard form problems.

The first case causes many more problems. It is not accidental that the examples from *Lie*(2) have property (*char* 2). As there, in groups satisfying (*char* 2) the centralizers of involutions have non-central normal 2-subgroups which in most cases are an obstruction to applying the centralizer method effectively. In this case the classification shifts from 2 to a properly chosen odd prime $r$. In fact, for groups in *Lie*(2) defined over not too small fields, $r$ divides the order of a maximal torus. Then the proof proceeds as before using a standard component theorem for the prime $r$ rather than 2. Unfortunately, this switch of primes cannot be executed in all cases. So one ends up with some unpleasant cases that have to be treated separately; for example in the Quasithin Group Theorem by Aschbacher-Smith 2004, [**AS**], and the Uniqueness Theorem by Aschbacher 1983, [**As3**].

[1] A $p$-local subgroup is the normalizer of a non-trivial $p$-subgroup; $O_p(M)$ is the largest normal $p$-subgroup of $M$.

The other two questions are more difficult to answer since there is as yet no classification using the geometric approach that would justify such an answer. But – similar to Aschbacher's Standard Component Theorem – one would expect an answer that gives a few cases that then can be treated independently. In addition, any of these cases should be inspired by properties of the generic examples involving the distinguished prime $p$.

Property ($char\,2$) is a good example for this. It reflects an important property that the groups in $Lie(2)$ have in common without using the terminology and conceptual background of groups of Lie type, so it also applies to finite groups in general, and it easily generalizes to arbitrary primes $p$.

We turn this into a definition. A finite group $G$ is of *local characteristic* $p$ if $G$ satisfies

($char\,p$) $\qquad C_G(O_p(M)) \leqslant O_p(M)$ for all $p$-local subgroups of $M$ of $G$.

In particular, the finite simple groups of local characteristic 2 are exactly the exceptions in the Standard Component Theorem that force the switch of primes. So even in this case alone, successfully carrying out a geometric approach for groups of local characteristic 2 would give an alternative proof for that part of the classification, avoiding not only the switch of primes but also the above mentioned cases where this switch fails.

There is another property which nearly all the generic examples share and which the authors believe is important for a classification following the geometric approach: the existence of a *large subgroup*. For any finite group $G$ a non-trivial $p$-subgroup $Q$ is *large* if

(i) $C_G(Q) \leqslant Q$, and
(ii) $N_G(U) \leqslant N_G(Q)$ for all $1 \neq U \leqslant C_G(Q)$.

Note that the first property is equivalent to $C_G(Q) = Z(Q)$. We will refer to the second property as the $Q$!-property, or just as $Q$!.

If $G \in Lie(p)$ and $S \in Syl_p(G)$, then $O_p(N_G(\Omega_1 Z(S)))$ is a large subgroup if and only if $\Omega_1 Z(S)$ is a root subgroup of $G$. Thus, every simple group of Lie type possesses such a large subgroup, except $Sp_{2n}(2^m)$, $n \geqslant 2$, $F_4(2^m)$ and $G_2(3^m)$.

From a group theoretic point of view, the concept of *groups with large subgroups* also generalizes the concept of *groups of* $GF(2)$*-type* introduced in [**GL**]. In particular, Timmesfeld's result, [**Ti**], on centralizers of involutions whose generalized Fitting subgroup is extraspecial, is an important part of the classification of the finite simple groups. But he has concentrated on the structure of the centralizer of a 2-central involution (which in our case is $N_G(Q)$), so at least in a formal sense he follows Brauer's centralizer approach. In contrast to this we will investigate every $p$-local subgroup *not* in $N_G(Q)$, where $Q$ is a large subgroup.

For several years the authors and various other collaborators have worked on a classification project for finite groups of local characteristic $p$ that uses the geometric approach; and the classification of the finite groups of local characteristic $p$ possessing a large subgroup is a major part of this project. An outline of this project can be found in [**MSS**]. There it is also demonstrated in which context large subgroups arise and what role the Local Structure Theorem plays in this classification.

Up to now several contributions to this project have been published or submitted for publication. For example, the local $C(G,T)$-Theorem [**BHS**], the $P$!-Theorem [**PPS**], the $\widetilde{P}$!-Uniqueness Theorem [**MMPS**], plus [**MeiStr3**], [**P1**] and [**P2**], and results about strongly $p$-embedded subgroups, [**PS1**] and [**PS2**], and as relevant background material about modules the Nearly Quadratic Module Theorem [**MS3**], the General FF-module Theorem [**MS5**] and its applications [**MS6**].

Some of these results rest upon properties or hypotheses derived from or justified by the Local Structure Theorem which is presented in this paper. In this sense the Local Structure Theorem is the cornerstone for the investigation of finite groups $G$ of local characteristic $p$ possessing a large subgroup $Q$. In fact, local characteristic $p$ is not really required in full strength for the proof of the Local Structure Theorem, but we will ignore this for the moment.

The Local Structure Theorem determines the action of $M$ on $\Omega_1 Z(O_p(M))$ for every $p$-local subgroup $M$ which contains a Sylow $p$-subgroup of $N_G(Q)$ and is not contained in $N_G(Q)$. Speaking in the geometric language of the generic examples, this information allows to determine the residues of the maximal parabolic subgroups different from the normalizer of a long root subgroup.

In a forthcoming paper the Local Structure Theorem will be used to prove the $H$-Structure Theorem, where under an additional assumption the structure of $N_G(Q)/Q$ is determined. Speaking again in the geometric language of the generic examples, the Local Structure Theorem and $H$-Structure Theorem combined give all the possibilities for the residues of maximal local parabolic subgroups of $G$. This then allows to determine up to isomorphism a parabolic subgroup $H$ of $G$ with $O_p(H) = 1$. If the residues resemble the residues of a group of Lie-type of rank at least three, this can be achieved via Tits' theory of buildings, see [**MSW**, Theorem 6.9]. Otherwise by a case-by-case discussion based on the detailed description of the maximal local parabolic subgroups of $G$ provided by the Local Structure Theorem and the $H$-Structure Theorem.

Having determined $H$ one proceeds by computing the group $G_0$ generated by all the $p$-local subgroups containing a given Sylow $p$-subgroup of $G$. Then one still has to show that $G = G_0$. But this part of the project has already been treated. A result of M. Salarian and G. Stroth [**SaS**], shows that $G_0$ is strongly $p$-embedded in $G$ if $G \neq G_0$, and results of Ch. Parker and G. Stroth, [**PS1**] and [**PS2**], show that this is impossible, so $G = G_0$.

Notation used in the Local Structure Theorem. We will now give the notation that is needed to state the Local Structure Theorem below. Some of this notation will be repeated and refined in the definitions given in later chapters.

In contrast to the Brauer method, where the centralizers of $p$-subgroups are of prime interest, in this paper we investigate the *non-trivial* action of $p$-local subgroups $M$ on suitable elementary abelian normal $p$-subgroups $V \trianglelefteq M$. The basic idea is to identify the group $M/C_M(V)$ and the $\mathbb{F}_pM$-module $V$ *at the same time*. This requires an inductive hypothesis that is called the $\mathcal{K}_p$-group Hypothesis.

A finite group $G$ is a $\mathcal{K}_p$*-group* if the simple sections of any $p$-local subgroup of $G$ are *known* simple groups (i.e., these sections are isomorphic to groups of prime order, groups of Lie type, alternating groups or one of the 26 sporadic groups). This hypothesis is related to (and compatible with) the proper $\mathcal{K}$-group Hypothesis used in the first and second generation proofs of the classification of the finite simple groups, which reflects the only inductive property needed in a minimal counterexample to the Classification Theorem.

This $\mathcal{K}_p$-group Hypothesis allows us to use module-theoretic results provided in [**MS3**], [**MS5**], [**MS6**], [**GM1**] and [**GM2**] for the identification of $M/C_M(V)$ and $V$.

Let $H$ be an arbitrary finite group. Then $H$ has *characteristic* $p$ if $C_H(O_p(H)) \leqslant O_p(H)$. Any subgroup of $H$ containing a Sylow $p$-subgroup of $H$ is a *parabolic subgroup* of $H$; and $H$ has *parabolic characteristic* $p$ if every $p$-local parabolic subgroup of $H$ has characteristic $p$. So the notion of *parabolic characteristic* generalizes the notion of *local characteristic* introduced earlier.

For $A \leqslant H$ we say that $H$ is $A$*-minimal* if $H = \langle A^H \rangle$, and $A$ is contained in a unique maximal subgroup of $H$; and $H$ is $p$*-minimal* if $H$ is $A$-minimal for $A \in Syl_p(H)$.

Let $A$ be an elementary abelian $p$-subgroup of $H$. We say that $A$ is *symmetric* in $H$ if there exists $g \in H$ such that
$$[A, A^g] \neq 1 \quad \text{and} \quad [A, A^g] \leqslant A \cap A^g;$$
otherwise $A$ is called *asymmetric* in $H$.

Let $T \in Syl_p(C_H(A))$. We say that $A$ is *tall* in $H$ if there exists $T \leqslant L \leqslant H$ such that $O_p(L) \neq 1$ and $A \nleqslant O_p(L)$; and $A$ is *char* $p$*-tall* in $H$ if there exists $T \leqslant L \leqslant H$ such that $A \nleqslant O_p(L)$ and $L$ has characteristic $p$. Note here that these definitions are independent of the choice of $T \in Syl_p(C_H(A))$.

Of prime interest in this paper will be the set
$$\mathcal{L}1_H := \{L \leqslant H \mid C_H(O_p(L)) \leqslant O_p(L) \text{ and } O_p(L) \neq 1\}.$$

By $\mathcal{M}_H$ we denote the set of maximal elements of $\mathcal{L}_H$ with respect to inclusion, and by $\mathcal{P}_H$ the set of $p$-minimal elements of $\mathcal{L}_H$. Moreover, for $K \leqslant H$

$$\mathcal{L}_H(K) := \{L \in \mathcal{L}_H \mid K \leqslant L\};$$

similarly we define $\mathcal{M}_H(K)$ and $\mathcal{P}_H(K)$.

By $Y_H$ we denote the largest $p$-reduced normal subgroup of $H$, i.e., the largest elementary abelian normal $p$-subgroup of $H$ satisfying $O_p(H/C_H(Y_H)) = 1$. (For the existence and elementary properties see [**MS4**, 2.2] and 1.24).

Let $\mathfrak{M}_H$ be the set of all $M \in \mathcal{L}_H$ such that

(i) $\mathcal{M}_H(M) = \{M^\dagger\}$ and $Y_M = Y_{M^\dagger}$, where $M^\dagger := MC_H(Y_M)$.
(ii) $C_M(Y_M)$ is $p$-closed and $C_M(Y_M)/O_p(M) \leqslant \Phi(M/O_p(M))$.

As above, for $K \leqslant H$ let $\mathfrak{M}_H(K) = \{M \in \mathfrak{M}_H \mid K \leqslant M\}$. In the following, if $M \in \mathfrak{M}_H$, we will refer to (i) and (ii) as the *basic property* of $M$.

If $A, B$ and $C$ are groups, then $A \sim B.C$ means that $A$ has a normal subgroup $B_1$ such that $B_1 \cong B$ and $A/B_1 \cong C$. $A \sim B{\cdot}C$ means that, in addition, there does not exists a complement to $B_1$ in $A$. If such an $A$ is unique up to isomorphism, we may also write $A \cong B{\cdot}C$.

Suppose that $V$ is a faithful $H$-module and $\mathcal{K}$ is a non-empty $H$-invariant set of subgroups of $H$. Then we say that $V$ is a natural $SL_2(q)$-wreath product module for $H$ with respect to $\mathcal{K}$ if

$$V = \bigoplus_{K \in \mathcal{K}} [V, K] \qquad \text{and} \qquad \langle \mathcal{K} \rangle = \bigtimes_{K \in \mathcal{K}} K,$$

and for each $K \in \mathcal{K}$, $K \cong SL_2(q)$ and $[V, K]$ is a natural $SL_2(q)$-module for $K$.

Note here that a natural $SL_2(q)$-module is a natural $SL_2(q)$-wreath product module with $|\mathcal{K}| = 1$.

If $V$ is a vector space over the finite field $\mathbb{K}$, then $\Lambda^2(V)$, $S^2(V)$ and $U^2(V)$, denote the exterior, symmetric and unitary square of $V$, that is, the set of symplectic, symmetric and unitary forms on the dual of $V$, respectively. For further details for our naming of modules see A.2.

The Local Structure Theorem. Suppose now that $G$ is a finite group and $Q$ is a (fixed) large subgroup of $G$. For $M \leqslant G$ we set

$$M^\circ := \langle Q^g \mid g \in G, Q^g \leqslant M \rangle,$$

and

$$Q^\bullet := O_p(N_G(Q)).$$

Let $Q \leqslant S \in Syl_p(G)$. Clearly, either $S$ is contained in a unique maximal $p$-local subgroup $M$ of $G$, or there exists a $p$-local subgroup $M$ of $G$ with $S \leqslant M$ and $Q \ntrianglelefteq M$. For the generic examples from $Lie(p)$, the first case corresponds to groups of Lie rank 1, the second to those of Lie rank larger than 1.

In general, in the first case $M$ contains the normalizer of every non-trivial characteristic subgroup of $S$. Then, at least if $G$ has local characteristic $p$, either $M$ is a strongly $p$-embedded subgroup of $G$ or the $p$-local structure of $G$ is well-understood and was investigated in [**BHS**]. Finally, if $G$ possesses a strongly $p$-embedded subgroup, the $p$-local analysis is no longer of any help. Fortunately, at least for $p = 2$, a Theorem of Bender, 1971 [**Be**], gives a complete classification, for odd primes such a theorem is not known.

In this paper we consider the second case, where $S$ is contained in more than one maximal $p$-local subgroup of $G$, and we investigate the action of $L$ on $Y_L$ for all $p$-local subgroups $L$ of $G$ with $Q$ not normal in $L$. We will prove:

THEOREM A (**Local Structure Theorem**). *Let $G$ be a finite $\mathcal{K}_p$-group and $S \in Syl_p(G)$. Suppose that $S$ is contained in at least two maximal $p$-local subgroups and that $Q$ is a large subgroup of $G$ in $S$. Let $L \leqslant G$ with $S \leqslant L$, $O_p(L) \neq 1$ and $Q \ntrianglelefteq L$.*

*Then there exist $M \in \mathfrak{M}_G(S)$ and $L^* \leqslant M$ with*

$$S \leqslant L^*,\ Y_L = Y_{L^*},\ LC_G(Y_L) = L^*C_G(Y_L), \ \text{ and } L^\circ = (L^*)^\circ.$$

*Moreover, for any such $L$ and $M$ one of the following holds, where $\widetilde{L} := L/C_L(Y_L)$ and $q$ is a power of $p$.*

(1) **The linear case.**
  (a) $\widetilde{L^\circ} \cong SL_n(q)$, $n \geqslant 3$, *and* $[Y_L, L^\circ]$ *is a corresponding natural module for* $\widetilde{L^\circ}$.
  (b) *If* $Y_L \neq [Y_L, L^\circ]$ *then* $\widetilde{L^\circ} \cong SL_3(2)$, $|Y_L/[Y, L^\circ]| = 2$, $[Y_L, L^\circ] \leqslant Q \leqslant Q^\bullet$, $Y_M = Y_L$ *and* $M^\circ = L^\circ$.

(2) **The symplectic case.**
  (a) $\widetilde{L^\circ} \cong Sp_{2n}(q)$, $n \geqslant 2$, *or* $Sp_4(q)'$ *(and* $q = 2$*), and* $[Y_L, L^\circ]$ *is the corresponding natural module for* $\widetilde{L^\circ}$
  (b) *If* $Y_L \neq [Y_L, L^\circ]$, *then* $p = 2$ *and* $|Y_L/[Y_L, L^\circ]| \leqslant q$.
  (c) *If* $Y_L \nleqslant Q^\bullet$, *then* $p = 2$ *and* $[Y_L, L^\circ] \nleqslant Q^\bullet$.
  (d) *Either* $L^\circ = M^\circ$ *and* $Y_L = Y_M$, *or one of following holds:*
    (1) $p = 2$, $\widetilde{L^\circ} \cong Sp_4(2)'$, $Y_L = [Y_L, L^\circ]$, $Y_L \nleqslant Q^\bullet$, $M^\circ/C_{M^\circ}(Y_M) \cong Mat_{24}$, *and* $Y_M$ *is the simple Golay code module of* $\mathbb{F}_2$*-dimension* $11$ *for* $M^\circ$.
    (2) $p = 2$, $\widetilde{L^\circ} \cong Sp_4(2)$, $|Y_L/[Y_L, L^\circ]| = 2$, $[Y_L, L^\circ] \nleqslant Q^\bullet$, $M^\circ/C_{M^\circ}(Y_M) \cong Aut(Mat_{22})$, *and* $Y_M$ *is the simple Todd module of* $\mathbb{F}_2$*-dimension* $10$ *for* $M^\circ$.

(3) **The Wreath Product Case.**
  (a) *There exists a unique* $\widetilde{L}$*-invariant set* $\mathcal{K}$ *of subgroups of* $\widetilde{L}$ *such that* $[Y_L, L^\circ]$ *is a natural* $SL_2(q)$*-wreath product module for* $\widetilde{L}$ *with respect to* $\mathcal{K}$. *Moreover,* $\widetilde{L^\circ} = O^p(\langle\mathcal{K}\rangle)\widetilde{Q}$ *and* $Q$ *acts transitively on* $\mathcal{K}$.
  (b) *If* $Y_L \neq [Y_L, L^\circ]$, *then* $p = 2$, $\widetilde{L} \cong \Gamma SL_2(4)$, $\widetilde{L^\circ} \cong SL_2(4)$ *or* $\Gamma SL_2(4)$, $|Y_L/[Y_L, L^\circ]| = 2$, $[Y_L, L^\circ] \nleqslant Q^\bullet$, $Y_M = Y_L$ *and* $MC_G(Y_L) = LC_G(Y_L)$.
  (c) *Either* $Y_M = Y_L$ *and* $M^\circ = L^\circ$ *or* $\widetilde{L^\circ} \cong SL_2(q)$.

(4) **The Weak Wreath Product Case.** $O^p(\widetilde{L^\circ})$ *is abelian and* $Y_L = [Y_L, L^\circ]$. *Let* $U_1, U_2, \ldots, U_s$ *be the Wedderburn components of* $O^p(L^\circ)$ *on* $Y_L$. *Then the following hold:*
  (a) $Y_L = U_1 \oplus \ldots \oplus U_s$, $O^p(L^\circ)/C_{O^p(L^\circ)}(U_i)$ *is cyclic of order dividing* $q-1$, *and* $q > 2$.
  (b) $Q$ *permutes the subgroups* $U_i$ *in (4:a) transitively.*
  (c) $Y_M$ *is a natural* $SL_2(q)$*-wreath product module for* $M/C_M(Y_M)$ *with respect to some* $\mathcal{K}$, $M^\circ/C_{M^\circ}(Y_M) \ncong SL_2(q)$, *and for the inverse image* $P^*$ *of* $\langle\mathcal{K}\rangle$ *in* $M$, $P^* \cap S \trianglelefteq L^\circ S$, $Y_L \leqslant C_{Y_M}(P^* \cap S)$, *and there exists an* $L$*-invariant partition* $\mathcal{K}_1, \mathcal{K}_2, \ldots, \mathcal{K}_s$ *of* $\mathcal{K}$ *with* $U_i = Y_L \cap [Y_M, \langle\mathcal{K}_i\rangle]$ *for all* $1 \leqslant i \leqslant s$.

(5) **The orthogonal case.** $Y_L \nleqslant Q^\bullet$, $\widetilde{L^\circ} \cong \Omega_n^\epsilon(q)$, $n \geqslant 5$, *where* $q$ *is odd if* $n$ *is odd, and* $Y_L$ *is a corresponding natural module for* $\widetilde{L^\circ}$. *Moreover, either* $Y_M = Y_L$ *and* $L^\circ = M^\circ$ *or one of the following holds:*
  (1) $\widetilde{L^\circ} \cong \Omega_6^+(q)$, *and* $Y_M$ *is the exterior square of a natural* $SL_m(q)$*-module for* $M^\circ$.
  (2) $p = 2$, $\widetilde{L^\circ} \cong \Omega_6^+(2)$ *and* $M^\circ/C_{M^\circ}(Y_M) \cong Mat_{24}$, *and* $Y_M$ *is the simple Todd- module of* $\mathbb{F}_2$*-dimension* $11$ *for* $M^\circ$.
  (3) $\widetilde{L^\circ} \cong \Omega_8^+(q)$ *and* $M^\circ/C_{M^\circ}(Y_M) \cong Spin_{10}^+(q)$, *and* $Y_M$ *is the half-spin module for* $M^\circ$.
  (4) $\widetilde{L^\circ} \cong \Omega_{10}^+(q)$ *and* $M^\circ/C_{M^\circ}(Y_M) \cong E_6(q)$, *and* $Y_M$ *is simple module of* $\mathbb{F}_q$*-dimension* $27$ *for* $M^\circ$.

(6) **The tensor product case.** $Y_L \nleqslant Q^\bullet$, *and there exist subgroups* $\widetilde{L}_1, \widetilde{L}_2$ *of* $\widetilde{L}$ *such that*
  (a) $\widetilde{L}_i \cong SL_{t_i}(q)$, $t_i \geqslant 2$, $[\widetilde{L}_1, \widetilde{L}_2] = 1$, *and* $\widetilde{L}_1\widetilde{L}_2 \trianglelefteq \widetilde{L}$,
  (b) $Y_L \cong Y_1 \otimes_{\mathbb{F}_q} Y_2$, *where* $Y_i$ *is a corresponding natural module for* $\widetilde{L}_i$ *(and* $\mathbb{F}_q$ *is a field of order* $q$*),*
  (c) $\widetilde{L} = \widetilde{L^\circ} \cong SL_2(2) \wr C_2$ *and* $p = 2$, *or* $\widetilde{L^\circ}$ *is one of* $\widetilde{L}_1, \widetilde{L}_2$, *or* $\widetilde{L}_1\widetilde{L}_2$,

(d) *Moreover, either $M$ fulfills the tensor product case, or $p = 2$, $\widetilde{L} = \widetilde{L}_1\widetilde{L}_2 \cong SL_2(2) \times SL_2(2)$, $M/C_M(Y_M) \cong 3{\cdot}Sym(6)$, and $Y_M$ is the simple module of $\mathbb{F}_2$-dimension 6 for $M$.*

(7) **The non-natural $SL_n(q)$-case.** $[Y_L, L^\circ] \nleqslant Q^\bullet$, *and one of the following holds:*

(1) $\widetilde{L^\circ} \cong SL_n(q)/\langle(-id)^{n-1}\rangle$, $n \geqslant 5$, *$Y_L$ is the exterior square of a natural $SL_n(q)$-module for $L^\circ$, and $Y_M$ is the exterior square of a natural $SL_m(q)$-module for $M^\circ$.*

(2) *$p$ is odd, $\widetilde{L^\circ} \cong SL_n(q)/\langle(-id)^{n-1}\rangle$, $n \geqslant 2$, and $Y_L$ is the symmetric square of a natural $SL_n(q)$ for $L^\circ$, and $Y_M$ is the symmetric square of a natural $SL_m(q)$-module for $M^\circ$.*

(3) $\widetilde{L^\circ} \cong SL_n(q)/\langle \lambda id \mid \lambda \in \mathbb{F}_q, \lambda^n = \lambda^{q_0+1} = 1\rangle$, $n \geqslant 2$, $q = q_0^2$, *and $[Y_L, L^\circ]$ is the unitary square of a natural $SL_n(q)$-module for $L^\circ$. Moreover, one of the following holds:*

(1) $Y_L = [Y_L, L^\circ]$, *and $Y_M$ is the unitary square of a natural $SL_m(q)$-module for $M^\circ$.*

(2) $p = 3$, $|Y_L/[Y_L, L^\circ]| = 3$, $\widetilde{L^\circ} \cong L_2(9)$, $M^\circ/C_{M^\circ}(Y_M) \cong Mat_{11}$, $Y_L = Y_M$, *and $Y_M$ is the simple Golay-code module of $\mathbb{F}_3$-dimension 5 for $M^\circ$.*

(3) $p = 2$, $Y_L = [Y_L, L^\circ]$, $\widetilde{L^\circ} \cong SL_2(4)$, $M^\circ/C_{M^\circ}(Y_M) \cong Mat_{22}$, *and $Y_M$ is the simple Golay-code module of $\mathbb{F}_2$-dimension 10 for $M^\circ$.*

(4) $p = 3$, $Y_L = [Y_L, L^\circ]$, $\widetilde{L^\circ} \cong L_2(3)$, $Y_L \nleqslant Q^\bullet$, $M^\circ/C_{M^\circ}(Y_M) \cong 2{\cdot}Mat_{12}$, *$Y_M$ is the simple Golay-code module of $\mathbb{F}_3$-dimension 6, and $Y_L \nleqslant Q^\bullet$.*

(8) **The exceptional case.** $Y_L \nleqslant Q^\bullet$, $Y_M = Y_L$, $M^\circ = L^\circ$, *and one of the following holds:*

(1) $\widetilde{L^\circ} \cong Spin^+_{10}(q)$, *and $Y_L$ is a half-spin module.*

(2) $\widetilde{L^\circ} \cong E_6(q)$, *and $Y_L$ is one of the (up to isomorphism) two simple $\mathbb{F}_pL^\circ$-modules of order $q^{27}$.*

(9) **The sporadic case.** $Y_L \nleqslant Q^\bullet$, $Y_L = Y_M$, $L^\circ = M^\circ$, *and one of the following holds:*

(1) $p = 2$, $\widetilde{L} \sim 3{\cdot}Sym(6)$, $\widetilde{L^\circ} \sim 3{\cdot}Alt(6)$ *or* $3{\cdot}Sym(6)$, *and $Y_L$ is a simple module of $\mathbb{F}_2$-dimension 6.*

(2) $p = 2$, $\widetilde{L^\circ} \cong Mat_{22}$, *and $Y_L$ is the simple Golay-code module of $\mathbb{F}_2$-dimension 10.*

(3) $p = 2$, $\widetilde{L^\circ} \cong Mat_{24}$, *and $Y_L$ is the simple Todd or Golay-code module of $\mathbb{F}_2$-dimension 11.*

(4) $p = 3$, $\widetilde{L^\circ} \cong Mat_{11}$, *and $Y_L$ is the simple Golay-code module of $\mathbb{F}_3$-dimension 5.*

(10) **The non-characteristic $p$ case.** *There exists $1 \neq y \in Y_L$ such that $C_G(y)$ is not of characteristic $p$, and one of the following holds:*

(1) *$Y_L$ is tall and asymmetric in $G$, but $Y_L$ is not char $p$-tall in $G$.*

(2) $p = 2$, $\widetilde{L^\circ} \cong Aut(Mat_{22})$, *$Y_L$ is the simple Todd module of $\mathbb{F}_2$-dimension 10, and $Y_L \nleqslant Q^\bullet$*

(3) $p = 3$, $\widetilde{L^\circ} \cong 2{\cdot}Mat_{12}$, *$Y_L$ is the simple Golay-code module of $\mathbb{F}_3$-dimension 6, and $Y_L \nleqslant Q^\bullet$.*

(4) $p = 2$, $\widetilde{L} \cong O^\epsilon_{2n}(2)$, $\widetilde{L^\circ} \cong \Omega^\epsilon_{2n}(2)$, $2n \geqslant 4$, $(2n, \epsilon) \neq (4, +)$, *$Y_L$ is a corresponding natural module and $Y_L \leqslant Q^\bullet$.*

(5) $p = 3$, $\widetilde{L^\circ} \cong \Omega^-_4(3)$, *$[Y_L, L^\circ]$ is the corresponding natural module,* $|Y_L/[Y_L, L^\circ]| = 3$, *$Y_L$ is isomorphic to the 5-dimensional quotient of a six dimensional permutation module for $\widetilde{L^\circ} \cong Alt(6)$, and $[Y_L, L^\circ] \nleqslant Q^\bullet$.*

(6) $p = 3$, $\widetilde{L^\circ} \cong \Omega_5(3)$, *$[Y_L, L^\circ]$ is the corresponding natural module,* $|Y_L/[Y_L, L^\circ]| = 3$, *and $[Y_L, L^\circ] \nleqslant Q^\bullet$.*

(7) $p = 2$, $\widetilde{L^\circ} \cong \Omega^+_6(2)$, *$[Y_L, L^\circ]$ is the corresponding natural module, and $|Y_L/[Y, L^\circ]| = 2$.*

(8) $p = 2$, $\widetilde{L^\circ} \cong Mat_{24}$, *$[Y_L, L^\circ]$ is the simple Todd-module of $\mathbb{F}_2$-dimension 11,* $|Y_L/[Y_L, L^\circ]| = 2$, *and $[Y_L, L^\circ] \nleqslant Q^\bullet$.*

*Moreover, either $Y_L = Y_M$ and $L^\circ = M^\circ$, or $\widetilde{L^\circ} \cong \Omega^+_6(2)$, $[Y_L, L^\circ] \nleqslant Q^\bullet$, $M^\circ/C_{M^\circ}(Y_M) \cong Mat_{24}$, $[Y_M, M^\circ]$ is the simple Todd-module of $\mathbb{F}_2$-dimension 11 and $|Y_M/[Y_M, M^\circ]| = 2$.*

Note that there is some overlap between the last case of the Local Structure Theorem and the previous cases: If $[Y_L, L^\circ]$ is a natural $\Omega_5(3)$, $\Omega^-_4(3)$ or $\Omega^+_6(2)$-module or the Todd module for $Mat_{24}$

TABLE 1. Examples for the Local Structure Theorem. Cases (1)–(9)

| | Case | $[Y_M, M^\circ]$ for $M^\circ$ | c | Remarks | examples for $G$ |
|---|---|---|---|---|---|
| | 1 | nat $SL_n(q)$ | 1 | - | $L_{n+1}(q)$ |
| | 1 | nat $SL_n(q)$ | 1 | $n = 7, 8$ | $E_n(q)$ |
| | 1 | nat $SL_3(2)$ | 1 | - | $Alt(9)$, $G_2(3)$, HS(.2), Ru, HN |
| | 1 | nat $SL_3(3)$ | 1 | - | $Fi'_{22,23,24}$, $F_4(2)$, ${}^2E_6(2)$, BM |
| | 1 | nat $SL_3(5)$ | 1 | - | Ly, BM, M |
| | 1 | nat $SL_4(2)$ | 1 | $N_G(Q) \leqslant M$ | $Mat_{24}$ |
| | 1 | nat $SL_5(2)$ | 1 | - | Th, BM |
| * | 1:b | nat $SL_3(2)$ | 2 | $G \neq G^\circ$ | $Aut(G_2(3))$ |
| | 2 | nat $Sp_4(2)$ | 1 | - | $Mat_{22}.2$,$PSO_6^-(3)$, $P\Omega_6^-(3)\langle\omega\rangle$ |
| | 2 | nat $Sp_4(2)'$ | 1 | - | $Mat_{22}$, $P\Omega_6^-(3)$, Suz |
| * | 2:b | nat $Sp_4(2)'$ or $Sp_4(2)$ | 2 | - | $P\Omega_6^-(3)\langle\omega\rangle$ or $PO_6^-(3)$ |
| * | 2 | nat $Sp_8(2)$ | 2 | - | BM |
| ** | 2 | nat $Sp_{2n}(q)$ | $\leqslant q$ | $(n, q) \neq (4, 2), (8, 2)$ | - |
| | 3 | nat $SL_2(q)$ | 1 | - | $L_3(q)$, $G_2(q)$ $p \neq 3$,${}^3D_4(q)$ ${}^2F_4(q)$ $p = 2$, $D_4(q)\Phi_3$ $p = 3$ |
| | 3 | nat $SL_2(2)$ | 1 | - | $Sp_4(2)'$, $G_2(2)'$,${}^2F_4(2)'$, $Mat_{12}(.2)$, $J_2$, $J_3$, $P\Omega_6^-(3).X$, $P\Omega_8^+(3).X$ |
| | 3 | nat $SL_2(3)$ | 1 | - | $Mat_{12}$, ${}^2F_4(2)'$, Th |
| | 3 | nat $SL_2(4)$ | 1 | - | $Mat_{22}$, $Mat_{23}$ |
| | 3 | nat $SL_2(5)$ | 1 | - | Ru, HN, Th |
| | 3 | nat $SL_2(7)$ | 1 | - | O'N, M |
| | 3 | nat $SL_2(13)$ | 1 | - | M |
| | 3 | nat $\Gamma SL_2(4)$ | 1 | - | $\Gamma L_3(4)$, $Mat_{22}$ |
| * | 3 | nat $SL_2(4)[.2]$ | 2 | $\overline{M} \cong \Gamma SL_2(4)$ | $Aut(Mat_{22})$ |
| * | 3 | nat $SL_2(q)$ wreath | 1 | $\|\mathcal{K}\| > 1$ | $(\Gamma)L_3(q) \wr$ 2-group, $q = 2, 4$. |
| | 5 | nat $\Omega_n^\epsilon(q)$ | 1 | - | $P\Omega_{n+2}^\epsilon(q)$ |
| | 5 | nat $\Omega_7(q)$ | 1 | - | $F_4(q)$, $p$ odd |
| | 5 | nat $\Omega_6^-(q)$ | 1 | - | ${}^2E_6(q)$ |
| | 5 | nat $\Omega_8^+(q)$ | 1 | - | $E_6(q)\Phi_2$ |
| | 5 | nat $\Omega_{14}^+(q)$ | 1 | - | $E_8(q)$ |
| | 6 | nat $SL_{t_1}(q)[\otimes SL_{t_2}(q)]$ | 1 | - | $L_{t_1+t_2}(q)$, $L_{2t_1+1}(q)\Phi_2$ $t_1 = t_2$ |
| | 6 | nat $SL_2(2))[\otimes SL_3(2)]$ | 1 | - | $Mat_{24}$ |
| | 1, 5, 7:1 | $\Lambda^2(\text{nat})SL_n(q^2)$ | 1 | $n \geqslant 3$ | $P\Omega_{2n}^+(q)$, $\Omega_{2n+1}(q)$ $p$odd, $P\Omega_{2n+2}^-(q)$, $O_{2n}^+(q)$ $p = 2$ |
| | 7:2 | $S^2(\text{nat})SL_n(q)$ | 1 | - | $PSp_{2n}(q)$ |
| | 7:3 | $U^2(\text{nat})SL_n(q_0^2)$ | 1 | - | $U_{2n}(q_0)$, $U_{2n+1}(q_0)$ |
| | 7:3 | $U^2(\text{nat})SL_2(9)$ | 1 | - | McL |
| | 8:1 | half-spin $Spin_{10}^+(q)$ | 1 | - | $E_6(q)$ |
| | 8:2 | $q^{27}$ for $E_6(q)$ | 1 | - | $E_7(q)$ |
| | 9:1 | $2^6$ for $3{\cdot}Alt(6)[.2]$ | 1 | $\overline{M} \sim 3{\cdot}Sym(6)$ | $Mat_{24}$ |
| | 9:2 | Golay $2^{10}$ for $Mat_{22}$ | 1 | - | $Co_2$ |
| | 9:3 | Golay $2^{11}$ for $Mat_{24}$ | 1 | - | $Co_1$ |
| | 9:3 | Todd $2^{11}$ for $Mat_{24}$ | 1 | - | $J_4$ |
| | 9:4 | Golay $3^5$ for $Mat_{11}$ | 1 | - | $Co_3$ |

one might have $Y_L = [Y_L, L^\circ]$ or $Y_L \neq [Y_L, L^\circ]$. Similarly, if $[Y_L, L]$ is a natural $\Omega_{2n}^\epsilon(2)$-module one might have $Y_L \nleqslant Q^\bullet$ or $Y_L \leqslant Q^\bullet$. But each time the second possibility can only occur if there exists $1 \neq y \in Y_L$ such that $C_G(y)$ is not of characteristic $p$.

TABLE 2. Examples for the Local Structure Theorem where (***char*** $\boldsymbol{Y_M}$) fails

| Case | $[Y_M, M^\circ]$ for $M^\circ$ | c | Remarks | examples for $G$ |
|---|---|---|---|---|
| 1:b | nat $SL_3(2)$ | 2 | $G \neq G^\circ$ | $Aut(G_2(3))$ |
| 2 | nat $Sp_4(2)'$ or $Sp_4(2)$ | 2 | - | $P\Omega_6^-(3)\langle\omega\rangle$ or $PO_6^-(3)$ |
| 2 | nat $Sp_8(2)$ | 2 | - | BM |
| 3 | nat $SL_2(q)$ wreath | 1 | $\lvert\mathcal{K}\rvert > 1$ | $(\Gamma)L_3(q) \wr 2$-group, $q = 2, 4$ |
| 3:b | nat $SL_2(4)[.2]$ | 2 | $\overline{M} \cong \Gamma SL_2(4)$ | $Aut(Mat_{22})$ |
| 3,6 | nat $SL_2(2) \otimes SL_2(2)$ | 1 | - | $Sym(9)$, $Alt(10)$ |
| 6 | nat $SL_2(2))[\otimes SL_2(2)]$ | 1 | - | $Alt(9)$ |
| 6 | nat $SL_2(3) \otimes SL_2(3)$ | 1 | - | HN |
| 7:2 | nat $\Omega_3(3)$ | 1 | - | $Sp_6(2), \Omega_8^-(2)$ |
| 7:2 | nat $\Omega_3(5)$ | 1 | - | $Co_1$ |
| 7:3 | nat $\Omega_4^-(2)$ | 1 | - | $L_4(3)$, $Alt(10)$ |
| 7:3, 10:5 | nat $\Omega_4^-(3)$ | $\leqslant 3$ | - | $U_6(2).c(.2)$ |
| 5 | nat $\Omega_5(3)$ | 1 | - | $Fi_{22}(.2)$ |
| 5,10:6 | nat $\Omega_5(3)$ | $\leqslant 3$ | - | ${}^2E_6(2).c(.2)$ |
| 5 | nat $\Omega_6^+(2)$ | 1 | - | $P\Omega_8^+(3)(.3)(.2)$ |
| 10:7 | nat $\Omega_6^+(2)$ | 2 | - | $P\Omega_8^+(3).2.(2)$, $P\Omega_8^+(3).Sym(3)$ |
| 5 | nat $\Omega_7(3)$ | 1 | - | $Fi'_{24}(.2)$ |
| 5 | nat $\Omega_{10}^+(2)$ | 1 | - | M |
| ** 10:4 | nat $\Omega_{2n}^\epsilon(2), (2n, \epsilon) \neq (4, +)$ | $\leqslant 2$ | $Y_M \leqslant Q^\bullet$ | - |
| 9:1 | $2^6$ for $3{\cdot}Sym(6)$ | 1 | $\overline{M} \sim 3{\cdot}Sym(6)$ | He |
| 9:3, 10:8 | Todd $2^{11}$ for $Mat_{24}$ | $\leqslant 2$ | - | $Fi'_{24}.c$ |
| 10:2 | Todd $2^{10}$ for $Aut(Mat_{22})$ | 1 | - | $Aut(Fi_{22})$ |
| 10:3 | Golay $3^6$ for $2{\cdot}Mat_{12}$ | 1 | - | $Co_1$ |
| ** 10:1 | ? | ? | tall, asymmetric, not *char* $p$-tall | - |

Note also that last case is not the only case of the Local Structure Theorem, where $C_G(y)$ may not be of characteristic $p$ for some $1 \neq y \in Y_L$. For example both $J_4$ and $Fi'_{24}$ contain a parabolic subgroup $M \sim 2^{11}Mat_{24}$, with $Y_M$ the Todd module. In $J_4$, $C_G(y)$ is of characteristic 2 for all $1 \neq y \in Y_M$, but this does not hold in $Fi'_{24}$. On the other hand, $M \sim 2^{11+1}Mat_{24}$ only occurs in $Fi_{24}$, matching the fact that the $2^{11+1}$ only appears in last case of the Local Structure Theorem.

The cases of the Local Structure Theorem are disjoint with one exception: The case $p = 2$, $O^2(\widetilde{L}) \cong C_3 \times C_3$ and $|Y_L| = 16$ appears in the wreath product and tensor product case. Combining the two cases we get the following possibilities:

- $\widetilde{L} \cong SL_2(2) \wr C_2$, $Y_L$ is a natural $O_4^+(2)$-module for $\widetilde{L}$, $\widetilde{Q} \cong C_4$ or $D_8$, $Y_M = Y_L$ and $M^\circ = L^\circ$. Both $Y_L \leqslant Q^\bullet$ and $Y_L \nleqslant Q^\bullet$ are possible.
- $\widetilde{L} \cong SL_2(2) \wr C_2$, $\widetilde{L^\circ} \cong SL_2(2) \times SL_2(2)$, and $Y_L$ is a natural $\Omega_4^+(2)$-module for $\widetilde{L}$. Both $Y_L \leqslant Q^\bullet$ and $Y_L \nleqslant Q^\bullet$ are possible. Either $Y_L = Y_M$ and $L^\circ = M^\circ$, or $Y_L \nleqslant Q^\bullet$ and $M$ fulfills the tensor product case with $M/C_M(Y_M) \cong SL_t(2) \wr C_2$, and $M^\circ/C_{M^\circ}(Y_M) \cong SL_t(2) \times SL_t(2)$.
- $\widetilde{L} = \widetilde{L^\circ} \cong SL_2(2) \times SL_2(2)$, $Y_L$ is a natural $\Omega_4^+(2)$-module for $\widetilde{L}$, and $Y_L \nleqslant Q^\bullet$. Either $Y_L = Y_M$ and $L^\circ = M^\circ$, or $M$ fulfills the tensor product case with $M/C_M(Y_M) \cong M^\circ/C_{M^\circ}(Y_M) \cong SL_{t_1}(2) \times SL_{t_2}(2)$, or $M/C_M(Y_M) \cong M^\circ/C_{M^\circ}(Y_M) \cong 3{\cdot}Sym(6)$ and $|Y_M| = 2^6$.
- $\widetilde{L} \cong SL_2(2) \times SL_2(2)$, $Y_L$ is a natural $\Omega_4^+(2)$-module for $\widetilde{L}$, $\widetilde{L^\circ} \cong SL_2(2)$, $Y_L$ is the direct sum of two natural $SL_2(2)$-modules for $\widetilde{L^\circ}$, and $Y_L \nleqslant Q^\bullet$. Either $Y_L = Y_M$ and $L^\circ = M^\circ$, or $M$ fulfills the tensor product case with $M^\circ/C_{\overline{M}^\circ}(Y_M) \cong SL_{t_1}(2)$, or $M/C_M(Y_M) \cong 3{\cdot}Sym(6)$, $M^\circ/C_{M^\circ}(Y_M) \cong 3{\cdot}Alt(6)$ and $|Y_M| = 2^6$.

Most of the cases listed in the Local Structure Theorem occur in interesting finite groups, see tables 1 and 2.

Consider the property

$$(char\ Y_M) \qquad\qquad C_G(y) \text{ is of characteristic } p \text{ for all } y \in Y_M^\sharp.$$

In those cases of the first table marked with '$*$' property ($char\ Y_M$) fails in the listed example, but we currently do not have a proof that ($char\ Y_M$) has to fail other than using the classification of finite simple groups to determine all the possible examples. For the case marked with '$**$' we do not know any example (with or without ($char\ Y_M$)). Showing that ($char\ Y_M$) fails in the '$*$' cases and that the '$**$' cases do not occur seems to require the determination of the whole structure of $M$ (and not only the action on $Y_M$) and sometimes even the structure of $G$, and will be done in separate papers. For example, case 1:b of Theorem A has already been treated in [**MeiStr3**] and case 3 (for $r > 1$ and $Y_M \leqslant Q^\bullet$) in [**PPS**].

In the table $c := |Y_M/[Y_M, M^\circ]|$ and $\Phi_i$ is a group of graph automorphism of order $i$. In the example $G = K.X$ with $K = P\Omega_6^-(3)$ or $P\Omega_8^+(3)$, $X \leqslant Out(K)$ such that $X$ acts transitively on the four elements of $\mathcal{P}_{N_K(Q)}(K \cap S)$. In the examples $G = P\Omega_6^-(3)\langle\omega\rangle$, $\omega$ is a reflection in $PO_6^-(3)$. An entry of the form $A[B]$ in the $[Y_M, M^\circ]$ column indicates that there exists more than one choice for $Q$ in the example $G$. Depending on this choice the structure of $[Y_M, M^\circ]$ as an $M^\circ$-module is either described by $A$ or $AB$.

The strategy for the proof of the Local Structure Theorem. Suppose that $G$ is a finite group possessing a large subgroup $Q$ with $Q \leqslant S \in Syl_p(G)$. In 1.55 it is shown that $G$ has parabolic characteristic $p$, and in 1.56 that for every $L \in \mathcal{L}_G(S)$ there exist $M \in \mathfrak{M}_G(S)$ and $L^* \leqslant M$ satisfying:

- $LC_G(Y_L) = L^*C_G(Y_{L^*})$, $L^\circ = (L^*)^\circ$ and $Y_L = Y_{L^*} \leqslant Y_M$.
- If $Q \ntrianglelefteq L$ then also $Q \ntrianglelefteq L^*$ and $Q \ntrianglelefteq M$.

In other words, the action of $L$ on $Y_L$ can be investigated via the action of the subgroup $L^*$ of $M$ on the submodule $Y_L$ of $Y_M$ since $L/C_L(Y_L) \cong L^*/C_{L^*}(Y_L)$. Hence, the structure of $Y_M$ and $M/C_M(Y_M)$ will also determine the possibilities for $Y_L$ and $L/C_L(Y_L)$. For this reason nearly the entire paper, Chapters 3 – 9, is devoted to the analysis of the action of $M/C_M(Y_M)$ on $Y_M$.

The global strategy. The basic idea is to find subgroups in $M/C_M(Y_M)$ that act in a "nice way" on $Y_M$ and then to identify $M/C_M(Y_M)$ and the $M$-module $Y_M$ via the action of these subgroups.

Of course, the crucial point is to find out what "nice way" should mean. On one side, it should be a property that arises naturally in the local analysis, and on the other side, it should be a property strong enough to allow to identify the action of $M$ on $Y_M$.

It turns out that in most cases being some kind of (non-trivial) offender, like quadratic offender, strong offender, etc., is the right property, and this then leads to one of the FF-Module Theorems from Appendix C. In other cases, when no non-trivial offenders are at hand, acting nearly quadratically or as a $2F$-offender is the property we work with, and again results are available that can be used; in particular, the classification of simple $2F$-modules for almost quasisimple groups by Guralnick and Malle, [**GM1**] and [**GM2**].

The list of possibilities for groups and modules in these results is usually much longer than the list we actually get as the final result of our analysis, so a major part of our proof is devoted to exclude groups and modules from such lists. Usually this is not done by beginning a case by case discussion right away, but by finding some general arguments first that allow to treat (some of) the cases in a uniform way. For example, the cases where $Y_M$ carries an $M$-invariant form usually can be treated uniformly using some general arguments from linear algebra.

The local strategy. It is obvious that one cannot get any information about $M$ and its action on $Y_M$ without discussing in one way or another the embedding of $M$ into $G$. But a priori, it is not clear at all what type of embedding properties one should study and how they would help to get this information. In the following we will describe in general terms the strategy we follow and which allows to subdivide the proof into a few cases which to a large extend are independent from each other.

Using the above definition of symmetry, it is clear that $Y_M$ is either symmetric or asymmetric in $G$, and this is the first major subdivision of the proof.

In Chapter 4 we treat the symmetric case, that is, $Y_M$ is symmetric in $G$, so there exists a conjugate $Y_M^g$ such that

$$1 \neq [Y_M, Y_M^g] \leqslant Y_M \cap Y_M^g.$$

Then $Y_M$ and $Y_M^g$ act quadratically and non-trivially on each other, and it is easy to see that $Y_M$ is a non-trivial quadratic offender on $Y_M^g$, or vice versa. In any case we can apply the General FF-Module Theorem C.2 to both, $M$ and $M^g$. The trick is now to use the $Q$!-property to show that $M \cap M^g$ contains a conjugate of the large subgroup $Q$. Now the action of such a "common" large subgroup allows to pin down the structure of $M/C_M(Y_M)$ and its action of $Y_M$.

The asymmetric case is much harder to handle. But here a fundamental property holds: $O_p(M)$ is a weakly closed subgroup of $G$ (see 2.6). As a consequence we get that $M^\dagger \cap H$ is a parabolic subgroup of $H$ for all subgroups $H$ containing $O_p(M)$. Since by the basic property of $M$, $O_p(M) \in Syl_p(C_G(Y_M))$, the properties "tall", "*char* $p$-tall" and "short" (here "short" means "not tall"), are tailored to further subdivide the asymmetric case.

In Chapter 5 we treat the short asymmetric case. Here $Y_M \leqslant O_p(P)$ for all $P \leqslant G$ with $O_p(M) \leqslant P$ and $O_p(P) \neq 1$. Asymmetry then implies that the closure $V := \langle Y_M^P \rangle$ is abelian. This property is used to show the existence of a symmetric pair $(Y_1, Y_2)$ of conjugates of $Y_M$ (see 2.19 and 2.23). In this pair no longer $Y_1$ and $Y_2$ act non-trivially on each other, as in the symmetric case, but abelian subgroups $V_1$ and $V_2$, where $V_i$ is the normal closure of $Y_i$ in a particularly chosen subgroup $L_i$.

The arguments used in the short asymmetric case are related to those used in the qrc-Lemma from [**MS4**].

The remaining case, the tall asymmetric case, is by far the hardest one. Here $Y_M$ is asymmetric, and there exists $P \leqslant G$ with $O_p(M) \leqslant P$, $O_p(P) \neq 1$ and $Y_M \nleqslant O_p(P)$. First of all, it may be that all such subgroups $P$ are not of characteristic $p$, in our notation, that $Y_M$ is tall but not *char* $p$-tall. The short Chapter 6 partially handles this case by showing that this cannot happen if $C_G(x)$ has characteristic $p$ for all $1 \neq x \in Y_M$.

Suppose that $Y_M$ is *char* $p$-tall. Then the Asymmetric L-Lemma 2.16 can be applied and provides us with a subgroup $L$ of characteristic $p$ such that $Y_M \leqslant L$ and $L/O_p(L) \cong SL_2(q)$, $Sz(q)$, or $D_{2r}$, where $p = 2$ in the last two case and $r$ is an odd prime, and $q = |Y_M/Y_M \cap O_p(L)|$.

It turns out that $\Omega_1 Z(O_p(L))$ is a non-trivial strong offender on $Y_M$ or $L$ normalizes a conjugate of $Q$. In the first case we can use the FF-Module Theorems from Appendix C; in the second case we show that $O_p(L)$ acts as a (non-trivial) nearly quadratic $2F$-offender on $Y_M$, and then [**MS2**] and the 2F-Module Theorems of Guralnick and Malle are the main tools in the investigation.

For more details see the introductions to Chapters 4 – 9.

In earlier publications the Local Structure Theorem is quoted under the name "Structure Theorem". In [**PPS**] the following earlier (weaker) version of the Local Structure Theorem was used, except that we correct a misprint , it should read $F^*(\overline{M}_0)$ rather than $F^*(\overline{M})$, and we added property (1:i) for better understanding.

Corollary B. *Let $G$ be a finite $\mathcal{K}_p$-group of local characteristic $p$ and $S \in Syl_p(G)$. Suppose that there exist $M$, $\widetilde{C} \in \mathcal{M}_G(S)$ such that the following hold for $Q := O_p(\widetilde{C})$:*

(i) $N_G(\Omega_1 Z(S)) \leqslant \widetilde{C}$.
(ii) $C_G(x) \leqslant \widetilde{C}$ *for every* $1 \neq x \in Z(Q)$.
(iii) $M \neq \widetilde{C}$, *and* $M = L$ *for every* $L \in \mathcal{M}_G(S)$ *with* $M = (M \cap L)C_M(Y_M)$.
(iv) $Y_M \leqslant Q$.

*Then for $M_0 := \langle Q^M \rangle C_S(Y_M)$ and $\overline{M} := M/C_M(Y_M)$ one of the following holds:*

(1) $F^*(\overline{M_0}) = \overline{M_0}'$, $\overline{M_0} \cong SL_n(p^m)$, $n \geqslant 2$, $Sp_{2n}(p^m)$, $n \geqslant 2$, *or* $Sp_4(2)'$ *(and* $p = 2$*), and* $[Y_M, M_0]$ *is a corresponding natural module for* $\overline{M_0}$*. Moreover,*
   (i) $Y_M = [Y_M, M_0]$ *or* $p = 2$ *and* $\overline{M_0} \cong Sp_{2n}(q)$, $n \geqslant 2$, *and*
   (ii) *either* $C_{M_0}(Y_M) = O_p(M_0)$, *or* $p = 2$ *and* $M_0/O_p(M_0) \cong 3^{\cdot}Sp_4(2)'$.

(2) $P_1 := M_0S \in \mathcal{P}_G(S)$, $Y_M = Y_{P_1}$, *and there exists a normal subgroup* $P_1^* \leqslant P_1$ *containing* $C_{P_1}(Y_{P_1})$ *but not* $Q$ *such that*
   (i) $\overline{P_1^*} = K_1 \times \cdots \times K_r$, $K_i \cong SL_2(p^m)$, $Y_M = V_1 \times \cdots \times V_r$, *where* $V_i := [Y_M, K_i]$ *is a natural* $K_i$*-module,*
   (ii) $Q$ *permutes the components* $K_i$ *of (i) transitively,*
   (iii) $O^p(P_1^*) = O^p(M_0)$, *and* $P_1^* C_M(Y_M)$ *is normal in* $M$,
   (iv) $C_{P_1}(Y_{P_1}) = O_p(P_1)$, *or* $r > 1$, $K_i \cong SL_2(2)$ *(and* $p = 2$*) and* $C_{P_1}(Y_{P_1})/O_2(P_1)$ *is a* 3*-group.*

The proof of this Corollary to the Local Structure Theorem is contained in Chapter 10.

We assume the reader to be familiar with the basic concepts of finite group theory, for example coprime action, components and the generalized Fitting subgroup. In addition, in Chapters 9 and 10 we assume basic knowledge of the parabolic subgroups of groups of Lie type and the sporadic simple groups and their action on some low dimensional modules. Most of this information can be found in [**Ca**], [**RS**] and [**MSt**]. Note also that the action of $\Omega_{10}^+(q)$ on the half spin modules and the action of $E_6(q)$ on the 27-dimensional modules can be seen inside the groups $E_6(q)$ and $E_7(q)$, respectively.

**Acknowledgement:** We would like to thank the referee for pointing out numerous inconveniences in the original manuscript which encouraged us to add more details and information for the reader.

CHAPTER 1

# Definitions and Preliminary Results

In this chapter we provide elementary group theoretic results needed in this paper. Some of them already indicate the kind of technical tools used throughout this paper.

In Section 1.2 some properties of $p$-reduced normal $p$-subgroups are given, since $p$-reduced subgroups are the typical modules for parabolic subgroups investigated in this paper. In Sections 1.3 and 1.4 we discuss $p$-irreducible and $Y$-minimal groups. They naturally occur as subgroups of $p$-local subgroups and belong to our most important tools.

In Section 1.6 we have a first look at large $p$-subgroups. In particular, we show that such subgroups are weakly-closed. Consequently, in Section 1.5 weakly closed subgroup are investigated.

Throughout this chapter $H$ always denotes a finite group and $p$ is a prime.

## 1.1. Elementary Properties of Finite Groups

DEFINITION 1.1. (a) $H$ is *$p$-irreducible* if $H$ is not $p$-closed and $O^p(H) \leqslant N$ for any normal subgroup $N$ of $H$ which is not $p$-closed.

(b) $H$ is *strongly $p$-irreducible* if $H$ is not $p$-closed and $O^p(H) \leqslant N$ for every normal subgroup $N$ of $H$ with $[N, H] \nleqslant O_p(H)$.

(c) $Z_H := \langle \Omega_1 Z(T) \mid T \in Syl_p(H) \rangle$.

(d) $\mathcal{A}_H$ is the set of elementary abelian $p$-subgroups of $H$ of maximal order, $J(H) := \langle \mathcal{A}_H \rangle$ is the *Thompson subgroup* of $H$ and

$$B(H) := \begin{cases} C_H(\Omega_1 Z(J(H))) & \text{if } H \text{ is a } p\text{-group} \\ \langle B(T) \mid T \in Syl_p(H) \rangle & \text{in general} \end{cases}$$

is the *Baumann subgroup* of $H$.

(e) Let $R \leqslant T \leqslant H$. Then $R$ is *weakly closed* in $T$ with respect to $H$ if $R$ is the only $H$-conjugate of $R$ contained in $T$; and a $p$-subgroup $R$ is a *weakly closed subgroup* of $H$ if $R$ is weakly closed with respect to $H$ in some Sylow $p$-subgroup of $H$.

(f) Let $A \leqslant H$. The *subnormal closure* of $A$ in $H$ is the intersection of all subnormal subgroups of $H$ containing $A$.

LEMMA 1.2. *Let $H$ be a finite group of characteristic $p$ and $T \in Syl_p(H)$. Then the following hold:*

(a) *Every subnormal subgroup of $H$ has characteristic $p$.*

(b) *Every subgroup containing $T$ has characteristic $p$.*

(c) *$H$ has local characteristic $p$.*

PROOF. [**MS6**, 1.2]. □

LEMMA 1.3. *Let $R$ be a $p$-subgroup of $H$ with $C_H(R) \leqslant R$. Let $L \leqslant N_G(R)$ and suppose that $L$ acts nilpotenly on $R$. Then $L$ is a $p$-group.*

PROOF. Since $L$ act nilpotenly in the $p$-group $R$, coprime actions shows that $O^p(L)$ centralizes $R$. By hypothesis, $C_H(R) \leqslant O_p(R)$. So $O^p(L) \leqslant R$ and $O^p(L)$ is $p$-group. Thus $O^p(L) = 1$, and $L$ is a $p$-group. □

LEMMA 1.4. *Let $L \leqslant H$. Suppose that $H$ has characteristic $p$.*

(a) *Suppose that $L$ acts nilpotently on $O_p(H)$. Then $L$ is a $p$-group.*
(b) *Suppose that $L \trianglelefteq\trianglelefteq H$ and $L$ acts nilpotently on $O_p(H)$. Then $L \leqslant O_p(H)$.*
(c) *Suppose that $L$ centralizes the factors of an $H$-invariant series*

$$1 = A_0 \leqslant A_1 \leqslant \ldots \leqslant A_{n-1} \leqslant A_n = O_p(H).$$

*Then $L \leqslant O_p(H)$.*

PROOF. (a): Since $H$ has characteristic $p$, $C_H(O_p(H)) \leqslant O_p(H))$. Thus 1.3 applied with $R = O_p(H)$ shows that $L$ is a $p$-group.

(b): By (a) $L$ is a $p$-group and since $L \trianglelefteq\trianglelefteq H$, this gives $L \leqslant O_p(H)$.

(c): Since $L$ centralizes $A_i/A_{i-1}$ and $H$ acts on $A_i/A_{i-1}$ also $\langle L^H \rangle$ centralizes $A_i/A_{i-1}$. Thus $\langle L^H \rangle$ acts nilpotently on $O_p(H)$, and (b) implies that $\langle L^H \rangle \leqslant O_p(H)$. □

LEMMA 1.5. *Suppose that $C_H(Y)$ has characteristic $p$ for some $Y \leqslant O_p(H)$. Then $H$ is of characteristic $p$.*

PROOF. Put $D := C_H(O_p(H))$. Note that $D \trianglelefteq H$ and since $Y \leqslant O_p(H)$, $D \leqslant C_H(Y)$. Thus

$$[O_p(C_H(Y)), D] \leqslant D \cap O_p(C_H(Y)) \leqslant O_p(D) \leqslant O_p(H) \leqslant C_H(D).$$

In particular, $[O_p(C_H(Y)), D, D] = 1$ and $D$ acts nilpotently on $O_p(C_H(Y))$. By hypothesis $C_H(Y)$ has characteristic $p$, and since $D \leqslant C_H(Y)$, 1.4 shows that $D$ is a $p$-group. Since $D \trianglelefteq H$ this gives $D \leqslant O_p(H)$, and so $H$ has characteristic $p$. □

LEMMA 1.6. *Let $M \in \mathcal{L}_H$ and $K \leqslant M$ with $O_p(M) \leqslant K$. Then*

$$\mathcal{L}_M(K) = \{L \mid K \leqslant L \leqslant M\} = \{L \in \mathcal{L}_H(K) \mid L \leqslant M\}.$$

PROOF. Let $L \leqslant H$.

Suppose that $L \in \mathcal{L}_M(K)$. Then $K \leqslant L \leqslant M$ by the definition of $\mathcal{L}_M(K)$.

Suppose that $K \leqslant L \leqslant M$. Then $O_p(M) \leqslant K \leqslant L$ and since $L \leqslant M$, $O_p(M) \leqslant O_p(L)$. Thus $C_H(O_p(L)) \leqslant C_H(O_p(M))$. Since $M \in \mathcal{L}_H$, $C_H(O_p(M)) \leqslant O_p(M)$ and so $C_H(O_p(L)) \leqslant O_p(M) \leqslant O_p(L)$. Hence $L \in \mathcal{L}_H(K)$ and $L \leqslant M$.

Suppose that $L \in \mathcal{L}_H(K)$ and $L \leqslant M$. Then $C_M(O_p(L)) \leqslant C_H(O_p(L)) \leqslant O_p(L)$ and so $L \in \mathcal{L}_M(K)$. □

LEMMA 1.7. (a) *Suppose that $O_p(H) = 1$. Then $\Phi(H) = \Phi(O^p(H))$.*
(b) *Suppose that $H = O^{p'}(H)$ and $O_p(H) = 1$. Then $Z(H) \leqslant \Phi(H)$.*

PROOF. (a): This the case $\pi = \{p\}$ of [**MS6**, 1.9].

(b): Since $O_p(H) = 1$, $Z(H)$ is a $p'$-group. Let $M \leqslant H$ such that $H = MZ(H)$. Then $M \trianglelefteq H$ and $H/M$ is a $p'$-group. As $H = O^{p'}(H)$ we get $H = M$. This shows that $Z(H)$ is contained in every maximal subgroup of $H$ and so $Z(H) \leqslant \Phi(H)$. □

LEMMA 1.8. *Let $Y$ be a finite $p$-group acting on $H$ and $L$ a $Y$-invariant subnormal subgroup of $F^*(H)$. Suppose that $O_p(H) = 1$.*

(a) $L = [L, Y]C_L(Y)$,
(b) $[L, Y] = [L, Y, Y]$.

PROOF. It is evident that (a) implies (b). Thus, it suffices to show (a).

Set $L_0 := [L, Y]C_L(Y)$. Note that $L \trianglelefteq\trianglelefteq H$ and so by [**KS**, 6.5.7b], $F^*(L) = F^*(H) \cap L = L$ and so $L = F(L)E(L)$, where $E(L)$ is the subgroup generated by the components of $L$. $O_p(F(L)) \leqslant O_p(H) = 1$ and $F(L)$ is nilpotent, $F(L)$ is a $p'$-group. Hence, the properties of coprime action show that $F(L) \leqslant L_0$.

Let $K$ be a component of $L$. Then $[Y, K] \trianglelefteq \langle K^Y \rangle \trianglelefteq\trianglelefteq L$, so by a fundamental property of components either $K \leqslant [Y, K]$ or $[Y, K, K] = 1$ (see for example 6.5.2 in [**KS**]). In the first case $K \leqslant L_0$, in the second case with the Three Subgroup Lemma $[Y, K] = 1$ since $K$ is perfect. Thus, also in this case $K \leqslant H_0$, and (a) follows. □

LEMMA 1.9. *Let $K$ be subgroup of $H$ with $O_p(K) = 1$ and $Y$ be a $p$-subgroup of $N_H(K)$ with $[K, Y, Y] = 1$. Then $[K, Y] = 1$.*

PROOF. Since $[K, Y, Y] = 1$,

$$Y \trianglelefteq Y[K, Y] = \langle Y^K \rangle = \langle Y^{YK} \rangle \trianglelefteq KY$$

and so $Y \leqslant O_p(KY)$. Thus

$$[K, Y] \leqslant O_p(KY) \cap K \leqslant O_p(K) = 1.$$

□

LEMMA 1.10. *Let $Y$ be a finite $p$-group acting on $H$, and let $A$ and $B$ be normal subgroups of $Y$. Suppose that $O_p(H) = 1$ and $[F^*(H), A, B] \neq 1$. Let $X$ be a $Y$-invariant subnormal subgroup of $F^*(H)$ minimal with respect to $[X, A, B] \neq 1$. Then*

$$X = [X, A] \quad \text{and} \quad X = [X, B].$$

PROOF. By 1.8(b) applied to $(X, A)$ in place of $(L, Y)$ we have $[X, A, A] = [X, A]$. Hence $[X, A, A, B] = [X, A, B] \neq 1$. So the minimal choice of $X$ gives $X = [X, A]$.

Suppose that $[X, A \cap B] \neq 1$. Then 1.8(b) applied with $Y = A \cap B$ shows

$$1 \neq [X, A \cap B] = [X, A \cap B, A \cap B] = [X, A \cap B, A \cap B, A \cap B] \leqslant [X, A \cap B, A, B].$$

Thus the minimal choice of $X$ implies that $X = [X, A \cap B]$ and so also $X = [X, B]$.

Suppose next that $[X, A \cap B] = 1$. Since $[A, B] \leqslant A \cap B$ this gives $[A, B, X] = 1$. Since $[X, A, B] \neq 1$, the Three Subgroups Lemma shows that $[X, B, A] \neq 1$. As above, 1.8(b) gives $[X, B, B] = [X, B]$ and so $[X, B, B, A] = [X, B, A] \neq 1$. Since $[A, B, [X, B]] \leqslant [[A, B], X] = 1$ another application of the Three Subgroups Lemma yields $[X, B, A, B] \neq 1$ and the minimal choice of $X$ implies $X = [X, B]$. □

LEMMA 1.11. *Let $A, B, K \leqslant H$ with $A = [A, B]$ and $B \leqslant K \trianglelefteq\trianglelefteq H$. Then $A \leqslant K$.*

PROOF. If $K = H$ the claim is obvious. In the other case there exists $L \trianglelefteq H$ such that $K \leqslant L \neq H$ since $K \trianglelefteq\trianglelefteq H$. Hence $A = [A, B] \leqslant [A, L] \leqslant L$. Since also $K \leqslant L$, we conclude that $A \leqslant K$ by induction on $|H|$. □

LEMMA 1.12. *Let $H$ be a group and $\mathcal{G}$ a function which assigns to each subgroup $X$ of $H$ a $N_H(X)$-invariant subgroup $\mathcal{G}(X)$ of $H$ such that $\mathcal{G}(X) \leqslant \mathcal{G}(Y)$ whenever $X \leqslant Y \leqslant H$.*

*Let $A \trianglelefteq\trianglelefteq B \leqslant H$ and suppose that $\mathcal{G}(A) = \mathcal{G}(C)$ for some $C \leqslant H$ with $N_H(\mathcal{G}(A)) \leqslant C$. Then $\mathcal{G}(A) = \mathcal{G}(B)$.*

PROOF. By induction on the subnormal length of $A$ in $B$ we may assume that $A \trianglelefteq B$. Then

$$B \leqslant N_H(A) \leqslant N_H(\mathcal{G}(A)) \leqslant C.$$

Thus $A \leqslant B \leqslant C$ and

$$\mathcal{G}(A) \leqslant \mathcal{G}(B) \leqslant \mathcal{G}(C) = \mathcal{G}(A).$$

□

LEMMA 1.13. *Let $A \leqslant H$ and $K$ be the subnormal closure of $A$ in $H$.*

(a) *$K = \langle A^K \rangle$ and $N_H(A) \leqslant N_H(K)$.*
(b) *$K = AO^p(K) = \langle A^{O^p(K)} \rangle$.*
(c) *If $A$ is a $p$-group, then $O^p(K) = [O^p(K), A]$.*

PROOF. (a): Note that $A \leqslant \langle A^K \rangle \trianglelefteq K \trianglelefteq\trianglelefteq H$ and so $K = \langle A^K \rangle$ by the minimality of $K$. The second statement should be evident.

(b): Note that $K/O^p(K)$ is a $p$-group, and so $AO^p(K)/O^p(K)$ is subnormal in $K/O^p(K)$. Hence $A \leqslant AO^p(K) \trianglelefteq\trianglelefteq K \trianglelefteq\trianglelefteq H$ and $K = AO^p(K)$ by minimality of $K$. Thus using (a)

$$K = \langle A^K \rangle = \langle A^{AO^p(K)} \rangle = \langle A^{O^p(K)} \rangle.$$

(c): By (c), $K = \langle A^{O^p(K)} \rangle = [O^p(K), A]A$. If $A$ is a $p$-group, then $O^p(K) \leqslant [O^p(K), A]$, and (c) holds. □

LEMMA 1.14. *Put $K := O^p(H)$. Suppose that $O_p(H) = 1$ and $K$ is quasisimple. Then*

(a) *$K = F^*(H)$.*
(b) *$K = [K, Y] \leqslant \langle Y^K \rangle$ for all non-trivial $p$-subgroups $Y$ of $H$.*
(c) *If $C \trianglelefteq H$ with $K \nleqslant C$, then $C$ is a $p'$-group. In particular, $H$ is $p$-irreducible.*

PROOF. (a): Note that $K$ is a component of $H$ and so $K \leqslant F^*(H)$. Since $O_p(H) = 1$, $F(H)$ is a $p'$-group. Thus $F^*(H) = F(H)E(H) \leqslant O^p(H) = K$ and so $K = F^*(H)$.

(b): In particular, $C_H(K) \leqslant Z(K)$ and so $C_H(K)$ is a $p'$-group. Thus $[Y, K] \neq 1$, and since $K$ is perfect, $[Y, K, K] \neq 1$. Hence $[Y, K] \nleqslant Z(K)$, and since $K$ is quasisimple, $K = [Y, K] \leqslant \langle Y^K \rangle$.

(c) follows immediately from (b). □

LEMMA 1.15. *Suppose that $O_p(H) = 1$, and let $Y$ be a $p$-subgroup of $H$. Then*

(a) *$[F^*(H), Y] = [F^*(K), Y] = [F^*(K), Y, Y]$ for every $K \trianglelefteq\trianglelefteq H$ with $Y \leqslant K$,*
(b) *If $[F^*(H), Y] = 1$ then $Y = 1$,*
(c) *If $Y_0 \leqslant Y$ with $[F^*(H), Y, Y_0] = 1$ then $Y_0 = 1$.*

PROOF. (a): Since $O_p(H) = 1$, 1.8(b) gives $[F^*(H), Y] = [F^*(H), Y, Y]$. Hence 1.11 implies $[F^*(H), Y] \leqslant K$ and so $[F^*(H), Y] \leqslant F^*(H) \cap K$. By [**KS**, 6.5.7b], $F^*(H) \cap K = F^*(K)$. Thus

$$[F^*(H), Y] = [F^*(H), Y, Y] \leqslant [F^*(K), Y] \leqslant [F^*(H), Y],$$

and (a) holds.

(b): Since $C_H(F^*(H)) \leqslant F^*(H)$, $[F^*(H), Y] = 1$ implies $Y \leqslant O_p(Z(F^*(H))) \leqslant O_p(H) = 1$.

(c): Note that $[F^*(H), Y_0, Y_0] \leqslant [F^*(H), Y, Y_0] = 1$. On the other hand, by 1.8, $[F^*(H), Y_0] = [F^*(H), Y_0, Y_0]$, so $[F^*(H), Y_0] = 1$, and (b) gives $Y_0 = 1$. □

LEMMA 1.16. *Let $N$ and $E$ be subnormal subgroups of $H$. Suppose that $E$ is a direct product of perfect simple groups. Then*

$$[N, E] = 1 \quad \Longleftrightarrow \quad [F^*(N), E] = 1 \quad \Longleftrightarrow \quad N \cap E = 1.$$

PROOF. Note that $F(E) = 1$ and $E$ is generated by its components. If $[N, E] = 1$, then also $[F^*(N), E] = 1$.

Suppose that $[F^*(N), E] = 1$. Since $F^*(N \cap E) \leqslant F^*(E) \cap N$ we conclude that $F^*(N \cap E)$ is abelian. Hence $F^*(N \cap E) = F(N \cap E) \leqslant F(E) = 1$ and so also $N \cap E = 1$.

Suppose that $N \cap E = 1$, and let $K$ be a component of $E$. Then $N \cap K = 1$ and so by [**KS**, 6.5.2], $[N, K] = 1$. Since $E$ is generated by its components, this gives $[N, E] = 1$. □

LEMMA 1.17. *Suppose that $O_p(H) = 1$. Let $Q$ be a $p$-subgroup of $H$, put $L := [F^*(H), Q]$, and let $F$ be the largest normal subgroup of $F^*(H)$ centralized by $Q$. Then the following hold:*

(a) *$F = C_{F^*(H)}(LQ)$.*
(b) *$L = [L, Q]$.*
(c) *$L \cap F \leqslant \Phi(L)$.*
(d) *If $B \leqslant N_H(Q)$ is a $p$-subgroup with $[L, B] \leqslant F$, then $[L, B] = 1$.*
(e) *$C_H(FL)$ is a $p'$-group.*

PROOF. (a): Note that $LQ = [F^*(H), Q]Q = \langle Q^{F^*(H)} \rangle$. Since $F \trianglelefteq F^*(H)$ and $[F, Q] = 1$ we conclude that $F \leqslant C_{F^*(H)}(LQ)$. On the other hand $C_{F^*(H)}(LQ)$ is a normal subgroup of $F^*(H)$ centralized by $Q$ and so $C_{F^*(H)}(LQ) \leqslant F$.

(b): Since $O_p(H) = 1$ we can apply 1.8(b) and conclude that $[F^*(H), Q] = [F^*(H), Q, Q]$. Thus (b) holds

(c): Let $N$ be a subgroup of $L$ with $L = N(L \cap F)$. It suffices to show that $N = L$. By (a) $L = NZ(L)$ and thus $L' = N'$. As $L \trianglelefteq F^*(H)$ and $O_p(H) = 1$, $O^p(L) = L$ and $L/L'$ is a $p'$-group. By (b) $[L/L', Q] = L/L'$ and since $L/L'$ is a $p'$-group we get $C_{L/L'}(Q) = 1$. So $F \cap L \leqslant L' \leqslant N$ and $N = L$.

(d): By hypothesis, $B \leqslant N_H(Q)$ and $[L, B] \leqslant F$. Hence $B$ normalizes $[F^*(H), Q] = L$ and $[L, B] \leqslant L \cap F$. By (c) $L \cap F \leqslant \Phi(L)$. Since $L \trianglelefteq F^*(H)$ we get from 1.8(a) that $L = [L, B]C_L(B)$. Thus $L = \Phi(L)C_L(B)$ and so $L = C_L(B)$.

(e): Observe that

$$[F^*(H), C_H(FL)] \leqslant F^*(H) \cap C_H(FL) =: F_0$$

Since $FL \trianglelefteq F^*(H)$ also $F_0 \trianglelefteq F^*(H)$. Hence 1.8(a) gives $F_0 = [F_0, Q]C_{F_0}(Q)$. Note that $[F_0, Q] \leqslant [F^*(H)Q] \leqslant L$ and $C_{F_0}(Q) = C_{F_0}(LQ)$. By (a), $C_{F_0}(LQ) \leqslant F$, so $F_0 \leqslant LF$. It follows that $[F^*(H), C_H(FL), C_H(FL)] = 1$. Let $Y$ be a $p$-subgroup of $C_H(FL)$. Then $[F^*(H), Y, Y] = 1$. Now 1.15(c) gives $Y = 1$ since $O_p(H) = 1$. Hence $C_H(FL)$ is a $p'$-group. □

LEMMA 1.18. *Suppose that $H$ acts on the finite $p$-group $P$ and $[P, H] \leqslant \Omega_1 Z(P)$. Then $[\Phi(P), H] = 1$.*

PROOF. Since $[P, H, P] = 1$, the Three Subgroups Lemma shows that $[P', H] = 1$, and since $[P, H]$ is elementary abelian and central in $P$,

$$(a^p)^h = (a^h)^p = (a[a, h])^p = a^p[a, h]^p = a^p \qquad \text{for all } a \in P \text{ and } h \in H,$$

and $[P^p, H] = 1$. By [**KS**, 5.2.8], $\Phi(P)$ is the smallest normal subgroup of $P$ that has elementary abelian factor group, so $\Phi(P) = P'P^p$, and the lemma follows. □

LEMMA 1.19. *Suppose that $H$ acts on a finite $p$-group $P$. Let $Y \leqslant C_H(P')$ such that $[P, Y]$ is elementary abelian. Then $O^p(\langle Y^H \rangle)$ centralizes $\Phi(P)$.*

PROOF. Put $\overline{P} = P/P'$ and $L = \langle Y^H \rangle$. Since $\overline{P}$ is abelian and $[P, Y]$ is elementary abelian we have $[\overline{P}, L] = \langle [\overline{P}, Y]^H \rangle \leqslant \Omega_1 Z(\overline{P})$. Thus by 1.18, $[\Phi(\overline{P}), L] = 1$. Note that $L$ centralizes $P'$ since $Y$ does. Since $\Phi(\overline{P}) = \Phi(P)/P'$ we conclude that $[\Phi(P), L, L] = 1$ and thus $[\Phi(P), O^p(L)] = 1$. □

LEMMA 1.20. *Let $A$ and $B$ be subgroups of $H$. Then $C_A(b) = C_A(B)$ for all $b \in B \backslash C_B(A)$ if and only if $C_B(a) = C_B(A)$ for all $a \in A \backslash C_A(B)$.*

PROOF. Both statements just say that $[a, b] \neq 1$ for all $a \in A \backslash C_A(B)$ and $b \in B \backslash C_B(A)$. □

For the next lemma recall from A.7(5) that $W$ is a root offender on $V$ if $W$ is an offender on $V$ and

$$C_V(W) = C_V(w) \quad \text{and} \quad [V, w] = [V, W] \qquad \text{for every } w \in W \backslash C_W(V).$$

LEMMA 1.21. *Let $V$ and $W$ be elementary abelian $p$-subgroups of $H$ with $[V, W] \leqslant V \cap W$. Then $V$ is a root offender on $W$ if and only if $W$ is a root offender on $V$.*

PROOF. We may assume that $W$ is a root offender on $V$. Then by A.37(a) $|V/C_V(W)| = |W/C_V(W)|$, and so $V$ is an offender on $W$. By A.37(b) $W$ is a strong dual offender on $V$. Hence $[v, W] = [V, W]$ for all $v \in V \backslash C_V(W)$. Moreover, by definition of a root offender $C_V(W) = C_V(w)$ for all $w \in W \backslash V$, and so by 1.20 also $C_W(V) = C_W(v)$ for all $v \in V \backslash C_V(W)$. Thus $V$ is a root offender on $W$. □

LEMMA 1.22. *Let $V_1$ and $V_2$ be elementary abelian $p$-subgroups of $H$ with $[V_1, V_2] \leqslant V_1 \cap V_2$, and let $\mathbb{K}_i$ is a subfield of $End_{\mathbb{F}_p}(V_i)$ with $|\mathbb{K}_i| > p$, $i = 1, 2$. Suppose that*

(i) *$V_j$ acts $\mathbb{K}_i$-semilinearly on $V_i$ for all $\{i, j\} = \{1, 2\}$.*
(ii) *$V_2$ does not act $\mathbb{K}_1$-linearly on $V_1$.*

*Then $p = 2$ and, for or all $\{i, j\} = \{1, 2\}$, $|\mathbb{K}_i| = |V_i| = 4$, $\dim_{\mathbb{K}_i} V_i = 1$, $|V_i/C_{V_i}(V_j)| = 2$, and $V_j$ does not act $\mathbb{K}_i$-linearly on $V_i$.*

PROOF. Let $\{i, j\} = \{1, 2\}$ and put $W_j := C_{V_j}(\mathbb{K}_j)$, so $W_j$ is the largest subgroup of $V_j$ acting $\mathbb{K}_i$-linearly on $V_i$ and $V_j/W_j$ is isomorphic to subgroup of $Aut(\mathbb{K}_i)$. Since $Aut(\mathbb{K}_i)$ is cyclic and $V_j$ is elementary abelian, we conclude that $|V_j/W_j| \leqslant p$. By hypothesis, $V_2$ does not act $\mathbb{K}_1$-linearly in $V_1$, so $|V_1/W_1| = p$. Note that $[V_1, W_2] \leqslant [V_1, V_2] \leqslant V_1 \cap V_2 \leqslant C_{V_1}(V_2)$. Since $W_2$ acts $\mathbb{K}_1$-linearly on $V_1$, $[V_1, W_2]$ is a $\mathbb{K}_1$-subspace of $V_1$ centralized by $V_2$. Since $V_2$ does not act $K_1$-linearly on $V_1$, this shows that $[V_1, W_2] = 1$. Observe that $C_{V_2}(V_1) \leqslant W_2$, so $W_2 = C_{V_1}(V_2)$. Thus $|V_2/C_{V_2}(V_1)| = |V_2/W_2| = p$.

Let $\mathbb{E}_i := C_{\mathbb{K}_i}(V_j)$, so $\mathbb{E}_i$ is the largest subfield of $\mathbb{K}_i$ such that $V_j$ acts $\mathbb{E}_i$-linearly on $V_i$. Then $C_{V_2}(V_1)$ is an $\mathbb{E}_2$-subspace of $V_2$, so $V_2/C_{V_2}(V_1)$ is an $\mathbb{E}_2$-space. As $|V_2/C_{V_2}(V_1)| = p$, this shows that $|\mathbb{E}_2| = p$. Since $|\mathbb{K}_2| > p$, we infer $\mathbb{E}_2 \neq \mathbb{K}_2$. So also $V_1$ does not act $\mathbb{K}_2$-linearly on $V_2$, and the setup is symmetric in 1 and 2. In particular, also $p = |V_1/W_1| \leqslant |Aut(\mathbb{K}_2)|$.

Note that any $\mathbb{E}_2$-hyperplane of $V_2$ contains a $\mathbb{K}_2$-hyperplane of $V_2$. In particular, $C_{V_2}(V_1)$ contains a $\mathbb{K}_2$-hyperplane $H_2$. As $V_1$ centralizes $H_2$ and does not act $\mathbb{K}_2$-linearly, we conclude that $H_2 = 1$. So $\dim_{\mathbb{K}_2} V_2 = 1$. In particular, the action of $V_1$ on $V_2$ is isomorphic to the action on $V_2$ on $\mathbb{K}_2$. It follows that $|C_{V_2}(V_1)| = |C_{\mathbb{K}_2}(V_1)| = |\mathbb{E}_2| = p$. As $|V_2/C_{V_2}(V_1)| = p$ this gives $|\mathbb{K}_2| = |V_2| = p^2$, so $|Aut(\mathbb{K}_2)| = 2$ and $p = 2$. By symmetry, $|\mathbb{K}_1| = |V_1| = p^2 = 4$, and the lemma is proved. □

LEMMA 1.23. *Let $\pi$ be a set of primes, and let $A$ and $B$ be subnormal subgroups of $H$. Suppose that $A$ is a $\pi$-group and $B = O^{\pi}(B)$. Then $A$ normalizes $B$ and $B = O^{\pi}(AB)$.*

PROOF. If $B = H$ then the claim is obvious. Assume that $B \neq H$. Then there exists $N \trianglelefteq H$ such that $B \leqslant N$ and $N \neq H$. As $A \trianglelefteq\trianglelefteq H$, we have $A \leqslant O_{\pi}(H) \trianglelefteq H$ and so $[A, B]$ is a $\pi$-group. Since $[A, B] \trianglelefteq \langle A, B\rangle \trianglelefteq\trianglelefteq H$ and $[A, B] \leqslant N$, we get that $[A, B]$, $B$ and $N$ satisfy the hypothesis in place of $A$, $B$ and $H$. Hence by induction on $|H|$, $B = O^{\pi}([A, B]B)$. Since $[A, B]B = \langle B^A\rangle$ is normalized by $A$, we conclude that $A$ normalizes $B$. Thus $AB/B$ is a $\pi$-group and so

$$O^{\pi}(B) \leqslant O^{\pi}(AB) \leqslant B = O^{\pi}(B).$$

□

## 1.2. The Largest $p$-Reduced Elementary Abelian Normal Subgroup

LEMMA 1.24. *Let $T \in Syl_p(H)$ and $L, M \leqslant H$ with $T \leqslant L \cap M$. Suppose that $L$ and $M$ are of characteristic $p$ and put $T_0 := C_T(Y_L)$ and $L_0 := N_L(T_0)$.*

(a) *$T_0 \in Syl_p(C_H(Y_L))$.*
(b) *Suppose that $LC_H(Y_M) = MC_H(Y_M)$. Then $Y_M \leqslant Y_L$.*
(c) *If $L \leqslant M$ and $Y$ is a $p$-reduced elementary abelian normal subgroup of $L$, then $\langle Y^M\rangle$ is a $p$-reduced elementary abelian normal $p$-subgroup of $M$.*
(d) *$Z_L$ is a $p$-reduced elementary abelian normal $p$-subgroup of $L$.*
(e) *$Z_L = \Omega_1 Z(L)[Z_L, O^p(L)]$ and $[Z_L, L] = [Z_L, O^p(L)]$.*
(f) *If $L \leqslant M$, then $Y_L \leqslant Y_M$.*
(g) *$O_p(L) \leqslant T_0 \leqslant C_L(Y_L)$ and $\Omega_1 Z(T) \leqslant Z_L \leqslant Y_L \leqslant \Omega_1 Z(O_p(L))$.*

(h) *Suppose that* $L \leqslant M$ *and* $M \subseteq LC_H(Y)$ *for some* $Y \leqslant H$ *with* $Y_M \leqslant Y$. *Then* $Y_L = Y_M$ *and* $LC_H(Y_L) = MC_H(Y_L)$.

(i) $L = L_0C_L(Y_L)$, $T_0 = O_p(L_0)$, $C_T(T_0) \leqslant T_0$, *and* $Y_L = \Omega_1 Z(T_0) = Y_{L_0}$.

(j) *Suppose that* $MC_H(Y_L) = LC_H(Y_L)$ *and put* $L^* = N_M(T_0)$. *Then* $Y_{L^*} = Y_L$ *and* $LC_H(Y_L) = L^*C_H(Y_L)$.

(k) *If* $C_L(Y_L)$ *is p-closed, then* $Y_L = \Omega_1 Z(O_p(L))$ *and* $O_p(L) = T_0 \in Syl_p(C_H(Y_L))$.

Proof. Note first that $Y_M \leqslant O_p(M) \leqslant T \leqslant L$. For the definition of a $p$-reduced module and nilpotent action see Definition A.4.

(a): Since $T \in Syl_p(H)$ and $T \leqslant L \leqslant N_H(Y_L)$, $T \in Syl_p(N_H(Y_L))$, and since $C_H(Y_L) \trianglelefteq N_H(Y_L)$ we conclude that $T_0 = C_T(Y_L) = T \cap C_H(Y_L) \in Syl_p(C_H(Y_L))$.

(b): Observe that $L$ normalizes $Y_M$ and since $Y_M \leqslant L$, $Y_M \leqslant O_p(L)$. We have

$$O_p(L/C_L(Y_M)) \cong O_p(LC_H(Y_M)/C_H(Y_M)) = O_p(MC_H(Y_M)/C_H(Y_M)) \cong O_p(M/C_M(Y_M)) = 1,$$

and so $Y_M$ is p-reduced for $L$. Thus $Y_M \leqslant Y_L$.

(c): This is [**MS4**, (2.2)(b)].

(d): Note that $\Omega_1 Z(T)$ is $p$-reduced for $T$, $T \leqslant L$ and $Z_L = \langle \Omega_1 Z(T)^L \rangle$. So (d) follows from (c).

(e): Note that $L$ normalizes $\Omega_1 Z(T)[Z_L, L]$. Since $Z_L = \langle \Omega_1 Z(T)^L \rangle$ we get $Z_L = \Omega_1 Z(T)[Z_L, L]$. Hence Gaschütz's Theorem gives $Z_L = C_{Z_L}(L)[Z_L, L] = \Omega_1 Z(L)[Z, L]$, see C.17. This implies $[Z_L, L] = [Z_L, L, L]$ and so $[Z_L, L] = [Z_L, O^p(L)]$, and (e) is proved.

(f): This is [**MS4**, (2.2)(c)].

(g): By the definition of $Z_L$ we have $\Omega_1 Z(T) \leqslant Z_L$. By (d), $Z_L$ is $p$-reduced for $L$ and so $Z_L \leqslant Y_L$. Since $Y_L$ is a normal $p$-subgroup of $L$, $Y_L \leqslant O_p(L)$. As $O_p(L/C_L(Y_L)) = 1$ we have $O_p(L) \leqslant C_L(Y_L)$. Thus $Y_L \leqslant \Omega_1 Z(O_p(L))$ and $O_p(L) \leqslant C_T(Y_L) = T_0$.

(h): By (f), $Y_L \leqslant Y_M$. By hypothesis $Y_M \leqslant Y$ and $M \subseteq LC_H(Y)$. Thus $C_H(Y) \leqslant C_H(Y_M)$ and so $M \subseteq LC_H(Y_M)$. As $L \leqslant M$ this implies $LC_H(Y_M) = MC_H(Y_M)$. So (b) gives $Y_M \leqslant Y_L$. Hence $Y_L = Y_M$ and (h) holds.

(i): Recall that $T_0 \leqslant L$. By (a) $T_0 \in Syl_p(C_H(Y_L))$ and so also $T_0 \in Syl_p(C_L(Y_L))$. A Frattini argument gives $L = L_0C_L(Y_L)$, and $T \leqslant L_0$ since $T_0 \trianglelefteq T$. Since $L$ is of characteristic $p$ and $O_p(L) \leqslant T_0 \leqslant O_p(L_0)$, also $L_0$ is of characteristic $p$. So (h) (applied with $L = L_0$, $M = L$ and $Y = Y_L$) implies that $Y_{L_0} = Y_L$. By (g) $Y_{L_0} \leqslant \Omega_1 Z(O_p(L_0))$. We record

$$Y_L = Y_{L_0} \leqslant \Omega_1 Z(O_p(L_0)).$$

Let $U$ be the largest normal subgroup of $L_0$ acting nilpotently on $\Omega_1 Z(O_p(L_0))$. Then $U$ acts nilpotently on $Y_{L_0}$. As $Y_{L_0}$ is $p$-reduced for $L_0$, A.10 implies that $U \leqslant C_{L_0}(Y_{L_0})$. So

$$O_p(L_0) \leqslant U \cap T \leqslant C_T(Y_{L_0}) = C_T(Y_L) = T_0 \leqslant O_p(L_0).$$

Therefore, $O_p(L_0) = T_0$. Note that $O^p(L_0) \leqslant C_{L_0}(\Omega_1 Z(O_p(L_0)))$ and thus

$$U = (U \cap T)C_{L_0}(\Omega_1 Z(O_p(L_0))) = C_{L_0}(\Omega_1 Z(O_p(L_0))).$$

Now A.10 shows that $\Omega_1 Z(O_p(L_0))$ is p-reduced for $L_0$ and thus

$$Y_L \leqslant \Omega_1 Z(T_0) = \Omega_1 Z(O_p(L_0)) \leqslant Y_{L_0} = Y_L.$$

Since $L_0$ is of characteristic $p$ and $T_0 = O_p(L_0)$, $C_{L_0}(T_0) \leqslant T_0$ and (i) is proved.

(j): By hypothesis $MC_H(Y_L) = LC_H(Y_L)$ and so $M$ normalizes $Y_L$. Hence $T_0$ is a Sylow $p$-subgroup of $C_M(Y_L)$ and $C_M(Y_L)$ is a normal subgroup of $M$. So by a Frattini argument $M = N_M(T_0)C_M(Y_L) = L^*C_M(Y_L)$. Thus $LC_H(Y_L) = MC_H(Y_L) = L^*C_H(Y_L)$. By (i), $Y_L = \Omega_1 Z(T_0)$ and $C_T(T_0) \leqslant T_0$. Since $T_0 \leqslant O_p(L^*)$ we conclude $Y_{L^*} \leqslant \Omega_1 Z(T_0) = Y_L$. By (b), $Y_L \leqslant Y_{L^*}$ and so $Y_L = Y_{L^*}$.

(k): By (g) $O_p(L) \leqslant T_0$. Since $C_L(Y_L)$ is $p$-closed and $T_0 \in Syl_p(C_L(Y_L))$, we have

$$T_0 = O_p(C_L(Y_L)) \leqslant O_p(L) \leqslant T_0.$$

So $T_0 = O_p(L)$, and (i) shows that $Y_L = \Omega_1 Z(T_0) = \Omega_1 Z(O_p(L))$. By (a), $T_0 \in Syl_p(C_H(Y_L))$ and so (k) is proved. □

LEMMA 1.25. *Suppose that $H$ is of parabolic characteristic $p$. Let $T \in Syl_p(H)$ and $L \in \mathcal{L}_H(T)$. Then there exist $M \in \mathfrak{M}_H(T)$ and $L^* \in \mathcal{L}_H(T)$ such that $L^* \leqslant M$, $Y_L = Y_{L^*}$, $Y_L \leqslant Y_M$ and $LC_H(Y_L) = L^*C_H(Y_L)$.*

PROOF. Put $T_0 := C_T(Y_L)$ and $L_1 := LC_H(Y_L)$. Then $Y_L \trianglelefteq L_1$ and $C_H(O_p(L_1)) \leqslant C_H(Y_L) \leqslant L_1$. Since $H$ is of parabolic characteristic $p$ we conclude that $C_H(O_p(L_1)) = C_{L_1}(O_p(L_1)) \leqslant O_p(L_1)$, so $L_1 \in \mathcal{L}_H(T)$. Note that $LC_H(Y_L) = L_1 = L_1C_H(Y_L)$ and so by 1.24(b), $Y_L \leqslant Y_{L_1}$. Thus $C_H(Y_{L_1}) \leqslant C_H(Y_L)$.

Suppose that there exist $M \in \mathfrak{M}_H(T)$ and $L_1^* \in \mathcal{L}_H(T)$ such that $L_1^* \leqslant M$, $Y_{L_1} = Y_{L_1^*} \leqslant Y_M$ and $L_1C_H(Y_{L_1}) = L_1^*C_H(Y_{L_1})$. As $C_H(Y_{L_1}) \leqslant C_H(Y_L)$ this gives $L_1C_H(Y_L) = L_1^*C_H(Y_L)$. Together with $L_1 = LC_H(Y_L)$ we get

$$LC_H(Y_L) = L_1 = L_1C_H(Y_L) = L_1^*C_H(Y_L).$$

Put $L^* = N_{L_1^*}(T_0)$. Since $T \leqslant L \cap L_1^*$ we can apply 1.24(j) with $(L_1^*, L)$ in place of $(M, L)$ and conclude that $Y_L = Y_{L^*}$ and $LC_L(Y_L) = L^*C_{L^*}(Y_L)$. Also $T \leqslant L^* \leqslant L_1^* \leqslant M$, and thus 1.24(f) with $(L^*, M)$ in place of $(L, M)$ yields $Y_{L^*} \leqslant Y_M$. So the lemma holds in this case.

Hence it suffices to prove the lemma for $L_1$ in place of $L$. Since $C_H(Y_{L_1}) \leqslant C_H(Y_L) \leqslant L_1$ we therefore may assume that $C_H(Y_L) \leqslant L$. By [**MS4**, Theorem 1.3] there exists a set $\mathcal{F}$ of parabolic subgroups of $H$ containing $T$ such that the following hold:

(i) For every $L \in \mathcal{L}_H(T)$ there exists $F \in \mathcal{F}$ such that $L \subseteq C_H(Y_L)F$ and $Y_L \leqslant Y_F$.
(ii) If $L \in \mathcal{L}_H(T)$ and $F \in \mathcal{F}$ with $F \subseteq C_H(Y_F)L$ and $Y_F \leqslant Y_L$, then $Y_L = Y_F$ and $L \leqslant F$.

According to (i) there exists $F \in \mathcal{F}$ with $L \subseteq C_H(Y_L)F$ and $Y_L \leqslant Y_F$. Since $C_H(Y_L) \leqslant L$, we get $L \leqslant C_L(Y_L)F$ and so $L = C_L(Y_L)(L \cap F) = (L \cap F)C_L(Y_L)$. In particular, by 1.24(h) (applied with $(L, L \cap F, Y_L)$ in place of $(M, L, Y)$) $Y_L = Y_{L\cap F}$.

Let $M \leqslant F$ be minimal with $T \leqslant M$ and $F = MC_F(Y_F)$. By 1.24(f), $Y_{L\cap F} \leqslant Y_F$ and so $Y_L = Y_{L\cap F} \leqslant Y_F$. Then $C_F(Y_F) \leqslant C_H(Y_L) \leqslant L$, $F = MC_{F\cap L}(Y_F)$ and $L \cap F = (L \cap M)C_{L\cap F}(Y_L)$. Thus $L = (L \cap F)C_H(Y_L) = (L \cap M)C_H(Y_L)$. Since $L \cap M \leqslant L$, 1.24(h) gives $Y_L = Y_{L\cap M}$ and since $L \cap M \leqslant M$, 1.24(f) gives $Y_{L\cap M} \leqslant Y_M$. Thus if $M \in \mathfrak{M}_H(T)$ then $L^* := L \cap M$ has the required properties. It remains to show that $M \in \mathfrak{M}_H(T)$.

By [**MS4**, 3.5] $F$ is the unique maximal $p$-local subgroup containing $M$. Since $H$ is of parabolic characteristic $p$, both $F$ and $M$ are of characteristic $p$. Since $M \leqslant F$ and $F = MC_F(Y_F)$ we conclude from 1.24(h) that $Y_M = Y_F$. Thus $F = MC_F(Y_M)$ and $\mathcal{M}(M) = \{F\}$. This shows condition (i) of the basic property for $M$.

Put $M_0 := N_M(C_T(Y_M))$. Since $O_p(M) \leqslant O_p(M_0)$ and

$$C_H(O_p(M_0)) \leqslant C_H(O_p(M)) \leqslant O_p(M) \leqslant O_p(M_0),$$

$M_0 \in \mathcal{L}_H(T)$. Moreover, the minimality of $M$ and a Frattini argument show that $M = M_0$. Thus $C_M(Y_M)$ is $p$-closed. In particular, by 1.24(k), $O_p(M) \in Syl_p(C_M(Y_M))$.

Let $X$ be a maximal subgroup of $M$ containing $O_p(M)$. Assume that $XC_M(Y_M) = M$. Since $F = MC_M(Y_F)$ and $Y_F = Y_M$ we get $F = XC_F(Y_F)$. In addition, $X$ contains a Sylow $p$-subgroup of $M$ since $O_p(M) \in Syl_p(C_M(Y_M))$. Hence without loss $T \leqslant X$, which contradicts the minimal choice of $M$. Thus $XC_M(Y_M) \neq M$, i.e. $C_M(Y_M) \leqslant X$, and so $C_M(Y_M)/O_p(M) \leqslant \Phi(M/C_M(Y_M))$. So also condition (ii) of the basic property holds for $M$. □

LEMMA 1.26. *Let $L \trianglelefteq\trianglelefteq H$. Then*

(a) $Y_H \cap L = Y_H \cap Y_L$ *is $p$-reduced for $L$.*
(b) $C_L(Y_L) = C_L(Y_H) = C_L(Y_H \cap L)$. *In particular* $[Y_L, L] = 1$ *if and only if* $[Y_H, L] = 1$.
(c) *Suppose that* $O^p(H) \leqslant L$. *Then* $[Y_L, L] = 1$ *if and only if* $[Y_H, H] = 1$.

PROOF. Let $R$ be the inverse image of $O_p(L/C_L(Y_H \cap L))$ in $L$, so $R \trianglelefteq L \trianglelefteq\trianglelefteq H$. Then $O^p(R)$ centralizes $Y_H \cap L$. Note that $O^p(R) = O^p(RY_H) \trianglelefteq RY_H$ since $R \trianglelefteq\trianglelefteq H$, and so $[O^p(R), Y_H] \leqslant O^p(R) \cap Y_H \leqslant Y_H \cap L$. Hence

$$[Y_H, O^p(R)] = [Y_H, O^p(R), O^p(R)] \leqslant [Y_H \cap L, O^p(R)] = 1.$$

Thus $R$ acts nilpotently on $Y_H$. Since $R \trianglelefteq\trianglelefteq H$ and $Y_H$ is a $p$-reduced $H$-module, A.10 now implies that $R$ centralizes $Y_H$. Hence

$$C_L(Y_H \cap L) \leqslant R \leqslant C_L(Y_H) \leqslant C_L(Y_H \cap L),$$

and thus

$$(*) \qquad C_L(Y_H \cap L) = R = C_L(Y_H).$$

In particular, $Y_H \cap L$ is $p$-reduced for $L$. Hence $Y_H \cap L \leqslant Y_L$ and $Y_H \cap L = Y_H \cap Y_L$. Thus (a) holds.

(b): By induction we may assume that $L \trianglelefteq H$. Then $H$ acts on $Y_L$ and by A.15(b) (applied with $V = Y_L$) $C_L(Y_{Y_L}(L)) = C_L(Y_{Y_L}(H))$, where $Y_{Y_L}(H)$ is the largest $p$-reduced $H$-submodule of $Y_L$. Since $Y_L$ is $p$-reduced for $L$, $Y_{Y_L}(L) = Y_L$, and since $Y_{Y_L}(H)$ is $p$-reduced for $H$, $Y_{Y_L}(H) \leqslant Y_H$. So

$$C_L(Y_H) \leqslant C_L(Y_{Y_L}(H)) = C_L(Y_{Y_L}(L)) = C_L(Y_L) \leqslant C_L(Y_H \cap Y_L) = C_L(Y_H),$$

where the last equality follows from $(*)$. Hence (b) holds.

(c): By (b), $[Y_L, L] = 1$ if and only if $[Y_H, L] = 1$. If $[Y_H, L] = 1$ then $[Y_H, O^p(H)] = 1$ and since $Y_H$ is $p$-reduced, also $[Y_H, H] = 1$. Hence $[Y_H, L] = 1$ if and only if $[Y_H, H] = 1$. $\square$

LEMMA 1.27. *Suppose that $H$ has characteristic $p$. Let $T \in Syl_p(H)$.*

(a) *(Kieler Lemma) Let $E$ be a subnormal subgroup of $H$. Then*

$$C_E(\Omega_1 Z(T)) = C_E(\Omega_1 Z(T \cap N)).$$

(b) *Let $V$ be an elementary abelian normal $p$-subgroup of $H$ containing $\Omega_1 Z(T)$. Then*

$$C_H(\Omega_1 Z(T)) = C_H\big([V, O^p(H)] \cap \Omega_1 Z(T)\big).$$

PROOF. (a): If $E = 1$ this if obvious. So suppose that $E \neq 1$. By 1.2(a), $E$ has characteristic $p$ and so $O_p(E) \neq 1$. In particular, $p$ divides $|E|$. Since $H$ has characteristic $p$, $H$ also has local characteristic $p$, see 1.2(c). Now (a) follows from [**MS6**, 1.5].

(b): By [**MS6**, 1,6] $C_E(C_V(T)) = C_E(C_{[V,E]}(T \cap E))$ for any subnormal subgroup $E$ of $H$. For $E = H$ this gives

$$(*) \qquad C_H(C_V(T)) = C_H(C_{[V,H]}(T)).$$

Put $[V, H, 1] = [V, H]$ and $[V, H, n] = [[V, H, n-1], H]$ for $n \geqslant 2$. Now an elementary induction on $n$ using $(*)$ gives

$$C_H(C_V(T)) = C_H(C_{[V,H,n]}(T)).$$

For $n$ large enough, $[V, H, n] = [V, O^p(H)]$ since $H$ acts nilpotently on $V/[V, O^p(H)]$. Thus

$$C_H(C_V(T)) = C_H(C_{[V,O^p(H)]}(T)).$$

Since $\Omega_1 Z(T) \leqslant V$ and $V$ is elementary abelian,

$$\Omega_1 Z(T) = C_V(T) \quad \text{and} \quad C_{[V,O^p(H)]}(T) = [V, O^p(H)] \cap \Omega_1 Z(T).$$

So (b) holds. $\square$

LEMMA 1.28. *Suppose that $H$ is of characteristic $p$ and $N \trianglelefteq\trianglelefteq H$.*

(a) $C_N(Z_H) = C_N(Z_N)$.

(b) *The following are equivalent:*

  (1) $[\Omega_1 Z(T), N] = 1$ *for some* $T \in Syl_p(H)$.

  (2) $[\Omega_1 Z(R), N] = 1$ *for some* $R \in Syl_p(N)$.

  (3) $[Z_N, N] = 1$.

(4) $[Z_H, N] = 1$.

PROOF. Let $T \in Syl_p(H)$. By the Kieler Lemma 1.27

$$(*) \qquad C_N(\Omega_1 Z(T)) = C_N(\Omega_1 Z(T \cap N)).$$

(a): Note that $Syl_p(N) = \{T \cap N \mid T \in Syl_p(H)\}$. So (a) follows from $(*)$ and the definition of $Z_H$ and $Z_N$.

(b): Since $T \cap N \in Syl_p(N)$ for $T \in Syl_p(H)$, $(*)$ shows that (b:1) implies (b:2). Since $N$ acts transitively on $Syl_p(N)$, (b:2) implies (b:3). By (a), (b:3) implies (b:4). Clearly (b:4) implies (b:1).

□

## 1.3. $p$-Irreducible Groups

LEMMA 1.29. *Suppose that $H$ is $p$-irreducible. Let $T \in Syl_p(H)$.*

(a) $H = \langle T^H \rangle = H'T$.
(b) $O^p(H) \leqslant H'$.
(c) $O^p(H) = [O^p(H), Y] \leqslant \langle Y^{O^p(H)} \rangle = \langle Y^H \rangle$ *for every* $Y \trianglelefteq T$ *with* $Y \nleqslant O_p(H)$.

PROOF. (a): Since $H$ is not $p$-closed, $T$ is not normal in $H$. Hence $\langle T^H \rangle$ is not $p$-closed. By definition of $p$-irreducible this gives $O^p(H) \leqslant \langle T^H \rangle$, and so $H = O^p(H)T = \langle T^H \rangle$. Since $H'T \trianglelefteq H$, we have $H = \langle T^H \rangle \leqslant H'T$, and thus $H = H'T$.

(b): This is an immediate consequence of $H = H'T$.

(c): Since $Y \nleqslant O_p(H)$, and $\langle Y^H \rangle \trianglelefteq H$, we get that $Y \nleqslant O_p(\langle Y^H \rangle)$. Hence $\langle Y^H \rangle$ is not $p$-closed, and since $H$ is $p$-irreducible, $O^p(H) \leqslant \langle Y^H \rangle$. Since $T$ normalizes $Y$ and $H = O^p(H)T$, we have $\langle Y^H \rangle = \langle Y^{O^p(H)} \rangle$ and so

$$O^p(H) \leqslant \langle Y^{O^p(H)} \rangle = [O^p(H), Y]Y.$$

Hence $O^p(H) \leqslant [O^p(H), Y]$, and (c) is proved. □

LEMMA 1.30. (a) *Let $D$ be a normal $p$-subgroup of $H$. Then $H$ is (strongly) $p$-irreducible if and only if $H/D$ is (strongly) $p$-irreducible.*
(b) *Let $K \leqslant H$ and $D$ a $K$-invariant $p$-subgroup of $H$. Then $K$ is (strongly) $p$-irreducible, if and only if $KD$ is (strongly) $p$-irreducible and if and only if $KD/D$ is (strongly) $p$-irreducible.*

PROOF. (a): Let $N \trianglelefteq H$ and put $\overline{H} := H/D$. Since $D$ is a $p$-group

$$N \ p\text{-closed} \quad \Longleftrightarrow \quad ND \ p\text{-closed} \quad \Longleftrightarrow \quad \overline{N} \ p\text{-closed}.$$

Moreover, since for every $X \leqslant H$, $O^p(X)$ does not have any non-trivial $p$-factor groups, one easily gets $O^p(N) = O^p(ND)$ and $O^p(\overline{H}) = \overline{O^p(H)}$. This gives

$$O^p(H) \leqslant N \quad \Longleftrightarrow \quad O^p(H) \leqslant ND \quad \Longleftrightarrow \quad O^p(\overline{H}) \leqslant \overline{N},$$

and

$$[N, H] \leqslant O_p(H) \quad \Longleftrightarrow [ND, H] \leqslant O_p(H) \quad \Longleftrightarrow [\overline{N}, \overline{H}] \leqslant O_p(\overline{H}).$$

Now (a) follows from the definition of (strongly) $p$-irreducible.

(b): Since $K \cap D$ is a normal $p$-subgroup of $K$, (a) shows that $K$ is (strongly) $p$-irreducible if and only if $K/K \cap D$ is (strongly) $p$-irreducible. Also $D$ is a normal $p$-subgroup of $KD$ and so $KD$ is (strongly) $p$-irreducible if and only $KD/D$ is (strongly) $p$-irreducible. Since $K/K \cap D \cong KD/D$, this gives (b). □

LEMMA 1.31. *Every strongly $p$-irreducible finite group is $p$-irreducible.*

PROOF. Suppose $H$ is strongly $p$-irreducible. Then $H$ is not $p$-closed. Let $N \trianglelefteq H$. If $[N, H] \leqslant O_p(H)$, then $N/N \cap O_p(H)$ is abelian and $N$ is $p$-closed. If $[N, H] \nleqslant O_p(H)$, then the definition of strongly $p$-irreducible gives $O^p(H) \leqslant N$. Thus $H$ is $p$-irreducible. □

LEMMA 1.32. *Suppose that there exists a non-empty $H$-invariant set of subgroups $\mathcal{K}$ of $H$ such that for $R := \langle \mathcal{K} \rangle$ and $E \in \mathcal{K}$:*

(i) *$H$ acts transitively on $\mathcal{K}$.*
(ii) *$O^p(H) \leqslant R$.*
(iii) *$EO_p(H) \trianglelefteq RO_p(H)$.*
(iv) *$E$ is strongly $p$-irreducible.*

*Then the following hold:*

(a) *For $E, K \in \mathcal{K}$ either $O^p\big(EO_p(H)\big) = O^p\big(KO_p(H)\big)$ or $[E, K] \leqslant O_p(H)$.*
(b) *$H$ is $p$-irreducible.*

PROOF. Put $\overline{H} := H/O_p(H)$. Then $\overline{H}$ and $\overline{\mathcal{K}}$ satisfy (i)–(iii). By (iv) $E$ is strongly $p$-irreducible and so by 1.30(b) also $\overline{E} = EO_p(H)/O_p(H)$ is strongly $p$-irreducible. Thus (iv) holds for $\overline{E}$. Hence $\overline{H}$ and $\overline{\mathcal{K}}$ satisfy (i) –(iv).

Moreover, if the claims (a) and (b) hold for $\overline{H}$ and $\overline{\mathcal{K}}$, then they also hold for $H$ and $\mathcal{K}$, again with the help of 1.30 in the case of (b). Thus, we may assume that $O_p(H) = 1$. Then $E \trianglelefteq R$. As $H$ acts transitively on $\mathcal{K}$, $R = \langle E^H \rangle$. Since $E$ is strongly $p$-irreducible, $E$ is $p$-irreducible by 1.31. Hence 1.29(b) gives $O^p(E) \leqslant E'$ and so

$$(*) \qquad O^p(R) = \langle O^p(E)^H \rangle \leqslant \langle E'^H \rangle \leqslant R'.$$

(a): Let $E, K \in \mathcal{K}$. Since $E$ and $K$ normalize each other, $D := [K, E] \leqslant K \cap E$, and $D$ is normal in $E$ and $K$. Since $E$ is strongly $p$-irreducible, either $D \leqslant Z(E)$ or $O^p(E) \leqslant D$ and by symmetry also $D \leqslant Z(K)$ or $O^p(K) \leqslant D$.

If $O^p(E) \leqslant D$ and $O^p(K) \leqslant D$, then $O^p(K) = O^p(D) = O^p(E)$, and (a) holds. Thus, we may assume without loss that $D \leqslant Z(E)$. Pick $T_E \in Syl_p(E)$. Since $O_p(E) \leqslant O_p(H) = 1$, both $Z(E)$ and $D$ are $p'$-groups. Note that $T_E$ centralizes $D$. We conclude that $DT_E = D \times T_E$ and $T_E = O_p(DT_E)$. Since $K$ normalizes $DT_E$, it also normalizes $T_E$, and $[T_E, K] \leqslant T_E \cap D = 1$. We conclude that $K$ centralizes every Sylow $p$-subgroup of $E$. Since $E$ is $p$-irreducible, 1.29(a) gives $E = \langle T_E^E \rangle$. Hence $[E, K] = 1$, and (a) is proved.

(b): Let $N$ be a normal subgroup of $H$. We need to show that $O^p(H) \leqslant N$ or $N$ is $p$-closed. Suppose first that $O^p(E) \leqslant N$. Then by $(*)$ $O^p(R) = \langle O^p(E)^H \rangle \leqslant N$. By (ii) $O^p(H) \leqslant R$ and so $O^p(H) \leqslant O^p(R) \leqslant N$.

Suppose next that $O^p(E) \nleqslant N$ for all $E \in \mathcal{K}$. Since $E$ is strongly $p$-irreducible by (iv), this gives $[E \cap N, E] \leqslant O_p(E) = 1$. Since $E \trianglelefteq R$, $[R \cap N, E] \leqslant E \cap N$. So $[R \cap N, E, E] = 1$ and with the Three Subgroups Lemma $[R \cap N, E'] = 1$. Since $R \cap N \trianglelefteq H$ we get $[R \cap N, \langle E'^H \rangle] = 1$. By $(*)$, $O^p(R) \leqslant \langle E'^H \rangle$, and thus $[R \cap N, O^p(R)] = 1$. In particular, $[R \cap N, O^p(R \cap N)] = 1$, so $R \cap N$ is $p$-closed. As $O_p(H) = 1$ this shows that $R \cap N$ is a $p'$-group. Since $[R \cap N, O^p(R)] = 1$, $R$ and so also $E$ acts a $p$-group on $N \cap R$. Thus coprime action gives $[R \cap N, E] = [R \cap N, E, E] = 1$. Since this holds for all $E \in \mathcal{K}$ and since $R = \langle \mathcal{K} \rangle$ this gives $[R \cap N, R] = 1$. Hence $[N, R, R] = 1$ and by the Three Subgroups Lemma, $[R', N] = 1$. By $(*)$ $O^p(R) \leqslant R'$ and thus $[O^p(R), N] = 1$. By (ii) $O^p(H) \leqslant O^p(R)$. It follows that $[O^p(H), N] = 1$, so $[O^p(N), N] = 1$, and $N$ is $p$-closed. □

LEMMA 1.33. *Suppose that $H$ is $p$-irreducible. Let $V$ be an $\mathbb{F}_pH$-module with $[V, O^p(H)] \neq 0$.*

(a) *$C_H(V)$ is $p$-closed.*
(b) *$C_T(V) \leqslant O_p(H)$ for all $p$-subgroups $T$ of $H$.*

PROOF. Note that $O^p(H) \nleqslant C_H(V)$. Hence (a) follows from the definition of $p$-irreducible, and (b) follows from (a). □

LEMMA 1.34. *Suppose that $H$ is $p$-irreducible. Let $V$ be an $\mathbb{F}_pH$-module with $[V, O^p(H)] \neq 1$ and $[V, O_p(H)] = 1$.*

(a) *$C_T(V) = T \cap O_p(H)$ for all $p$-subgroups $T$ of $H$.*
(b) *$V$ is $p$-reduced for $H$.*

(c) *Let $U$ be an $H$-submodule of $V$ minimal with $[U, O^p(H)] \neq 0$. Then $U$ is a quasisimple $H$-module.*

Proof. (a): By 1.33(a), $C_T(V) \leqslant O_p(H)$. Since $[V, O_p(H)] = 0$, this gives (a).

(b): Let $R/C_H(V) = O_p(H/C_H(V))$. Then $O^p(R) \leqslant C_H(V)$. Since $O^p(O^p(H)) = O^p(H) \nleqslant C_H(V)$ this gives $O^p(H) \nleqslant R$. The definition of $p$-irreducible now shows that $R$ is $p$-closed. Since $O_p(R) \leqslant O_p(H) \leqslant C_H(V)$ we conclude that $R/C_R(V)$ is a $p'$-group. Thus $R = C_H(V)$ and $V$ is $p$-reduced.

(c): Recall from the definitions, see A.2, that $U$ is a perfect $H$-module if $0 \neq U = [U, H]$ and that $U$ is a quasisimple $H$-module if $U$ is perfect and $p$-reduced for $H$ and $U/C_U(O^p(H))$ is a simple $H$-module.

Since $[U, O^p(H)] \neq 0$ we have $[U, O^p(H)] \nleqslant C_U(O^p(H))$. By minimality of $U$, $C_U(O^p(H))$ is the unique maximal $H$-submodule of $U$ and so $U = [U, O^p(H)]$ and $U/C_U(O^p(H))$ is simple. In particular, $U = [U, H]$ and thus $U$ is a perfect $H$-module. By (b) applied to $U$, $U$ is $p$-reduced for $H$. Thus $U$ is $H$-quasisimple. □

Lemma 1.35. *Suppose that $H$ is $p$-irreducible and of characteristic $p$. Then either*

$$Y_H = \Omega_1 Z(O_p(H)) \quad \textit{and} \quad [Y_H, O^p(H))] \neq 1,$$

*or*

$$[Y_H, H] = 1 \quad \textit{and} \quad [\Omega_1 Z(O_p(H)), O^p(H)] = 1.$$

Proof. Put $V := \Omega_1 Z(O_p(H))$. Recall from 1.24(g) that $Y_H \leqslant V$.

Assume first that $[V, O^p(H)] \neq 1$. Then 1.34(b) shows that $V$ is $p$-reduced for $H$. Hence $V \leqslant Y_H$. Since $Y_H \leqslant V$ this gives $V = Y_H$.

Assume next that $[V, O^p(H)] = 1$. Then $[Y_H, O^p(H)] = 1$ since $Y_H \leqslant V$, and $H/C_H(Y_H)$ is a $p$-group. Since $Y_H$ is $p$-reduced this gives $[Y_H, H] = 1$. □

## 1.4. $Y$-Minimal Groups

Recall from the introduction:

Definition 1.36. $H$ is *$Y$-minimal* for $Y \leqslant H$, if $H = \langle Y^H \rangle$ and $Y$-is contained in unique maximal subgroup of $H$; and $H$ is *$p$-minimal* if $H$ is $T$-minimal for $T \in Syl_p(H)$.

Lemma 1.37. *Suppose that $H$ is $p$-minimal. Then $H$ is $p$-irreducible.*

Proof. Let $T \in Syl_p(H)$. By the definition of $p$-minimality, $H = \langle T^H \rangle$ and $T$ is contained in a unique maximal subgroup $M$ of $H$. Hence $T \leqslant M < H$, $\langle T^H \rangle \nleqslant M$ and $T \ntrianglelefteq H$. So $H$ is not $p$-closed.

Let $N \trianglelefteq H$. Then either $NT = H$ or $N \leqslant M$. In the first case $O^p(H) \leqslant N$. In the second case by a Frattini argument $H = NN_H(N \cap T)$, so $T \leqslant N_H(N \cap T) \nleqslant M$ and thus $N_H(T \cap N) = H$. Hence $T \cap N \leqslant O_p(H)$, and $N$ is $p$-closed. □

Lemma 1.38. *Suppose that $H$ is $p$-minimal and $N \trianglelefteq H$. Then either $H/N$ is a $p$-group or $H/N$ is $p$-minimal.*

Proof. Let $T \in Syl_p(H)$. Since $H$ is $p$-minimal, $H = \langle T^H \rangle$ and $T$ is contained in a unique maximal subgroup $M$ of $H$. If $N \leqslant M$, then $H/N = \langle (TN/N)^{H/N} \rangle$ and $M/N$ is the unique maximal subgroups of $H/N$ containing $TN/N$, and so $H/N$ is $p$-minimal. So suppose $N \nleqslant M$. Since $T \leqslant NT$ and $NT \nleqslant M$, $NT$ is not contained in any maximal subgroup of $H$. Thus $NT = H$ and $H/N$ is a $p$-group. □

Lemma 1.39. *Suppose that there exists a non-empty $H$-invariant set of subgroups $\mathcal{K}$ of $H$ such that for $R := \langle \mathcal{K} \rangle$ and $E \in \mathcal{K}$:*

(i) *$H$ acts transitively on $\mathcal{K}$.*

(ii) *$O^p(H) \leqslant R$.*
(iii) *$EO_p(H) \trianglelefteq RO_p(H)$.*
(iv) *$E$ is $p$-minimal .*

*Then $H$ is $p$-minimal.*

PROOF. Put $\overline{H} := H/O_p(H)$. Clearly, $\overline{E}$ is $p$-minimal since $E$ is. Hence $\overline{H}$ and $\overline{\mathcal{K}}$ satisfy (i) – (iv). Moreover, $\overline{H}$ is $p$-minimal if and only if $H$ is $p$-minimal. Thus, we may assume that $O_p(H) = 1$. In particular, by (iii), $E \trianglelefteq R$ for $E \in \mathcal{K}$.

Since $O^p(H) \leqslant R$, $H = RT$. Since $E \trianglelefteq R$ we know that $R$ acts trivially on $\mathcal{K}$, while by (i) $H = RT$ acts transitively on $\mathcal{K}$. Hence $T$ acts transitively on $\mathcal{K}$.

Let $E \in \mathcal{K}$. Then $E \trianglelefteq R \trianglelefteq H$ and so $T \cap E \in Syl_p(E)$. Since $E$ is $p$-minimal, $T \cap E$ is contained in a unique maximal subgroup $E_T$ of $E$ and $E = \langle (T \cap E)^E \rangle \leqslant \langle T^H \rangle$. Thus $R \leqslant \langle T^H \rangle$ and $H = RT = \langle T^H \rangle$. Put $D := \bigcap_{E \in \mathcal{K}} N_R(E_T)$. Suppose that $H = DT$. Then $O^p(H) \leqslant D$ and so also $O^p(E) \leqslant D$. Hence

$$E = O^p(E)(E \cap T) = O^p(E)E_T \leqslant DE_T \leqslant N_R(E_T).$$

But then $\langle (E \cap T)^E \rangle \leqslant E_T$, a contradiction since $E$ is $p$-minimal. Thus $H \neq DT$.

We will show that $DT$ is the unique maximal subgroup of $H$ containing $T$. For this let $M \leqslant H$ with $T \leqslant M$. Suppose first that $M \cap E \nleqslant E_T$ for some $E \in \mathcal{K}$. Since $T \cap E \leqslant M \cap E$ and $E_T$ is the unique maximal subgroup of $E$ containing $T \cap E$, this gives $E = M \cap E \leqslant M$. The transitivity of $T$ on $\mathcal{K}$ now shows that $R = \langle \mathcal{K} \rangle = \langle E^T \rangle \leqslant M$ and $H = RT = M$.

Suppose next that $M \cap E \leqslant E_T$ for all $E \in \mathcal{K}$. Since $T \cap E \leqslant M \cap E$, $E_T$ is the unique maximal subgroup of $E$ containing $M \cap E$. Note that $M \cap R$ normalizes $E$ and $M \cap E$ and so $M \cap R$ normalizes $E_T$. Since this holds for all $E \in \mathcal{K}$, $M \cap R \leqslant D$. From $T \leqslant M \leqslant H = RT$ we have $M = (M \cap R)T$ and so $M \leqslant DT$.

We have proved that $DT$ is the unique maximal subgroup of $H$ containing $T$ and that $H = \langle T^H \rangle$. Thus $H$ is $p$-minimal. □

LEMMA 1.40. *Let $L$ be a group acting on a group $V$. Suppose that $X \leqslant L$ and $g \in L$ such that $[V, X, X] = 1$ and $L = \langle X, X^g \rangle$. Then for $W := [V, L]$*

$$W = [V, X][V, X^g],\ C_W(X) = [V, X] \text{ and } C_W(L) = [V, X] \cap [V, X^g].$$

PROOF. Clearly

$$W = [V, L] = [V, \langle X, X^g \rangle] = [V, X][V, X^g]$$

and

$$C_W(L) = C_W(\langle X, X^g \rangle) = C_W(X) \cap C_W(X^g).$$

Thus, it remains to show that $C_W(X) = [V, X]$.

Since $[V, X, X] = 1$, $[V, X] \leqslant C_W(X)$. As $W = [V, X][V, X^g]$, this implies

$$C_W(X) = [V, X]([V, X^g] \cap C_W(X)).$$

Moreover,

$$[V, X^g] \cap C_W(X) = [V, X^g] \cap C_W(X) \cap C_W(X^g) = [V, X^g] \cap C_W(L) \leqslant [V, X^g] \cap C_W(g) \leqslant [V, X].$$

This shows that $C_W(X) = [V, X]$. □

LEMMA 1.41 (L-Lemma). *Suppose that $H$ is $p$-minimal. Let $T \in Syl_p(H)$, and $A \leqslant T$ such that $A \nleqslant O_p(H)$. Also let $M$ be the unique maximal subgroup of $H$ containing $T$. Then there exists a subgroup $L \leqslant H$ with $AO_p(H) \leqslant L$ satisfying:*

(a) *$AO_p(L)$ is contained in a unique maximal subgroup $L_0$ of $L$, and $L_0 = L \cap M^g$ for some $g \in H$.*
(b) *$L = \langle A, A^x \rangle O_p(L)$ for every $x \in L \backslash L_0$.*

PROOF. This is the $L$-Lemma on page 34 of [**PPS**]. Note that although formally the L-Lemma was proved under Hypothesis 1 of Section 3 in [**PPS**], this hypothesis was never used in the proof. □

LEMMA 1.42. *Let $L$ be a finite group and $L_0$ a maximal subgroup of $L$, and let $Y \leqslant T \in Syl_p(L_0)$. Suppose that $L$ is $Y$-minimal. Then the following hold:*

(a) $Y \nleqslant O_p(L)$.
(b) $N_L(T) \leqslant L_0$ *and* $O_p(L) \leqslant L_0$. *In particular,* $T \in Syl_p(L)$.
(c) $N_L(L_0) = L_0$.
(d) $\bigcap L_0^L/O_p(L) = \Phi(L/O_p(L))$. *In particular,* $\bigcap L_0^L$ *is* $p$*-closed.*
(e) *Let* $N \trianglelefteq L$ *with* $N \leqslant L_0$. *Then* $N/O_p(N)$ *is a nilpotent* $p'$*-group. In particular,* $N$ *is* $p$*-closed.*
(f) $L = \langle Y, Y^g\rangle$ *for each* $g \in L\backslash L_0$.
(g) $Y \cap Y^g = C_Y(L)$ *if* $Y$ *is abelian and* $g \in L\backslash L_0$.

PROOF. (a): By [**MS6**, Lemma 2.5(b)] $Y$ is not subnormal in $L$ and so $Y \nleqslant O_p(L)$.

(b): By [**MS6**, Lemma 2.5(h)], $L_0$ contains the normalizer of a Sylow $p$-subgroup of $L$. Hence $N_L(T) \leqslant L_0$ and $O_p(L) \leqslant L_0$.

(c): See [**MS6**, Lemma 2.5(b)].

(d): Put $D := \bigcap L_0^L$. By [**MS6**, Lemma 2.7(c)] applied to $L/O_p(L)$, $D/O_p(L)$ is a $p'$-group and $D/O_p(L) = \Phi(L/O_p(L))$. In particular, $D$ is $p$-closed and so (d) holds.

(e): Since $N \leqslant \bigcap L_0^L$, this follows from (d).

(f): See [**MS6**, Lemma 2.5(c)].

(g): Let $Y$ be abelian and $g \in L\backslash L_0$. By (f) $L = \langle Y, Y^g\rangle$. Thus $Y \cap Y^g \leqslant C_Y(L)$, and clearly $C_Y(L) \leqslant Y \cap Y^g$. □

LEMMA 1.43. *Let $L$ be a finite group and $L_0$ a maximal subgroup of $L$, and let $Y$ be an elementary abelian $p$-subgroup of $L_0$. Suppose that*

(i) *$L$ is $Y$-minimal and of characteristic $p$, and*
(ii) $O_p(L) \leqslant N_L(Y)$.

*Put*

$$A := \langle (O_p(L) \cap Y)^L\rangle \text{ and } \overline{L} := L/C_Y(L),$$

*and let $B$ be an $L$-invariant subgroup of $A$. Then the following hold for every $g \in L\backslash L_0$:*

(a) $\Phi(A) = A' = [A \cap Y, A] = [A \cap Y, A \cap Y^g] \leqslant C_Y(L)$.
(b) $Y \cap Y^g = (A \cap Y) \cap (A \cap Y^g) = C_Y(L) = C_A(L)$.
(c) $C_L(\overline{a}) \leqslant L_0$ *for every* $1 \neq \overline{a} \in \overline{A \cap Y}$.
(d) $\overline{B \cap Y} = C_{\overline{B}}(Y) = C_{\overline{B}}(y) = [\overline{B}, y]$ *for every* $y \in Y\backslash O_p(L)$.
(e) $\overline{B} = \overline{B \cap Y} \times \overline{B \cap Y^g}$, $B = (B \cap Y)(B \cap Y^g)$ *and* $|B/B \cap Y| = |B \cap Y/C_{B\cap Y}(L)|$.
(f) *If* $\overline{B} \neq 1$ *and* $b \in B\backslash Y$, *then* $C_Y(\overline{b}) = C_Y(\overline{B}) = A \cap Y$ *and* $C_Y(B) \leqslant C_Y(b) \leqslant A \cap Y$.
(g) $B \cap Y = C_B(Y) = C_B(y) = [B, Y]C_{B\cap Y}(L) = [B, y]C_{B\cap Y}(L)$ *for every* $y \in Y\backslash O_p(L)$.
(h) $C_{\overline{B}}(L) = 1$ *and* $C_B(O^p(L)) = C_B(L) = B \cap C_Y(L) = B \cap Y \cap Y^g$.
(i) $[a, Y] \cap C_Y(L) = 1$ *for all* $a \in Z(A)$.
(j) $\overline{A} \neq 1$, $C_A(A \cap Y) = Z(A)(A \cap Y)$ *and* $C_Y(A) = Z(A) \cap Y$.
(k) $L_0^g \cap Y = A \cap Y$.
(l) $|Y/C_Y(B)| = |Y/A \cap Y||A \cap Y/C_Y(B)| \leqslant |B/C_B(Y)||A \cap Y/C_Y(B)|$ *if* $\overline{B} \neq 1$.
(m) $A/C_A(Y)$ *is elementary abelian,* $[Y, A] \neq 1$ *and* $A$ *acts nearly quadratically* [1] *on* $Y$.
(n) $|Y/C_Y(A)| \leqslant |A/C_A(Y)|^2$.
(o) *If* $B \leqslant Z(A)$, *then* $B$ *is a strong offender*[2] *on* $Y$.
(p) *$L$ has no central chief factor on* $\overline{A}$.
(q) $Z(A) = \Omega_1 Z(A)$.

PROOF.

1°. $L = \langle Y, Y^g\rangle$.

---

[1] for the definition of nearly quadratic see A.1(4)

[2] for the definition of a strong offender see A.7(4)

This holds by 1.42(f).

2°. $C_Y(L) = Y \cap Y^g = (A \cap Y) \cap (A \cap Y^g)$.

By 1.42(g) $Y \cap Y^g = C_Y(L)$. Also $C_Y(L) \leqslant Y \cap O_p(L) \leqslant A$ and so (2°) follows.

3°. $A = (A \cap Y)(A \cap Y^g)$.

Since $A \leqslant O_p(L) \leqslant N_L(Y) \cap N_L(Y^g)$, we get $[A, Y] \leqslant A \cap Y$ and $[A, Y^g] \leqslant A \cap Y^g$. As $L = \langle Y, Y^g \rangle$, we have $[A, L] = [A, Y][A, Y^g]$. Thus

$$A = \langle (O_p(L) \cap Y)^L \rangle = \langle (A \cap Y)^L \rangle = (A \cap Y)[A, L] = (A \cap Y)[A, Y][A, Y^g] = (A \cap Y)(A \cap Y^g),$$

and (3°) is proved.

4°. $\Phi(A) = [A \cap Y, A \cap Y^g] \leqslant Y \cap Y^g = C_Y(L)$. *In particular,* $\overline{A}$ *is elementary abelian.*

Since $A \cap Y$ and $A \cap Y^g$ are elementary abelian, the first equality follows from (3°). The inequality holds since $A$ normalizes $Y$ and $Y^g$, and the last equality follows from (2°).

5°. $\overline{A} = \overline{A \cap Y} \times \overline{A \cap Y^g}$. *In particular,* $|\overline{A}| = |\overline{A \cap Y}|^2$.

Since $\overline{A}$ is abelian, this follows from (2°) and (3°).

6°. $|\overline{B}/C_{\overline{B}}(y)| = |[\overline{B}, y]| \leqslant |C_{\overline{B}}(y)|$ *for* $y \in Y$. *In particular,* $|\overline{B}| = |[\overline{B}, y]||C_{\overline{B}}(y)|$.

Since $B \leqslant O_p(L) \leqslant N_L(Y)$ we get $[B, Y, Y] \leqslant [Y, Y] = 1$, and $Y$ acts quadratically on the abelian group $\overline{B}$. Thus

$$\phi : \overline{B} \to C_{\overline{B}}(y) \text{ defined by } \bar{b} \mapsto [\bar{b}, y]$$

is a homomorphism with $\ker \phi = C_{\overline{B}}(y)$ and so (6°) holds.

7°. $[\overline{A}, y] = [\overline{A}, Y] = C_{\overline{A}}(y) = \overline{A \cap Y} = C_{\overline{A}}(Y)$ *for each* $y \in Y \backslash O_p(L)$.

By 1.42(d), $\bigcap L_0^L$ is $p$-closed. Since $y \notin O_p(L)$, this implies $y \notin \bigcap L_0^L$, and there exists $h \in L$ such that $y \notin L_0^h$. Hence by 1.42(f) $L = \langle Y^h, Y^{hy} \rangle$ and thus also $L = \langle y, Y^h \rangle$. In particular, $h \notin L_0$, and (5°) applied to $h$ in place of $g$ gives

$$\overline{A} = \overline{A \cap Y} \times \overline{A \cap Y^h}.$$

Since $Y^h$ is abelian, $\overline{A \cap Y^h} \leqslant C_{\overline{A}}(Y^h)$. Thus

$$C_{\overline{A}}(y) \cap \overline{A \cap Y^h} \leqslant C_{\overline{A}}(\langle y, Y^h \rangle) = C_{\overline{A \cap Y^h}}(L) \leqslant \overline{A \cap Y} \cap \overline{A \cap Y^h} = 1,$$

and using that $\overline{A \cap Y} \leqslant C_{\overline{A}}(y)$,

$$C_{\overline{A}}(y) = C_{\overline{A}}(y) \cap (\overline{A \cap Y} \times \overline{A \cap Y^h}) = \overline{A \cap Y}\big(C_{\overline{A}}(y) \cap \overline{A \cap Y^h}\big) = \overline{A \cap Y}.$$

By (5°) we get $|\overline{A}| = |\overline{A \cap Y}|^2 = |C_{\overline{A}}(y)|^2$ and by (6°) applied to $A$ in place of $B$, $|\overline{A}| = |[\overline{A}, y]||C_{\overline{A}}(y)|$. Thus $|[\overline{A}, y]| = |C_{\overline{A}}(y)|$.

Moreover, the quadratic action of $Y$ on $\overline{A}$ gives

$$[\overline{A}, y] \leqslant [\overline{A}, Y] \leqslant C_{\overline{A}}(Y) \leqslant C_{\overline{A}}(y) = \overline{A \cap Y}.$$

As $|[\overline{A}, y]| = |C_{\overline{A}}(y)|$, equality holds everywhere and (7°) is proved.

(a): This is (4°).

(b): Note that $\overline{C_A(L)} \leqslant C_{\overline{A}}(Y)$. By (7°) $C_{\overline{A}}(Y) = \overline{A \cap Y}$ and so $C_A(L) \leqslant A \cap Y \cap Y^g$. By (2°) $A \cap Y \cap Y^g = Y \cap Y^g = C_Y(L) \leqslant C_A(L)$. Hence $C_A(L) = A \cap Y \cap Y^g$, and (b) holds.

(c): Pick $1 \neq \overline{a} \in \overline{A \cap Y}$. Then $Y \leqslant C_L(\overline{a})$ and so either $C_L(\overline{a}) \leqslant L_0$ or $C_L(\overline{a}) = L$. In the second case $\overline{a} \in \overline{A \cap Y^g}$, and (5°) yields $\overline{a} = 1$, a contradiction.

(d) and (e): By (7°), $[\overline{A}, y] = [\overline{A}, Y] = C_{\overline{A}}(y) = \overline{A \cap Y}$, and intersecting with $B$ gives

$$[\overline{B}, y] \leqslant [\overline{B}, Y] \leqslant C_{\overline{B}}(y) = \overline{B \cap Y},$$

By (5°) $\overline{A} = \overline{A \cap Y} \times \overline{A \cap Y^g}$ and so $\overline{B \cap Y} \cap \overline{B \cap Y^g} = 1$. Thus

$$|\overline{B \cap Y}|^2 = |\overline{B \cap Y}\,\overline{B \cap Y^g}| \leqslant |\overline{B}|.$$

In addition, by (6°)

$$|[\overline{B}, y]||C_{\overline{B}}(y)| = |\overline{B}|.$$

Combining the last three displayed equations we get

$$|\overline{B}| = |[\overline{B}, y]||C_{\overline{B}}(y)| \leqslant |\overline{B \cap Y}|^2 = |\overline{B \cap Y}\,\overline{B \cap Y^g}| \leqslant |\overline{B}|,$$

and so (d) and (e) follow.

(f): Let $b \in B \backslash Y$ and $y \in Y \backslash A$. By (d) $C_{\overline{B}}(y) = \overline{B \cap Y}$, so $y \notin C_Y(\bar{b})$ and $C_Y(\bar{b}) \leqslant A \cap Y$. By (a) $[A, A \cap Y] \leqslant C_Y(L)$ and so $A \cap Y \leqslant C_Y(\bar{b})$. Hence $C_Y(\bar{b}) = A \cap Y$, and (f) holds.

(g): This follows from (d) by taking preimages in $B$.

(h): We have

$$C_{\overline{B}}(L) \leqslant C_{\overline{B}}(Y) \cap C_{\overline{B}}(Y^g) \overset{(d)}{=} \overline{B \cap Y} \cap \overline{B \cap Y^g} \overset{(e)}{=} 1.$$

Hence also $C_{\overline{B}}(O^p(L)) = 1$ and

$$C_B(O^p(L)) \leqslant C_Y(L) \cap B \leqslant C_B(L) \leqslant C_B(O^p(L)).$$

(i): Let $y \in Y$ and $a \in Z(A)$ with $[a, y] \in C_Y(L)$. If $y \in O_p(L)$, then $y \in A$ and $[a, y] = 1$. If $y \notin O_p(L)$, then (e) gives $\overline{a} \in C_{\overline{A}}(y) = \overline{A \cap Y}$. So $a \in Y$ and again $[a, y] = 1$. As $Y$ is abelian, $[a, Y] = \{[a, y] \mid y \in Y\}$, and we conclude that $[a, Y] \cap C_Y(L) = 1$. Hence (i) holds.

(j): By (3°) $A = (A \cap Y)(A \cap Y^g)$ and so

$$C_A(A \cap Y) = (A \cap Y)C_{A \cap Y^g}(A \cap Y) = (A \cap Y)Z(A).$$

Since $[O_p(L), Y] \leqslant O_p(L) \cap Y \leqslant A$, we get that $C_Y(\overline{A})$ centralizes the factors of the normal $L$-series $1 \leqslant C_Y(L) \leqslant A \leqslant O_p(L)$. Since by Hypothesis (i) $L$ has characteristic $p$, 1.4(c) shows that $C_Y(\overline{A}) \leqslant O_p(L)$. As $Y \nleqslant O_p(L)$ we conclude that $\overline{A} \neq 1$. Moreover, since $C_Y(A) \leqslant C_Y(\overline{A}) \leqslant O_p(L)$

$$C_Y(A) = A \cap C_Y(A) = Z(A) \cap Y.$$

(k): Choose $T_0 \in Syl_p(L_0^g)$ with $L_0^g \cap Y \leqslant T_0$. Then choose $x \in L_0^g$ with $Y^{gx} \leqslant T_0$. Note that $L_0^{gx} = L_0^g$ and $gx \notin L_0$. So replacing $g$ by $gx$ we may assume that $Y^g \leqslant T_0$. If $L_0^g \cap Y \leqslant O_p(L)$, then $L_0^g \cap Y = O_p(L) \cap Y = A \cap Y$ and (k) holds. Assume that $L_0 \cap Y \nleqslant O_p(L)$. Since $\langle L_0^g \cap Y, Y^g \rangle \leqslant T_0$

$$C_{\overline{A}}(T_0) \leqslant C_{\overline{A}}(L_0^g \cap Y) \cap C_{\overline{A}}(Y^g) \overset{(7^\circ)}{=} \overline{A \cap Y} \cap \overline{A \cap Y^g} \overset{(5^\circ)}{=} 1,$$

which is impossible since $T_0$ and $\overline{A}$ are $p$-groups and $\overline{A} \neq 1$ by (j).

(l): Suppose that $\overline{B} \neq 1$. Then

$$C_Y(L) \leqslant C_Y(B) \overset{(f)}{\leqslant} A \cap Y \quad \text{and} \quad |B/C_B(Y)| \overset{(e)}{=} |B \cap Y/C_{B \cap Y}(L)| = |\overline{B \cap Y}|.$$

Since $\overline{B} \neq 1$ we can pick $b \in B \backslash Y$. Then

$$|Y/A \cap Y| \overset{(f)}{=} |Y/C_Y(\bar{b})| \overset{(6^\circ)}{=} |[\bar{b}, Y]| \leqslant |\overline{B \cap Y}| = |B/C_B(Y)|.$$

Thus

$$|Y/C_Y(B)| = |Y/A \cap Y||A \cap Y/C_Y(B)| \leqslant |B/C_B(Y)||A \cap Y/C_Y(B)|.$$

(m): By (a) $A' = \Phi(A) \leqslant C_Y(L) \leqslant C_A(Y)$, and so $A/C_A(Y)$ is elementary abelian. By (j), $\overline{A} \neq 1$ and so $[A, L] \neq 1$. Since $L = \langle A^L \rangle$ this gives $[A, Y] \neq 1$. Note that $[Y, A] \leqslant A$ and, as seen above, $[A, A] = A' \leqslant C_Y(L) \leqslant C_Y(A)$. Thus $A$ acts cubically on $Y$. By (g) $A \cap Y = [Y, A]C_Y(L) \leqslant [Y, A]C_Y(A)$ and by (f) $C_Y(A) \leqslant A \cap Y$. So

$$[Y, A]C_Y(A) = A \cap Y.$$

Let $y \in Y \backslash [Y, A]C_Y(A)$. We conclude that $y \in Y \backslash A = Y \backslash O_p(L)$, and (g) gives $[A, y]C_Y(L) = A \cap Y$. Since $[Y, A]C_Y(A) = A \cap Y$ this implies

$$[A, y]C_Y(A) = [Y, A]C_Y(A).$$

Hence $A$ acts nearly quadratically on $Y$.

(n): By bb $C_A(Y) = A \cap Y$ and so $C_Y(L) \leqslant C_A(Y) = A \cap Y$. We get

$$|A \cap Y/C_A(Y)| \leqslant |A \cap Y/C_Y(L)| \overset{(e)}{=} |A/A \cap Y|,$$

and so

$$|Y/C_Y(A)| \overset{(1)}{\leqslant} |A/A \cap Y||A \cap Y/C_Y(A)| \leqslant |A/A \cap Y||A \cap Y/C_Y(L)| = |A/A \cap Y|^2 = |A/C_A(Y)|^2.$$

(o): Suppose that $B \leqslant Z(A)$. If $\overline{B} = 1$, then $[B, Y] = 1$ and (o) holds. So suppose that $\overline{B} \neq 1$. Then by (f) $C_Y(B) \leqslant A \cap Y$. Since $B \leqslant Z(A)$ this gives $C_Y(B) = A \cap Y$. Thus

$$|Y/C_Y(B)| \overset{(1)}{\leqslant} |B/C_B(Y)||A \cap Y/C_Y(B)| = |B/C_B(Y)|,$$

so $B$ is an offender on $Y$. Let $b \in B \backslash C_B(Y)$. Then $b \notin Y$. Thus

$$C_Y(B) \leqslant C_Y(b) \overset{(f)}{\leqslant} A \cap Y = C_Y(B),$$

so $B$ is a strong offender on $Y$.

(p): Suppose that $L$ has a central chief factor on $\overline{A}$. Then there exists an $L$-invariant subgroup $B$ of $A$ with $C_Y(L) \leqslant B$ and $[\overline{B}, L] < \overline{B}$. But by (d), $\overline{Y \cap B} = [\overline{B}, y] \leqslant [\overline{B}, L]$ and so by (e) $\overline{B} = \overline{B \cap Y} \times \overline{B \cap Y^g} \leqslant [\overline{B}, L]$, a contradiction.

(q): By (e) $Z(A) = (Z(A) \cap Y)(Z(A) \cap Y^g)$. Since $Z(A)$ is abelian and $Y$ is elementary abelian we conclude that $Z(A)$ is elementary abelian. □

## 1.5. Weakly Closed Subgroups

In this section $Q$ is a fixed non-trivial $p$-subgroup of $H$. Recall that $Q$ is a weakly closed subgroup of $H$ if every Sylow $p$-subgroup of $H$ contains exactly one $H$-conjugate of $Q$.

NOTATION 1.44. For $L \leqslant H$

$$L^\circ := \langle P \in Q^G \mid P \leqslant L \rangle \text{ and } L_\circ = O^p(L^\circ).$$

(So $L^\circ$ is the weak closure of $Q$ in $L$ with respect to $H$.)

LEMMA 1.45. *The following statements are equivalent:*

(a) *$Q$ is a weakly closed subgroup of $H$.*
(b) *$Q = P$ for all $P \in Q^G$ with $[Q, P] \leqslant Q \cap P$.*
(c) *$Q \trianglelefteq N_H(R)$ for all $p$ subgroups $R$ of $H$ with $Q \leqslant R$.*

PROOF. (a) $\Longrightarrow$ (b): Let $P \in Q^H$ with $[Q, P] \leqslant Q \cap P$. Then $QP$ is a $p$-group, and since $Q$ is weakly closed in $H$, $PQ$ contains only one conjugate of $Q$ in $H$. Thus $P = Q$, and (b) holds.

(b) $\Longrightarrow$ (c): Let $R$ be $p$-subgroup with $Q \leqslant R$, and let $r \in N_H(N_R(Q))$. Then both $Q$ and $Q^r$ are normal in $N_R(Q)$ and so (b) shows $Q = Q^r$. Thus $N_H(N_R(Q)) \leqslant N_H(Q)$. In particular, $N_R(N_R(Q)) = N_R(Q)$. Hence $N_R(Q) = R$ and $N_H(R) \leqslant N_H(Q)$.

(c) $\Longrightarrow$ (a): Let $Q \leqslant T \in Syl_p(H)$ and $P \in Q^H$ with $P \leqslant T$. By (c) both $P$ and $Q$ are normal in $N_H(T)$. In particular, $Q$ and $P$ are normal in $T$ and so by Burnside's Lemma [**KS**, 7.1.5], $P = Q^h$ for some $h \in N_H(T)$. Thus $P = Q^h = Q$ and $Q$ is a weakly closed subgroup of $H$. □

LEMMA 1.46. *Let $Q$ be a weakly closed $p$-subgroup of $H$, $Q \leqslant K \leqslant H$ and $N \trianglelefteq H$. Then the following hold:*

(a) *$Q$ is a weakly closed subgroup of $K$.*
(b) *Let $g \in H$ with $Q^g \leqslant K$, then $Q^g = Q^k$ for some $k \in K$.*
(c) *$Q^K = Q^{K^\circ}$ and $K^\circ = \langle Q^K \rangle = \langle Q^{K^\circ} \rangle$.*
(d) *$K^\circ$ is the subnormal closure of $Q$ in $K$. In particular, $K^\circ = K_\circ Q = \langle Q^{O^p(K)} \rangle = \langle Q^{K_\circ} \rangle$.*
(e) *$K_\circ = [K_\circ, Q]$.*
(f) *$K^\circ \trianglelefteq\trianglelefteq H$ iff $K^\circ = H^\circ$ iff $Q^H = Q^K$ iff $H = KN_H(Q)$.*
(g) *$N_K(Q)$ is a parabolic subgroup of $K$, in particular $N = N_N(Q)O^p(N)$.*

(h) $N = N_N(Q)[N, Q]$.
(i) $[N, Q] = (Q \cap [N, Q])[N, Q, Q]$.
(j) *$QN/N$ is a weakly closed subgroup of $H/N$.*

PROOF. (a): This is an immediate consequence of the definition of a weakly closed subgroup.

(b): Let $Q \leqslant T \in Syl_p(K)$ and choose $k \in H$ with $Q^g \leqslant T^k$. Since $Q$ is weakly closed in $T$ with respect to $H$, $Q^g = Q^k$.

(c): Let $\mathcal{Q} = \{Q^g \mid g \in H, Q^g \leqslant K\}$. By (b), $\mathcal{Q} = Q^K$ and by (b) applied to $K^\circ$ in place of $K$, $\mathcal{Q} = Q^{K^\circ}$. Hence $Q^K = Q^{K^\circ}$ and $K^\circ = \langle \mathcal{Q} \rangle = \langle Q^K \rangle = \langle Q^{K^\circ} \rangle$.

(d): Since $K^\circ = \langle Q^K \rangle = \langle Q^{K^\circ} \rangle$, $K^\circ$ is the subnormal closure of $Q$ in $K$. Now 1.13 shows that $K^\circ = O^p(K^\circ)Q = K_\circ Q$ and $K^\circ = \langle Q^{O^p(K^\circ)} \rangle = \langle Q^{K_\circ} \rangle$. Note that $K_\circ \leqslant O^p(K) \leqslant K$ and so

$$K^\circ = \langle Q^{K_\circ} \rangle \leqslant \langle Q^{O^p(K)} \rangle \leqslant \langle Q^K \rangle = K^\circ.$$

Thus $K^\circ = \langle Q^{O^p(K)} \rangle$, and (d) is proved.

(e): By (d) $K^\circ = \langle Q^{K_\circ} \rangle = [K_\circ, Q]Q$ and so $K_\circ = O^p(K^\circ) \leqslant [K_\circ, Q]$. Hence $K_\circ = [K_\circ, Q]$.

(f): Suppose that $K^\circ \trianglelefteq\trianglelefteq H$. By (d) $K^\circ$ is the subnormal closure of $Q$ in $K$, and since $K^\circ \trianglelefteq\trianglelefteq H$, $K^\circ$ is also the subnormal closure of $Q$ in $H$. Thus $K^\circ = H^\circ$.
If $K^\circ = H^\circ$, then by (b) applied to $K$ and $H$, $Q^K = Q^{K^\circ} = Q^{H^\circ} = Q^H$.
If $Q^H = Q^K$ then (c) gives $H^\circ = \langle Q^H \rangle = \langle Q^K \rangle = \langle K^Q \rangle = K^\circ$ and so $K^\circ \trianglelefteq H$ and $K^\circ \trianglelefteq\trianglelefteq H$.

So the first three statements in (f) are equivalent. By a Frattini argument, $H = N_H(Q)K$ if and only if $Q^H = Q^K$. Hence (f) holds.

(g): Let $Q \leqslant T \in Syl_p(K)$. Then $T \leqslant N_K(Q)$ and so $N_K(Q)$ is a parabolic subgroup of $K$.

(h): Note that $Q[N, Q] \trianglelefteq NQ$. So

$$Q^{[N,Q]} = Q^{Q[N,Q]} \overset{\text{(f)}}{=} Q^{QN} = Q^N,$$

and thus (h) follows from a Frattini argument.

(i): By (h),

$$[N, Q] = [N_N(Q)[N, Q], Q] = [N_N(Q), Q][N, Q, Q] \leqslant (Q \cap [N, Q])[N, Q, Q].$$

(j): Put $\overline{H} := H/N$ and let $\overline{S} \in Syl_p(\overline{H})$ with $\overline{Q} \leqslant \overline{S}$ and $h \in H$ with $\overline{Q}^{\overline{h}} \leqslant \overline{S}$. Pick $R \in Syl_p(H)$ with $Q \leqslant R$ and $\overline{R} = \overline{S}$. Then $Q \trianglelefteq R$ and $Q^h \leqslant RN$. Hence by (b) $Q^h \in Q^{RN} = Q^N$ and so $\overline{Q}^{\overline{h}} = \overline{Q}$. □

LEMMA 1.47. *Let $Q$ be a weakly closed subgroup of $H$. Suppose that $H_1$ and $H_2$ are normal subgroups of $H^\circ$ such that*
(i) *$H^\circ = H_1 H_2$, and*
(ii) *$[H_1, H_2] \leqslant N_H(Q)$.*
*Let $i \in \{1, 2\}$ and set $K_i := (H_i Q)_\circ$. Then*
(a) *$K_i = [K_i, Q] = [K_i, H_i] \leqslant H_i'$ and $K_i \trianglelefteq H^\circ$,*
(b) *$H_\circ = K_1 K_2$ and $[K_1, K_2] \leqslant [K_1, H_2][H_2, K_1][H_1 \cap H_2, H^\circ] \leqslant O_p(H^\circ)$.*
(c) *Let $N \trianglelefteq H$. Then $F^*(H/N)$ normalizes $K_i N/N$.*

PROOF. Let $\{i, j\} = \{1, 2\}$. By hypothesis $H_i \trianglelefteq H^\circ$ and so

$$K_i = (H_i Q)_\circ = O^p((H_i Q)^\circ) \leqslant O^p(H_i Q) \leqslant O^p(H_i);$$

in particular, $K_i \trianglelefteq H_i$. Put $Z := [H_1, H_2]$. We first show:

$1^\circ$. *$O_p(Z) \trianglelefteq H^\circ$, $K_i Z \trianglelefteq H^\circ$ and $[Z, H^\circ] \leqslant O_p(Z)$.*

By (i) $H^\circ = H_1H_2$, so $Z \trianglelefteq H_1H_2 = H^\circ$ and thus also $O_p(Z) \trianglelefteq H^\circ$, and by (ii) $Z \leqslant N_H(Q)$. Thus

$$[K_i, H^\circ] = [K_i, H_iH_j] \leqslant [K_i, H_i][K_i, H_j] \leqslant K_iZ \quad \text{and} \quad [Z, Q] \leqslant Z \cap Q \leqslant O_p(Z).$$

Since $Z \trianglelefteq H^\circ$ the first chain of inequalities gives $K_iZ \trianglelefteq H^\circ$, and since by 1.46(c), $H^\circ = \langle Q^{H^\circ}\rangle$, the second one gives $[Z, H^\circ] \leqslant O_p(Z)$.

2°. $R_i := [K_i, H_i]O_p(Z) \trianglelefteq H^\circ$.

By (1°) $K_iZ$ and $O_p(Z)$ are normal in $H^\circ$. Since also $H_i \trianglelefteq H^\circ$, we get $[K_iZ, H_i]O_p(Z) \trianglelefteq H^\circ$. Again by (1°) $[Z, H_i] \leqslant [Z, H^\circ] \leqslant O_p(Z)$ and so

$$[K_iZ, H_i]O_p(Z) = [K_i, H_i][Z, H_i]O_p(Z) = [K_i, H_i]O_p(Z) = R_i.$$

Thus (2°) holds.

3°. $K_i \leqslant R_i$.

By 1.46(e), $K_i = [K_i, Q]$ and so

$$K_i = [K_i, Q] \leqslant [K_i, H^\circ] = [K_i, H_1H_2] = [K_i, H_i][K_i, H_j] \leqslant [K_i, H_i]Z \leqslant R_iZ.$$

Thus

$$K_i = [K_i, Q] \leqslant [R_iZ, Q] = [R_i, Q][Z, Q] \leqslant R_iO_p(Z) = R_i.$$

(a): Recall that $K_i \trianglelefteq H_i$, so $[K_i, H_i] \leqslant K_i \cap H_i'$. Hence $R_i \leqslant K_iO_p(Z)$, and by (3°), $R_i = K_iO_p(Z)$.

Since $Z \leqslant H_1 \cap H_2 \leqslant H_i$, $O_p(Z)$ normalizes $K_i$ and $[K_i, H_i]$. Thus

$$O^p([K_i, H_i]) = O^p\big([K_i, H_i]O_p(Z)\big) = O^p(R_i) = O^p(K_iO_p(Z)) = O^p(K_i) = K_i.$$

Since by (2°) $R_i \trianglelefteq H^\circ$, this shows that $K_i \trianglelefteq H^\circ$ and $K_i \leqslant [K_i, H_i]$. As $[K_i, H_i] \leqslant K_i$ we get $K_i = [K_i, H_i] \leqslant H_i'$, and (a) is proved.

(b): Again by 1.46(e) $H_\circ = [H_\circ, Q]$. Since $H^\circ = H_1H_2$ we have $H_\circ = O^p(H_1)O^p(H_2)$. By 1.46(d), $(H_iQ)^\circ = (H_iQ)_\circ Q = K_iQ$ and so $[O^p(H_i), Q] \leqslant (H_iQ)^\circ \leqslant K_iQ$. Hence

$$H_\circ = [H_\circ, Q] = [O^p(H_1)O^p(H_2), Q] = [O^p(H_1), Q][O^p(H_2), Q] \leqslant K_1QK_2Q = K_1K_2Q,$$

and as $K_1K_2 \leqslant H_\circ = O^p(H_\circ)$ and by (a) $K_1K_2 \trianglelefteq H^\circ$, $H_\circ = O^p(K_1K_2Q) = K_1K_2$.

Note that by (1°), $[H_j, H_i, H_i] = [Z, H_i] \leqslant O_p(Z)$. Hence, the Three Subgroups Lemma shows that $[H_i, H_i, H_j] = [H_i', H_j] \leqslant O_p(Z)$. Since by (a) $K_i \leqslant H_i'$, we get

$$[K_i, K_j] \leqslant [K_i, H_j] \leqslant [H_i', H_j] \leqslant O_p(Z) \leqslant O_p(H^\circ).$$

As $H_\circ = K_1K_2$, we also get $[H_1 \cap H_2, H_\circ] = [H_1 \cap H_2, K_1][H_1 \cap H_2, K_2] \leqslant O_p(H^\circ)$, and (b) is proved.

(c): Put $\overline{H} = H/N$. By 1.46(j) $\overline{Q}$ is a weakly closed subgroup of $\overline{H}$. Hence $\overline{H}, \overline{H_1}, \overline{H_2}, \overline{Q}$ fulfill the hypothesis of the lemma and $\overline{K_i} = (\overline{H_1}\,\overline{Q})_\circ$. So replacing $H$ by $H/N$ we may assume that $N = 1$. Put $L_i := O^{p'}(H_i)$. We first show:

4°. $K_i = (L_iQ)_\circ$.

Note that $H^\circ = O^{p'}(H^\circ)$. Since $H^\circ = H_1H_2$ we get $H^\circ = L_1L_2$. As $L_i \leqslant H_i$ we conclude that $H_i = L_i(H_1 \cap H_2)$. By (b), $[H_1 \cap H_2, Q] \leqslant [H_1 \cap H_2, H^\circ] \leqslant O_p(H^\circ)$. So $H_1 \cap H_2$ normalizes $O_p(H^\circ)Q$. Since $Q$ is weakly closed, this shows that $H_1 \cap H_2 \leqslant N_H(Q)$ and $H_i = L_i(H_1 \cap H_2) = L_iN_{H_i}(Q)$. Hence 1.46(f) gives $(H_iQ)^\circ = (L_iQ)^\circ$. Thus also $K_i = (L_iQ)_\circ$.

Observe that $F^*(H) = E(H)O_p(H)D$, where $E(H)$ is the product of the components of $H$ and $D := O_{p'}(F(H))$. Thus, to prove (c) it suffices to show that each of the factors $E(H)$, $O_p(H)$ and $D$ normalizes $K_1$.

Note that $K_1$ is a subnormal subgroup of $H$. Thus, by [**KS**, 5.5.7(c)] $E(H) = E(K_1)C_{E(H)}(K_1)$ and so $E(H) \leqslant N_H(K_1)$. Moreover, since $K_1 = O^p(K_1)$, 1.23 (with $\pi = \{p\}$) shows that also $O_p(H) \leqslant N_H(K_1)$.

The coprime action of $Q$ on $D$ gives $D = C_D(Q)[D,Q]$, and by (a) $[D,Q] \leqslant H^\circ \leqslant N_H(K_1)$. Since $L_1 \trianglelefteq\trianglelefteq H$ and $L_1 = O^{p'}(L_1)$, $D$ normalizes $L_1$ by 1.23. It follows that $C_D(Q)$ normalizes $L_1$, $Q$ and $(L_1Q)_\circ$. By (4°) $K_1 = (L_1Q)_\circ$ and so $C_D(Q) \leqslant N_H(K_1)$. This shows that also $D = C_D(Q)[D,Q]$ normalizes $K_1$, and (c) is proved. □

LEMMA 1.48. *Suppose that $Q$ is a weakly closed subgroup of $H$.*

(a) *Let $X \subseteq Z(Q)$ and $h \in H$ with $X^h \subseteq Z(Q)$. Then there exists $g \in N_H(Q)$ with $x^g = x^h$ for all $x \in X$.*

(b) *$x^H \cap Z(Q) = x^{N_H(Q)}$ for every $x \in Z(Q)$.*

PROOF. (a): Note that $\langle Q, Q^h\rangle \leqslant C_H(X^h)$ and so by 1.46(b) there exists $c \in C_H(X^h)$ such that $Q^{hc} = Q$. Hence $hc \in N_H(Q)$ and $x^{hc} = x^h$ for all $x \in X$. Thus (a) holds.

(b) follows from (a) applied with $X = \{x\}$. □

LEMMA 1.49. *Let $Q$ be a weakly closed $p$-subgroup of $H$, and let $Q \leqslant L \leqslant H$. Suppose that $C_H(Q) \leqslant Q$ and $H$ is of characteristic $p$. Then $[Q, C_H(O_p(L))] \leqslant Q \cap O_p(L)$, $C_H(O_p(L))$ is a $p$-group, and $L$ is of characteristic $p$.*

PROOF. Put $D := C_H(O_p(L))$. Since $Q$ is weakly closed, $O_p(H)$ normalizes $Q$. Thus

$$[O_p(H), Q] \leqslant O_p(H) \cap Q \leqslant O_p(H) \cap L \leqslant O_p(H) \cap O_p(L).$$

Hence $Q$ centralizes $O_p(H)/O_p(H) \cap O_p(L)$. Since $D$ centralizes $O_p(H) \cap O_p(L)$, we conclude that $[Q,D]$ centralizes the factors of the series

$$1 \leqslant O_p(H) \leqslant O_p(L) \leqslant O_p(H).$$

Since $H$ is of characteristic $p$, 1.4 shows that $[Q,D]$ is a $p$-group It follows that $Q[Q,D]$ is a $p$-group normalized by $D$ and since $Q$ is weakly closed this implies that $D$ normalizes $Q$. Thus

$$[Q,D] \leqslant Q \cap D \leqslant Q \cap O_p(D) \leqslant Q \cap L \cap O_p(D) \leqslant Q \cap O_p(L) \leqslant O_p(L) \leqslant C_H(D).$$

Hence $[Q,D,D] = 1$, and 1.3 shows that $D$ is a $p$-group. Hence also $C_L(O_p(L))$ is a $p$-group, so $C_L(O_p(L)) \leqslant O_p(L)$ and $L$ is of characteristic $p$. □

COROLLARY 1.50. *Let $Q$ be a weakly closed $p$-subgroup of $H$, and let $Q \leqslant L \leqslant H$. Suppose that $C_H(Q) \leqslant Q$ and that $C_H(y)$ is of characteristic $p$ for some $1 \neq y \in C_{O_p(L)}(Q)$. Then $[Q, C_H(O_p(L))] \leqslant Q \cap O_p(L)$, $C_H(O_p(L))$ is a $p$-group, and $L$ has characteristic $p$.*

PROOF. Put $K := N_H(O_p(L))$ and note that $Q \leqslant C_K(y) \leqslant C_H(y)$. By hypothesis, $C_H(y)$ is of characteristic $p$. Since $Q$ is also a weakly closed subgroup of $C_H(y)$, we can apply 1.49 with $C_H(y)$ and $C_K(y)$ in place of $H$ and $L$. Then $C_K(y)$ is of characteristic $p$. Note that $y \in O_p(L) \leqslant O_p(K)$ and so 1.5 shows that $K$ has characteristic $p$. Now 1.49 (with $K$ in place of $H$) shows that

$[Q, C_K(O_p(L))] \leqslant Q \cap O_p(L)$, $C_K(O_p(L))$ is a $p$-group, and $L$ is of characteristic $p$. As $C_H(O_p(L)) \leqslant K$ we have $C_H(O_p(L)) = C_K(O_p(L))$ and so the corollary is proved. □

## 1.6. Large Subgroups

In this section $Q$ is a fixed non-trivial $p$-subgroup of $H$.

DEFINITION 1.51. Recall from the introduction: $Q$ is *large* (in $H$) if $C_H(Q) \leqslant Q$ and

$$(Q!) \qquad N_H(U) \leqslant N_H(Q) \text{ for every } 1 \neq U \leqslant C_H(Q).$$

We will refer to this property as the $Q!$-property, or shorter just $Q!$.

Moreover

$$Q^\bullet := O_p(N_G(Q)), \quad M^\circ := \langle Q^g \mid g \in G, Q^g \leqslant M\rangle, \quad M_\circ := O^p(M^\circ).$$

Note that according to 1.52(b) below $Q$ is a weakly closed subgroup of $G$, so the notions $M^\circ$ and $M_\circ$ correspond to those introduced in 1.44 for weakly closed subgroups.

LEMMA 1.52. *Let $Q$ be large in $H$ and $Q \leqslant L \leqslant H$ and let $Y$ be a non-trivial $p$-subgroup of $H$ normalized by $L$. Then the following hold:*

(a) *$N_H(T) \leqslant N_H(Q)$ for every $p$-subgroup $T$ of $H$ with $Q \leqslant T$.*
(b) *$Q$ is a weakly closed subgroup of $H$.*
(c) *$L^\circ = (LC_H(Y))^\circ$ and $[L^\circ, C_H(Y)] \leqslant O_p(L^\circ)$. In particular, $C_H(Y)$ normalizes $L^\circ$.*
(d) *Let $\widetilde{L} := L/O_p(L)$. Suppose that $O^p(L) \leqslant L^\circ$ and $L = O^{p'}(L)$. Then $\widetilde{C_L(Y)} \leqslant Z(\widetilde{L^\circ}) \leqslant \Phi(\widetilde{L}) = \Phi(\widetilde{L_\circ})$.*
(e) *$C_H(Q) \cap C_H(Q^g) = C_H(Q^\bullet) \cap C_H(Q^{\bullet g}) = Z(Q) \cap Z(Q^g) = Z(Q^\bullet) \cap Z(Q^{\bullet g}) = 1$ for every $g \in H \backslash N_H(Q)$; in particular, $N_H(Q) = N_H(Q^\bullet)$, and $Q^\bullet$ is a large subgroup of $H$.*

PROOF. (a): Let $Q \leqslant T$, $T$ a $p$-subgroup of $H$. Then $N_H(T) \leqslant N_H(Z(T)) \leqslant N_H(Q)$ since $Z(T) \leqslant C_H(Q)$.

(b): By 1.45 the condition in (a) is equivalent to $Q$ being a weakly closed subgroup of $H$.

(c): We may assume that $H = LC_H(Y)$. Note that $C_Y(Q) \neq 1$ since $Y \neq 1$, and so by $Q!$, $C_H(Y) \leqslant N_H(C_Y(Q)) \leqslant N_H(Q)$. Thus $H = LN_H(Q)$. Since $Q$ is a weakly closed subgroup of $H$, 1.46(f) gives $L^\circ = H^\circ = (LC_H(Y))^\circ$.

In addition

$$[C_H(Y), Q] \leqslant Q \cap C_H(Y) \trianglelefteq C_{L^\circ}(Y) \trianglelefteq L^\circ,$$

so $[C_H(Y), Q] \leqslant O_p(L^\circ)$. Since $Q$ is a weakly closed subgroup of $H$, 1.46(c) implies $L^\circ = \langle Q^{L^\circ} \rangle$ and so conjugation with $L^\circ$ gives $[C_H(Y), L^\circ] \leqslant O_p(L^\circ)$.

(d): Since $O^p(L) \leqslant L^\circ$, $O^p(L) = O^p(L^\circ) = L_\circ$ and thus also $O^p(\widetilde{L}) = \widetilde{L_\circ}$. Put $D := C_L(Y)$. By (c) $[L^\circ, D] = [L^\circ, C_L(Y)] \leqslant O_p(L^\circ) \leqslant O_p(L)$ and so $[\widetilde{L^\circ}, \widetilde{D}] = 1$. Since $O^p(D) \leqslant L^\circ$, this shows $O^p(\widetilde{D}) \leqslant Z(\widetilde{D})$, therefore $\widetilde{D}$ is nilpotent. As $O_p(\widetilde{D}) \leqslant O_p(\widetilde{L}) = 1$, we conclude that $\widetilde{D}$ is a $p'$-group and thus $\widetilde{D} \leqslant O^p(\widetilde{L}) = \widetilde{L_\circ} \leqslant \widetilde{L^\circ}$. Hence $\widetilde{C_L(Y)} = \widetilde{D} \leqslant Z(\widetilde{L^\circ})$. Since $L^\circ$ is generated by $p$-elements, $\widetilde{L^\circ} = O^{p'}(\widetilde{L^\circ})$. Thus 1.7(b) applied to $\widetilde{L^\circ}$ gives $Z(\widetilde{L^\circ}) \leqslant \Phi(\widetilde{L^\circ})$. By 1.7(a) $\Phi(\widetilde{L^\circ}) = \Phi(\widetilde{L_\circ})$, and so (d) holds.

(e): By definition of a large subgroup, $Q$ contains its centralizer in $H$. Hence $C_H(Q^\bullet) \leqslant C_H(Q) \leqslant Q \leqslant Q^\bullet$ and $Z(Q^\bullet) \leqslant Z(Q)$ since $Q \leqslant Q^\bullet$. Moreover, $C_H(Q) = Z(Q)$ and $C_H(Q^\bullet) = Z(Q^\bullet)$.

Let $g \in H$ with $Z(Q) \cap Z(Q)^g \neq 1$. By $Q!$, $Q$ and $Q^g$ are normal in $N_H(Z(Q) \cap Z(Q)^g)$. Since $Q$ is a weakly closed subgroup of $H$, this gives $Q = Q^g$ and thus $g \in N_H(Q)$. Hence $Z(Q^\bullet) \cap Z(Q^\bullet)^g \leqslant Z(Q) \cap Z(Q)^g = 1$ for all $g \in H \backslash N_H(Q)$ and $N_H(Q^\bullet) \leqslant N_H(Z(Q^\bullet)) \leqslant N_H(Q)$. Clearly $N_H(Q) \leqslant N_H(Q^\bullet)$ and so $N_H(Q^\bullet) = N_H(Q)$.

Let $1 \neq X \leqslant C_H(Q^\bullet)$, Then $X \leqslant C_H(Q)$ and by $Q!$, $N_H(X) \leqslant N_H(Q) = N_H(Q^\bullet)$. Moreover, as seen above, $C_H(Q^\bullet) \leqslant Q^\bullet$, and so $Q^\bullet$ is a large subgroup of $H$. □

LEMMA 1.53. *Let $Q$ be large in $H$ and $H = H^\circ S$ for $Q \leqslant S \in Syl_p(H)$. Suppose that there exists $R \trianglelefteq H$ such that $R \leqslant N_H(Q)$ and $H/R$ is $p$-minimal. Then $H$ is $p$-minimal.*

PROOF. Since $H/R$ is $p$-minimal, there exists a unique maximal subgroup $H_0$ of $H$ containing $SR$. Let $H_1$ be any maximal subgroup of $H$ containing $S$. Assume $H_1 \neq H_0$. Then $R \nleqslant H_1$ and so $H = H_1 R$. Since $R \leqslant N_H(Q)$, $H = H_1 N_H(Q)$. Since $Q$ is a weakly closed subgroup of $H$, 1.46(f) gives $H_1^\circ = H^\circ$, and so $H = H^\circ S \leqslant H_1$, which contradicts $H_1 \neq H$. □

LEMMA 1.54. *Suppose that $Q$ is a large $p$-subgroup of $H$. Let $U$ be a non-trivial elementary abelian $p$-subgroup of $H$ and $Q \leqslant E \leqslant N_H(U)$. Suppose that $Q \ntrianglelefteq E$, $O_p(E/C_E(U)) = 1$ and $O^p(E)C_E(U)/C_E(U)$ is quasisimple. Then the following hold:*

(a) *$O^p(E)C_E(U) = E_\circ C_E(U)$ and $E_\circ = E_\circ' = O^p(E_\circ)$.*
(b) *$E_\circ/C_{E_\circ}(U)$, $E_\circ/O_p(E_\circ)$ and $E_\circ/[O_p(E_\circ), E_\circ]$ all are quasisimple.*
(c) *$E_\circ = [E_\circ, Y] \leqslant \langle Y^{E_\circ} \rangle$ for all $p$-subgroups $Y$ of $E$ with $[U, Y] \neq 1$.*

PROOF. Put $\overline{E} = E/C_E(U)$. Then $O_p(\overline{E}) = 1$ and $\overline{O^p(E)}$ is quasisimple.

(a): Since $Q \ntrianglelefteq E$, $Q!$ shows that $[U,Q] \neq 1$. So $\overline{Q} \neq 1$ and, as $\overline{O^p(E)}$ is quasisimple and $O_p(\overline{E}) = 1$, 1.14(b) gives $O^p(\overline{E}) \leqslant \langle \overline{Q}^{O^p(\overline{E})} \rangle \leqslant \overline{E^\circ}$. Also $E^\circ = O^p(E^\circ) = O^p(E_\circ)$. Thus $O^p(\overline{E}) = \overline{E_\circ}$, and the first statement in (a) holds. In particular, $\overline{E_\circ}$ is quasisimple and so perfect. By 1.46(d) $E^\circ = E_\circ Q$. Since $\overline{E_\circ}$ is perfect, $E_\circ = E'_\circ C_{E_\circ}(U)$ and so $E^\circ = E'_\circ Q C_{E^\circ}(U)$. By 1.52(c),

$$E^\circ = \big(E'_\circ Q C_{E^\circ}(U)\big)^\circ = (E'_\circ Q)^\circ \leqslant E'_\circ Q,$$

and so $E_\circ \leqslant E'_\circ$. Thus $E_\circ$ is perfect, and (a) is proved.

(b): As seen above, $E^\circ / C_{E^\circ}(U) \cong \overline{E_\circ} = O^p(\overline{E})$ is quasisimple. By 1.52(c), $[E^\circ, C_E(U)] \leqslant O_p(E)$ and so $[E_\circ, C_{E_\circ}(U)] \leqslant O_p(E_\circ)$. Let $L$ be the inverse image of $Z(E_\circ/C_{E_\circ}(U))$ in $L$. Then $[L, E_\circ] \leqslant C_{E_\circ}(U)$ and $[C_{E_\circ}(U), E_\circ] \leqslant O_p(E_\circ)$. Since $E_\circ$ is perfect the Three Subgroups Lemma gives $[L, E_\circ] \leqslant [E_\circ, O_p(E_\circ)]$. Thus $L/O_p(E_\circ) = Z\big(E_\circ/O_p(E_\circ)\big)$ and $L/[O_p(E_\circ), E_\circ] = Z\big(E_\circ/[E_\circ, O_p(E_\circ)]\big)$. Since $E_\circ/L$ is simple and $E_\circ$ is perfect, this shows that $E_\circ/O_p(E_\circ)$ and $E_\circ/[O_p(E_\circ), E_\circ]$ are quasisimple. So (b) holds.

(c): By 1.14(b) $\overline{E_\circ} = [\overline{E_\circ}, \overline{Y}]$. Since $E_\circ/O_p(E_\circ)$ is quasisimple this gives $E_\circ = [E_\circ, Y]O_p(E_\circ)$. As $E_\circ = O^p(E_\circ)$, we conclude that $E_\circ = [E_\circ, Y] \leqslant \langle Y^{E_\circ} \rangle$. □

LEMMA 1.55. *Let $Q$ be large in $H$ and $L \leqslant H$ with $Q \leqslant L$ and $O_p(L) \neq 1$. Then*

(a) *$C_H(O_p(L))$ is a $p$-group; in particular, $L$ has characteristic $p$.*
(b) *Let $R$ be a parabolic subgroup of $H$ with $O_p(R) \neq 1$. Then $C_H(O_p(R)) \leqslant O_p(R)$.*
(c) *$H$ has parabolic characteristic $p$.*
(d) *Either $L^\circ = Q$ or $C_H(L^\circ) = 1$.*

PROOF. (a): To show that $C_H(O_p(L))$ is a $p$-group, it suffices to verify the hypothesis of 1.50. Note that $C_H(Q) \leqslant Q$ since $Q$ is large and that $Q$ is a weakly closed subgroup by 1.52(b). Since $O_p(L) \neq 1$ and $Q$ normalizes $O_p(L)$, there exists $1 \neq y \in C_{O_p(L)}(Q)$. So it remains to show that $C_H(y)$ has characteristic $p$.

Put $Y := \langle y \rangle$. Then by $Q!$, $N_H(Y) \leqslant N_H(Q)$ and so $N_H(Y)$ is a local subgroup of $N_H(Q)$. Since $C_H(Q) \leqslant Q$, $N_H(Q)$ has characteristic $p$ and so by 1.2(c) $N_H(Q)$ has local characteristic $p$. Thus $N_H(Y)$ has characteristic $p$. Since $C_H(y) = C_H(Y) \trianglelefteq N_H(Y)$, also $C_H(y)$ has characteristic $p$ (see 1.2(a)).

(b): Since $R$ is parabolic subgroup of $H$, $R$ contains a Sylow $p$-subgroup $T$ of $H$ and so also a conjugate of $Q$. So by (a) $C_H(O_p(R))$ is a $p$-group. Observe that $T$ normalizes $C_H(O_p(R))$ and so $C_H(O_p(R)) \leqslant T \leqslant R$. As $R$ normalizes $C_H(O_p(R))$ this gives $C_H(O_p(R)) \leqslant O_p(R))$.

(c) follows from (b).

(d): If $C_H(L^\circ) \neq 1$, then $Q!$ implies that $Q \trianglelefteq N_H(C_H(L^\circ))$ and so $Q \trianglelefteq L$. As $L^\circ = \langle Q^L \rangle$ by 1.46(c), this gives $L^\circ = Q$. □

LEMMA 1.56. *Let $Q$ be large in $H$ and $Q \leqslant S \in Syl_p(H)$, and let $L \in \mathcal{L}_H(S)$.*

(a) *There exist $M \in \mathfrak{M}_H(S)$ and $L^* \in \mathcal{L}_H(S)$ such that $L^* \leqslant M$, $LC_H(Y_L) = L^*C_H(Y_L)$, $L^\circ = (L^*)^\circ \leqslant M^\circ$ and $Y_L = Y_{L^*} \leqslant Y_M$.*
(b) *Suppose that $Q \ntrianglelefteq L$, and let $M$ and $L^*$ be as in (a). Then $Q \ntrianglelefteq L^*$ and $Q \ntrianglelefteq M$.*
(c) *Either $\mathcal{M}_H(S) = \{N_H(Q)\}$ or there exists $M \in \mathfrak{M}_H(S)$ with $Q \ntrianglelefteq M$.*

PROOF. (a): By 1.55 $H$ has parabolic characteristic $p$. Hence 1.25 shows that there exists $M \in \mathfrak{M}_H(S)$ and $L^* \in \mathcal{L}_H(S)$ with $L^* \leqslant M$, $LC_H(Y_L) = L^*C_H(Y_L)$ and $Y_L = Y_{L^*} \leqslant Y_M$. Thus 1.52(c) gives

$$L^\circ = \big(LC_H(Y_L)\big)^\circ = \big(L^*C_H(Y_L)\big)^\circ = (L^*)^\circ,$$

and (a) holds.

(b): $Q!$ shows that $C_H(Y_L) \leqslant C_H(C_{Y_L}(Q)) \leqslant N_H(Q)$. Since $Q \ntrianglelefteq L$ and $LC_H(Y_L) = L^*C_H(Y_L)$ we conclude that $Q \ntrianglelefteq L^*$ and so also $Q \ntrianglelefteq M$.

(c): Suppose that $\mathcal{M}_H(S) \neq \{N_H(Q)\}$. Then there exists $L \in \mathcal{L}_H(S)$ with $Q \ntrianglelefteq L$ and so by (b) there exists $M \in \mathfrak{M}_H(S)$ with $Q \ntrianglelefteq M$. $\square$

LEMMA 1.57. *Let $Q$ be large in $H$. Suppose that $M \leqslant H$ with $Q \leqslant M$ and $V$ is a non-trivial elementary abelian $M$-invariant $p$-subgroup of $H$. Then the following hold:*

(a) *$N_M(A) \leqslant N_M(Q)$ for every $1 \neq A \leqslant C_V(Q)$.*
(b) *Suppose that $M \nleqslant N_H(Q)$. Then $V$ is a faithful $Q!$-module [3] for $M/C_M(V)$ with respect to $QC_M(V)/C_M(V)$.*
(c) *Let $U \leqslant M$ be transitive on $V$. Then $M^\circ = \langle Q^U\rangle$.*

PROOF. (a): This is a direct consequence of the $Q!$-property.

(b): Since $M \nleqslant N_H(Q)$, $Q \ntrianglelefteq H$. Together with (a) this shows that $V$ is a $Q!$-module for $H$ with respect to $Q$. Now (b) follows from A.51.

(c): Let $1 \neq v \in C_V(Q)$. By a Frattini argument $M = UC_M(v)$, and $Q!$ implies $C_M(v) \leqslant N_M(Q)$. So $M = UN_M(Q)$, and 1.46(f) gives $M^\circ = \langle Q^U\rangle$. $\square$

LEMMA 1.58. *Let $Q$ be large in $H$, let $S \in Syl_p(H)$ with $Q \leqslant S$, and let $L \in \mathcal{L}_H(S)$. Put $P := L^\circ S$ and $\widetilde{L} := L/C_L(Y_L)$. Let $\mathcal{K}$ be a non-empty $P$-invariant set of subgroups of $\widetilde{P}$ and suppose that $Y_L$ is a natural $SL_2(q)$-wreath product module for $\widetilde{P}$ with respect to $\mathcal{K}$. Then $\mathcal{K}$ is uniquely determined by that property. Moreover, the following hold, where $P^*$ is the inverse image of $\langle\mathcal{K}\rangle$ in $P$.*

(a) *$Q$ acts transitively on $\mathcal{K}$.*
(b) *$Y_L = Y_P$, $Y_P$ is a simple $P$-module, and $O_p(P) = C_S(Y_L)$.*
(c) *$O^p(P) = O^p(P^*) = L_\circ$, and $\widetilde{P^*}$ is normal in $\widetilde{L}$.*
(d) *$P_1 = P^*$ for all $P_1 \trianglelefteq P$ with $O_p(P) \leqslant P_1$ and $\widetilde{P_1} = \langle\mathcal{K}\rangle$.*
(e) *$P \in \mathcal{P}_H(S)$.*
(f) *One of the following holds:*
   (1) *$C_P(Y_P) = O_p(P)$.*
   (2) *$p = 2 = |\mathcal{K}|$, $\widetilde{Q} \cong C_4$, and, for any $T \in Syl_3(L^\circ)$, $T$ is extraspecial of order $3^3$, $[Z(T), L^\circ] \leqslant O_2(P)$. and $L_\circ = TO_2(L_\circ)$.*

PROOF. Since $Y_L$ is a faithful natural $SL_2(q)$-wreath product module for $\widetilde{P}$ with respect to $\mathcal{K}$, A.25 gives

1°. *$Y_L = \times_{K\in K}[Y_L, K]$ and $\widetilde{P^*} = \times_{K\in\mathcal{K}} K$, and for $K \in \mathcal{K}$, $K \cong SL_2(q)$ and $[Y_L, K]$ is a natural $SL_2(q)$-module for $K$.*

In particular, $O^p(\widetilde{P^*}) \neq 1$ and thus also $L^\circ \neq Q$. Hence by 1.55(d) $C_{Y_L}(\langle K^Q\rangle) = 1$, and so

2°. *$\mathcal{K} = K^Q$ for $K \in \mathcal{K}$, and (a) holds.*

Thus $\widetilde{P}$ and $Y_L$ satisfy the hypothesis of A.28 in place of $H$ and $V$, and A.28(b) gives:

3°. *$\widetilde{P}$ is $p$-minimal.*

By 1.52(c)

4°. *$[C_L(Y_L), L^\circ] \leqslant O_p(L^\circ) \leqslant O_p(L) \leqslant O_p(P)$.*

Since $Q$ is weakly closed, 1.46(e) gives

5°. *$L_\circ = [L_\circ, Q]$.*

By 1.24(f) $Y_P \leqslant Y_L$. Since $[Y_L, K]$ is a simple $K$-module for $K \in \mathcal{K}$ and $Q$ acts transitively on $\mathcal{K}$, $Y_L$ is a simple $P$-module, so $Y_P = Y_L$ and $O_p(\widetilde{P}) = 1$. Hence $O_p(P) \leqslant C_S(Y_L)$, and by (4°) $[C_S(Y_L), L^\circ] \leqslant O_p(P) \leqslant C_S(Y_L)$. Since $P = L^\circ S$ we conclude that $C_S(Y_L) \trianglelefteq P$. Hence $C_S(Y_P) = O_p(P)$. We have proved:

[3]See A.5 for the definition of a $Q!$-module

$6^\circ$. *$Y_P = Y_L$, $Y_L$ is a simple $P$-module and $C_S(Y_P) = O_p(P)$. In particular, (b) holds.*

Let $P_1 \trianglelefteq P$ with $O_p(P) \leqslant P_1$ and $\widetilde{P_1} = \langle \mathcal{K} \rangle$. The $p$-minimality of $\widetilde{P}$ implies that either $\widetilde{S} \cap \widetilde{P_1} \leqslant O_p(\widetilde{P})$ or $\widetilde{P} = \widetilde{P_1}\widetilde{S}$. The first case is clearly impossible since $O_p(\widetilde{P}) = 1$ and $\langle \mathcal{K} \rangle$ is not a $p'$-group. Hence $\widetilde{P} = \widetilde{P_1}\widetilde{S}$. As $P = L_\circ S$ we have $O^p(P) = O^p(L_\circ) = L_\circ$, and we conclude that

$$O^p(\widetilde{P_1}) = O^p(\widetilde{P}) = \widetilde{L_\circ}.$$

In particular, $O^p(P_1) \leqslant L_\circ C_P(Y_L)$. Since $O^p(P_1) \leqslant O^p(P) = L_\circ$ this gives

$$L_\circ = O^p(P_1)\big(L^\circ \cap C_P(Y_L)\big).$$

So

$$L_\circ \overset{(5^\circ)}{=} [L_\circ, Q] = [O^p(P_1)(L^\circ \cap C_P(Y_L)), Q] \leqslant [O^p(P_1), Q][C_P(Y_L), L^\circ] \overset{(4^\circ)}{\leqslant} O^p(P_1)O_p(P).$$

Hence $L_\circ = O^p(L_\circ) = O^p(P_1)$. Note that $O_p(P) \leqslant P^*$ and $\widetilde{P^*} = \langle \mathcal{K} \rangle$. So $P^*$ fulfills the assumptions on $P_1$, and we conclude

$7^\circ$. *$O^p(P) = L_\circ = O^p(P_1) = O^p(P^*)$; in particular $P = O^p(P_1)S$.*

Thus $O^p(\langle \mathcal{K} \rangle) = O^p(\widetilde{P^*}) = \widetilde{L_\circ} \trianglelefteq \widetilde{L}$. Hence by A.27 any subgroup $E$ of $\widetilde{L}$ such that $[Y_L, E]$ is a faithful natural $SL_2(q)$-module for $E$ is contained in $\mathcal{K}$. It follows that

$8^\circ$. *$\mathcal{K}$ is uniquely determined and $\widetilde{P^*} = \langle \mathcal{K} \rangle \trianglelefteq \widetilde{L}$. In particular, (c) holds.*

Put $T := S \cap P^*$. Then $\widetilde{T} \in Syl_p(\widetilde{P^*}) = Syl_p(\widetilde{P_1})$, and since $C_T(Y_L) \leqslant C_S(Y_L) = O_p(P) \leqslant P_1$, $T \in Syl_p(P_1)$. By ($7^\circ$), $O^p(P_1) = L_\circ$ and so $P_1 = O^p(P_1)T = L_\circ T$. This result also applies to $P^*$. Thus

$9^\circ$. *$P_1 = L_\circ T = P^*$. In particular, (d) holds.*

Set

$$Q^* := Q \cap P^*, \quad \widehat{P} := P/O_p(P), \quad r := |\mathcal{K}|, \quad \text{and} \quad \{K_1, \dots, K_r\} := \mathcal{K},$$

and let $1 \leqslant i \leqslant r$. Then $K_i \cong SL_2(q)$ and $\widetilde{S} \cap K_i \neq 1$.

We claim that $\widetilde{Q^*} \neq 1$. Since $Q \trianglelefteq S$, $[\widetilde{S} \cap K_i, \widetilde{Q}] \leqslant \widetilde{Q^*}$. If $r > 1$, the transitive action of $Q$ on $\mathcal{K}$ shows that $[\widetilde{S} \cap K_i, \widetilde{Q}] \neq 1$ and so $\widetilde{Q^*} \neq 1$. If $r = 1$ and $\widetilde{Q} \nleqslant K_1$, then $\widetilde{Q}$ induces some non-trivial field automorphism on $K_1$ and hence $[\widetilde{S} \cap K_1, \widetilde{Q}] \neq 1$ and $\widetilde{Q^*} \neq 1$. If $r = 1$ and $\widetilde{Q} \leqslant K_1$, then $\widetilde{Q^*} = \widetilde{Q} \neq 1$. So indeed $\widetilde{Q^*} \neq 1$.

Recall that $\widetilde{P}$ is $p$-minimal and thus also $p$-irreducible. Hence 1.29(c) shows that $O^p(\widetilde{P}) = [O^p(\widetilde{P}), \widetilde{Q^*}]$. Also

$$(*) \qquad O^p(\widetilde{P}) = O^p(\widetilde{P^*}) = O^p\Big(\bigtimes_{i=1}^{r} K_i\Big) = \bigtimes_{i=1}^{r} O^p(K_i).$$

As $O^p(\widetilde{P}) = [O^p(\widetilde{P}), \widetilde{Q^*}]$ and $\widetilde{Q^*}$ normalizes each $O^p(K_i)$, this gives $[O^p(K_i), \widetilde{Q^*}] = O^p(K_i)$.

Let $K_i^* \leqslant P^*$ be minimal with $\widetilde{K_i^*} = O^p(K_i)$ and $[K_i^*, Q^*] \leqslant K_i^*$. Observe that

$$[\widetilde{K_i^*, Q_i^*}] = [\widetilde{K_i^*}, \widetilde{Q_i^*}] = [O^p(K_i), \widetilde{Q_i^*}] = O^p(K_i),$$

and the minimality of $K_i^*$ gives $K_i^* = [K_i^*, Q_i^*]$ and $K_i^* = O^p(K_i^*)$. By (c) $O^p(P^*) = L_\circ$ and so $K_i^* \leqslant L_\circ$. Thus by ($4^\circ$) $[C_P(Y_L), Q^* K_i^*] \leqslant [C_L(Y_L), L^\circ] \leqslant O_p(P)$. With $R := \widehat{K_i^*}$ and $D := C_R(Y_L)$ this gives $D \leqslant Z(R\widehat{Q^*})$. Observe that $R/R'$ is an abelian $p'$-group. Since $[R/R', Q^*] = R/R'$, coprime actions shows $C_{R/R'}(Q^*) = 1$ and since $Q^*$ centralizes $D$, we get $D \leqslant R'$. Hence $R$ is a non-split central extension of $R/D \cong \widetilde{R} \cong O^p(SL_2(q)) = SL_2(q)'$ by the $p'$-group $D$.

If $q \leqslant 3$ then $R/D \cong C_3$ or $Q_8$. By [**Hu**, V.25.3] the Schur multiplier of cyclic and quaternion groups is trivial and so $D = 1$. If $q > 3$ then $R/D \cong SL_2(q)' = SL_2(q)$. As the Schur multiplier of $SL_2(q)$ is a $p'$ -group (cf. [**Hu**, V.25.7]) we again have $D = 1$. We have proved:

$10^\circ$. *$O^p(K_i^*) = K_i^*$, $\widehat{K_i^*} \cong O^p(K_i)$ and $C_{K_i^*}(Y_L) \leqslant O_p(P)$.*

Next we show:

11°. *Put* $K^* := \langle K_i^* \mid 1 \leqslant i \leqslant r\rangle$. *Then* $O^p(\widehat{P}) = \widehat{K^*}$, $P = K^*S$ *and* $[\widehat{K_i}, \widehat{K_j}] \leqslant C_{\widehat{K^*}}(Y_L) \leqslant Z\big(\widehat{K^*Q^*}\big)$ *for all* $1 \neq i < j \leqslant r$.

Note that $O^p(\widehat{P^*}) = C_{\widehat{P^*}}(Y_L)\widehat{K^*}$. As $Q^*$ centralizes $C_{\widehat{P^*}}(Y_L)$ and $[K_i^*, Q^*] = K_i^*$ we conclude $\widehat{K^*} = [\widehat{P^*}, Q^*]$ and $K^* \leqslant L^\circ$. In particular, $S$ normalizes $\widehat{K^*}$. Hence $S$ normalizes $K^*O_p(P)$, and (7°), applied to $P_1 = K^*(S \cap P^*)$, gives $L_\circ = O^p\big(K^*(S \cap P^*)\big) \leqslant O_p(P)K^*$ and $P = L_\circ S = O^p\big((K^*(S \cap P^*)\big)S = K^*S$. By (10°), $O^p(K_i^*) = K_i^*$ and so also $O^p(K^*) = K^*$. Since $S$ normalizes $\widehat{K^*}$ and $P = K^*S$ we conclude that $O^p(\widehat{P}) = \widehat{K^*}$. By (1°) $[K_i, K_j] = 1$ and so $[K_i^*, K_j^*] \leqslant C_{K^*}(Y_L)$. As $K^*Q^* \leqslant L^\circ$ and by (4°) $[C_L(Y_L), L^\circ] \leqslant O_p(P)$ we have $C_{\widehat{K^*}}(Y_L) \leqslant Z\big(\widehat{K^*Q^*}\big)$.

12°. *Suppose that* $[\widehat{K_i^*}, \widehat{K_j^*}] = 1$ *for all* $1 \leqslant i < j \leqslant r$. *Then* $C_P(Y_P) = O_p(P)$ *and* $P$ *is* $p$*-minimal. In particular, (e) and (f) hold.*

Since $[\widehat{K_i^*}, \widehat{K_j^*}] = 1$ we have $|\widehat{K^*}| \leqslant \big|\prod_{i=1}^r \widehat{K_i^*}\big| \leqslant |\widehat{K_i^*}|^r$. Moreover, by (10°) $|\widehat{K_i^*}| = |O^p(K_i)|$ and by (11°) $\widehat{K^*} = O^p(\widehat{P})$. Now $(*)$ implies $|\widetilde{K^*}| = |O^p(\widehat{P})| = |O^p(K_i)|^r$ and so $|\widehat{K^*}| \leqslant |\widetilde{K^*}|$. Since $\widetilde{K^*}$ is a factor group of $\widehat{K^*}$, we get that $|\widetilde{K^*}| = |\widehat{K^*}|$ and $C_{\widehat{K^*}}(Y_L) = 1$. As $O^p(\widehat{P}) = \widehat{K^*}$, it follows that $\widehat{C_P(Y_P)}$ is a $p$-group and so $C_P(Y_P) = O_p(P)$. Hence $P/O_p(P) = \widetilde{P}$. By (3°) $\widetilde{P}$ is $p$-minimal and so also $P$ is $p$-minimal. Therefore (e) and (f) hold.

We now distinguish the cases $r = 1$, $r = 2$ and $r \geqslant 3$. If $r = 1$, we are done by (12°). Assume next that $r = 2$. Since $Q$ acts transitively on $\mathcal{K}$ by (a), we have $p = 2$. Suppose that $q > 2$. Then $q \geqslant 4$, $K_i$ is perfect, and $\widehat{K_i^*}$ is a component of $\widehat{P}$. Thus $[\widehat{K_1}, \widehat{K_2}] = 1$ and we are done by (12°).

Suppose that $q = 2$. Then $|\widehat{K_i^*}| = 3$ and by (12°) we may assume that $[\widehat{K_1^*}, \widehat{K_2^*}] \neq 1$. By (11°) $[\widehat{K_1^*}, \widehat{K_2^*}] \leqslant Z(\widehat{K^*})$. Since $r = 2$, $\widehat{K^*} = \big\langle \widehat{K_1^*}, \widehat{K_2^*}\big\rangle$, and we conclude that that $\widehat{K^*}$ is extra special of order $3^3$. By (11°) $\widehat{K^*} = O^p(\widehat{P})$, and so (f) holds. As $\widehat{P}/\Phi(\widehat{K^*}) \cong \widetilde{P}$ is $p$-minimal, so are $\widehat{P}$ and $P$.

Since $\widehat{K^*}$ is extra special of order $3^3$, any involution in $Aut(\widehat{K^*})$ which centralizes $\Phi(\widehat{K^*})$ inverts $\widehat{K^*}/\Phi(\widehat{K^*})$. By (f) $\widehat{Q}$ centralizes $\Phi(\widehat{K^*})$, and so $\widehat{Q}$ contains only one involution. As $\widetilde{Q^*}$ is non-trivial and elementary abelian and has index 2 in $\widetilde{Q}$ we conclude that $\widetilde{Q} \cong C_4$. Thus (e) holds.

Assume finally that $r \geqslant 3$. Let $i, j, k$ be three different elements in $\{1, \ldots, r\}$. Pick $z \in S \cap K_i \backslash O_p(P)$. Since $Q$ acts transitively on $\mathcal{K}$ we can choose $y \in Q$ with $K_i^y = K_k$. Put $x := [y, z] = z^{-1y}z$. Then $x \in Q^*$, $\widetilde{x} \in K_iK_k$ and $\widetilde{x} \in \widetilde{z}K_k$. Since $K_i \cong SL_2(q)$, we have $K_i = \langle \widetilde{z}^{K_i}\rangle$. Now $[K_i, K_k] = 1$ implies

$$[O^p(K_i), x] = [O^p(K_i), z] = O^p(K_i),$$

and since $[K_iK_k, K_j] = 1$ and $\widetilde{x} \in K_iK_k$, we also have $[K_j, x] = 1$. Recall that $Q^*$ normalizes $\widehat{K_i}$, $\widetilde{K_i^*} = O^p(K_i)$ and $C_{\widehat{K_i}}(Y_L) = 1$. Thus

$$[\widehat{K_i}, \widehat{x}] = \widehat{K_i} \qquad \text{and} \qquad [\widehat{K_j}, \widehat{x}] = 1.$$

In particular,

$$[\widehat{K_j}, \widehat{x}, \widehat{K_i}] = 1.$$

By (11°) $[\widehat{K_i}, \widehat{K_j}] \leqslant Z(\widehat{K}\widehat{Q}^*)$ and so

$$[\widehat{K_i}, \widehat{K_j}, \widehat{x}] = 1.$$

With the Three Subgroups Lemma $[\widehat{x}, \widehat{K_i}, \widehat{K_j}] = 1$, and since $[\widehat{x}, \widehat{K_i}] = \widehat{K_i}$,

$$[\widehat{K_i}, \widehat{K_j}] = 1.$$

Another reference to (12°) completes the proof of the lemma. □

CHAPTER 2

# The Case Subdivision and Preliminary Results

In this chapter we give the relevant definitions that allow to subdivide the proof of our main result stated in the introduction. This partition of the proof enables us to treat the different parts independently and sometimes under a slightly more general hypothesis.

We believe that concepts like symmetry, asymmetry, shortness and tallness can also be useful in other situations. In a certain sense they reflect the general behavior of conjugates of (abelian) subgroups in finite groups. In the amalgam method these concepts have already proved their relevance (without getting particular names). For example, symmetry is closely related (and generalizes) the "$b$ even"-case of the amalgam method, while tallness corresponds to the "$b = 1$"-case.

In Section 2.2 general properties of asymmetry are investigated. Most of these properties are elementary, the exception being 2.15 where the Quadratic L-Lemma of [**MS6**] is used and so also a $\mathcal{K}_p$-group Hypothesis is needed.

Finally in Section 2.3 symmetric pairs are introduced. It is probably our most complicated and technical definition. Also the existence of symmetric pairs requires a rather long and sophisticated argument, see 2.22 and 2.23.

In this chapter $G$ is a finite group, $S \in Syl_p(G)$, and $Q$ is a large $p$-subgroup of $G$ contained in $S$. Moreover, $M \in \mathfrak{M}_G(S)$ and $M^\dagger = MC_G(Y_M)$. So $M$ fulfills the basic property defined in the Introduction.

### 2.1. Notation and Elementary Properties

NOTATION 2.1. Recall from the introduction that $Q^\bullet = O_p(N_G(Q))$ and that $L$ is $Y_M$*-minimal* if $L = \langle Y_M^L \rangle$ and $Y_M$ is contained in a unique maximal subgroup of $L$.

Let $A$ be an abelian $p$-subgroup of $G$. Then

- $A$ is *symmetric* in $G$ if there exist $A_1, A_2 \in A^G$ such that $1 \neq [A_1, A_2] \leqslant A_1 \cap A_2$,
- $A$ is *asymmetric* in $G$ if $A$ is not symmetric in $G$.

Let $\mathcal{N}$ be a set of subgroups of $G$. Then

- $A$ is $\mathcal{N}$*-tall* if there exist $T \in Syl_p(C_G(A))$ and $L \in \mathcal{N}$ such that $T \leqslant L$ and $A \nleqslant O_p(L)$,
- $A$ is $\mathcal{N}$*-short* if $A \leqslant O_p(L)$ for all $T \in Syl_p(C_G(A))$ and $L \in \mathcal{N}$ with $T \leqslant L$. (So $A$ is $\mathcal{N}$-short if and only if $A$ is not $\mathcal{N}$-tall.)
- $A$ is *tall (short)* if $A$ is $\mathcal{N}$-tall ($\mathcal{N}$-short), where $\mathcal{N}$ is the set of all subgroups $L$ of $G$ with $O_p(L) \neq 1$,
- $A$ is *char p-tall (char p-short)* if $A$ is $\mathcal{N}$-tall ($\mathcal{N}$-short), where $\mathcal{N}$ is the set of all subgroups of characteristic $p$ of $G$,
- A is $Q$*-tall* ($Q$*-short)* if $A$ is $\mathcal{N}$-tall ($\mathcal{N}$-short), where $\mathcal{N} = N_G(Q)^G$.

For $K \leqslant G$ with $O_p(M) \leqslant K$ let $\mathfrak{H}_K(O_p(M))$ be the set of subgroups $H$ of $K$ such that

(i) $H$ is of characteristic $p$,
(ii) $O_p(M) \leqslant H$ and $Y_M \nleqslant O_p(H)$, and
(iii) $Y_M \leqslant O_p(P)$ whenever $P$ is proper subgroup of $H$ containing $O_p(M)$.

For $K \leqslant G$ with $Y_M \leqslant K$ let $\mathfrak{L}_K(Y_M)$ be the set of subgroups $L$ of $K$ such that

(i) $Y_M \leqslant L$ and $O_p(L) = \langle (Y_M \cap O_p(L))^L \rangle \leqslant N_L(Y_M)$,
(ii) $L$ is $Y_M$-minimal and of characteristic $p$,

(iii) $N_L(Y_M)$ is the unique maximal subgroup of $L$ containing $Y_M$, and
(iv) $L/O_p(L) \cong SL_2(q)$, $Sz(q)$, $q := |Y_M/Y_M \cap O_p(L)|$ or $L/O_p(L) \cong D_{2r}$, where $p = 2$ in the last two cases and $r$ is an odd.

Note that, since $Y_M = \Omega_1 Z(O_p(M))$, $\mathfrak{H}_K(O_p(M))$ only depends on $K$ and $O_p(M)$.

We use the following subdivision:

*The Symmetric Case.* $Y_M$ is symmetric in $G$.
*The Short Asymmetric Case.* $Y_M$ is short and asymmetric in $G$.
*The Tall char p-Short Asymmetric Case.* $Y_M$ is tall, *char* $p$-short and asymmetric in $G$.
*The char p-Tall Q-short Asymmetric Case.* $Y_M$ is *char* $p$-tall, $Q$-short and asymmetric in $G$.
*The Q-Tall Asymmetric Case.* $Y_M$ is $Q$-tall and asymmetric in $G$.

LEMMA 2.2. (a) *$C_G(O_p(M)) \leqslant O_p(M)$.*
(b) *$Q$ is a weakly closed subgroup of $G$.*
(c) *$N_G(K) \leqslant M^\dagger$ for all $1 \neq K \trianglelefteq M$, in particular, $N_G(O_p(M)) \leqslant M^\dagger$.*
(d) *$M^\dagger = N_G(Y_M) = MC_G(Y_M)$.*
(e) *$Y_M = \Omega_1 Z(O_p(M))$.*
(f) *$O_p(M) \in Syl_p(C_G(Y_M))$; in particular, $C_S(Y_M) = O_p(M)$.*
(g) *$Q \trianglelefteq M$ if and only if $Q \trianglelefteq M^\dagger$.*
(h) *$M^\circ = (M^\dagger)^\circ$.*

PROOF. (a): We have $M \in \mathfrak{M}_G(S) \subseteq \mathfrak{M}_G \subseteq \mathcal{L}_G$ and so $C_G(O_p(M)) \leqslant O_p(M)$ by definition of $\mathcal{L}_G$.

(b): Since $Q$ is a large subgroup of $G$, 1.52(b) shows that $Q$ is a weakly closed subgroup of $G$.

(c) Put $R := N_G(K)$. Since $M$ has characteristic $p$ and $K \trianglelefteq M$, also $K$ has characteristic $p$, see 1.2(a). In particular, $O_p(K) \neq 1$ and so also $O_p(R) \neq 1$. Note that $S \leqslant M \leqslant R$ and so $R$ is a parabolic subgroup of $G$. Thus 1.55(a) implies that $C_G(O_p(R)) \leqslant O_p(R)$ and so $R \in \mathcal{L}_G$. Let $R^*$ be maximal in $\mathcal{L}_G$ with $R \leqslant R^*$. Since $M \leqslant R \leqslant R^*$, $R^* \in \mathcal{M}_G(M)$. By the basic property of $M \in \mathfrak{M}_G$, we have $\mathcal{M}_G(M) = \{M^\dagger\}$ and so $R^* = M^\dagger$ and $R \leqslant M^\dagger$.

(d): By (b), $N_G(Y_M) \leqslant M^\dagger$, and by the basic property of $M$, $M^\dagger = MC_G(Y_M)$ and $Y_M = Y_{M^\dagger}$. So $M^\dagger \leqslant N_G(Y_M)$ and (d) holds.

(e),(f): By the basic property of $M$, $C_M(Y_M)$ is $p$-closed. Thus 1.24(k) gives $Y_M = \Omega_1 Z(O_p(M))$ and $O_p(M) \in Syl_p(C_G(Y_M))$.

(g), (h): By $Q!$, $C_G(Y_M) \leqslant N_G(Q)$ and so $M^\dagger = MC_G(Y_M) = MN_{M^\dagger}(Q)$. Thus $Q \trianglelefteq M$ if and only if $Q \trianglelefteq M^\dagger$. Moreover, by 1.52(c) $M^\circ = \big(MC_G(Y_M)\big)^\circ = (M^\dagger)^\circ$. □

LEMMA 2.3. *Let $A \leqslant Z(Q)$. Then the following hold:*
(a) *Let $g \in G$ and $\widetilde{A} \leqslant Z(Q^g)$ such that $[A, \widetilde{A}] \leqslant A \cap \widetilde{A}$. Then $[A, \widetilde{A}] = 1$.*
(b) *$A$ is asymmetric in $G$.*
(c) *Suppose that $B \leqslant G$ is a $Q$-short abelian $p$-subgroup, $A \leqslant Z(Q^\bullet)$ and $A \cap B \neq 1$. Then $[A, B] = 1$.*

PROOF. (a): If $g \in N_G(Q)$, then $A\widetilde{A} \leqslant Z(Q)$ and $[A, \widetilde{A}] = 1$. If $g \notin N_G(Q)$, then 1.52(e) gives $Z(Q) \cap Z(Q^g) = 1$. Thus
$$[A, \widetilde{A}] \leqslant A \cap \widetilde{A} \leqslant Z(Q) \cap Z(Q^g) = 1.$$
(b): This is a direct consequence of (a) and the definition of asymmetric.

(c): Assume that $R := A \cap B \neq 1$. Then $R \leqslant A \leqslant C_G(Q)$ and $Q!$ implies $N_G(R) \leqslant N_G(Q)$. Since $B$ is abelian, $B \leqslant C_G(B) \leqslant N_G(R) \leqslant N_G(Q)$. In particular, $N_G(Q)$ contains a Sylow $p$-subgroup of $C_G(B)$, and as $B$ is $Q$-short, we conclude that $B \leqslant O_p(N_G(Q)) = Q^\bullet$. So $[B, A] = 1$. □

LEMMA 2.4. (a) *$O_p(M)$ is a weakly closed subgroup of $M^\dagger$.*

(b) *Let $B$ be a $p$-subgroup of $M^\dagger$ with $O_p(M) \leqslant B$. If $N_G(B) \nleqslant M^\dagger$, then $Y_M$ is symmetric in $G$.*

PROOF. (a): By the basic property of $M$, $Y_M = Y_{M^\dagger}$ is normal in $M^\dagger$. Hence $C_G(Y_M) = C_{M^\dagger}(Y_M) \trianglelefteq M^\dagger$. By 2.2(f) $O_p(M) \in Syl_p(C_G(Y_M))$. Sylow subgroups of normal subgroups are clearly weakly closed (even strongly closed) subgroups of the whole group.

(b): Since $S$ is a Sylow $p$-subgroup of $M^\dagger$, there exists $g \in M^\dagger$ with $B^g \leqslant S$. Then $O_p(M)^g \leqslant B^g \leqslant S$, and since $O_p(M)$ is a weakly closed subgroup of $M^\dagger$, $O_p(M) = O_p(M)^g$. So replacing $B$ by $B^g$ we may assume that $B \leqslant S \leqslant M$. We will now verify that the assumptions of E.16(a) are fulfilled. Note that $O_p(M) \leqslant B \trianglelefteq N_G(B)$ and $Y_M$ is a non-trivial normal $p$-subgroup of $M$. By 2.2(f), $O_p(M) \in Syl_p(C_G(Y_M))$ and so $C_M(Y_M)$ is $p$-closed. By assumption, $N_G(B) \nleqslant M^\dagger$. By 2.2(c), $N_G(K) \leqslant M^\dagger$ for all $1 \neq K \trianglelefteq M$, and so no non-trivial $p$-subgroup of $M \cap N_G(B)$ is normal in $M$ and $N_G(B)$.

Thus indeed all assumptions of E.16(a) are fulfilled for $H_1 := M$, $H_2 := N_G(B)$, $A_1 := Y_M$ and $H := G$. Hence there exists $h \in H = G$ with $1 \neq [A_1, A_1^h] \leqslant A_1 \cap A_1^h$, and so $A_1 = Y_M$ is symmetric in $G$. □

## 2.2. Asymmetry

LEMMA 2.5. *Suppose that $Y_M$ is asymmetric in $G$. Let $Y_M \leqslant R \leqslant M^\dagger$. Then $\langle Y_M^{N_G(R)} \rangle$ is elementary abelian.*

PROOF. Recall from the basic property that $Y_M = Y_{M^\dagger}$, so $Y_M \trianglelefteq R$ since $R \leqslant M^\dagger$. Thus $Y_M^x$ is normal in $R$ for every $x \in N_G(R)$. Hence, the claim is an immediate consequence of the definition of asymmetry. □

LEMMA 2.6. *Suppose that $Y_M$ is asymmetric in $G$. Then the following hold:*

(a) *Let $L$ be a $p$-subgroup of $G$ with $O_p(M) \leqslant L$. Then $N_G(L) \leqslant N_G(O_p(M)) \leqslant M^\dagger$.*
(b) *$O_p(M)$ is a weakly closed subgroup of $G$.*
(c) *Let $O_p(M) \leqslant L \leqslant G$. Then $L \cap M^\dagger$ is a parabolic subgroup of $L$.*
(d) *$x^G \cap Y_M = x^M$ for every $x \in Y_M$.*
(e) *$Y_M$ is $Q$-tall if and only if $Y_M \nleqslant O_p(N_G(Q))$.*

PROOF. (a): Put $B := M^\dagger \cap L$. Since $Y_M$ is asymmetric, 2.4(b) implies that $N_G(B) \leqslant M^\dagger$. In particular, $N_L(B) \leqslant M^\dagger \cap L = B$ and so $L = B$ since $L$ is a $p$-group. By 2.4(a) $O_p(M)$ is a weakly closed subgroup of $M^\dagger$. Thus

$$N_G(L) = N_G(B) = N_{M^\dagger}(B) \leqslant N_{M^\dagger}(O_p(M)) \leqslant N_G(O_p(M)) \overset{2.2(c)}{\leqslant} M^\dagger,$$

and (a) is proved.

(b): By (a) $O_p(M) \trianglelefteq N_G(L)$ for all $p$-subgroups $L$ of $G$ containing $O_p(M)$. Thus 1.45 shows that $O_p(M)$ is a weakly closed subgroup of $G$.

(c): Since $O_p(M)$ is a weakly closed subgroup of $G$, 1.46(g) shows that $N_L(O_p(M))$ is a parabolic subgroup of $L$, and since $N_L(O_p(M)) \leqslant L \cap M^\dagger$, also $L \cap M^\dagger$ is a parabolic subgroup of $L$.

(d): Since $O_p(M)$ is a weakly closed subgroup of $G$ and $Y_M \leqslant Z(O_p(M))$, 1.48(b) shows that $x^G \cap Y_M = x^{N_G(O_p(M))}$. By 2.2(c) $N_G(O_p(M)) \leqslant M^\dagger = C_G(Y_M)M$, and so (d) holds.

(e): Recall from 2.2(f) that $O_p(M) \in Syl_p(C_G(Y_M))$.

Suppose that $Y_M \nleqslant O_p(N_G(Q))$. Since $O_p(M) \leqslant N_G(Q)$, we conclude that $Y_M$ is $Q$-tall.

Suppose that $Y_M$ is $Q$-tall. Then there exists $g \in G$ such that $O_p(M) \leqslant N_G(Q^g)$ and $Y_M \nleqslant O_p(N_G(Q^g))$. Since $O_p(M)$ is a weakly closed subgroup of $G$ by (b), $Q^g \leqslant N_G(O_p(M))$ and since $Q$ is a weakly closed subgroup of $G$ by 1.52(b), $Q^{gh} = Q$ for some $h \in N_G(O_p(M)) \leqslant N_G(Y_M)$. Thus $Y_M \nleqslant O_p(N_G(Q))$. □

LEMMA 2.7. *Suppose that $Y_M$ is asymmetric in $G$.*

(a) *Let $g \in G$ with $C_{Y_M}(Q^g) \neq 1$. Then $Q^g \leqslant M^\circ$.*
(b) *Let $1 \neq U_0 \leqslant U \leqslant Y_M$. Put $E_U := \langle Q^g \mid g \in G, C_U(Q^g) \neq 1\rangle$. Then $N_G(U_0)^\circ \leqslant E_U \leqslant M^\circ$.*

PROOF. (a): Let $g \in G$ with $C_{Y_M}(Q^g) \neq 1$. Since $1 \neq C_{Y_M}(Q^g) \leqslant Y_M \leqslant C_G(O_p(M))$, $Q!$ implies $O_p(M) \leqslant N_G(Q^g)$. By 2.6(b) $O_p(M)$ is a weakly closed subgroup of $G$ and so $Q^g \leqslant N_G(O_p(M))$. By 2.2(c) $N_G(O_p(M)) \leqslant M^\dagger$. By 2.2(h), $(M^\dagger)^\circ = M^\circ$ and so $Q^g \leqslant M^\circ$.

(b): Let $h \in G$ with $Q^h \leqslant N_G(U_0)$. Then $C_{U_0}(Q^h) \neq 1$, so $C_U(Q^h) \neq 1$ and $Q^h \leqslant E_U$. Thus $N_G(U_0)^\circ \leqslant E_U$. By (a) $E_U \leqslant M^\circ$, and so (b) holds. □

LEMMA 2.8. *Let $F$ be the inverse image of $F^*(M^\dagger/C_{M^\dagger}(Y_M))$ in $M^\dagger$. Suppose that $Y_M$ is asymmetric in $G$, $F \leqslant H \leqslant G$ and $H$ is of characteristic $p$. Then $Y_M \leqslant Y_H$.*

PROOF. Since $Y_M$ is asymmetric in $G$ and $O_p(M) \leqslant F \leqslant H$, 2.6(c) implies that $H \cap M^\dagger$ contains a Sylow $p$-subgroup of $H$. Thus by 1.24(f), $Y_{M^\dagger \cap H} \leqslant Y_H$.

Now let $\overline{M^\dagger} := M^\dagger/C_{M^\dagger}(Y_M)$. Then $O_p(\overline{F}) \leqslant O_p(\overline{M^\dagger}) = 1$ and $C_{\overline{M^\dagger \cap H}}(\overline{F}) \leqslant \overline{F}$. Note that $\overline{F} \trianglelefteq \overline{M^\dagger \cap H}$ and so $[O_p(\overline{M^\dagger \cap H}), \overline{F}] \leqslant O_p(\overline{F}) = 1$. It follows that $O_p(\overline{M^\dagger \cap H}) = 1$. Thus $Y_M$ is $p$-reduced for $M^\dagger \cap H$ and so $Y_M \leqslant Y_{M^\dagger \cap H} \leqslant Y_H$. □

LEMMA 2.9. *Suppose that $Y_M$ is asymmetric in $G$ and that there exists a subgroup $H^*$ of characteristic $p$ such that $O_p(M) \leqslant H^*$ and $Y_M \nleqslant O_p(H^*)$. Let $H \leqslant H^*$ be minimal with $O_p(M) \leqslant H$ and $Y_M \nleqslant O_p(H)$. Then $H \in \mathfrak{H}_G(O_p(M))$.*

PROOF. By 2.2(a) $C_G(O_p(M)) \leqslant O_p(M)$. Since $Y_M$ is asymmetric in $G$, 2.6(b) shows that $O_p(M)$ is a weakly closed subgroup of $G$. Thus the hypothesis of 1.49 are fulfilled and we conclude that $H$ is of characteristic $p$. Let $O_p(M) \leqslant P < H$. Then the minimal choice of $H$ implies that $Y_M \leqslant O_p(P)$ and so $H \in \mathfrak{H}_G(O_p(M))$. □

LEMMA 2.10. *Suppose that $Y_M$ is char $p$-tall and asymmetric in $G$. Then $\mathfrak{H}_G(O_p(M)) \neq \varnothing$.*

PROOF. Since $Y_M$ is *char* $p$-tall there exists $H^* \leqslant G$ such that $H^*$ is of characteristic $p$, $Y_M \nleqslant O_p(H^*)$, and $H^*$ contains a Sylow $p$-subgroup of $C_G(Y_M)$. By 2.2(e), $O_p(M) \in Syl_p(C_G(Y_M))$ and after conjugation in $C_G(Y_M)$ we may assume that $O_p(M) \leqslant H^*$. Then by 2.9 $\mathfrak{H}_G(O_p(M)) \neq \varnothing$. □

LEMMA 2.11. *Suppose that $Y_M$ is char $p$-tall and asymmetric in $G$. Let $H \in \mathfrak{H}_G(O_p(M))$ and put $\overline{H} := H/O_p(H)$. Then the following hold:*

(a) *Let $T_H \in Syl_p(H \cap M^\dagger)$. Then $T_H \in Syl_p(H)$. In particular, $H = O^p(H)T_H$.*
(b) *$O_p(H)$ normalizes $O_p(M)$ and $Y_M$.*
(c) *$O^p(H) = [O^p(H), Y_M]$ and $H = O^p(H)O_p(M)$.*
(d) *$O^p(H)Y_M = \langle Y_M^{O^p(H)}\rangle = \langle Y_M^H\rangle \trianglelefteq H$ and $H = \langle O_p(M)^H\rangle$.*
(e) *Let $N \trianglelefteq H$. Then either $O^p(H) \leqslant N$, or $N$ is $p$-closed and $[\overline{N}, \overline{Y}_M\overline{O^p(H)}] = 1$. In particular, $H$ is $p$-irreducible.*
(f) *$Z(\overline{O^p(H)Y_M})$ is a normal $p'$-subgroup of $\overline{H}$.*
(g) *$\Phi(\overline{H}) = \Phi(\overline{O^p(H)}) = Z(\overline{O^p(H)Y_M})$, and $\overline{O^p(H)}/\Phi(\overline{H})$ is a minimal normal subgroup of $\overline{H}/\Phi(\overline{H})$.*
(h) *Either $\overline{O^p(H)}$ is a $q$-group for some prime $q \neq p$, or $\overline{O^p(H)}$ is a product of components, which are permuted transitively by $\overline{O_p(M)}$.*
(i) *If $Y_M \leqslant L \leqslant H$ and $L = \langle Y_M^L\rangle$, then $[O_p(H), L] \leqslant O_p(L)$, and $L$ is of characteristic $p$.*

Proof. Put $H_0 := O^p(H)$, and let $T_H \in Syl_p(H)$ with $O_p(M) \leqslant T_H$.

(a): By 2.6(c) $H \cap M^\dagger$ is a parabolic subgroup of $H$ and so (a) holds.

(b): Since $O_p(M)O_p(H)$ is $p$-group containing $O_p(M)$, 2.6(a) shows that

$$O_p(M)O_p(H) \leqslant N_G(O_p(M)) \leqslant M^\dagger = N_G(Y_M).$$

(c): By (a) $H = H_0T_H$, so $[H_0, Y_M]Y_M$ is normal in $H$. Hence $O_p([H_0, Y_M]Y_M) \leqslant O_p(H)$ and thus $Y_M \nleqslant O_p([H_0, Y_M]Y_M)$ since $Y_M \nleqslant O_p(H)$. Now the definition of $\mathfrak{H}_G(O_p(M))$ implies $[H_0, Y_M]O_p(M) = H$, so $H_0 = O^p(H) \leqslant [H_0, Y_M] \leqslant H_0$, and (c) follows.

(d): By (c) $H = H_0O_p(M)$. Thus $H_0Y_M$ is normal in $H$ and so $\langle Y_M^H\rangle \leqslant H_0Y_M$. Also by (c) $H_0 = [H_0, Y_M]$. We get

$$H_0Y_M = [H_0, Y_M]Y_M = \langle Y_M^{H_0}\rangle \leqslant \langle Y_M^H\rangle \leqslant H_0Y_M.$$

Hence equality holds everywhere and the first statement in (d) is proved. Similarly,

$$H = H_0O_p(M) = [H_0, Y_M]O_p(M) \leqslant [H, O_p(M)]O_p(M) \leqslant \langle O_p(M)^H\rangle \leqslant H,$$

and so $H = \langle O_p(M)^H\rangle$.

(e): By the definition of $\mathfrak{H}_G(O_p(M))$, $H = NT_H$ or $Y_M \leqslant O_p(NT_H)$. In the first case $H_0 \leqslant N$. In the second case

$$[N, Y_M] \leqslant N \cap O_p(NT_H) \leqslant O_p(N) \leqslant O_p(H),$$

so $[\overline{N}, \overline{Y_M}] = 1$. Since by (d) $\overline{H_0Y_M} = \langle \overline{Y_M}^{\overline{H_0}}\rangle$, we conclude that $[\overline{N}, \overline{H_0Y_M}] = 1$; in particular, $H_0$ centralizes $T_H \cap N$. By (a) $H = H_0T_H$. Thus $T_H \cap N$ is normal in $H$ and $N$ is $p$-closed. So (e) is proved.

(f): Put $\overline{D} := Z(\overline{H_0Y_M})$. By (a) $H = H_0T_H$, so $\overline{D} \trianglelefteq \overline{H}$, and since $\overline{D}$ is abelian and $O_p(\overline{H}) = 1$, $\overline{D}$ is a $p'$-group.

(g): Since $O_p(\overline{H}) = 1$, 1.7(a) shows that $\Phi(\overline{H_0}) = \Phi\big(\overline{O^p(H)}\big) = \Phi(\overline{H})$. In particular, $\overline{H_0} \nleqslant \Phi(\overline{H})$. Hence (e) shows that $\Phi(\overline{H}) \leqslant Z(\overline{H_0Y_M}) =: \overline{D}$.

Suppose that $\overline{D} \nleqslant \Phi(\overline{H})$. Then there exists a maximal subgroup $\overline{K}$ of $\overline{H}$ with $\overline{D} \nleqslant \overline{K}$. By (f) $\overline{D}$ is a normal $p'$-subgroup of $\overline{H}$, so $\overline{H} = \overline{DK}$ and $\overline{K}$ contains a Sylow $p$-subgroup of $\overline{H}$. Thus we may choose $\overline{K}$ such that $\overline{O_p(M)} \leqslant \overline{K}$. Hence the definition of $\mathfrak{H}_G(O_p(M))$ gives $\overline{Y_M} \leqslant O_p(\overline{K})$. Since $[\overline{D}, \overline{Y_M}] = 1$ this shows that $\langle \overline{Y_M}^{\overline{H}}\rangle = \langle \overline{Y_M}^{\overline{DK}}\rangle = \langle \overline{Y_M}^{\overline{K}}\rangle$ is $p$-group, a contradiction to $O_p(\overline{H}) = 1$. Thus $\overline{D} = \Phi(\overline{H})$, and the first part of (g) is proved.

By (e) every normal subgroup of $\overline{H}$ properly contained in $\overline{H_0}$ is contained in $\overline{D}$. Hence $\overline{H_0}/\overline{D}$ is a minimal normal subgroup of $\overline{H}$ and (g) is proved.

(h): By (g) $\overline{H_0}/\overline{\Phi(H)}$ is a minimal normal subgroup of $\overline{H}$. So either $\overline{H_0}/\overline{\Phi(H)}$ is a $q$-group for some prime $q$ or $\overline{H_0}/\overline{\Phi(H)}$ is the direct product of non-abelian simple groups transitively permuted by $\overline{H}$. By (g) $\Phi(\overline{H_0}) = \Phi(\overline{H}) \leqslant Z(\overline{H_0})$. So in the first case, $\overline{H_0}$ is nilpotent and

$$\overline{H_0} = \Phi(\overline{H})O_q(\overline{H_0}) = \Phi(\overline{H_0})O_q(\overline{H_0}),$$

so $\overline{H_0} = O_q(\overline{H_0})$ is a $q$-group.

In the second case, $\overline{H_0} = \Phi(\overline{H_0})\overline{H_0}'$, $\overline{H_0} = \overline{H_0}'$ and $\overline{H_0}$ is the product of components transitively permuted by $\overline{H}$. In particular, each component of $\overline{H_0}$ is normal in $\overline{H_0}$, and since $\overline{H} = \overline{H_0}\overline{O_p(M)}$, already $\overline{O_p(M)}$ permutes the components transitively.

(i): By (b) $O_p(H)$ normalizes $Y_M$ and so $[O_p(H), Y_M] \leqslant Y_M \leqslant L$. Since $L = \langle Y_M^L\rangle$ this gives

$$[O_p(H), L] = [O_p(H), \langle Y_M^L\rangle] \leqslant O_p(H) \cap L \leqslant O_p(H) \cap O_p(L).$$

It follows that $C_L(O_p(L))$ acts quadratically on $O_p(H)$. By the definition of $\mathfrak{H}_G(O_p(M))$, $H$ is of characteristic $p$, and so 1.4(a) shows that $C_L(O_p(L))$ is a $p$-group. Hence $L$ is of characteristic $p$. $\square$

Lemma 2.12. *Let $H \in \mathfrak{H}_G(O_p(M))$. Suppose that $Y_M$ is asymmetric in $G$ and that there exists $g \in G$ such that $H \leqslant N_G(Q^g)$. Then the following hold:*

(a) $Y_M$ *is* $Q$*-tall.*
(b) $Q^g \leqslant O_p(HQ^g) \leqslant N_G(O_p(M)) \leqslant N_G(Y_M) = M^\dagger$.
(c) $H \trianglelefteq HQ^g$*; in particular* $O^p(H) = O^p(HQ^g)$.
(d) $HQ^g$ *is* $p$*-irreducible.*

PROOF. (a): By the definition of $\mathfrak{H}_G(O_p(M))$, $Y_M \nleqslant O_p(H)$ and $O_p(M) \leqslant H$. Since $H \leqslant N_G(Q^g)$ this gives $Y_M \nleqslant O_p(N_G(Q^g))$ and $O_p(M) \leqslant N_G(Q^g)$. By 2.2(f), $O_p(M) \in Syl_p(C_G(Y_M))$ and so $Y_M$ is $Q$-tall.

(b): Clearly $Q^g \leqslant O_p(HQ^g)$ and $O_p(M)O_p(HQ^g)$ is a $p$-group. By 2.6(b) $O_p(M)$ is a weakly closed subgroup of $G$, and we conclude that $Q^g \leqslant O_p(HQ^g) \leqslant N_G(O_p(M))$. By 2.2(c),(d), $N_G(O_p(M)) \leqslant M^\dagger = N_G(Y_M)$, and so (b) is proved.

(c): By (b), $Q^g$ normalizes $O_p(M)$ and thus also every $N_G(Q^g)$-conjugate of $O_p(M)$. Since by 2.11(d) $H = \langle O_p(M)^H \rangle$ and $H \leqslant N_G(Q^g)$, $Q^g$ normalizes $H$ and $H \trianglelefteq HQ^g$.

(d): By 2.11(e) $H$ is $p$-irreducible. Since $H$ normalizes $Q^g$, 1.30(b) shows that $HQ^g$ is $p$-irreducible. □

Recall the definition of a minimal asymmetric module from Definition A.4 for the next lemma.

LEMMA 2.13. *Suppose that* $Y_M$ *is char* $p$*-tall and asymmetric in* $G$*. Let* $H \in \mathfrak{H}_G(O_p(M))$ *and let* $V$ *be a non-central* $H$*-chief factor of* $O_p(H)$*. Put* $\widetilde{H} := H/C_H(V)$*,* $A := \widetilde{Y_M}$ *and* $B := \widetilde{O_p(M)}$*. Then* $V$ *is a faithful simple minimal asymmetric* $\mathbb{F}_p\widetilde{H}$*-module with respect to* $A$ *and* $B$*.*

PROOF. We have to verify A.4 (i) – (iv). By 2.6(b) $O_p(M)$ is a weakly closed subgroup of $G$ and so by 1.46(j) $B = \widetilde{O_p(M)}$ is a weakly closed subgroup of $\widetilde{H}$. Hence A.4(i) holds.

By 2.11(b), $O_p(H)$ normalizes $Y_M$ and $O_p(M)$. Therefore,

$$[O_p(H), Y_M] \leqslant Y_M \leqslant C_G(O_p(M)) \text{ and } [O_p(H), O_p(M)] \leqslant O_p(M) \leqslant C_G(Y_M).$$

Thus

$$[O_p(H), Y_M, O_p(M)] = 1 \text{ and } [O_p(H), O_p(M), Y_M] = 1,$$

and Property A.4(ii) holds.

Assume for a contradiction that $\langle Y_M^H \rangle$ acts nilpotently on $V$. Since $V$ is a chief factor and so a simple $H$-module, $[V, \langle Y_M^H \rangle] = 1$. By 2.11(c), $O^p(H) \leqslant \langle Y_M^H \rangle$ and thus $[V, O^p(H)] = 1$. But then $V$ is a central $H$-chief factor, a contradiction. So $\langle Y_M^H \rangle$ does not act nilpotently on $V$, and A.4(iii) holds.

Finally let $C_H(V) \leqslant P \leqslant H$ such that $B \leqslant \widetilde{P} < \widetilde{H}$. Then $P$ is a proper subgroup of $H$ containing $O_p(M)$, so by the definition of $\mathfrak{H}_G(O_p(M))$, $Y_M \leqslant O_p(P)$. By 2.6(a) $O_p(P) \leqslant M^\dagger$ and thus by 2.5 $\langle Y_M^P \rangle$ is elementary abelian. Let $W$ be the inverse image of $V$ in $H$. Then

$$[W, \langle Y_M^P \rangle] \leqslant W \cap \langle Y_M^P \rangle \text{ and } [W, \langle Y_M^P \rangle, \langle Y_M^P \rangle] = 1.$$

This gives A.4(iv). □

LEMMA 2.14. *Let* $L \in \mathfrak{L}_G(Y_M)$ *and put* $A := O_p(L)$*. Then* $Y_M A/A$ *is the unique non-trivial elementary abelian normal* $p$*-subgroup of* $N_L(Y_M)/A$*.*

PROOF. Let $T \in Syl_p(N_L(Y))$. By definition of $\mathfrak{L}_G(Y_M)$,

$$L/A \cong SL_2(q), Sz(q), \text{ or } Dih_{2r},$$

where $p = 2$ in the last two cases, $r$ is an odd prime, and $N_L(Y_M)$ is the unique maximal subgroup $L$ containing $Y_M$. If $L/A \cong SL_2(q)$ or $Sz(q)$, then $N_L(Y)/A$ is a Borel subgroup of $L/A$ and $T/A = O_p(N_L(Y)/A)$.

In the $SL_2(q)$-case $T/A$ is elementary abelian and $N_L(Y)$ acts simply on $T/A$. Thus $YA/A = T/A$ and the lemma holds.

In the $Sz(q)$-case all involutions of $T/A$ are contained in $Z(T/A)$, and $N_L(Y)$ acts simply on $Z(T/A)$. Thus $YA/A = Z(T/A)$, and the lemma holds.

In the $Dih_{2r}$-case, $N_L(Y) = T$ and $|T/A| = 2$, and the lemma holds. □

For the next lemma recall the definition of $\mathfrak{L}_K(Y_M)$ from 2.1 and the definition of a $\mathcal{CK}$-group from C.1.

LEMMA 2.15. *Let $Y_M \leqslant L \leqslant G$ and suppose that $L$ is a $\mathcal{CK}$-group of characteristic $p$. Then $L \in \mathfrak{L}_G(Y_M)$ if and only if $L$ is $Y_M$-minimal and $N_L(Y_M)$ is a maximal subgroup of $L$.*

PROOF. If $L \in \mathfrak{L}_G(Y_M)$, then by definition, $L$ is $Y_M$-minimal and $N_L(Y_M)$ is a maximal subgroup of $L$.

Suppose now that $L$ is $Y_M$-minimal and $N_L(Y_M)$ is a maximal subgroup of $L$. Let $T \in Syl_p(N_L(Y_M))$ with $Y_M \leqslant T$. By 1.42(b), $N_L(T) \leqslant N_L(Y_M)$ and $O_p(L) \leqslant T \in Syl_p(L)$. In particular, $O_p(L) \leqslant N_L(Y_M)$.

Let $V$ be the direct sum of the $L$-chief factors on $O_p(L)$ (in a given chief series). Since $L$ is of characteristic $p$, 1.4(c) shows that $C_L(V) \leqslant O_p(L)$, and since $O_p(L) \leqslant C_L(V)$, we get $C_L(V) = O_p(L)$. Hence $V$ is a faithful $\mathbb{F}_pL/O_p(L)$-module.

As $O_p(L)$ normalizes $Y_M$, we have $[O_p(L), Y_M] \leqslant Y_M$ and $[O_p(L), Y_M, Y_M] = 1$. It follows that $Y_M$ acts quadratically on $V$, and we can apply the Quadratic $L$-Lemma [**MS6**, Lemma 2.9] to $L/O_p(L)$ and $V$. This gives

$$L/O_p(L) \;\cong SL_2(q), Sz(q) \text{ or } Dih_{2r^k},$$

where $q$ is a power of $p$, $r$ is an odd prime, and $p = 2$ in the last two cases.

Set $X := \langle (Y_M \cap O_p(L))^L \rangle$ and $\widehat{L} := L/X$. Suppose that $|\widehat{Y_M}| = 2$. Then $\widehat{L}$ is a dihedral group, but not a 2-group. So there exists $\widehat{Y_M} \leqslant \widehat{D} \leqslant \widehat{L}$ with $\widehat{D} \cong Dih_{2r}$, $r$ an odd prime. Then $\widehat{Y}_M \ntrianglelefteq \widehat{D}$ and so $D \nleqslant L \cap M^\dagger$. Since $L$ is $Y_M$-minimal with $L \cap M^\dagger$ being the maximal subgroup containing $Y_M$, we conclude that $\widehat{D} = \widehat{L}$ and $X = O_2(L)$.

Hence we may assume that $|\widehat{Y_M}| > 2$. In particular, $L/O_p(L) \cong SL_2(q)$ or $Sz(q)$. As seen above, $N_L(T) \leqslant N_L(Y_M)$ and $T \in Syl_p(L)$. It follows that $N_L(Y_M)//O_p(L)$ is a Borel subgroup of $L/O_p(L)$ and normalizes the elementary abelian group $Y_M O_p(L)/O_p(L)$. Thus the structure of $SL_2(q)$ and $Sz(q)$ shows that $q = |Y_M O_p(L)/O_p(L)| = |\widehat{Y_M}|$. It remains to show that $O_p(L) = X$.

Suppose for a contradiction that $O_p(L) \neq X$. Since $[O_p(L), Y_M] \leqslant Y_M \cap O_p(L) \leqslant X$, $\widehat{L}$ is a non-trivial central extension of $L/O_p(L)$ by a $p$-group. Hence [**Gr1**] shows that either $q = 9$ and $\widehat{L} \sim 3{\cdot}SL_2(9)$ or $q = 8$ and $\widehat{L} \sim 2^a{\cdot}Sz(8)$, $1 \leqslant a \leqslant 2$. In both cases, since $\widehat{T}$ is a Sylow $p$-subgroup of $\widetilde{L}$, $O_p(\widehat{L}) = Z(\widehat{T})$. In particular $\widehat{Y_M} \cap Z(\widehat{T}) = 1$, a contradiction as $\widehat{T}$ normalizes $\widehat{Y_M}$. □

LEMMA 2.16 (Asymmetric L-Lemma). *Suppose that $Y_M$ is char $p$-tall and asymmetric in $G$. Let $H \in \mathfrak{H}_G(O_p(M))$ and $L$ be minimal among all subgroups $L \leqslant H$ satisfying $Y_M \leqslant L$ and $Y_M \nleqslant O_p(L)$. Then the following hold:*

(a) $H = \langle L, O_p(M) \rangle = \langle Y_M^h, O_p(M) \rangle$ *for all* $h \in L \backslash N_L(Y_M)$.
(b) *$L$ is $Y_M$-minimal and of characteristic $p$, and $N_L(Y_M)$ is the unique maximal subgroup of $L$ containing $Y_M$.*
(c) $[V, O^p(L)] \neq 1$ *for all non-central chief factors $V$ of $H$ on $O_p(H)$.*
(d) $\langle (O_p(L) \cap Y_M)^L \rangle \leqslant O_p(H)$.
(e) *Suppose that $L$ is a $\mathcal{CK}$-group. Then $L \in \mathfrak{L}_H(Y_M)$ and $O_p(L) \leqslant O_p(H)$.*

PROOF. Define

$$H^* := \langle Y_M^H \rangle,\; B := [O_p(H), H^*] = \langle [O_p(H), Y_M]^H \rangle, \;\text{ and }\; P := N_L([O_p(H), Y_M]) \cap N_L(C_B(Y_M))$$

1°. $L = \langle Y_M, Y_M^g \rangle$ *for some* $g \in L$. *In particular,* $L = \langle Y_M^L \rangle$.

Suppose that $\langle Y_M, Y_M^g \rangle$ is a $p$-group for all $g \in L$. Then Baer's Theorem [**KS**, 6.7.6] shows that $Y_M \leqslant O_p(L)$, a contradiction to the choice of $L$. Thus there exists $g \in L$ such that $\langle Y_M, Y_M^g \rangle$ is not $p$-group. Then $Y_M \nleqslant O_p(\langle Y_M, Y_M^g \rangle)$ and the minimal choice of $L$ gives $\langle Y_M, Y_M^g \rangle = L$.

In the following we fix $g \in L$ such hat $L = \langle Y_M, Y_M^g \rangle$.

2°. $H = \langle L, O_p(M) \rangle = \langle Y_M^g, O_p(M) \rangle$.

Note that $O_p(M) \leqslant \langle L, O_p(M)\rangle$ and $Y_M \nleqslant O_p(\langle L, O_p(M)\rangle$. So the definition of $\mathfrak{H}_G(O_p(M))$ gives $H = \langle L, O_p(M)\rangle = \langle Y_M^g, O_p(M)\rangle$.

3°. *$[O_p(H), Y_M] \leqslant Y_M \cap B \leqslant C_B(O_p(M)) \leqslant C_B(Y_M)$ and $[B, O_p(M)] \leqslant B \cap O_p(M) \leqslant C_B(Y_M)$.*

By 2.11(b) $O_p(H)$ normalizes $Y_M$ and $O_p(M)$. By definition of $B$, $[O_p(M), Y_M] \leqslant B \leqslant O_p(H)$ and so (3°) holds.

4°. *$L$ has characteristic $p$.*

By (1°), $L = \langle Y_M^L\rangle$, and so (4°) follows from 2.11(i).

5°. *$B = [O_p(H), Y_M^g]C_B(Y_M)$.*

Since $[B, Y_M^g] \leqslant [O_p(H), Y_M^g]$ and by (3°), $[B, O_p(M)] \leqslant C_B(Y_M)$, both $Y_M^g$ and $O_p(M)$ normalize $[O_p(H), Y_M^g]C_B(Y_M)$. As $H = \langle Y_M^g, O_p(M)\rangle$ and $B = \langle [O_p(H), Y_M^g]^H\rangle$, this gives $B = [O_p(H), Y_M^g]C_B(Y_M)$.

6°. *$C_B(Y_M) = [O_p(H), Y_M]C_B(L)$.*

Since $[O_p(H), Y_M^g] \leqslant C_B(Y_M^g)$ and $L = \langle Y_M, Y_M^g\rangle$, (5°) implies

$$C_B(Y_M^g) = [O_p(H), Y_M^g]\big(C_B(Y_M) \cap C_B(Y_M^g)\big) = [O_p(H), Y_M^g]C_B(L).$$

7°. *$P = N_L(Y_M) = N_L(O_p(M))$.*

By 2.2(e), $Y_M = \Omega_1 Z(O_p(M))$ and so $N_L(O_p(M)) \leqslant N_L(Y_M)$. Clearly $N_L(Y_M)$ normalizes $[O_p(H), Y_M]$ and $C_B(Y_M)$. Thus $N_L(Y_M) \leqslant P$. So it remains to show that $P \leqslant N_L(O_p(M))$.

Both, $P$ and $O_p(M)$ normalize the series

$$1 \leqslant [O_p(H), Y_M] \leqslant C_B(Y_M) \leqslant B \leqslant O_p(H).$$

By (3°) $[B, O_p(M)] \leqslant C_B(Y_M)$ and $[O_p(H), Y_M] \leqslant Y_M$, so $O_p(M)$ centralizes $[O_p(H), Y_M]$ and $B/C_B(Y_M)$. As $L = \langle Y_M, Y_M^g\rangle \leqslant H^*$, we know that $L$ centralizes $O_p(H)/B$. By (6°) $C_B(Y_M) = [O_p(H), Y_M]C_B(L)$. Since $P \leqslant L$ and $P$ normalizes $[O_p(H), Y_M]$, we conclude that $P$ centralizes $C_B(Y_M)/[O_p(H), Y_M]$. It follows that $[P, O_p(M)]$ centralizes all factors of the above series and so acts nilpotently on $O_p(H)$. As $H$ is of characteristic $p$, 1.4(a) implies that that $[P, O_p(M)]$ is a $p$-group. So $[P, O_p(M)]O_p(M)$ is a $p$-group normalized by $P$ and since $O_p(M)$ is a weakly closed subgroup of $G$, $P \leqslant N_L(O_p(M))$.

8°. *$Y_M$ is a weakly closed subgroup of $L$.*

Let $r \in L$ with $[Y_M, Y_M^r] \leqslant Y_M \cap Y_M^r$. By 1.45(b) it suffices to show that $Y_M = Y_M^r$.

As $Y_M$ is asymmetric in $G$, $[Y_M, Y_M^r] \leqslant Y_M \cap Y_M^r$ implies $[Y_M, Y_M^r] = 1$. By (3°) $[O_p(H), Y_M] \leqslant Y_M$ and so $[O_p(H), Y_M] \leqslant C_B(Y_M^r)$. Now (6°) gives $C_B(Y_M) \leqslant C_B(Y_M^r)$ and so $C_B(Y_M) = C_B(Y_M)^r$. So $r$ normalizes $C_B(Y_M)$. Put $W := [O_p(H), L]$. Since $L = \langle Y_M, Y_M^g\rangle$, 1.40 shows that $[O_p(H), Y_M] = C_W(Y_M)$. Note that $W \leqslant B$, and so $W \cap C_B(Y_M) = C_W(Y_M) = [O_p(H), Y_M]$. Thus $r$ also normalizes $[O_p(H), Y_M]$ and so $r \in P$. By (7°) $P = N_L(Y_M)$. Hence $Y_M^r = Y_M$, and (8°) is proved.

9°. *$L$ is $Y_M$-minimal, and $N_L(Y_M)$ is the unique maximal subgroup of $L$ containing $Y_M$.*

Let $Y_M \leqslant U < L$. By the minimal choice of $L$, $Y_M \leqslant O_p(U)$. By (8°) $Y_M$ is a weakly closed subgroup of $L$ and so $Y_M \trianglelefteq U$. Thus $N_L(Y_M)$ is the unique maximal subgroup of $L$ containing $Y_M$. By (1°) $L = \langle Y_M^L\rangle$ and thus $L$ is $Y_M$-minimal.

10°. *$O_p(L) \cap Y_M \leqslant O_p(H)$.*

By (9°) $O_p(L) \leqslant N_L(Y_M)$ and so also $O_p(L) \leqslant N_L(Y_M^g)$. By (3°), $[O_p(H), Y_M^g] \leqslant Y_M^g$ and thus

$$\big[[O_p(H), Y_M^g], O_p(L) \cap Y_M\big] \leqslant Y_M \cap Y_M^g \cap B \leqslant C_B(\langle Y_M^g, O_p(M)\rangle) = C_B(H).$$

By (5°) $B = [O_p(H), Y_M^g]C_B(Y_M)$ and so $[B, O_p(L) \cap Y_M] \leqslant C_B(H)$. Hence $O_p(L) \cap Y_M$ centralizes all factor of the $H$-invariant series

$$1 \leqslant C_B(H) \leqslant B \leqslant O_p(H).$$

Since $H$ is of characteristic $p$, 1.4(c) shows that $O_p(L) \cap Y_M \leqslant O_p(H)$. So (10°) holds.

(a), (b), (d): This follows from (2°), (4°) and (9°), and (10°), respectively.

(c): Let $V$ be a non-central $H$-chief factor on $O_p(H)$ and assume that $O^p(L) \leqslant C_H(V)$. By (1°) $L = \langle Y_M^L \rangle$, so $L = O^p(L)Y_M$. Thus (2°) implies

$$H = \langle L, O_p(M)\rangle = \langle O^p(L)Y_M, O_p(M)\rangle \leqslant C_H(V)O_p(M),$$

and $[V, O^p(H)] = 1$. But then $V$ is a central $H$-factor, a contradiction.

(e): By (9°) $L$ is $Y_M$-minimal and $N_L(Y_M)$ is a maximal subgroup of $L$. Thus 2.15 shows that $L \in \mathfrak{L}_G(Y_M)$. In particular, $O_p(L) = \langle (O_p(L) \cap Y_M)^L \rangle$. By (10°) $O_p(L) \cap Y_M \leqslant O_p(H)$ and so $O_p(L) \leqslant O_p(H)$. □

For the next lemma recall the definition of a quasisimple module from A.2.

LEMMA 2.17. *Suppose that $G$ is a $\mathcal{K}_p$-group, $Y_M$ is char $p$-tall and asymmetric in $G$ and there exists $H \in \mathfrak{H}_G(O_p(M))$ with $[\Omega_1 Z(O_p(H)), O^p(H)] \neq 1$. Then there exist $L \in \mathfrak{L}_H(Y_M)$ and a quasisimple $H$-submodule $V$ of $Y_H$. Moreover, the following holds for any such $L$ and $V$ and $W := [V, L]$:*

(a) $H = \langle O_p(M), L\rangle$.
(b) $W \leqslant Z(O_p(L))$ *and* $[O_p(H), L] \leqslant O_p(L) \leqslant O_p(H) \leqslant N_H(Y_M)$.
(c) $1 \neq W = [W, L] = [W, O^p(L)] = [V, O^p(L)]$, $V = WC_V(Y_M) = WC_V(L)$, *and $W$ is a non-trivial strong offender on $Y_M$.*
(d) $C_V(O^p(L)) = C_V(L)$, *and* $C_V(O^p(H)) = C_V(\langle Y_M^H\rangle)$.
(e) $W \cap C_{Y_M}(L) = C_W(O^p(L)) = C_W(O^p(H)) = C_W(H)$.
(f) $[W, Y_M] = [W, X]$ *for every* $X \leqslant Y_M$ *with* $|X/C_X(W)| > 2$.

PROOF. Let $V_0$ be minimal in $\Omega_1 Z(O_p(H))$ with $[V_0, O^p(H)] \neq 1$. By 2.11(g), $H$ is $p$-irreducible and so by 1.34(c), $V_0$ is quasisimple. In particular, $V_0$ is $p$-reduced for $H$ and so $V_0 \leqslant Y_H$ by definition of $Y_H$. By definition of $\mathfrak{H}_G(O_p(M))$, $Y_M \nleqslant O_p(H)$ and so we can choose $L_0 \leqslant H$ minimal with respect to $Y_M \leqslant L_0$ and $Y_M \nleqslant O_p(L_0)$. By 2.16(e) $L_0 \in \mathfrak{L}_H(Y_M)$. This shows the existence of $L$ and $V$.

Now let $V$ be any quasisimple $H$-submodule of $Y_H$ and $L \in \mathfrak{L}_H(Y_M)$. Then $V/C_V(O^p(H))$ is a non-central chief factor for $H$ on $O_p(H)$. Let $Y_M \leqslant R < L$. By definition of $\mathfrak{L}_G(Y_M)$, $L$ is $Y$-minimal and $N_L(Y_M)$ is the unique maximal subgroup of $L$ containing $Y_M$. Thus $R \leqslant N_L(Y_M)$ and so $Y_M \trianglelefteq R$ and $Y_M \leqslant O_p(R)$. So $L$ satisfies the assumptions of 2.16. We conclude that $H = \langle L, O_p(M)\rangle$, $O_p(L) \leqslant O_p(H)$ and $[V, O^p(L)] \neq 1$. In particular, (a) holds and $W := [V, L] \neq 1$.

(b): Since $W \leqslant V \leqslant Y_H \leqslant Z(O_p(H))$ and $O_p(L) \leqslant O_p(H)$ we have $W \leqslant Z(O_p(L))$. By 2.11(i), $[O_p(H), L] \leqslant O_p(L)$, and by 2.11(b), $O_p(H)$ normalizes $Y_M$. Thus

$$W \leqslant [O_p(H), L] \leqslant O_p(L) \leqslant O_p(H) \leqslant N_H(Y_M),$$

and (b) holds.

(c): By 1.43(o) $W$ is a strong offender on $Y_M$. Let $h \in L \backslash N_L(Y_M)$. By 1.42(f), $L = \langle Y_M, Y_M^h\rangle$, and by 2.16(a), $H = \langle Y_L^h, O_p(M)\rangle$. As $V$ is a perfect [1] $H$-module, $V = [V, H] = [V, Y_M^h][V, O_p(M)]$. By 2.11(b), $O_p(H)$ normalizes $O_p(M)$, so $[V, O_p(M)] \leqslant V \cap O_p(M) \leqslant C_V(Y_M)$, and we conclude that

$$V = [V, Y_M^h]C_V(Y_M).$$

In particular, $V = WC_Y(Y_M)$. Moreover, $[V, Y_M] \leqslant V \cap Y_M \leqslant C_V(Y_M)$, and since also $h^{-1} \in L \backslash N_L(Y_M)$, $V = [V, Y_M]C_V(Y_M^h)$, so

$$C_V(Y_M) = [V, Y_M]\big(C_V(Y_M) \cap C_V(Y_M^h)\big) = [V, Y_M]C_V(L).$$

[1] for the definition of a perfect module see A.2

Hence $V = [V, Y_M^h][V, Y_M]C_V(L) = WC_V(L)$. It follows that $W = [V, L] = [W, L]$, $W = [W, O^p(L)] = [V, L, O^p(L)] = [V, O^p(L)]$ and $[W, Y_M] = [V, Y_M]$. Since $V$ is quasisimple, we have $[V, O^p(H)] \neq 1$. As $O^p(H) \leqslant \langle Y_M^H \rangle$ by 2.11(d), this gives $[W, Y_M] = [V, Y_M] \neq 1$. So (c) holds.

(d): Since $V = WC_V(L)$, $C_V(O^p(L)) = C_W(O^p(L))C_V(L)$. By 1.43(h), $C_W(O^p(L)) = C_W(L)$ and so $C_V(O^p(L)) = C_V(L)$. Since $O^p(H) \leqslant \langle Y_M^H \rangle$ we have $C_V(\langle Y_M^H \rangle) \leqslant C_V(O^p(H))$. Also

$$C_V(O^p(H)) \leqslant C_V(O^p(L)) = C_V(L) \leqslant C_V(Y_M),$$

and so $\langle Y_M^H \rangle$ centralizes $C_V(O^p(H))$.

(e): By 1.43(h) $C_W(O^p(L)) = W \cap C_{Y_M}(L)$, and so

$$C_W(H) \leqslant C_W(O^p(H)) \leqslant C_W(O^p(L)) = W \cap C_{Y_M}(L) \leqslant C_W(\langle L, O_p(M) \rangle) \overset{\text{(a)}}{=} C_W(H).$$

Hence (e) holds.

(f): If $|Y_M/Y_M \cap O_p(L)| = p$, there is nothing to prove. Thus, we may assume that $q := |Y_M/Y_M \cap O_p(L)| > p$. Since $L \in \mathfrak{L}_H(Y_M)$, we get $L/O_p(L) \cong SL_2(q)$ or $Sz(q)$; in particular, $L/O_p(L)$ is quasisimple.

Assume that $q$ is odd. Then $L/O_p(L) \cong SL_2(q)$ and $V = [V, Z(L/O_p(L))] \times C_V(Z(L/O_p(L)))$. Put $V_1 := C_V(Z(L/O_p(L)))$. Then $L/C_L(V_1)$ has dihedral Sylow 2-subgroups. As $[V_1, Y_M, Y_M] \leqslant [Y_M, Y_M] = 1$, [**Gor**, Theorem 8.1.2] shows that $Y_M \leqslant C_L(V_1)$ and $V_1 = C_V(L)$. Hence $W = [V, L] = [V, Z(L/O_p(L))]$ and $C_W(L) = 1$. On the other hand 1.43(d) shows that $[W/C_W(L), x] = [W/C_W(L), Y_M]$ for all $x \in Y_M \backslash C_{Y_M}(X)$, and so (f) holds.

Assume now that $q$ is even. Let $X \leqslant Y_M$ such that $|X/X \cap O_2(L)| \geqslant 4$. Then there exists $y \in Y_M$ and $g \in L$ such that

$$L = \langle X, y^g \rangle O_2(L) = \langle X, X^g \rangle O_2(L) = \langle Y_M, Y_M^g \rangle O_2(L).$$

Put $\overline{L} := L/C_L(W)$. Since by (b) $O_p(L) \leqslant C_L(W)$, we get $\overline{L} = \langle \overline{X}, \overline{X}^{\overline{g}} \rangle$ and so are allowed to apply 1.40 with $\overline{L}$, $W$ and $\overline{X}$ in place of $L$, $V$ and $X$. This gives $C_{[W,L]}(X) = [W, X]$. As $[W, X] \leqslant [W, Y_M] \leqslant C_{[W,L]}(X)$ we conclude $[W, Y_M] = [W, X]$, and (f) is proved. □

LEMMA 2.18. *Let $L \in \mathfrak{L}_G(Y_M)$ and put $A := O_p(L)$, $Y := Y_M$ and $\tilde{q} := |Y/Y \cap A|$.*

(a) *Let $h \in L$. If $h$ is not a $p$-element then $h$ acts fixed-point freely on $A/C_Y(L)$.*
(b) *Let $U$ be any chief factor for $N_L(Y)$ on $AY/C_Y(L)$. Then $|U| = \tilde{q}$, and if $\tilde{q} > 2$, then $U = [U, N_L(Y)]$.*
(c) *Let $U$ be any $N_L(Y)$-invariant section of $AY/C_Y(L)$. Then $|U|$ is a power of $\tilde{q}$.*

PROOF. (a): Recall form the definition of $\mathfrak{L}_G(Y_M)$ that $L/A \cong Dih_{2r}$, $r$ an odd prime, $SL_2(\tilde{q})$ or $Sz(\tilde{q})$. Moreover, by 1.43(p) $L$ has no central chief factors in $A/C_Y(L)$. Thus, the claim is obvious if $L/A \cong Dih_{2r}$.

So suppose that $L/A \cong SL_2(\tilde{q})$ or $Sz(\tilde{q})$. Then by C.15 every chief factor is a natural module for $L/A$. As the non-trivial $p'$-elements of $L/A$ act fixed-point freely on these modules, they also act fixed-point freely on $A/C_Y(L)$.

(b): Now let $U$ be a chief factor for $N_L(Y)$ on $A/C_Y(L)$. Then the $p'$-elements of $N_L(Y)$ acts fixed-point freely on $U$, so $U$ is a faithful simple module for $N_L(Y)/O_p(N_L(Y))$ over $\mathbb{F}_p$. Since $N_L(Y)/O_p(N_L(Y))$ is cyclic of order $\tilde{q} - 1$, we get that $|U| = \tilde{q}$, and if $\tilde{q} > 2$, $U = [U, N_L(Y)]$.

Since $YA/A$ is a simple $N_L(Y)$-module of order $\tilde{q}$ we conclude that (b) holds.

(c) follows immediately from (b). □

## 2.3. Symmetric Pairs

In this section we study how $Y_M$ is embedded in parabolic subgroups of $G$ if $Y_M$ is short and asymmetric in $G$.

DEFINITION 2.19. Let $Y$ be a conjugate of $Y_M$ in $G$. A subgroup $L \leqslant G$ is a *$Y$-indicator* if either

(1) $L$ is $p$-group and $Y \trianglelefteq L$, or
(2) $L$ is $p$-minimal, $Y \leqslant O_p(L)$, $N_L(Y)$ is a maximal and parabolic subgroup of $L$ (so $Y \ntrianglelefteq L$), and one of the following holds:
 (i) There exists $Q_0 \in Q^G$ with $Q_0^\bullet \leqslant N_G(Y)$ and $L \leqslant N_G(Q_0)$.
 (ii) There exists $T \in Syl_p(N_G(Y))$ such that $T \cap L \in Syl_p(N_L(Y))$, $[\Omega_1 Z(T), O^p(L)] \neq 1$, and $[Y, O^p(L)] \nleqslant [\Omega_1 Z(T), O^p(L)]$.

A pair $(Y_1, Y_2)$ of conjugates of $Y_M$ is a *symmetric pair* if there exist $Y_i$-indicators $L_i$, $i = 1, 2$, such that for $V_i := \langle Y_i^{L_i} \rangle$

$$V_1 V_2 \leqslant L_1 \cap L_2 \qquad \text{and} \qquad [V_1, V_2] \neq 1.$$

LEMMA 2.20. *Suppose that $Y_M$ is asymmetric in $G$. Let $Y$ be a conjugate of $Y_M$ and $L$ be a $Y$-indicator. Then $\langle Y^L \rangle$ is elementary abelian.*

PROOF. Without loss of generality we may assume that $Y = Y_M$. We discuss the cases given in 2.19. Observe that in every case $Y_M \leqslant O_p(L)$. In case 2.19(1) $Y_M \trianglelefteq L$, so the lemma holds in this case.

In case 2.19(2) $N_L(Y_M)$ is a parabolic subgroup of $L$, so $O_p(L) \leqslant N_L(Y_M) \leqslant M^\dagger$. Now 2.5 yields the assertion. □

LEMMA 2.21. *Suppose that $Y_M$ is $\mathcal{P}_G(S)$-short and asymmetric in $G$ and that $\mathcal{M}_G(S) \neq \{M^\dagger\}$. Then there exists $\widetilde{P} \in \mathcal{P}_G(S)$ such that $\widetilde{P} \cap M^\dagger$ is a maximal subgroup of $\widetilde{P}$. Moreover, for any such $\widetilde{P}$:*

(a) $O_p(\langle M, \widetilde{P} \rangle) = 1$.
(b) $Y_M \leqslant O_p(\widetilde{P}) \leqslant S \leqslant M^\dagger$.
(c) $\langle Y_M^{\widetilde{P}} \rangle$ *is an elementary abelian $p$-group.*

PROOF. Since $\mathcal{M}_G(S) \neq \{M^\dagger\}$ there exists $\widetilde{P} \in \mathcal{L}_G(S)$ with $\widetilde{P} \nleqslant M^\dagger$. We choose $\widetilde{P}$ minimal with this property. Since $O_p(\widetilde{P}) \leqslant S \leqslant \widetilde{P} \cap M^\dagger$, 1.6 shows that $\{U \mid \widetilde{P} \cap M^\dagger \leqslant U \leqslant \widetilde{P}\} \subseteq \mathcal{L}_G(S)$. Thus, the minimal choice of $\widetilde{P}$ implies that $\widetilde{P} \cap M^\dagger$ is a maximal subgroup of $\widetilde{P}$.

Since $Y_M$ is asymmetric in $G$, 2.6(a) implies that $N_G(S) \leqslant M^\dagger$, so $S \ntrianglelefteq \widetilde{P}$. Now the minimality of $\widetilde{P}$ shows that $\widetilde{P} \in \mathcal{P}_G(S)$. This shows the existence of $\widetilde{P}$.

Now suppose that $\widetilde{P} \in \mathcal{P}_G(S)$ such that $\widetilde{P} \cap M^\dagger$ is a maximal subgroup of $\widetilde{P}$. Then $\widetilde{P} \nleqslant M^\dagger$, and since $\mathcal{M}_G(M) = \{M^\dagger\}$, (a) holds.

Note that $O_p(M) \leqslant S \leqslant \widetilde{P}$, $O_p(M) \in Syl_p(C_G(M))$ and $Y_M$ is $\mathcal{P}_G(S)$-short. Thus $Y_M \leqslant O_p(\widetilde{P}) \leqslant S \leqslant M^\dagger$, and (b) is proved.

As $Y_M$ is asymmetric, 2.5 now shows that $\langle Y_M^{\widetilde{P}} \rangle$ is an elementary abelian $p$-group. □

LEMMA 2.22. *Let $L \in \mathcal{P}_G(S)$ such that $L \cap M^\dagger$ is a maximal subgroup of $L$. Suppose that $Q \ntrianglelefteq M^\dagger$ and that $Y_M$ is short and asymmetric in $G$. Then $L$ is a $Y_M$-indicator.*

PROOF. Since $L \in \mathcal{P}_G(S) \subseteq \mathcal{L}_G$, $L$ is of characteristic $p$. Thus 1.24(g) gives

1°. $\quad \Omega_1 Z(S) \leqslant Y_L \leqslant \Omega_1 Z(O_p(L))$.

By 2.21(b), $Y_M \leqslant O_p(L) \leqslant S \leqslant M^\dagger$. Suppose that $Q \leqslant O_p(L)$. Then $L \leqslant N_G(Q)$ by 1.52(a). As $Q^\bullet \leqslant S \leqslant N_G(Y_M)$ we conclude that $L$ satisfies 2.19(2:i) with $Y = Y_M$ and $Q_0 = Q$, so $L$ is a $Y_M$-indicator in this case. Thus we may assume that $Q \nleqslant O_p(L)$. If $[\Omega_1 Z(S), L] = 1$, then $L \leqslant N_G(Q)$ by $Q!$ and $Q \leqslant O_p(L)$. Hence

$2^\circ$. $[\Omega_1 Z(S), O^p(L)] \neq 1$.

So if $[Y_M, O^p(L)] \nleqslant [\Omega_1 Z(S), O^p(L)]$, then $L$ satisfies 2.19(2:ii), with $Y = Y_M$ and $T = S$ and $L$ is a $Y_M$-indicator. Hence we may assume for the rest of the proof that

$3^\circ$. *$Q \nleqslant O_p(L)$ and $[Y_M, O^p(L)] \leqslant [\Omega_1 Z(S), O^p(L)] \leqslant Y_L$.*

In particular,

$4^\circ$. *$Y_M Y_L \trianglelefteq L$.*

By 2.21(a),

$5^\circ$. *$O_p(\langle M, L\rangle) = 1$.*

If $O_p(M) \leqslant O_p(L)$, then 2.6(a) gives $L \leqslant N_G(O_p(L)) \leqslant M^\dagger$, a contradiction. Thus

$6^\circ$. *$O_p(M) \nleqslant O_p(L)$.*

Next we show:

$7^\circ$. *$Y_M \leqslant \Omega_1 Z(O_p(L))$.*

Assume that $R := [Y_M, O_p(L)] \neq 1$. By ($4^\circ$) $Y_M Y_L \trianglelefteq L$ and so $R \trianglelefteq L$ and $C_L(R) \trianglelefteq L$. Since $[R, O_p(M)] = 1$ and $O_p(M) \nleqslant O_p(L)$, $C_L(R)$ is not $p$-closed. Hence 1.37 implies that $O^p(L) \leqslant C_L(R)$. Thus $L = O^p(L)S \leqslant N_G(C_R(Q))$, and since $C_R(Q) \neq 1$, $Q!$ shows that $L \leqslant N_G(Q)$ and $Q \leqslant O_p(L)$, a contradiction to ($3^\circ$).

Let $V := \Omega_1 Z(O_p(L))$, $J := J_L(V)$ (for the definition see A.7), and $\overline{L} := L/C_L(V)$.

$8^\circ$. *$C_S(V) = O_p(L)$, and $V$ is a $p$-reduced $L$-module.*

Put $\overline{N} := O_p(\overline{L}))$ and let $N$ be the inverse image of $\overline{N}$ in $L$. By ($1^\circ$) $\Omega_1 Z(S) \leqslant V$ and by ($2^\circ$) $[\Omega_1 Z(S), O^p(L)] \neq 1$, so $[V, O^p(L)] \neq 1$. Hence $O^p(L) \nleqslant N$ and 1.37 gives that $N$ is $p$-closed. Thus $N \cap S = O_p(L))$, $C_S(V) = O_p(L)$, $\overline{N} = 1$ and $V$ is a $p$-reduced $L$-module.

By A.40 each $A \in \mathcal{A}_{O_p(M)}$ induces a best offender on $V$. In particular $\overline{J_{O_p(M)}(V)} \neq 1$ if $J(O_p(M)) \nleqslant C_L(V)$.

$9^\circ$. *$O_p(L) \leqslant O_p(M)$ and $J(O_p(L)) \nleqslant C_L(V)$; in particular, $\overline{J_{O_p(M)}(V)} \neq 1$.*

By ($7^\circ$) $O_p(L) \leqslant C_S(Y_M) = O_p(M)$, and by ($5^\circ$) $O_p(\langle M, L\rangle) = 1$. Hence no non-trivial characteristic subgroup of $O_p(M)$ is normal in $L$. In particular $J(O_p(M)) \ntrianglelefteq L$. Since $O_p(L) \leqslant O_p(M)$ this gives $J(O_p(M)) \nleqslant O_p(L)$. By ($8^\circ$) $O_p(L) = C_S(V)$ and so $\overline{J_{O_p(M)}(V)} \neq 1$.

$10^\circ$. *There exists subgroups $E_1, \dots, E_k$ in $L$ such that for $i = 1, \dots, k$ and $U_i := E_i C_L(V) \cap M^\dagger$:*

(a) *$\overline{J} = \overline{E_1} \times \cdots \times \overline{E_k}$, $L = JS$, $V = [V, E_1] \times \cdots \times [V, E_k]$, and $\overline{E_1}', \dots, \overline{E_k}'$ are the $J_{\overline{L}}(V)$-components of $\overline{L}$.*

(b) *$\overline{E_i} \cong SL_2(q)$, $q = p^n$, or $p = 2$ and $\overline{E_i} \cong Sym(5)$, and $[V, E_i]$ is the corresponding natural module for $\overline{E_i}$.*

(c) *$\overline{Q}$ is transitive on $\{\overline{E_1}, \dots, \overline{E_k}\}$.*

(d) *$\overline{E_i} \cong SL_2(q)$ and $\overline{U_i} = N_{\overline{E_i}}(\overline{S} \cap \overline{E_i})$, or $\overline{E_i} \cong Sym(5)$ and $\overline{U_i} \cong Sym(4)$.*

Since $V$ is a $Q!$-module for $L$, (a), (b) and (c) are a straightforward application of C.13 and C.24. Note here that the case $\overline{E_i} \cong Sym(2^n + 1)$ in C.13 only appears for $n = 2$ via $Sym(5) \cong O_4^-(2)$ in C.24. Moreover, (d) follows from the structure of the groups given in (b) and the fact that $L$ is $p$-minimal with $L \cap M^\dagger$ being the unique maximal subgroup containing $S$.

In the following we use the notation of ($10^\circ$) and put

$$W_i := [V, E_i], \quad J_0 := J(O_p(M)), \quad R_i := [W_i, J_0].$$

$11^\circ$. *$\overline{E_i} \cong SL_2(q)$, $\overline{O_p(U_i)} = \overline{J_0} \cap \overline{E_i}$, $V \leqslant J_0$, and $|R_i| = q$, $i = 1, \dots, k$. Moreover, $Y_M \cap W_i = R_i$ if $q > 2$.*

By (9°) $J_0 \nleqslant C_L(V)$. Let $A \in \mathcal{A}_{J_0}$ such that $A \nleqslant C_L(V)$ and $|A/C_A(V)|$ is minimal with this property. By C.13

$$\overline{A} = \overline{A} \cap \overline{E_1} \times \cdots \times \overline{A} \cap \overline{E_r} \text{ and } |W_i/C_{W_i}(\overline{A} \cap \overline{E_i})| = |\overline{A} \cap \overline{E_i}|.$$

Hence $|A| = |A_i| = |VC_A(V)|$, where $A_i := (A \cap U_i)C_V(A \cap U_i)$. Since by (9°) $V \leqslant O_p(M)$, we get $VC_A(V) \in \mathcal{A}_{O_p(M)}$ and $A_i \in \mathcal{A}_{O_p(M)}$. In particular, $V \leqslant J_0$. Moreover, the minimality of $A$ implies that $A = A_l$ for some $l \in \{1, \dots, k\}$, so $\overline{A} \leqslant \overline{E_l}$ and $\overline{A} \leqslant O_p(\overline{U_l})$. Suppose that $\overline{E_l} \cong Sym(5)$. Since $\overline{A}$ is an offender on $W_l$, C.4(g) shows that $\overline{A}$ is generated by transpositions, a contradiction since $\overline{U_l} \cong Sym(4)$ and so $O_p(\overline{U_l})$ contains no transpositions.

Thus $\overline{E_i} \cong SL_2(q)$ and by (10°) $\overline{U_i} = N_{\overline{E_i}}(\overline{S} \cap \overline{E_i})$. From the structure of $SL_2(q)$ we get that $\overline{U_i} = (\overline{S} \cap \overline{E_i})\overline{K_i}$, $\overline{K_i} \cong C_{q-1}$. Since $A$ is an offender on $W_l$ we have $|W_l/C_{W_l}(A)| = |\overline{A}| = q$, and $\overline{A} \in Syl_p(\overline{E_l})$. In particular, $\overline{A} = O_p(\overline{U_l}) = \overline{J_0} \cap \overline{E_l}$ and $\overline{J_0} = \overline{A} \times C_{\overline{J_0}}(W_l)$, and $R_l = [W_l, J_0] = C_{W_l}(A)$ is a 1-dimensional $\mathbb{F}_q$-subspace of $W_l$. Since $S$ normalizes $O_p(M)$ and acts transitively on $\{\overline{E_1}, \dots \overline{E_k}\}$ we conclude that the first sentence in (11°) holds.

Assume now that in addition $q > 2$, so $\overline{K_l} \neq 1$. Since $U_l \leqslant M^\dagger$ and $Y_M \leqslant C_V(A)$, $[Y_M, U_l] \leqslant Y_M \cap C_{W_l}(A)$, and either $[Y_M, U_l] = 1$ or $[Y_M, U_l] = C_{W_l}(A) = R_l$. In the latter case the last part of (11°) holds. In the first case $[Y_M, \overline{K_l}] = 1$, and the action of $\overline{K_l}$ on $W_l$ shows that $C_V(\overline{K_l}) = C_V(\overline{E_l})$. But then $Y_M \leqslant C_V(E_l)$ and $E_l \leqslant C_G(Y_M) \leqslant M^\dagger$. Hence $\overline{E_l} = \overline{U_l}$, a contradiction.

12°. $O_p(L) = V = C_L(V)$.

By (11°) $\overline{E_i} \cong SL_2(q)$ and $\overline{O_p(U_i)} = \overline{J_0} \cap \overline{E_i}$ and by (10°)(d) $\overline{U_i} = N_{\overline{E_i}}(\overline{S} \cap \overline{E_i})$. So $\overline{J_0} \cap \overline{E_i} \in Syl_p(\overline{E_i})$ and thus $\overline{J_0} \in Syl_p(\overline{J})$. According to (9°) $O_p(L) \leqslant O_p(M)$ and so $O_p(M) \in Syl_p(O_p(M)J)$. By (10°)(a) $L = JS$ and so $\langle M, L\rangle = \langle M, J\rangle$. Thus by (5°) $O_p(\langle M, J\rangle) = 1$ and so no non-trivial characteristic subgroup of $O_p(M)$ is normal in $O_p(M)J$. Moreover, by (10°)(a) $Z(O_p(M)J) = 1$. Hence, the $C(G,T)$-Theorem [**BHS**], applied to $O_p(M)J$, shows that $[O_p(L), O^p(J)] \leqslant V$ and $[\Phi(O_p(L)), O^p(J)] = 1$. As $O^p(J) = O^p(L)$ and $Z(O_p(M)J) = 1$, we get $\Phi(O_p(L)) = 1$ and so $V = O_p(L)$. Since $L$ is of characteristic $p$, also $V = C_L(V)$.

13°. *$q > p$ and $k = 1$.*

Let $\Omega := \{R_i \mid i = 1, \dots, k\}$. By (11°) $|R_i| = q$, and by (10°)(c) $Q$ is transitive on $\Omega$. We will show that $M$ acts on $\Omega$. For this let $x \in M$ and $i \in \{1, \dots, k\}$. Note that $W_i^x \leqslant V^x \leqslant J_0^x = J_0 \leqslant L$, and so $[W_i^x, J_0] = R_i^x$.

Suppose that $[W_i^x, V] = 1$. By (12°) $C_L(V) = V$ and so $W_i^x \leqslant V = W_1 \times \ldots \times W_r$. Since $[W_i^x, J_0] \neq 1$ we can choose $j \in \{1, \dots, r\}$ such that the projection of $W_i^x$ to $W_j$ is not centralized by $\overline{J_0}$. Then

$$R_j = [W_i^x, \overline{J_0} \cap \overline{E_j}] \leqslant [W_i^x, J_0] = R_i^x.$$

Hence $R_j = R_i^x \in \Omega$.

Suppose that $[W_i^x, V] \neq 1$. Then there exists $j \in \{1, \dots, k\}$ such that $[W_i^x, W_j] \neq 1$. Hence $R_j = [W_i^x, W_j] \leqslant [W_i^x, J_0] = R_i^x$, so again $R_j = R_i^x \in \Omega$.

We have shown that $M$ acts on $\Omega$. Let $\Lambda \subseteq \Omega$ be an orbit of $O_p(M)$ and $R_0 := \prod_{R_\ell \in \Lambda} R_\ell$. Observe that $O_p(M) \leqslant N_G(R_0)$ and $O_p(M) \in Syl_p(C_G(Y_M))$. Hence $\Omega_1 Z(S) \leqslant Y_M \leqslant O_p(N_G(R_0))$ since $Y_M$ is short[2].

Assume that $\Lambda \neq \Omega$. Then there exists $i \in \{1, \dots, k\}$ such that $R_i \nleqslant R_0$. Note that

$$(*) \qquad [R_l, E_j] = 1 \text{ for all } 1 \leqslant l, j \leqslant k \text{ with } l \neq j.$$

Hence $[R_0, E_i] = 1$, so $E_i \leqslant N_G(R_0)$ and $W_i = [\Omega_1 Z(S), E_i] \leqslant O_p(N_G(R_0))$. On the other hand, by (5°) $O_p(\langle M, L\rangle) = 1$. So $V \ntrianglelefteq M$, and there exists $y \in M$ such that $V^y \neq V$. Then $V^y \leqslant J_0$. Moreover, by (10°)(c) $Q$ is transitive on $\Omega$, so $y$ can be chosen such that $[W_i, V^y] = R_i$. Since $M$ acts on $\Omega$, $R_i^{y^{-1}} \in \Omega$ and so there exists $j \in \{1, \dots, k\}$ such that $[W_i, V^y] = R_i = R_j^y$. Conjugating $(*)$ by $y$ gives

$$[R_l^y, E_j^y] = 1 \text{ for all } 1 \leqslant l, j \leqslant k \text{ with } l \neq j.$$

[2]This is the unique place in the proof of this lemma where shortness is needed and not only *char* $p$-shortness

Since $R_j^y = R_i$ and $y$ acts in $\Omega$ this shows $[R_l, E_j^y] = 1$ for all $1 \leqslant l \leqslant k$ with $l \neq i$. Then also $[R_0, E_j^y] = 1$, so $E_j^y \leqslant N_G(R_0)$. Since $W_i \leqslant O_p(N_G(R_0))$ this shows that $[W_i, E_j^y]$ is a $p$-group. But $W_i \leqslant J_0 = J_0^y \leqslant J^y$ and $[V^y, W_i] = R_j^y$. So the action of $J^y$ on $V^y$ implies that $[E_j^y, W_i]$ is not a $p$-group, a contradiction.

We have shown that $\Lambda = \Omega$. Hence $O_p(M)$ is transitive on $\Omega$ and thus also on the groups $\overline{E_1}, \ldots, \overline{E_k}$. Suppose that $q = p$, then the transitivity of $O_p(M)$ shows that $|Y_M| = |C_V(O_p(M))| = p$. Thus $Y_M \leqslant C_G(Q)$ and $Q!$ gives $M^\dagger \leqslant N_G(Y_M) \leqslant N_G(Q)$, a contradiction since $Q \ntrianglelefteq M^\dagger$ by assumption. Therefore $q > p$. Now (11°) shows that $W_i \cap Y_M \neq 1$ and thus $\overline{O_p(M)} \leqslant C_{\overline{L}}(W_i \cap Y_M) \leqslant N_{\overline{L}}(\overline{E_i})$. Hence, the transitivity of $O_p(M)$ gives $k = 1$.

14°. *$J/V \cong SL_2(q)$, $q > p$, and $V$ is a natural $SL_2(q)$-module for $J$.*

By (12°) $V = C_L(V)$, and by (11°) and (13°) $J/V \cong SL_2(q)$, $q > p$. Moreover, by (10°)(b) $V = W_1$ is a natural $SL_2(q)$-module.

15°. *$J_0 \in Syl_p(J)$, and there exists $x \in M \backslash L$ with $J_0 = VV^x$.*

By (5°) $O_p(\langle M, L\rangle) = 1$ and so $M \nleqslant N_G(V)$. Pick $x \in M \backslash N_G(V)$. Then $V \neq V^x$ and so by (12°), $V^x \nleqslant C_L(V)$. Since $M$ normalizes $J_0$, $V^x \leqslant J_0 \leqslant J$. By (14°) $V$ is a natural $SL_2(q)$)-module for $J$, and we conclude that $|C_V(V^x)| = q$. So $|V \cap V^x| \leqslant q$, $|V^xV/V| \geqslant q$ and $VV^x = J_0 \in Syl_p(J)$.

16°. *$O_p(M) = J_0$, $Y_M = C_V(O_p(M))$, $|Y_M| = q$, and $M \cap J$ acts transitively on $Y_M$.*

By (7°) $Y_M \leqslant V$, so $Y_M \leqslant C_V(J_0)$. By (14°) $V$ is a natural $SL_2(q)$-module for $J$, and so $U_i$ acts transitively on $C_V(J_0)$. As $U_i \leqslant M^\dagger$, $U_i$ normalizes $Y_M$ and hence $Y_M = C_V(J_0)$.

It remains to be shown that $O_p(M) = J_0$. Since $O_p(M)$ centralizes the $\mathbb{F}_q$-subspace $Y_M = C_V(J_0)$ of $V$, $O_p(M)$ acts $\mathbb{F}_q$-linearly on $V$. As $GL_2(q)/SL_2(q)$ is a $p'$-group, this gives $\overline{O_p(M)} \leqslant \overline{J}$ and so $O_p(M) \leqslant J$. Since $J_0 \leqslant O_p(M)$ and $J_0 \in Syl_p(J)$, this shows that $O_p(M) = J_0$.

17°. *$p$ is odd.*

Assume that $p = 2$. By (15°) and (16°) $O_2(M) = J_0 = VV^x$. As $V$ is a natural $SL_2(q)$-module for $J$, this implies that $V$ and $V^x$ are the only maximal elementary abelian subgroups of $O_2(M)$, so $|M/N_M(V)| = 2$. But this contradicts the fact that $V$ is normalized by the Sylow 2-subgroup $S$ of $M$.

18°. *$Q \nleqslant J$.*

Assume that $Q \leqslant J$. By (15°) and (16°) $O_p(M) = J_0 \in Syl_p(J)$ and so $Q \leqslant O_p(M)$. Thus by 1.52(a), $N_G(O_p(M)) \leqslant N_G(Q)$. Hence $Q \trianglelefteq M$ and by 2.2(g), $Q \trianglelefteq M^\dagger$, a contradiction to the assumption.

19°. *$M_\circ \leqslant M \cap J$, and $N_G(V)^\circ = (QJ)^\circ$.*

By (16°) $M \cap J$ acts transitively on $Y_M$ and so by 1.57(c), $M^\circ = \langle Q^{M \cap J}\rangle \leqslant Q(M \cap J)$. Thus $M_\circ = O^p(M^\circ) \leqslant M \cap J$. Since $J$ acts transitively on $V$ another application of 1.57(c) gives $N_G(V)^\circ = \langle Q^J\rangle = (QJ)^\circ$.

Put $B := M^\dagger \cap J$ and $\breve{B} := B/C_B(Y_M)$.

20°. *$B = N_J(O_p(M))$, $C_B(Y_M) = O_p(M) = O_p(B)$, $\breve{B}$ is cyclic of order $q-1$ and acts regularly on $Y_M^\sharp$. In particular, $\breve{M_\circ}$ acts fixed-point freely on $Y_M$.*

By (15°), $J_0 \in Syl_p(J)$, and by (16°), $O_p(M) = J_0$ and $Y_M = C_V(O_p(M_0))$. In particular, $O_p(M) \in Syl_p(J)$. By (14°) $J/V \cong SL_2(q)$ and $V$ is a natural $SL_2(q)$-module. It follows that $B = N_J(O_p(M))$, $C_B(Y_M) = O_p(M) = O_p(B)$ and $\breve{B}$ is cyclic of order $q-1$. By (19°), $M_\circ \leqslant B$ and so also the last statement holds.

21°. *$\sqrt{q} + 1 < |\breve{M_\circ}|$.*

The elements of $Q\backslash J$ induce field automorphisms on $J/V$, and by (18°) $Q \nleqslant J$. This shows that $|C_{\breve{B}}(Q)| = q_0 - 1$, where $q_0$ is a power of $p$ with $q_0^p \leqslant q$. By 2.2(c) $B \leqslant N_G(O_p(M)) \leqslant M^\dagger$ and by 2.2(h) $M^\circ = (M^\dagger)^\circ$. So $[B, Q] \leqslant M^\circ$. Since $[\breve{B}, Q]$ is a $p'$-group and $M^\circ/M_\circ$ is a $p$-group, this gives $[\breve{B}, Q] \leqslant \breve{M}_\circ$. Hence $|[\breve{B}, Q]| \leqslant |\breve{M}_\circ|$ and $\breve{B} = [\breve{B}, Q] \times C_{\breve{B}}(Q)$, so

$$q - 1 = |\breve{B}| = |[\breve{B}, Q]||C_{\breve{B}}(Q)| \leqslant |\breve{M}_\circ|(q_0 - 1).$$

As $q_0^p \leqslant q$ and $p > 2$, we have $q_0 < \sqrt{q}$ and we conclude

$$|\breve{M^\circ}| \geqslant \frac{q-1}{q_0 - 1} > \frac{q-1}{\sqrt{q}-1} = \sqrt{q} + 1.$$

22°. *$Y_M$ is a simple $\breve{M}_\circ$-module.*

By (20°) $\breve{M}_\circ$ acts fixed-point freely on $Y_M$ and by (21°) $|\breve{M}_\circ| > \sqrt{q} + 1$. Thus any non-trivial $M^\circ$-invariant section of $Y_M$ has order larger than $\sqrt{q} + 1$. As $|Y_M| = q$ by (16°) we conclude that a composition series for $M^\circ$ on $Y_M$ has at most one factor and so (22°) holds.

We now derive a final contradiction. Put $\mathbb{F} := End_{M_\circ}(Y_M)$, $\mathbb{K} := End_J(V)$ and $\widetilde{\mathbb{K}} := End_{J^x}(V^x)$, where $x$ is as in (15°). By (22°) $\mathbb{F}$ is a finite field, and by (14°) $V$ is the natural $SL_2(q)$-module for $J$, so $\mathbb{K}$ is a finite field of order $q$. By (19°) $M_\circ \leqslant J$, and since $Y_M = C_V(O_p(M))$, $M_\circ$ acts $\mathbb{K}$-linearly on $Y_M$. Thus $\mathbb{F}$ contains a field isomorphic to $\mathbb{K}$. Since $|Y_M| = q = |\mathbb{K}|$ and $|\mathbb{F}| \leqslant |Y_M|$ this show that $\mathbb{F}$ is a field isomorphic to $\mathbb{K}$, indeed $\mathbb{F}$ is the restriction of $\mathbb{K}$ to $Y_M$. Note that $\mathbb{F}$ is invariant under $x$ and so also $\widetilde{\mathbb{K}}$ is a isomorphic to $\mathbb{F}$, and $\mathbb{F}$ is the restriction of $\widetilde{\mathbb{K}}$ to $Y_M$. Moreover, since $\breve{M}_\circ$ is abelian, $\breve{M}_\circ$ embeds into $\mathbb{F}$ via its action on $Y_M$.

Pick $y \in V^x\backslash V$, $v \in V\backslash Y_M$, and $d \in M_\circ$. Then there exists $\mu \in \mathbb{F}$ such that $d$ acts on $Y_M$ as multiplication by $\mu$. Let $\lambda \in \mathbb{K}$ and $\widetilde{\lambda} \in \widetilde{\mathbb{K}}$ such that

$$\lambda|_{Y_M} = \mu = \widetilde{\lambda}|_{Y_M}.$$

Then

$$v^d \in v\lambda^{-1} + Y_M \text{ and } y^d \in y\widetilde{\lambda}^{-1} + Y_M$$

since the action of $d$ on $V$ and $V^x$ has determinant 1. The mappings

$$\begin{array}{lllll} V/Y_M & \to & Y_M & \text{with} & w + Y_M \mapsto [w, y], \quad \text{and} \\ V^x/Y_M & \to & Y_M & \text{with} & w + Y_M \mapsto [v, w]. \end{array}$$

are $\mathbb{K}$- and $\widetilde{\mathbb{K}}$-linear, respectively. It follows that

$$[v, y]\mu = [v, y]^d = [v^d, y^d] = [v\lambda^{-1}, y\widetilde{\lambda}^{-1}] = [v, y]\lambda^{-1}|_{Y_M}\widetilde{\lambda}^{-1}|_{Y_M} = [v, y]\mu^{-2}.$$

This shows that $\mu = \mu^{-2}$ and $\mu^3 = 1$. Since the multiplicative group of $\mathbb{F}$ is cyclic, we get that $|\breve{M}_\circ| \leqslant 3$. Hence (21°) implies that $\sqrt{q} + 1 < 3$, so $q < 4$. On the other hand, by (13°) $p < q$ and by (17°) $p$ is odd, a contradiction. □

LEMMA 2.23. *Suppose that $Y_M$ is short and asymmetric in $G$, that $\mathcal{M}_G(S) \neq \{M^\dagger\}$ and that $Q \ntrianglelefteq M^\dagger$. Then $G$ possesses a symmetric pair.*

PROOF. Note that the assumptions of 2.21 are fulfilled and so we can choose $\widetilde{P} \in \mathcal{P}_G(S)$ as there. Since $Q \ntrianglelefteq M^\dagger$, $\widetilde{P}$ satisfies the hypothesis of 2.22 in place of $L$. Hence $\widetilde{P}$ is a $Y_M$-indicator. We will now verify the assumptions of E.16(b) for $(G, Y_M, M, \widetilde{P})$ in place of $(H, A_1, H_1, H_2)$.

Observe that $Y_M$ is a non-trivial normal $p$-subgroup of $M$ and by 2.2(f) $C_M(Y_M)$ is $p$-closed. By 2.21(a), $O_p(\langle M, \widetilde{P}\rangle) = 1$ and so no nontrivial normal $p$-subgroup of $M \cap \widetilde{P}$ is normal in $M$ and in $\widetilde{P}$. Since $S \leqslant M \cap \widetilde{P}$, $M \cap \widetilde{P}$ is parabolic subgroup of $M$ and $\widetilde{P}$. By 2.21(b) $Y_M \leqslant O_p(\widetilde{P})$, and as $\widetilde{P} \in \mathcal{P}_G(S)$, $\widetilde{P}$ is $p$-minimal and so by 1.37 $p$-irreducible.

We have shown that $(G, Y_M, M, \widetilde{P})$ satisfy the hypothesis of E.16(b) in place of $(H, A_1, H_1, H_2)$. Hence there exist $i \in \{1, 2\}$ and $h \in G$ with $1 \neq [A_i, A_i^h] \neq A_i \cap A_i^h$ and $A_iA_i^h \leqslant H_i \cap H_i^g$, where $A_2 := \langle A_1^{H_2}\rangle = \langle Y_M^{\widetilde{P}}\rangle$. Since $Y_M$ is asymmetric in $G$, we conclude that $i \neq 1$. So $i = 2$. As already observed, $\widetilde{P}$ is a $Y_M$-indicator and thus $(Y_M, Y_M^h)$ is a symmetric pair with indicators $\widetilde{P}$ and $\widetilde{P}^h$. □

## 2.4. Tall Natural Symplectic Modules

LEMMA 2.24. *Let $I$ be a non-trivial normal $p$-subgroup of $M$ and $Y \leqslant O_p(M^\dagger)$. Let $L \leqslant G$ and let $A$ be a normal $p$-subgroup of $L$. Suppose that $I \leqslant A \leqslant M$ and $Y \leqslant L$. If $[Y,A] \leqslant [I,A]$, then $Y \leqslant O_p(L)$.*

PROOF. Let $H$ be the subnormal closure of $Y$ in $L$. Put $W := \langle [I,A]^H\rangle$. Since $[Y,A] \leqslant [I,A]$ and $H = \langle Y^H\rangle$ (see 1.13), we get $[H,A] \leqslant W$, and $H$ acts trivially on $A/W$. Since $I \leqslant A$ we conclude that $H$ normalizes $IW$ and so

$$W = \langle [I,A]^H\rangle = [\langle I^H\rangle, A] \leqslant [IW,A] \leqslant I[W,A].$$

Since $A \leqslant M$, $A$ acts in $IW/I$, and we conclude that $IW/I = [IW/I,A]$. so $IW = I$, and $H$ normalizes $I$. As $I \trianglelefteq M$, 2.2(c) gives $N_G(I) \leqslant M^\dagger$. Thus $H \leqslant M^\dagger$. In particular, since $Y \trianglelefteq\trianglelefteq O_p(M^\dagger) \trianglelefteq M^\dagger$, $Y \trianglelefteq\trianglelefteq H$. Hence $Y = H$ and $Y \trianglelefteq\trianglelefteq L$, so $Y \leqslant O_p(L)$. $\square$

LEMMA 2.25. *Suppose that $p = 2$ and $Y$ is an $M$-submodule of $Y_M$ such that $I := [Y, M_\circ]$ is natural $Sp_{2m}(2^k)'$-module for $M_\circ$, $m \geqslant 1$.*

(a) *Let $L \leqslant G$ and $A$ a normal $p$-subgroup of $L$. Suppose that $I \leqslant A \leqslant M$ and $Y \leqslant L$. Then $Y \leqslant O_p(L)$.*
(b) *If $I \leqslant Q^\bullet$, then $Y \leqslant Q^\bullet$.*

PROOF. (a): Put $q = 2^k$. By 1.55(d), $C_G(M^\circ) = 1$. In particular, $C_Y(M^\circ) = 1$ and so also $C_Y(M_\circ) = 1$.

We claim that $C_M(Y) = C_M(I)$. Indeed by 1.52(c), $[M^\circ, C_M(I)] \leqslant O_p(M) \leqslant C_M(Y)$. Thus $[M_\circ, C_M(I), Y] = 1$. Also $[Y, M_\circ, C_M(I)] = [I, C_M(I)] = 1$, and so the Three Subgroups Lemma implies $[Y, C_M(I), M_\circ] = 1$. Since $C_Y(M_\circ) = 1$, this gives $[Y, C_M(I)] = 1$ and $C_M(Y) = C_M(I)$.

Suppose that $\overline{M^\circ} \not\cong Sp_2(2)'$ and put $\mathbb{K} := End_{M_\circ}(I)$. Then $\mathbb{K}$ is a finite field of order $q$ and $\dim_{\mathbb{K}} I \geqslant 2$. Put $D := \langle I^L\rangle$. Then $D \leqslant A \leqslant M \leqslant N_G(I)$, and so $[D,I] \leqslant I$. Suppose that $I^x$ does not act $\mathbb{K}$-linearly on $I$. Then $|\mathbb{K}| > 2$, and 1.22 shows that $\dim_{\mathbb{K}} I = 1$, a contradiction. Hence $I^x$ and so also $D$ acts $\mathbb{K}$-linearly in $I$. Note that the set of $M_\circ$-invariant symplectic forms on $I$ form a 1-dimensional $\mathbb{K}$-space, on which $D$ acts $\mathbb{K}$-linearly and so trivially. We conclude that $D$ leaves all these forms invariant. Thus $I$ is a natural $Sp_{2m}(q)$- or $Sp_{2m}(q)'$-module for $M_\circ D$. Note that the same statement holds if $\overline{M^\circ} \cong Sp_2(2)'$.

Now C.20 shows that $[I,D] = [Y,D]$, and so 2.24, applied with $D$ in place of $A$, gives $Y \leqslant O_p(L)$.

(b): Just apply (a) with $L = N_G(Q)$ and $A = Q^\bullet$. $\square$

LEMMA 2.26. *Suppose that $X$ is an $M$-submodule of $Y_M$ and a natural $Sp_{2m}(p^k)$-module for $M^\circ$, $2m \geqslant 4$ and $p$ odd. Then $X \leqslant Q^\bullet$.*

PROOF. Note that $X$ is an $\mathbb{F}_q M^\circ$-module equipped with a non-degenerate $M^\circ$-invariant symplectic form, where $q := p^k$. Put $\widetilde{M} = M/C_M(X)$, $X_0 := C_X(Q)$ and $X_1 := [X,Q]$. Note that $X_1 = X_0^\perp$ in the symplectic space $X$. By B.37 $X_0$ is 1-dimensional over $\mathbb{F}_q$ and

$$\widetilde{Q} = C_{\widetilde{M^\circ}}(X_1/X_0) \cap C_{\widetilde{M^\circ}}(X_0).$$

In particular, $[X_1, Q] = X_0$. Moreover, by B.28(b:a)

$$(*) \qquad Z(\widetilde{Q}) = C_{\widetilde{Q}}(X_1).$$

Put

$$H := N_G(Q) \qquad \text{and} \qquad W := \langle X_1^H\rangle.$$

Suppose first that $W$ is non-abelian. Then $[X_1, W] \neq 1$ and we can choose $g \in H$ with $[X_1, X_1^g] \neq 1$. From $X_1 \leqslant Q$ we conclude $X_1^g \leqslant Q$. As $[X_1, Q] = X_0$ and $X_0$ is 1-dimensional we get $[X_1, X_1^g] = X_0$. This gives

$$[X_1, Q] = [X_1, X_1^g] = X_0 = X_0^g = [X_1^g, Q].$$

Thus $[X_1^g, Q] \leqslant X_0 \leqslant C_M(X)$ and $\widetilde{X_1}^g \leqslant Z(\widetilde{Q})$. As $Z(\widetilde{Q}) = C_{\widetilde{Q}}(X_1)$ by $(*)$ this gives $\widetilde{X_1^g} \leqslant C_{\widetilde{Q}}(X_1)$ and so $[X_1^g, X_1] = 1$, a contradiction to the choice of $g$.

Suppose now that $W$ is abelian. Then $W \leqslant C_Q(X_1)$ and so $\widetilde{W} \leqslant C_{\widetilde{Q}}(X_1) = Z(\widetilde{Q})$. Thus $[W,Q] \leqslant C_M(X)$ and $[W,Q,X] = 1$. As $X_1 \leqslant C_X(W)$ we have $[W,X] \leqslant X_1^\perp = X_0 = [X_1,Q] \leqslant [W,Q]$. Also $[Q,X] = X_1 \leqslant W$ and so $X$ centralizes all factors of the series

$$1 \leqslant [W,Q] \leqslant W \leqslant Q.$$

This series is $H$-invariant and so also $\langle X^H \rangle$ centralizes these factors. Hence $\langle X^H \rangle$ acts nilpotently on $Q$. Since $C_G(Q) \leqslant Q$ this implies that $\langle X^H \rangle$ is a $p$-group, see 1.3, and so $X \leqslant \langle X^H \rangle \leqslant O_p(H) = Q^\bullet$.
□

CHAPTER 3

# The Orthogonal Groups

In this chapter we treat a particular situation, which arises in Chapter 4 and in Chapter 5. In this situation, $[Y_M, O^2(M)]$ is a natural $O^\epsilon_{2n}(2)$-module some $M \in \mathfrak{M}_G$. Natural $O^\epsilon_{2n}(2)$-modules for $p = 2$ are the only examples of simple $Q$!-modules $V$ with a non-trivial offender $A$ such that $[V, A]$ does not contain non-trivial 2-central elements of $M$. This forces us to look at centralizers of non-2-central involutions and requires a line of arguments quite different from those of later chapters.

THEOREM C. *Let $G$ be a finite $\mathcal{K}_2$-group and $S \in Syl_2(G)$, and let $Q \leqslant S$ be a large 2-subgroup of $G$. Let $M \in \mathfrak{M}_G(S)$ and suppose that the following hold:*

(i) $M/C_M(Y_M) \cong O^\epsilon_{2n}(2), n \geqslant 2$.
(ii) *$[Y_M, O^2(M)]$ is a natural $O^\epsilon_{2n}(2)$-module for $M/C_M(Y_M)$.*
(iii) *$C_G(y) \nleqslant M^\dagger$ for all non-singular elements $y \in [Y_M, O^2(M)]$.*
(iv) *$Q \ntrianglelefteq M$.*

*Then $C_G(y)$ is not of characteristic 2 for all non-singular elements $y \in [Y_M, O^2(M)]$.*

Here an element of a natural $O^\epsilon_{2n}(2)$-module $V$ is singular if $h(v) = 0$, where $h$ is the $M$-invariant quadratic form on $V$. For the definition of $M^\circ$ see 1.51. Recall from 1.52(b) that $Q$ is a weakly closed subgroup of $G$. In particular, by 1.46(c) $M^\circ = \langle Q^M \rangle$.

### 3.1. Notation and Elementary Properties

In this section we assume the hypothesis of Theorem C apart from C(iii). The first lemma collects elementary facts about a natural $O^\epsilon_{2n}(2)$-module $V$ with quadratic form $h$ and associate symplectic bilinear form $f$.

LEMMA 3.1. *Let $V$ be a natural $O^\epsilon_{2n}(2)$-module for $X = O^\epsilon_{2n}(2)$, $n \geqslant 2$.*

(a) *$X$ is transitive on the non-singular elements of $V$ and on the non-trivial singular elements of $V$.*
(b) *Let $0 \neq z \in V$ be singular. Then $C_X(z) = AK$, where*
  (a) *$K \cong O^\epsilon_{2n-2}(2)$ and $A$ is a natural $O^\epsilon_{2n-2}(2)$-module for $K$.*
  (b) *$[z^\perp, A] = \langle z \rangle$, $C_X(z^\perp) = 1$ and $A$ induces $Hom(z^\perp/\langle z \rangle, \langle z \rangle)$ on $z^\perp$.*
  (c) *$C_X(z)$ is a parabolic subgroup of $X$.*
  (d) *If $(2n, \epsilon) \neq (4, +)$, then $O_2(C_X(z)) = A \leqslant \Omega^\epsilon_{2n}(2)$.*
(c) *Let $y \in V$ be non-singular. Then $C_X(y) = T \times E$, where*
  (a) *$T \cong C_2$, $E \cong Sp_{2n-2}(2)$, $y^\perp$ is a natural $O_{2n-1}(2)$-module for $E$, and $y^\perp/\langle y \rangle$ is a natural $Sp_{2n-2}(2)$-module for $E$.*
  (b) *$T = C_X(y^\perp)$, $[X, T] = \langle y \rangle$, $y^\perp = C_X(T)$, and $T \nleqslant \Omega^\epsilon_{2n}(2)$.*
  (c) *Let $\mathcal{Z}$ be the set of non-trivial singular elements of $y^\perp$. Then $y^\perp = \langle \mathcal{Z} \rangle$, and $E$ acts transitively on $\mathcal{Z}$.*
(d) *Let $0 \neq v \in V$. If $X = O^+_4(2)$ suppose that $v$ is singular. Then $C_X(v)$ is a maximal subgroup of $X$.*

PROOF. (a): Note that $h(v) = 1 = h(w)$ for any two non-singular vectors. It follows that any two non-singular and any two singular vectors are isometric. Thus (a) follows from B.16.

(b): Put $Z := \langle z\rangle$ and $A := C_X(Z) \cap C_X(Z^\perp/Z)$. Let $v \in V\backslash Z^\perp$ and put $K := C_X(z) \cap C_X(v)$. Then B.25(c) shows that (b:a) holds. It follows from B.24(a), that $A$ induces $Hom(Z^\perp/Z)$ on $Z^\perp$, via the commutator map. So (b:b) holds. By B.12(c) any 2-subgroup of $X$ centralizes a non-trivial singular vector and so $C_X(z)$ is a parabolic subgroups of $X$.

Suppose that $n \geqslant 4$ or $\epsilon = -$. Then $O_2(K) = 1$ and so $O_2(C_X(z)) = A \leqslant \Omega^\epsilon_{2n}(2)$.

(c): Since $y^{\perp\perp} = \langle y\rangle$ and $y$ is non-singular, $y^\perp$ is a non-degenerate orthogonal space. By Witt's Lemma B.15 $C_X(y)$ induces $O(y^\perp) = O_{2n-1}(2)$ on $y^\perp$, and by B.14 $O_{2n-1}(2) \cong Sp_{2n}(2)$. Put $T := C_X(y^\perp)$. Then $T = \langle \omega_y\rangle$ where $\omega_y$ is the reflection associated to $y$. In particular, $|T| = 2$, $[X,T] = \langle y\rangle$, $C_X(T) = y^\perp$, and $T \nleqslant \Omega^\epsilon_{2m}(2)$.

Put $E := C_X(y) \cap \Omega^\epsilon_{2n}(2)$. Since $\Omega^\epsilon_{2n}(2)$ has index 2 in $X$, $C_X(y) = T \times E$. In particular, $E$ acts faithfully on $y^\perp$, $y^\perp$ is natural $O_{2n-1}(2)$-module for $E$ and $E \cong Sp_{2n-2}(2)$.

Now B.16 shows that $C_X(y)$ acts transitively on $\mathcal{Z}$. By B.13 $y^\perp = \langle \mathcal{Z}\rangle$. Thus (c) is proved.

(d): By Witt's Lemma B.15 $C_X(v)$ has at most three orbits on $v^X$, namely

$$\begin{aligned} &\{v\},\\ &\mathcal{T}_0(v) := \{w \in V^\sharp \mid h(w) = h(v), f(v,w) = 0, v \neq w\}, \text{ and}\\ &T_1(v) := \{w \in V^\sharp \mid h(w) = h(v), f(v,w) = 1\}. \end{aligned}$$

Suppose that $C_X(v) < H < X$. Then $\{v\} \neq v^H \neq v^X$, so both $\mathcal{T}_0(v)$ and $\mathcal{T}_1(v)$ are non-empty, and $\Delta := v^H = \{v\} \cup \mathcal{T}_i(v)$ for some $i \in \{0,1\}$. In particular, $H$ acts 2-transitively on $\Delta$, and $\Delta = \{u\} \cup \mathcal{T}_i(u)$ for all $u \in \Delta$. Let $\{1,2\} =: \{i,j\}$. Since $H$ is transitive on $\Delta$ and leaves invariant $\mathcal{T}_j(v)$, we have $\mathcal{T}_j(v) = \mathcal{T}_j(u)$ for all $u \in \Delta$. Put $W := \langle\Delta\rangle = \langle v\rangle + W_i$ and, for $k = 0,1$, $W_k := \langle \mathcal{T}_k(v)\rangle$ and $W := \langle\Delta\rangle = \langle v\rangle + W_i$.

We claim that $v^\perp = \langle v\rangle + W_0$. Clearly $\langle v\rangle + W_0 \leqslant v^\perp$. Suppose first that $v$ is singular. Then $\{v\} \cup \mathcal{T}_0(v)$ is the set of singular vectors of $v^\perp$. On the other hand, since $\mathcal{T}_0(v) \neq \varnothing$, there exist singular vectors in $v^\perp\backslash\langle v\rangle$. Thus, by B.13 $v^\perp$ is generated by its singular vectors, and so $v^\perp = \langle v\rangle + W_0$. Suppose next that $v$ is non-singular. Then $\{v\} \cup \mathcal{T}_0(v)$ is the set of non-singular vectors of $v^\perp$. Let $w \in v^\perp$ be non-zero and singular. Then $h(v) = h(v+w)$ and $v + w \in \mathcal{T}_0(v)$, so $w = (v+w) - v \in \langle v\rangle + W_0$. Thus $\langle v\rangle + W_0$ contains all singular and non-singular vectors of $v^\perp$ and again $v^\perp = \langle v\rangle + W_0$

Assume that $i = 0$. Then $W = \langle v\rangle + W_0 = v^\perp$ and so $\langle v\rangle = W^\perp$. Since $H$ normalizes $W$ this gives $H \leqslant C_X(v)$, a contradiction.

Hence $i = 1$ and so $j = 0$. Thus, as seen above, $\mathcal{T}_0(v) = \mathcal{T}_0(u)$ for $u \in \Delta$ and so $W_0 = \langle \mathcal{T}_0(v)\rangle = \langle \mathcal{T}_0(u)\rangle \leqslant u^\perp$. Thus $W_0 \leqslant W^\perp$. Therefore, $W \leqslant W_0^\perp$ and so $W \cap v^\perp \leqslant (\langle v\rangle + W_0)^\perp = v^{\perp\perp} = \langle v\rangle$. Thus $|W| \leqslant 4$. Let $u \in \mathcal{T}_1(v)$. Then $W = \langle u, v\rangle$ has order 4 and since $u \notin v^\perp$, $W \cap W^\perp = 0$.

Let $d \in W^\perp$ be singular. Then $h(d+u) = h(u)$ and so $d+u \in \mathcal{T}_1(v) \subseteq W$. Thus $d \in W \cap W^\perp = 0$. Hence $W^\perp$ does not contain any non-zero singular vectors. Since $\mathcal{T}_0(v) \subseteq W_0 \leqslant W^\perp$ this shows that $v$ is not singular. Also B.19(c) shows that $\dim_{\mathbb{F}_2} W^\perp \leqslant 2$. Since $\dim_{\mathbb{F}_2} W + \dim_{\mathbb{F}_2} W^\perp = \dim_{\mathbb{F}_2} V \geqslant 4$, this gives $\dim_{\mathbb{F}_2} W^\perp = 2$ and $\dim_{\mathbb{F}_2} V = 4$. Let $v'$ and $u'$ be distinct non-zero elements in $W^\perp$. Then $\langle v+v', u+u'\rangle$ is a singular subspace of dimension 2, and so $V$ has Witt index 2. Hence $X = O_4^+(2)$, and (d) is proved. □

Notation 3.2. Let $\overline{M^\dagger} := M^\dagger/C_{M^\dagger}(Y_M)$ and recall from 1.1 that

$$Z_M = \langle \Omega_1 Z(X) \mid X \in Syl_2(M)\rangle.$$

By our hypothesis $[Y_M, O^2(M)]$ is a natural $O^\epsilon_{2n}(2)$-module for $M$, and we will use the corresponding orthogonal structure for the following notation.

We choose $y, z \in [Y_M, O^2(M)]^\sharp$ such that

$$\text{z is singular, } y \text{ is non-singular, and } y \perp z.$$

Recall from 3.1 that $z$ is 2-central in $M$. Thus we can fix our notation such that $z \in \Omega_1 Z(S)$ and $C_S(y) \in Syl_2(C_M(y))$.

Since $z \in C_G(Q)$, $Q!$ implies $C_G(z) \leqslant N_G(Q)$. Hence, we can define

$$Q_{z^g} := Q^g \text{ for } g \in G.$$

We further put

$$F_0 := C_M(y),\ \ T^* := C_S(y),\ \ Y := y^\perp \text{ (in } [Y_M, O^2(M)]),\ \ F := \langle (Q_z \cap F_0)^{F_0} \rangle,\ \ T := C_S(Y).$$

Note that $T^* \in Syl_2(F_0)$.

LEMMA 3.3. (a) *$C_G(M^\circ) = 1$ and $Z(M)$=1.*

(b) *$[Y_M, O^2(M)] = Z_M$. In particular, $Z_M$ is a natural $O^\epsilon_{2n}(2)$-module for $\overline{M}$.*

(c) *Either $\overline{Q_z} = O_2(C_{\overline{M}}(z))$, or $(2n, \epsilon) = (4, +)$, and $\overline{Q_z} \cong C_4, D_8$ or $C_2 \times C_2$, with $\overline{Q_z} \leqslant \Omega^+_4(2)$ in the last case. In all cases $C_{Z_M}(Q_z)$ is a 1-dimensional singular subspace and $|Z_M/[Y_M, O^2(M), Q_z]| = 2$.*

(d) *$\overline{M^\circ} \sim \Omega^\epsilon_{2n}(2)$, $3^2C_4$ or $3^2D_8$.*

(e) *Suppose that $(2n, \epsilon) = (4, +)$. Then $TQ_z = S$.*

(f) *Either $Y_M = Z_M$, or $(2n, \epsilon) = (6, +)$ and $Y_M$ is the factor module of order $2^7$ of the natural permutation module for $Sym(8) \cong O^+_6(2)$.*

(g) *$[Y_M, T] = \langle y \rangle$.*

(h) *$M/O_2(M) \cong O^\epsilon_{2n}(2)$; or $M/O_2(M) \sim 3^{\cdot}O^+_4(2)$, $O_3(M/O_2(M))$ is extra-special of order $3^3$ and exponent 3 and $\overline{Q} \cong C_4$.*

(i) *$M = M^\circ S$.*

PROOF. (a): By Hypothesis C(iv) $Q \ntrianglelefteq M$. So $M^\circ \neq Q$, and 1.55(d) shows that $C_G(M^\circ) = 1$. In particular, $Z(M) = 1$.

(b): Since $[Y_M, O^2(M)]$ is a simple $M$-module, we have $[Y_M, O^2(M)] \leqslant Z_M$. By 1.24(e) $Z_M = \Omega_1 Z(M)[Z_M, O^2(M)]$. Also $Z(M) = 1$ by (a), and so (b) holds.

(c) and (d) : By Hypothesis C(iv) $Q \ntrianglelefteq M$ and since $Q$ is large, 1.57(b) shows that $Z_M$ is a $Q$!-module for $\overline{M^\circ}$ with respect to $\overline{Q}$. Thus we can apply B.37. Since $O^+_4(2)$ does not have a normal subgroup isomorphic to $SL_2(2)$, Case B.37(4) does not occur. Hence (c) and (d) follow from B.37.

(e): Note that $|\overline{S}| = 8$ and $\overline{T} \nleqslant \Omega^+_4(2)$. Now (c) implies $|\overline{Q_z}| = 8$ or $|\overline{Q_z}| = 4$ and $\overline{T} \nleqslant \overline{Q_z}$. Thus $\overline{S} = \overline{TQ_z}$. As $C_S(Y_M) \leqslant C_S(Y) = T$ this gives $S = TQ_z$.

(f): By C.18 , $H^1\big(\overline{O^2(M)}, Z_M\big) = 1$, unless $(2n, \epsilon) = (6, +)$, in which case it has order 2. Since $Z_M = [Y_M, O^2(M)]$ and $C_{Y_M}(O^2(M)) = 1$, this implies (f).

(g): Note that by 3.1(c:b), $[Z_M, T] = \langle y \rangle$. So if $Y_M = Z_M$, (g) holds. Otherwise, (f) shows that $Y_M$ is quotient of the natural $Sym(8)$-permutation module. As $[Z_M, T] = \langle y \rangle$, $\overline{T}$ is generated by a transposition and so again $[Y_M, T] = \langle y \rangle$.

Put $\breve{M} := M/O_2(M)$ and $D := C_M(Y_M)$.

(h): By the basic property of $M$, $\breve{D} \leqslant \Phi(\breve{M})$. From $\Phi(O^\epsilon_{2n}(2)) = 1$, we conclude that $\Phi(\breve{M}) = \breve{D}$. By 1.7(a) $\Phi(\breve{M}) = \Phi(O^2(\breve{M}))$ and thus $\breve{D} = \Phi(\breve{M^\circ})$. By 2.2(f) $O_2(M) \in Syl_2(C_M(Y_M))$ and so $\breve{D}$ has odd order.

Moreover, by 1.52(d) $[M^\circ, C_M(Y_M)] \leqslant O_2(M^\circ) \leqslant O_2(M)$. Thus $M^\circ$ centralizes $\breve{D}$, and $\breve{M}_\circ$ is a non-split central extension of $\breve{M}_\circ/\breve{D}$ by a group of odd order. If $(2n, \epsilon) \neq (4, +)$, then $\breve{M}_\circ/\breve{D} \cong \Omega^\epsilon_{2n}(2)$ is simple. Also the odd part of the Schur multiplier of $\Omega^\epsilon_{2n}(2)$ is trivial in this case (see [**Gr1**]), and (h) follows.

Assume now that $(2n, \epsilon) = (4, +)$. By (f) $Y_M = Z_M$. It follows that $Y_M$ is a natural $SL_2(2)$-wreath product module for $M$. So we can apply 1.58(f) and conclude that (h) also holds in the $O^+_4(2)$-case.

(i): Note that (d) implies that $O^2(\overline{M}) \leqslant \overline{M^\circ}$ and so $\overline{M} = \overline{M^\circ S}$. By the basic property of $M$, $\breve{D} \leqslant \Phi(\breve{M})$ and so $\breve{M} = \breve{M^\circ S}$. As $O_p(M) \leqslant S$ this gives $M = M^\circ S$. $\square$

LEMMA 3.4. *The following hold:*

(a) $|Q_z/C_{Q_z}(y)| = 2$, $[y, Q_z] = \langle z\rangle$ *and* $y^{Q_z} = \{y, yz\}$.
(b) $TM^\circ = M$ *and* $\overline{F_0} = \overline{FT}$. *Moreover, either* $C_M(Y_M) = O_2(M)$ *and* $F_0 = FT$, *or* $M/O_2(M) \sim 3{\cdot}O_4^+(2)$ *and* $|F_0/FT| = 3$.
(c) $C_F(Y_M) \leqslant O_2(M)$, $F/F \cap T \cong Sp_{2n-2}(2)$, $Y$ *is a natural* $O_{2n-1}(2)$*-module for* $FT/T$, $Y/\langle y\rangle$ *is a natural* $Sp_{2n-2}(2)$*-module for* $FT/T$, *and* $[Y_M, T] = \langle y\rangle$. *In particular,* $T = O_2(FT)$, $|T/O_2(M)| = 2$, *and* $T^* \in Syl_2(FT)$.
(d) *Suppose that* $n \geqslant 6$. *Then* $\overline{Q_z \cap F}$ *is a natural* $O_{2n-3}(2)$*-module for* $C_F(z)$. *In particular,* $\overline{Q_z \cap F}$ *is elementary abelian of order* $2^{2n-3}$ *and* $[Q_z \cap F, C_F(z)] \nleqslant T$.
(e) $N_{M^\dagger}(T) \leqslant C_{M^\dagger}(y)$ *and* $O_2(N_{M^\dagger}(T)) = T$.
(f) $\langle y, z\rangle = \Omega_1 Z(T^*) = C_{Y_M}(T^*) = C_{Y_M}(Q_z \cap T^*)$.
(g) $\Omega_1 Z(S) = \langle z\rangle$.
(h) *There exists* $M_1 \leqslant M$ *such that*
  (a) $T \in Syl_2(M_1)$ *and* $M_1/O_2(M) \cong Sym(3)$;
  (b) *if* $2n = 4$, *then* $T^*$ *normalizes* $M_1$ *and* $\langle M_1, S\rangle = \langle M_1, N_M(T^*)\rangle = M$; *and*
  (c) *if* $(2n, \epsilon) \neq (4, +)$, *then* $\langle M_1, N_M(T)\rangle = M$.
(i) $\langle Q_z, F\rangle = M^\circ$ *and, if* $(2n, \epsilon) \neq (4, +)$, $F_0$ *is a maximal subgroup of* $M$.
(j) *Suppose that* $(2n, \epsilon) \neq (4, +)$ *and* $A \leqslant O_2(C_M(z))$ *is an offender on* $Y_M$. *Then* $A \leqslant O_2(M)$.
(k) $F \trianglelefteq C_{M^\dagger}(y) = FTC_{M^\dagger}(Y_M)$ *and* $F = \langle (Q_z \cap F)^F\rangle$.

PROOF. (a): Suppose first that $(2n, \epsilon) \neq (4, +)$. Then by 3.3(c) $\overline{Q}_z = O_2(C_{\overline{M}}(z))$ and so by 3.1(b:b) $Q_z$ induces $Hom(z^\perp/\langle z\rangle, \langle z\rangle)$ on $Z_M$. This gives (a). Suppose next that $(2n, \epsilon) = (4, +)$. Then $y$ and $yz$ are the only non-singular vectors in $z^\perp$. By 3.1(c:a), $C_{\overline{M}}(y) \cong Sp_2(2) \times C_2$. As $T \leqslant C_M(y)$ and $\overline{T} \nleqslant \Omega_4^+(2)$, we conclude that $\overline{T^*} \cong C_2 \times C_2$ and $\overline{T^*} \nleqslant \Omega_4^+(2)$. Hence by 3.3(h) $\overline{Q} \nleqslant \overline{T^*}$ and so $Q$ does not centralize $y$. Since $Q_z$ acts on the non-singular vectors of $z^\perp$, (a) follows.

(b) – (d): By 3.1(c), $\overline{F_0} = C_{\overline{M}}(y) \cong C_2 \times Sp_{2n-2}(2)$, $|C_{\overline{M}}(Y)| = 2$ and $Y$ is natural $O_{2n-1}(2)$-module for $F_0$. In particular, $\overline{T} = C_{\overline{S}}(Y) = C_{\overline{M}}(Y) = O_2(\overline{F_0})$, $[Y_M, T] = [Z_M, T] = \langle y\rangle$, $\overline{F_0}/\overline{T} \cong Sp_{2n-2}(2)$, $Y/\langle y\rangle$ is a natural $Sp_{2n-2}(2)$-module for $F_0$ and $\overline{T}\,\overline{M^\circ} = \overline{M}$.

Suppose now that $(2n, \epsilon) \neq (4, +)$. Then 3.3(h) gives $M/O_2(M) \cong O_{2n}^\epsilon(2)$, so $C_M(Y_M) = O_2(M)$ and $T = C_M(Y)$. By 3.3(c) $\overline{Q_z} = O_2(C_{\overline{M}}(z))$ and so by 3.1(a) $\overline{Q_z}$ is a natural $O_{2n-2}^\epsilon(2)$-module for $C_{\overline{M}}(z)$. Observe that $\overline{Q_z \cap F}$ is the hyperplane corresponding to a non-singular vector of $\overline{Q_z}$. Thus by 3.1(c:a) applied to $C_{\overline{M}}(z)/\overline{Q_z}$ and $\overline{Q_z}$ in place of $X$ and $V$, we have $C_{\overline{F_0}}(z)/\overline{Q_z \cap F} \cong C_2 \times Sp_{2n-4}(2)$, and $\overline{Q_z \cap F}$ is a natural $O_{2n-3}(2)$-module for $C_{F_0}(z)$. Note also that $\overline{T} \nleqslant \overline{Q_z}$ and $T = C_M(Y) = C_{F_0}(Y)$. Thus $\overline{Q_z \cap F}$ acts faithfully on $Y$ and $C_{F_0}(z)/(Q_z \cap F)C_{F_0}(Y) \cong Sp_{2n-4}(2)$. It follows that

$$(Q_z \cap F)C_{F_0}(Y)/C_{F_0}(Y) = O_2\big(C_{F_0}(z)/C_{F_0}(Y)\big).$$

As $F_0/C_{F_0}(Y) \cong Sp_{2n-2}(2)$, we have $F_0/C_{F_0}(Y) = \langle O_2\big(C_{F_0}(z)/C_{F_0}(Y)\big)^{F_0}\rangle$. Since $F = \langle (Q_z \cap F)^{F_0}\rangle$, this gives $F_0 = FC_{F_0}(Y) = FT$. Hence (b) – (d) hold for $(2n, \epsilon) \neq (4, +)$.

Suppose next that $(2n, \epsilon) = (4, +)$. By 3.3(e), $\overline{S} = \overline{TQ_z}$. Thus $\overline{T^*} = \overline{T}(\overline{Q_z} \cap \overline{T^*})$ and so $\overline{Q_z} \cap \overline{F}_0 \nleqslant \overline{T}$. Since $\overline{F_0}/\overline{T} \cong F_0/C_{F_0}(Y) \cong Sp_2(2)$, this gives $\overline{F_0} = \overline{FT}$. If $C_M(Y_M) = O_2(M)$ we conclude that (b) and (c) hold. Assume that $C_M(Y_M) \neq O_2(M)$. Then by 3.3(h) $M/O_2(M) \sim 3{\cdot}O_4^+(2)$, $O_3(M/O_2(M))$ is extra-special of order $3^3$ and $\overline{Q} \cong C_4$. In particular, $C_M(Y_M)/O_2(M) \leqslant M^\circ O_2(M)$ and so $M = M^\circ T$.

Also $F_0/O_2(M) \sim 3.(C_2 \times Sym(3))$. Since $T$ neither centralizes nor inverts $O_3(M/O_2(M))$, $T$ inverts $C_M(Y_M)/O_2(M)$. As $C_M(Y) = TC_M(Y_M)$, this gives $C_M(Y)/O_2(M) \cong Sym(3)$. By 1.52(c), $[C_M(Y), M^\circ] \leqslant O_2(M^\circ) \leqslant O_2(M)$, and so $Q_z \cap F_0$ centralizes $C_M(Y_M)/O_2(M)$. It follows that $F/O_2(M) \cong Sym(3)$, $F_0/O_2(M) \cong Sym(3) \times Sym(3)$, $FT/O_2(M) \cong C_2 \times Sym(3)$, $|F_0/FT| = 3$ and $O_2(FT) = T$. Also $T(Q_z \cap F_0) \in Syl_2(F_0)$ and so $T^* = T(Q_z \cap F_0) \in Syl_2(F)$. Thus again (b) and (c) hold.

(e): Note that $C_S(Y_M)$ is a Sylow 2-subgroup of $C_{M^\dagger}(Y_M)$ and $C_S(Y_M) \leqslant C_S(Y) = T$. Hence $T$ is Sylow 2-subgroup of $C_{M^\dagger}(Y_M)T$. Also

$$FT \leqslant N_{M^\dagger}(T) \leqslant N_{M^\dagger}([Y_M, T]) = C_{M^\dagger}(y) = C_{M^\dagger}(Y_M)FT,$$

and so $O_2(\overline{N_{M^\dagger}(T)}) \leqslant O_2(\overline{FT}) = \overline{T}$. Thus $O_2(N_{M^\dagger}(T)) = T$.

(f): By 2.2(e) $\Omega_1 Z(O_2(M)) = Y_M$. Since $O_2(M) \leqslant T^*$ and $\Omega_1 Z(T^*) \leqslant C_G(O_2(M)) \leqslant O_2(M)$, we conclude that $\Omega_1 Z(T^*) \leqslant Y_M$.

Assume first that $\Omega_1 Z(T^*) \leqslant Z_M$. Suppose that $(n, \epsilon) = (4, +)$. Then $|Z_M| = 2^4$ and $\overline{S} \cong D_8$, so $\overline{U} := Z(\overline{S})$ has order 2. Hence $\overline{U} \leqslant \overline{Q_z}$ since $\overline{Q_z} \trianglelefteq \overline{S}$, and $\overline{U} = \overline{S'} \leqslant \Omega_4^+(2)$. In particular, $\overline{U}$ does not act as a transvection group on $Z_M$ and thus $|C_{Z_M}(\overline{U})| \leqslant 4$. Since $y$ and $yz$ are the only non-singular vectors in $z^\perp$, $S$ acts on $\{y, yz\}$ and since $\overline{U} \leqslant \overline{S'}$, $\overline{U}$ centralizes $y$. Hence $C_{Z_M}(\overline{U}) = \langle y, z\rangle$. Since $\overline{U} \leqslant \overline{Q_z}$ it follows that

$$\langle y, z\rangle \leqslant C_{Z_M}(T^*) \leqslant C_{Z_M}(T^* \cap Q_z) \leqslant C_{Z_M}(\overline{U}) = \langle y, z\rangle,$$

and (f) holds on this case.

Suppose that $(n, \epsilon) \neq (4, +)$. Put $\overline{A} := O_2(C_{\overline{M}}(z))$ and $\overline{A_u} := C_{\overline{A}}(u)$ for $u \in z^\perp$. Then by 3.1(c:a) $\overline{A} = \overline{Q}_z$, and by 3.1(b:b) $\overline{A}$ induces $Hom(z^\perp/\langle z\rangle, \langle z\rangle)$ on $y^\perp$. Hence

$$\overline{A_y} = \overline{A}_w \iff w \in y + \langle z\rangle.$$

Thus $C_{Z_M}(\overline{A_y}) = \langle y, z\rangle = C_{Z_M}(T^*)$, and again (f) follows.

Assume now that $\Omega_1 Z(T^*) \nleqslant Z_M$ and pick $v \in \Omega_1 Z(T^*)\backslash Z_M$. By 3.3(f) $Y_M$ is the 7-dimensional quotient of the natural permutation module for $\overline{M} \cong O_6^+(2) \cong Sym(8)$. Hence $C_{\overline{M}}(v) \cong Sym(7)$ or $Sym(3) \times Sym(5)$. By 3.3(d) $\overline{M^\circ} = \Omega_6^+(2) \cong Alt(8)$ and so the Sylow 2-subgroups of $C_{\overline{M^\circ}}(v)$ are dihedral of order 8. On the other hand $Q_z \cap F \leqslant Q_z \cap T^* \leqslant C_{M^\circ}(v)$ and by (d), $\overline{Q_z \cap F}$ is elementary abelian of order $2^{6-3} = 8$, a contradiction.

(g): Note that $\Omega_1 Z(S) \leqslant \Omega_1 Z(T^*)$. By (f) $\Omega_1 Z(T^*) = \langle z, y\rangle$ and by (a) $[y, Q_z] \neq 1$. Thus $\Omega_1 Z(S) = \langle z\rangle$.

(h): Let $q \in Q_z$ with $y^q = yz$. Then $Y \neq Y^q$ and so there exists $y_1 \in Y^q\backslash Y$. Replacing $y_1$ by $y_1(yz)$ if necessary, we may assume that $y_1$ is non-singular. Thus $y_1 = y^u$ for some $u \in M$. Note that $\langle y, y^u\rangle \leqslant Y^q = C_{Z_M}(T^q)$ and so $T^q$ centralizes $\overline{T^u}$. As $T^u \in Syl_2(T^u C_M(Y_M))$ we can choose $u$ such that $T^q$ normalizes $T^u$.

Put $M_1 := \langle T, T^u\rangle$ and $W := \langle y, y^u\rangle$. Since $T = C_S(y^\perp)$ we have $[Z_M, T] = \langle y\rangle$, and so $[Z_M, M_1] = W$. As $Z_M = W \oplus W^\perp$, we conclude that $M_1$ centralizes $W^\perp$ and $M_1/C_{M_1}(Z_M) \cong SL_2(2) \cong Sym(3)$. Together with $C_M(Y_M) = C_M(Z_M)$ this gives $\overline{M_1} \cong Sym(3)$. If $C_M(Y_M) = O_2(M)$, then obviously $M_1/O_2(M) \cong Sym(3)$. If $C_M(Y_M) \neq O_2(M)$, then by 3.3(h), $O^2(M/O_2(M))$ is extra-special of exponent 3. Since $M_1/O_2(M)$ is a dihedral group, we conclude again that $M_1/O_2(M) \cong Sym(3)$. Thus (h:a) holds.

Suppose that $2n = 4$. Then $T \trianglelefteq T^* = TT^q \trianglelefteq S$. As $T^q$ normalizes $T^u$ we conclude that $T^*$ normalizes $M_1 = \langle T, T^u\rangle$. Note that $\overline{M}$ is 2-minimal and so $\overline{M} = \langle \overline{M_1}, \overline{S}\rangle$. As $O_2(M) \leqslant S$ and $C_M(Y_M)/O_2(M) \leqslant \Phi(M/O_2(M))$, we get $M = \langle M_1, S\rangle = \langle M_1, N_M(T^*)\rangle$. So (h:b) is proved.

Suppose that $(2n, \epsilon) \neq (4, +)$. Then by 3.3(h) $C_M(Y_M) = O_2(M)$ and so $T = C_M(Y) \trianglelefteq F_0$. Also by 3.1(d) $F_0$ is maximal subgroup of $M$. Thus $M = \langle M_1, F_0\rangle = \langle M_1, M_M(T)\rangle$, and (h:c) holds.

(i): Put $L = \langle F, Q_z\rangle$. Note that $L \leqslant M^\circ$ and $T$ normalizes $L$. If $(2n, \epsilon) = (4, +)$, then by 3.3(h), $S = Q_z T$ and $\overline{S}$ is a maximal subgroup of $\overline{M}$, so $\overline{LS} = \overline{M}$. As above, $O_2(M) \leqslant S$ and $C_M(Y_M)/O_2(M) \leqslant \Phi(M/O_2(M))$ give $M = LS$, and so $M = LQ_z T = LT$. Hence $L \trianglelefteq M$ and $M^\circ = \langle Q_z^M\rangle \leqslant L \leqslant M^\circ$. So (i) holds in this case.

Suppose that $(2n, \epsilon) \neq (4, +)$. Then by (b) $C_M(Y_M) = O_2(M)$ and $F_0 = FT$. By 3.1(d), $\overline{F_0}$ is a maximal subgroup of $\overline{M}$. Thus $FT$ is maximal subgroup of $M$ and again $LT = M$, and (i) holds.

(j): This follows for example from C.8.

(k): By 1.52(c), $[C_G(Y), M^\circ] \leqslant O_2(M^\circ) \leqslant O_2(M^\circ)$. Since $F \leqslant M^\circ$, we get $[F, C_{M^\dagger}(Y_M)] \leqslant O_2(M) \leqslant F$. As $M^\dagger = C_{M^\dagger}(Y_M)M$, we have $C_{M^\dagger}(y) = C_{M^\dagger}(Y_M)C_M(y) = C_{M^\dagger}(Y_M)F_0$. By (b) $\overline{F_0} = \overline{TF}$, and we conclude that

$$C_{M^\dagger}(y) = C_{M^\dagger}(Y_M)TF = C_{M^\dagger}(\langle y, z\rangle)F = C_{M^\dagger}(\langle y, z\rangle)F_0.$$

Note that $C_{Q_z}(y) = Q_z \cap F_0 = Q_z \cap F$ and so $C_{M^\dagger}(\langle y, z\rangle)$ normalizes $Q_z \cap F_0$. Hence

$$F = \langle (Q_z \cap F_0)^{F_0}\rangle = \langle (Q_z \cap F_0)^{C_{M^\dagger}(y)}\rangle = \langle (Q_z \cap F_0)^F\rangle = \langle (Q_z \cap F)^F\rangle.$$

Thus (k) holds. $\square$

### 3.2. The Proof of Theorem C

In this section we will prove Theorem C. For this we assume that $(G, M)$ is a counterexample to Theorem C. Thus $C_G(x)$ is of characteristic 2 for some non-singular $x \in Z_M$. We continue to use the notation introduced in section 3.1. By 3.1(a) $M$ acts transitively on the non-singular elements of $Z_M$ and so $C_G(y)$ is of characteristic 2. We will derive a contradiction in a sequence of lemmas.

LEMMA 3.5. *Suppose that $[O_2(M), O^2(M)] \leqslant Y_M$. Then $Y_M = O_2(M) = C_G(Y_M)$ and $M^\dagger = M$.*

PROOF. Since $[O_2(M), O^2(M)] \leqslant Y_M \leqslant \Omega_1 Z(O_2(M))$, 1.18 implies that $[\Phi(O_2(M)), O^2(M)] = 1$. As $Z(M) = 1$ by 3.3(a), we conclude that $\Phi(O_2(M)) = 1$ and $O_2(M) = Y_M = O_2(M^\dagger)$. In particular $C_G(Y_M) = Y_M$ since $M^\dagger$ is of characteristic 2. Thus $M^\dagger = MC_G(Y_M) = M$. □

LEMMA 3.6. (a) *If $2n = 4$, then $N_G(T^*) \leqslant M^\dagger$ and $T^* \in Syl_2(C_G(y))$. In particular, $y$ is not 2-central.*

(b) *If $(2n, \epsilon) \neq (4, +)$, then $N_G(T) \leqslant N_G(B(T)) \leqslant M^\dagger$ and $T = O_2(N_{M^\dagger}(T))$.*[1]

PROOF. (a): Let $M_1$ be as in 3.4(h). Since $2n = 4$, $T^*$ normalizes $M_1$. Put $M^* = T^* M_1$. Note that $T^* \in Syl_2(M^*)$ and $M^*/O_2(M^*) \cong Sym(3)$.

We claim that $N_G(T^*) \leqslant M^\dagger$. For this, suppose first that no non-trivial characteristic subgroup of $B(T^*)$ is normal in $B(M^*)$. By the Baumann argument (see for example [**PPS**, 2.8 and 2.9(a)]) $B(T^*) \in Syl_2(B(M^*))$. Note that $B(M^*)/O_2(B(M^*)) \cong Sym(3)$, and so the pushing up result for $Sym(3)$, see [**Gl2**], shows that $B(M^*)$ has a unique non-central chief factor in $O_2(B(M^*))$. Since $O^2(M^*) \leqslant B(M^*)$, the same holds for $M^*$, and since $[Y_M, O^2(M^*)] \neq 1$ we conclude that

$$[O_2(M), O^2(M^*)] \leqslant [O_2(M^*), O^2(M^*)] \leqslant Y_M.$$

Hence also $[O_2(M), O^2(M)] \leqslant Y_M$ and by 3.5 $Y_M = O_2(M)$.

A straightforward computation shows that $\mathcal{A}_{T^*} = \{Y_M, A, A_1, A_2\}$, where $|A_i Y_M/Y_M| = 2 = |Y_M/Y_M \cap A_i|$, and $|AY_M/Y_M| = 4 = |Y_M/Y_M \cap A|$. So $\{A_1, A_2\}$ and $\{Y_M, A\}$ are the only pairs of elements of $\mathcal{A}_{T^*}$ which intersect in a group of order 4. Hence $N_G(T^*)/C_G(\mathcal{A}_{T^*})$ is a 2-group. Since $N_G(T^*) \cap M^\dagger$ contains the Sylow 2-subgroup $S$ of $G$ and $C_G(\mathcal{A}_{T^*}) \leqslant N_G(Y_M) = M^\dagger$, we conclude that $N_G(T^*) \leqslant M^\dagger$, and the claim holds in this case.

Suppose next that $K$ is non-trivial characteristic subgroup of $B(T^*)$ which is normal in $M^*$. By 3.4(h:b) $M = \langle M_1, N_M(T^*)\rangle = \langle M^*, N_M(T^*)\rangle$ and so $K \trianglelefteq M$. Thus by 2.2(c), $N_G(K) \leqslant M^\dagger$. Since $K$ is a characteristic subgroup of $T^*$ this implies $N_G(T^*) \leqslant M^\dagger$.

We have shown that $N_G(T^*) \leqslant M^\dagger$. Note that $C_S(Y_M) \in Syl_2(C_G(Y_M))$. Since $M^\dagger = C_G(Y_M)M$ and $C_S(Y_M) \leqslant C_S(y) = T^* \in Syl_2(C_M(y))$ we have $T^* \in Syl_2(C_{M^\dagger}(y))$. Let $T_1 \in Syl_2(C_G(y))$ with $T^* \leqslant T_1$. Then $N_{T_1}(T^*) \leqslant N_{C_{M^\dagger}(y)}(T^*)$ and so $T^* = N_{T_1}(T^*)$ and $T_1 = T^*$. Thus (a) holds.

(b): Suppose now that $(2n, \epsilon) \neq (4, +)$ and that $L := N_G(B(T)) \nleqslant M^\dagger$. We derive a contradiction using a similar pushing up argument as in the proof of (a). Let $K$ be non-trivial characteristic subgroup of $B(T)$ normal in $M_1$. By 3.4(h) $M = \langle M_1, N_M(T)\rangle$ and so $K \trianglelefteq M$. Thus by 2.2(c), $N_G(K) \leqslant M^\dagger$, contrary to $L \nleqslant M^\dagger$. Hence no non-trivial characteristic subgroup of $B(T)$ is normal in $B(M_1)$.

The same pushing up argument as in (a) shows that $[O_2(M), O^2(M)] \leqslant Y_M$. So by 3.5 $Y_M = O_2(M) = C_M(Y_M)$ and $M = M^\dagger$. Let $t \in L \backslash M^\dagger$. Then $Y_M \neq Y_M^t$ and so $Y_M Y_M^t = T = B(T)$. Note that all involutions in $T$ are contained in $Y_M \cup Y_M^t$ and thus $\mathcal{A}_T = \{Y_M, Y_M^t\}$. Since $N_L(Y_M) = L \cap M^\dagger = L \cap M = N_M(T) = FT$ we conclude that $|L/FT| = 2$ and $FT \trianglelefteq L$. Let $T_1 \in Syl_2(L)$ with $T^* \leqslant T_1$. Then $L = (FT)T_1 = FT_1$ and so $T_1 \nleqslant M$. By 3.4(i) $\langle FT, Q_z\rangle = M^\circ T = M$. Since $N_G(M) \leqslant M^\dagger = M$ we conclude that $T_1 \nleqslant N_G(Q_z)$ and so $[z, T_1] \neq 1$.

Put $H := N_G(T^*)$. Since $T^* = C_S(y) = C_S(\langle z, y\rangle)$ and $y^{Q_z} = \{y, yz\}$ we have $Q_z \leqslant H$. Also $T_1 \leqslant H$. By 3.4(f), $\Omega_1 Z(T^*) = \langle z, y\rangle$. Since $\Omega_1 Z(T^*) \trianglelefteq H$, the action of $Q_z$ and $T_1$ on

[1] For the definition of $B(T)$ see 1.1.

$\Omega_1 Z(T^*)$ shows that $H/C_H(\Omega_1 Z(T^*)) \cong Sym(3)$, and $H$ acts transitively on $\Omega_1 Z(T^*)$. So there exists $h \in H$ with $z^h = y$. Thus $y \in z^G$. Since $z$ is 2-central, (a) gives $2n \geqslant 6$. Also $Q_z^h = Q_y$ and $|Q_y/Q_y \cap T^*| = |Q_z/Q_z \cap T^*| = 2$. Since

$$Q_y \cap T^* \leqslant Q_y \cap M \leqslant O_2(C_M(y)) = O_2(F_0) = O_2(FT) = T,$$

we have $|Q_y/Q_y \cap T| \leqslant 2$. As $|T/Y_M| = |T/O_2(M)| = 2$ this gives $|Q_y/Q_y \cap Y_M| \leqslant 4$ and so

$$|Q_y| \leqslant 4|Q_y \cap Y_M| \leqslant 4|Y_M|.$$

Since $2n \geqslant 6$, 3.3(c) implies $|Q_z Y_M/Y_M| = |Q_z O_2(M)/O_2(M)| = |O_2(C_{\overline{M}}(z))| = 2^{2n-2}$. By 3.3(c) $|Z_M/Z_M \cap Q_z| \leqslant |Z_M/[Z_M, Q_z]| = 2$ and by 3.3(f) $|Y_M/Z_M| \leqslant 2$. Therefore

$$|Q_z| = 2^{2n-2}|Y_M \cap Q_z| \geqslant 2^{2n-2}|Z_M \cap Q_z| \geqslant 2^{2n-4}|Y_M| \geqslant 4|Y_M|.$$

Since $|Q_y| = |Q_z|$ equality must hold in each of the last two displayed inequalities. Thus $Y_M \cap Q_y = Y_M$ and $Y_M \cap Q_z = [Z_M, Q_z]$. Hence $Y_M \leqslant Q_y$ and $Y_M \nleqslant Q_z$. Therefore $Y_M^{h^{-1}}$ is an elementary abelian normal subgroup of order $|Y_M|$ in $Q_z$ and $Y_M \neq Y_M^{h^{-1}}$. Since $Y_M = C_M(Y_M)$ we conclude from A.40 that $Q_z Y_M$ contains a non-trivial offender on $Y_M$, a contradiction to 3.4(j).

We have shown that $N_G(B(T)) \leqslant M^\dagger$. Since $N_G(T) \leqslant N_G(B(T))$ this gives $N_G(T) = N_{M^\dagger}(T)$. By 3.4(e), $O_2(N_{M^\dagger}(T)) = T$, and so (b) holds. □

LEMMA 3.7. $z^G \cap Y_M = z^M = z^G \cap Z_M$.

PROOF. Suppose that there exists $u \in z^G \cap Y_M$ with $u \notin z^M$. Assume first that $u \in Z_M$. By 3.1(a) $M$ has two orbits on $Z_M^\sharp$, and since $u \notin Z^m$, we have $u = y^m$ for some $m \in M$, so $y \in z^G$. If $2n = 4$ then 3.6(a) shows that $y$ is not 2-central, a contradiction. Thus $2n \neq 4$. By $Q$! we have $Q_y \trianglelefteq C_G(y)$. By 3.6(b) and 3.4(e) $N_G(T) = N_{M^\dagger}(T) \leqslant C_{M^\dagger}(y) \leqslant C_G(y)$ and so $N_G(T)$ normalizes $Q_y$. Thus $N_{Q_y}(T) \leqslant O_2(N_G(T))$. By 3.6(b) $O_2(N_G(T)) = T$ and so $N_{Q_y}(T) \leqslant T$. It follows that $Q_y \leqslant T \leqslant S$. By 2.2(b), $Q$ is weakly closed in $S$ with respect to $G$, so $Q_z = Q_y$. In particular $[Q_z, y] = 1$, which contradicts 3.4(a).

Assume now that $u \in Y_M \backslash Z_M$. By 3.3(f), $(2n, \epsilon) = (6, +)$ and $Y_M$ is the 7-dimensional quotient of the natural permutation module for $\overline{M} \cong O_6^+(2) \cong Sym(8)$. Hence $C_{\overline{M}}(v) \cong Sym(7)$ or $Sym(3) \times Sym(5)$. In both cases $O_2(C_{M^\dagger}(u)) \leqslant C_{M^\dagger}(Y_M)$ and thus

$$O_2(C_{M^\dagger}(u)) = O_2(C_{M^\dagger}(Y_M)) = O_2(M^\dagger) \leqslant O_2(M).$$

Since $N_G(O_2(M)) \leqslant M^\dagger$ by 2.2(c) and $Q_u \leqslant O_2(C_G(u))$, we conclude that

$$N_{Q_u}(O_2(M)) \leqslant O_2(C_{M^\dagger}(u)) \leqslant O_2(M),$$

and so $Q_u \leqslant O_2(M)$. Since $Q$ is a weakly closed subgroup of $G$, this implies $Q_u = Q^g$ for all $g \in M$ and so $Q \trianglelefteq M$, a contradiction to Hypothesis C(iv). □

LEMMA 3.8. *The following hold:*

(a) $\Omega_1 Z(T) = C_{Y_M}(T)$ *and* $Y = C_{Z_M}(T) = \Omega_1 Z(T) \cap Z_M$.
(b) *Let* $\mathcal{Z} = \{u \in Y \mid 1 \neq u \text{ is singular in } Z_M\}$. *Then* $N_G(T) \leqslant N_G(\Omega_1 Z(T)) = N_G(\mathcal{Z}) = N_G(Y) = FTC_{M^\dagger}(Y_M) \leqslant M^\dagger$.
(c) $O_2(N_G(T)) = T$ *and* $C_G(Y) = TC_{M^\dagger}(Y_M)$.

PROOF. (a): By 2.2(e), $\Omega_1 Z(O_2(M)) = Y_M$. Since $O_2(M) \leqslant T$ we get $\Omega_1 Z(T) = C_{Y_M}(T)$. Thus $C_{Z_M}(T) = \Omega_1 Z(T) \cap Z_M$. By 3.1(c:b) $C_{Z_M}(T) = y^\perp = Y$, and so (a) is proved.

(b): Observe that $z \in \mathcal{Z}$ and so

$$Q_z \leqslant L := \langle Q_u \mid u \in \mathcal{Z} \rangle \leqslant M^\circ.$$

By 3.1(c:c)

$$\langle \mathcal{Z} \rangle = Y;$$

in particular, $N_G(\mathcal{Z}) \leqslant N_G(Y)$. Since $M$ acts transitively on the non-trivial singular vectors in $Z_M$, $\mathcal{Z} = Y \cap z^M$. By (a) $Y = \Omega_1 Z(T) \cap Z_M$ and since $z^M \subseteq Z_M$ we get $\mathcal{Z} = \Omega_1 Z(T) \cap z^M$. By 3.7 $z^M = z^G \cap Y_M$, and since $\Omega_1 Z(T) \subseteq Y_M$, we conclude that $\mathcal{Z} = \Omega_1 Z(T) \cap z^G = Y \cap z^G$. Hence

$$N_G(T) \leqslant N_G(\Omega_1 Z(T)) \leqslant N_G(\mathcal{Z}) \leqslant N_G(Y) \leqslant N_G(\mathcal{Z}) \leqslant N_G(L).$$

In particular, $N_G(\mathcal{Z}) = N_G(Y)$.

Since $F_0 \leqslant N_M(Y)$, we get $F = \langle (Q_z \cap F_0)^{F_0} \rangle \leqslant L$ and so by 3.4(i), $M^\circ = \langle Q_z, F \rangle \leqslant L \leqslant M^\circ$. Hence $L = M^\circ$ and $N_G(\mathcal{Z}) \leqslant N_G(M^\circ) = M^\dagger$. Thus $N_G(Y) = N_G(\mathcal{Z}) = N_{M^\dagger}(\mathcal{Z}) = N_{M^\dagger}(Y)$. Since $Y = y^\perp$, $N_{M^\dagger}(Y) = C_{M^\dagger}(y)$. By 3.4(k) we have $C_{M^\dagger}(y) = C_{M^\dagger}(Y_M)FT$, and (b) is proved.

(c): By (b) $N_G(T) = N_{M^\dagger}(T)$ and $C_G(Y) = C_{M^\dagger}(Y)$. Hence 3.4(e) gives the first part of (c), and 3.1(c:b) the second part. □

Notation 3.9. By Hypothesis C(iii) $C_G(y) \not\leqslant M^\dagger$, and so there exists a subgroup $L \leqslant C_G(y)$ with $FT \leqslant L$ and $L \not\leqslant M^\dagger$. Among all such subgroups we choose $L$ such that $|L|$ is minimal.

Observe that the minimality of $L$ implies that $L \cap M^\dagger$ is the unique maximal subgroup of $L$ containing $FT$. By 3.4(c), $T^* \in Syl_2(FT)$ and we can pick $T_0 \in Syl_2(L)$ such that $T^* \leqslant T_0$. We set

$$D := L \cap M^\dagger, \quad Z_L := \langle \Omega_1 Z(T_0)^L \rangle, \quad P := C_L(z), \quad \text{and} \quad P^* := O^{2'}(P).$$

Lemma 3.10. *The following hold:*

(a) $O_2(\langle Q_z, L \rangle) = 1$.
(b) $[Q_z, P] \leqslant Q_z \cap P = Q_z \cap L = Q_z \cap F \leqslant O_2(P)$.
(c) $O_2(\langle L, L^t \rangle) = 1$ *for* $t \in Q_z \backslash L$.
(d) $Z_L Z_L^t \ntrianglelefteq L$ *and* $Z_L Z_L^t \ntrianglelefteq L^t$ *for* $t \in Q_z \backslash L$.
(e) $O_2(C_G(y)) \leqslant O_2(L) \leqslant T$.
(f) $\Omega_1 Z(T_0) = \Omega_1 Z(T^*) = \langle y, z \rangle$.
(g) $P = C_L(\Omega_1 Z(T_0))$, *so* $P^*$ *is a point-stabilizer for* $L$ *on* $Y_L$ *and on* $Z_L$.[2]
(h) $Z_L = \langle Y^L \rangle$ *and* $Y \leqslant C_{Z_L}(T) \leqslant C_{Y_M}(T)$.
(i) $D = FTC_D(Y_M) = FC_D(Y)$. *In particular,* $Y$ *is a natural* $O_{2n-1}(2)$*-module and* $Y/\langle y \rangle$ *is a natural* $Sp_{2n-2}(2)$*-module for* $D$.
(j) $F$ *is normal in* $D$.
(k) *If* $2n = 4$, *then* $T^* = T_0 \in Syl_2(L)$.

Proof. (a): By 3.4(b) and (i), $M = M^\circ T = \langle Q_z, TF \rangle \leqslant \langle Q_z, L \rangle$. Since $L \not\leqslant M^\dagger$, $\mathcal{M}_G(M) = \{M^\dagger\}$ implies $O_2(\langle Q_z, L \rangle) = 1$.

(b): Note that $C_{Q_z}(y) = Q_z \cap F = Q_z \cap L = Q_z \cap P$ and that by 3.4(a) $|Q_z/C_{Q_z}(y)| = 2$. By $Q!$, $P$ normalizes $Q_z$ and so also $Q_z \cap P$, and (b) follows.

(c): Since $|Q_z/Q_z \cap L| = 2$ we conclude that $\langle L, t \rangle = \langle L, Q_z \rangle$, $t^2 \in Q_z \cap L$ and $t$ normalizes $\langle L, L^t \rangle$. So (c) follows from (a).

(d): Note that $t$ normalizes $Z_L Z_L^t$. Thus (d) follows from (c).

(e): Put $U := O_2(L)O_2(C_G(y))$. Then $FT$ normalizes $U$. By 3.8(b) $N_G(T) \leqslant C_{M^\dagger}(Y_M)FT$ and by 3.4(c) $O_2(\overline{FT}) = \overline{T}$, so $\overline{N_U(T)} \leqslant O_2(\overline{FT}) = \overline{T}$. Hence $N_U(T) \leqslant C_{M^\dagger}(Y_M)T$. Since $T$ is a Sylow 2-subgroup of $C_{M^\dagger}(Y_M)T$ and $T$ normalizes $N_U(T)$ we get $N_U(T) \leqslant T$. It follows that $U \leqslant T \leqslant L$. Since $L$ normalizes $O_2(C_G(y))$, this gives $O_2(C_G(y)) \leqslant O_2(L) \leqslant T$.

(f): Choose $g \in G$ with $T_0 \leqslant S^g$. Since $G$ is a counterexample to Theorem C $C_G(y)$ is of characteristic 2 and so

$$C_G(O_2(C_G(y))) \leqslant O_2(C_G(y)).$$

By (e)

$$O_2(C_G(y))) \leqslant T \leqslant T^* \leqslant T_0 \leqslant S^g.$$

Thus

$$\Omega_1 Z(S^g)\Omega_1 Z(T_0) \leqslant C_G(O_2(C_G(y))) \leqslant T \leqslant T^* \leqslant T_0,$$

[2] For the definition of a point-stabilizer on a module see A.3

and so

$$\Omega_1 Z(S^g) \leqslant \Omega_1 Z(T_0) \leqslant \Omega_1 Z(T^*).$$

By 3.4(e) $\Omega_1 Z(T^*) = \langle y, z\rangle$, and by 3.4(g) $\Omega_1 Z(S) = \langle z\rangle$. Hence $\Omega_1 Z(S^g) = \langle z^g\rangle$ and

$$\langle y, z^g\rangle \leqslant \Omega_1 Z(T_0) \leqslant \langle y, z\rangle.$$

By 3.7 $Y_M \cap z^G = z^M$. Thus $y \notin z^G$, $z^g \neq y$ and $\Omega_1 Z(T_0) = \langle y, z\rangle$.

(g): By (f), $\Omega_1 Z(T_0) = \langle y, z\rangle$. Since $L \leqslant C_G(y)$ we have $C_L(\Omega_1 Z(T_0)) = C_L(z) = P$.

(h): Let $\mathcal{Z}$ be the set of non-trivial singular vectors in $Y$. By3.1(c:c) $Y = \langle \mathcal{Z}\rangle$ and $C_{\overline{M}}(y)$ acts transitively on $\mathcal{Z}$. By 3.4(b) $C_{\overline{M}}(y) = \overline{F_0} = \overline{FT}$, and we conclude that $Y = \langle z^F\rangle = \langle \Omega_1 Z(T_0)^F\rangle$. Therefore $Z_L = \langle Y^L\rangle$. Moreover, by (e) $Z_L \leqslant T$ and so

$$Y \leqslant C_{Z_L}(T) \leqslant \Omega_1 Z(T).$$

By 3.8(a) $\Omega_1 Z(T) = C_{Y_M}(T)$, and so (h) holds.

(i): By 3.4(k), $C_{M^\dagger}(y) = FTC_{M^\dagger}(Y_M)$. Since $FT \leqslant D = M^\dagger \cap L \leqslant C_{M^\dagger}(y)$, this gives $D = FT(D \cap C_{M^\dagger}(Y_M)) = FTC_D(Y_M)$. Since $T$ centralizes $Y$ we get $D = FC_D(Y)$. By 3.4(c), $Y$ is a natural $O_{2n-1}(2)$-module and $Y/\langle y\rangle$ is a natural $Sp_{2n}(2)$-module for $FT$ and so also for $D$. Thus (i) holds.

(j): By 3.4(k). $F \trianglelefteq C_{M^\dagger}(y)$ and since $D = M^\dagger \cap L \leqslant C_{M^\dagger}(y)$ we get $F \trianglelefteq D$.

(k): Suppose that $2n = 4$. Then by 3.6(b), $T^* \in Syl_2(C_G(y))$. Since $T^* \leqslant T_0 \in Syl_2(L)$ and $L \leqslant C_G(y)$ this gives $T^* = T_0$. □

LEMMA 3.11. *$L$ is of characteristic 2, $C_L(Z_L) = O_2(L) = C_L(Y_L)$ and $Y_L = \Omega_1 Z(O_2(L))$.*

PROOF. Since $G$ is a counterexample to Theorem C, $C_G(y)$ is of characteristic 2. Moreover, by 3.10(e) $O_2(C_G(y)) \leqslant O_2(L)$ and so $L$ is of characteristic 2.

By 3.10(h) $Y \leqslant Z_L$, and 3.8(c) implies $C_L(Z_L) \leqslant C_L(Y) \leqslant TC_{M^\dagger}(Y_M)$, so $O^2(C_L(Z_L)) \leqslant C_{M^\dagger}(Y_M)$. On the other hand, by 1.52(c) $[M^\circ, C_{M^\dagger}(Y_M)] \leqslant O_2(M^\circ) \leqslant O_2(M^\dagger)$, and thus $Q_z$ normalizes $O^2(C_L(Z_L))O_2(M^\dagger)$. But $O^2(C_L(Z_L)) = O^2(C_L(Z_L)O_2(M^\dagger))$ since $O_2(M^\dagger) \leqslant T \leqslant L$. Hence $O^2(C_L(Z_L))$ is normalized by $Q_z$ and $L$. As $O_2(\langle Q_z, L\rangle) = 1$ by 3.10(a), we get $O_2(O^2(C_L(Z_L))) = 1$. This yields $O^2(C_L(Z_L)) = 1$ since $L$ is of characteristic 2. Hence $C_L(Z_L) = O_2(L)$.

Put $U := \Omega_1 Z(O_2(L))$. Since $Z_L \leqslant Y_L \leqslant U$ this implies

$$O_2(L) \leqslant C_L(U) \leqslant C_L(Y_L) \leqslant C_L(Z_L) = O_2(L).$$

Hence $O_2(L) = C_L(U)$ and $O_2(L/C_L(U)) = 1$. Thus $U$ is 2-reduced for $L$, so $U \leqslant Y_L$ and $Y_L = U$. □

LEMMA 3.12. *Let $N_0$ be a subnormal subgroup of $D$.*

(a) *Suppose that $O^2(F) \nleqslant N_0$. Then $O^{2'}(O^2(N_0)) \leqslant TC_{M^\dagger}(Y_M)$ and if in addition $N_0 \trianglelefteq D$, then $N_0 \leqslant TC_{M^\dagger}(Y_M)$ and $O^2(N_0) \leqslant C_{M^\dagger}(Y_M)$.*

(b) *If $N_0$ is subnormal in $L$, then either $O^2(F) = O^2(N_0)$ or $N_0 \leqslant O_2(L)$.*

PROOF. From 3.4(k) we get that $F \trianglelefteq C_{M^\dagger}(y)$ and from 3.4(c) that

$$\overline{D} = \overline{FT} \cong C_2 \times Sp_{2n-2}(2) \tag{I}$$

and

$$F/O_2(F) \cong Sp_{2n-2}(2). \tag{II}$$

(a): By (II) either $O^2(F) \leqslant X$ or $[O^2(F), X] \leqslant O_2(F)$ for every subnormal subgroup $X$ of $D$. By the hypothesis of (a), $O^2(F) \nleqslant N_0$, and hence $[O^2(F), N_0] \leqslant O_2(F)$. By (I) $\overline{N_0} \leqslant \overline{T}$, or $2n = 4$ and $O^2(\overline{N_0}) = O^2(\overline{F}) \cong C_3$. The first case gives $N_0 \leqslant TC_{M^\dagger}(Y_M)$, while the second case gives $O^{2'}(O^2(N_0)) \leqslant TC_{M^\dagger}(Y_M)$.

Moreover, if $N_0$ is normal in $D$, then $[F, N_0] \leqslant F \cap N_0$. Since $O^2(F) \nleqslant N_0$, we conclude that $[F, N_0] \leqslant O_2(F)$. But then also in this case $\overline{N_0} \leqslant \overline{T}$.

(b): Assume now that $N_0$ is subnormal in $L$. Note that if (b) holds for $\langle N_0^D\rangle$ in place of $N_0$, then (II) shows that (b) also holds for $N_0$. So we may assume that $N_0 \trianglelefteq D$. We first treat the case

$$(*) \qquad O^2(F) \nleqslant N_0.$$

By 3.10(f) $\Omega_1 Z(T_0) = \langle y, z\rangle \leqslant Z_M$ and by (a) $O^2(N_0) \leqslant C_{N_0}(Y_M)$. Since $T_0 \cap N_0 \in Syl_2(N_0)$, this gives

$$N_0 = (N_0 \cap T_0)O^2(N_0) = (N_0 \cap T_0)C_{N_0}(Y_M) \leqslant C_G(\Omega_1 Z(T_0)).$$

Thus by 1.28(b), $[Z_L, N_0] = 1$, and 3.11 implies $N_0 \leqslant C_L(Z_L) \leqslant O_2(L)$.

Assume next that $O^2(F) \leqslant N_0$. By (I), $\overline{D} = \overline{FT}$ and so $O^2(N_0) \leqslant O^2(F)C_{M^\dagger}(Y_M)$. As $O^2(F) \leqslant N_0$, we get

$$\text{(III)} \qquad O^2(N_0) = O^2(F)\big(O^2(N_0) \cap C_{M^\dagger}(Y_M)\big).$$

Note that $O^2(N_0) \cap C_{M^\dagger}(Y_M)$ is subnormal in $L$, normal in $D$ and satisfies $(*)$ in place of $N_0$. As we have seen already, $O^2(N_0) \cap C_{M^\dagger}(Y_M) \leqslant O_2(L)$, and so by (III) $O^2(N_0) \leqslant O^2(F)O_2(L)$. Thus $O^2(N_0) \leqslant O^2(F)$. By (III) $O^2(F) \leqslant O_2(N_0)$ and so $O^2(N_0) = O^2(F)$. □

Lemma 3.13. *Let $N_0$ be a normal subgroup of $L$. Then $O^2(F) \leqslant N_0$ or $T_0 \cap N_0 = T \cap N_0$.*

Proof. By 3.4(c), $FT/T \cong Sp_{2n}(2)$. Since $FT \cap N_0 \trianglelefteq FT$ we conclude that either $O^2(FT) \leqslant FT \cap N_0$ or $FT \cap N_0 \leqslant T$. In the first case we are done. So we may assume that $FT \cap N_0 \leqslant T$. Since $T^* \leqslant FT$ also

$$(*) \qquad T^* \cap N_0 \leqslant T,$$

in particular, $[N_{T_0 \cap N_0}(T^*), T^*] \leqslant T^* \cap N_0 \leqslant T$. It follows that $N_{T_0\cap N_0}(T^*) \leqslant N_{T_0 \cap N_0}(T)$. By 3.8(b) $N_G(T) \leqslant M^\dagger$ and thus $N_{T_0\cap N_0}(T^*) \leqslant T_0 \cap M^\dagger \leqslant T^*$. This shows that $T_0 \cap N_0 \leqslant T^* \cap N_0$ and by $(*)$ $T_0 \cap N_0 = T \cap N_0$. □

Lemma 3.14. *Let $t \in Q_z \backslash L$.*

(a) *$J(O_2(L)O_2(L^t)) \nleqslant O_2(L)$.*
(b) *If $(2n, \epsilon) \neq (4, +)$ then $J(T) \nleqslant O_2(L)$.*
(c) *$O_2(L)O_2(L^t) \trianglelefteq P$ and $O_2(L)O_2(L^t) \leqslant O_2(P) = O_2(P^*)$.*
(d) *There exists $A \leqslant O_2(P^*)$ such that $A$ is a minimal non-trivial quadratic best offender on $Y_L$.*

Proof. (a): Assume that $J(O_2(L)O_2(L^t)) \leqslant O_2(L)$. Then $J(O_2(L)O_2(L^t)) = J(O_2(L)) = J(O_2(L^t))$, and so $t$ normalizes $J(O_2(L))$. A contradiction, since $O_2(\langle L, L^t\rangle) = 1$ by 3.10(c). Hence (a) holds.

(b): Assume now that $J(T) \leqslant O_2(L)$ and $(2n, \epsilon) \neq (4, +)$. By 3.10(e) $O_2(L) \leqslant T$ and so $J(T) = J(O_2(L))$. Since $Z_L \leqslant Z(J(O_2(L))$ we conclude that $Z_L \leqslant Z(J(T))$. As $C_L(Z_L) = O_2(L)$ by 3.11, this gives $B(T) \leqslant C_L(Z_L) = O_2(L)$. Thus $B(T) = B(O_2(L))$ and $B(T)$ is normal in $L$, a contradiction, since by 3.6(b) $N_G(B(T)) \leqslant M^\dagger$.

(c): By 3.10(b) $Q_z$ and so also $t$ normalizes $P$. Since $O_2(L) \trianglelefteq P$ we get $O_2(L^t) \trianglelefteq P$. Since $O_2(L)O_2(L^t)$ is a 2-group, this gives $O_2(L)O_2(L^t) \leqslant O_2(P)$. Recall that $P^* = O^{2'}(P)$, so $O_2(P^*) = O_2(P)$.

(d): By (a) we can choose $B \in \mathcal{A}_{J(O_2(L)O_2(L^t))}$ such that $B \nleqslant O_2(L)$. By (c), $B \leqslant O_2(P^*)$. Since by 3.11 $C_L(Y_L) = O_2(L)$, $[Y_L, B] \neq 1$. Thus by A.40, $C_B([Y_L, B])$ is a non-trivial quadratic best offender on $Y_L$. Hence there also exists such a minimal offender $A$ in $C_B([Y_L, B])$ and (d) holds. □

Notation 3.15. Recall from 3.11 that $C_L(Z_L) = O_2(L)$. So $\widetilde{L} := L/O_2(L)$ is faithful on $Z_L$. According to 3.14(d) we can choose $A \leqslant O_2(P^*)$ such that

$$\widetilde{A} \text{ is a minimal non-trivial quadratic offender on } Y_L.$$

Put $H := \langle A^L\rangle O_2(L)$ and $Y_L^+ := Y_L/C_{Y_L}(H)$. For $X \subseteq Y_L$, let $X^+ := XC_{Y_L}(H)/C_{Y_L}(H)$, the image of $X$ in $Y_L^+$.

LEMMA 3.16. *There exist subgroups $H_i$, $i = 1, \dots, m$, of $H$ such that for $V_i := [Z_L, H_i]$:*

(a) $O_2(H) \leqslant H_i \trianglelefteq H$.
(b) $\widetilde{H} = \widetilde{H_1} \times \widetilde{H_2} \times \dots \times \widetilde{H_m}$.
(c) $Z_L^+ = V_1^+ \times V_2^+ \times \dots V_m^+$.
(d) $\widetilde{H_i} \cong SL_l(2^k)$, $l \geqslant 2$, $Sp_{2l}(2^k)$, $l \geqslant 2$, $G_2(2^k)$ *or* $Sym(l)$, $l > 6$, $l \equiv 2, 3 \pmod 4$. *Moreover* $V_i^+$ *is a corresponding natural module.*
(e) $L$ *acts transitively on* $\{H_1, H_2, \dots, H_m\}$.

PROOF. Let $H_1$ be the smallest subnormal subgroup of $L$ containing $AO_2(L)$ and put

$$H_1^L =: \{H_1, \dots, H_m\}.$$

By Gaschütz' Theorem $C_{Z_L}(T_0 \cap H) \leqslant C_{Z_L}(H)[Z_L, H]$, see C.17. Since $\Omega_1 Z(T_0) \leqslant C_{Z_L}(T_0 \cap H)$ and $Z_L = \langle \Omega_1 Z(T_0)^L \rangle$, this gives $Z_L = C_{Z_L}(H)[Z_L, H]$ and so $Z_L^+ = [Z_L^+, H]$. The lemma now follows from C.9. □

NOTATION 3.17. In the following we will use the notation introduced in 3.16.

LEMMA 3.18. *$H \nleqslant D$, $L = HFT$ and $T_0 = (H \cap T_0)T^*$.*

PROOF. Suppose that $H \leqslant D$. Then we can apply 3.12(b) with $N_0 = H$. Since $H \nleqslant O_2(L)$ we conclude $O^2(H) = O^2(F)$ and so $O^2(H)/O_2(O^2(H)) \cong Sp_{2n-2}(2)'$. Now 3.16 shows that $m = 1$ and

$$[Z_L, O^2(H)]C_{Z_L}(H)/C_{Z_L}(H)$$

is a simple $O^2(H)$-module. From 3.10(h) we get $Y \leqslant Z_L$. Since $[Y, O^2(F)] \neq 1$ this gives $[Z_L, O^2(H)] = [Y, O^2(F)]$. Hence $Y = \langle y \rangle [Y, O^2(F)]$ is normal in $L$, a contradiction since $N_G(Y) \leqslant M^\dagger$ by 3.8(b).

Thus $H \nleqslant D$ and the minimal choice of $L$ implies $L = HFT$. Since $T^* \in Syl_2(FT)$ and $T^* \leqslant T_0$ we conclude that $T_0 \leqslant HT^*$ and so $T_0 = (H \cap T_0)T^*$. □

LEMMA 3.19. *$\widetilde{H_i} \ncong SL_2(2^k)$.*

PROOF. Suppose for a contradiction that $\widetilde{H_i} \cong SL_2(2^k)$. We will first show

1°. $\quad Y^+ \neq 1$.

Otherwise $H \leqslant C_G(Y)$. By 3.8(b) $C_G(Y) \leqslant M^\dagger$ and so $H \leqslant L \cap M^\dagger = D$, a contradiction to 3.18.

2°. $\quad O^2(F) \leqslant H$.

Assume that $O^2(F) \nleqslant H$. Then by 3.13, $H \cap T_0 = H \cap T$ and by 3.18 $T_0 = (H \cap T_0)T^* = T^*$. In particular, $H \cap T_0 \trianglelefteq FT$. Since $D$ is the unique maximal subgroup of $L$ containing $FT$ we conclude that $N_L(H \cap T_0) \leqslant D$. On the other hand by 3.12(a), applied with $N_0 = D \cap H$, $O^2(D \cap H) \leqslant C_L(Y)$, so $O^2(N_H(H \cap T_0))$ centralizes $Y$. For $k \neq 1$ this yields a contradiction since by (1°) $Y^+ \neq 1$ while $O^2(N_H(H \cap T_0))$ acts fixed-point freely on the direct sum $Z_L^+$ of natural $SL_2(2^k)$-modules.

Thus $k = 1$. Then $O^2(\widetilde{H})$ is an abelian 3-group, $D \cap H = H \cap T_0$, $[Z_L, H] \cap \Omega_1 Z(H) = 1$ and, for $1 \leqslant i \leqslant m$, $V_i$ is a natural $SL_2(2)$-module for $H_i$. In particular, $|C_{V_i}(T_0 \cap H)| = 2$. Let $1 \neq v_i \in C_{V_i}(T_0 \cap H)$, and put $v = \prod_{i=1}^m v_i$. Then $D = N_H(T_0 \cap H)$ centralizes $v$ and $1 \neq v \in [Z_L, H]$. Since $FT \leqslant D$, $v \in \Omega_1 Z(FT)$. By 3.4(f) $\Omega_1 Z(T^*) = \langle y, z \rangle$. Hence $\Omega_1 Z(FT) \leqslant \Omega_1 Z(T^*) = \langle y, z \rangle$ and since $[z, F] \neq 1$, $\Omega_1 Z(FT) = \langle y \rangle \leqslant \Omega_1 Z(H)$. This shows that $v = y$ and so $v \in [Z_L, H] \cap \Omega_1 Z(H) = 1$, a contradiction.

3°. $\quad L = HT^*$, $2n = 4$, $T^* = T_0$, $D = N_L(T_0 \cap H)$ *and* $k > 1$.

By 3.18, $L = HFT$, and by (2°), $O^2(F) \leqslant H$. As $F = O^2(F)T^*$, we get $L = HT^*$. Since $\widetilde{H}_i \cong SL_2(2^k)$, $\widetilde{H}$ does not have any section isomorphic to $Sp_{2t}(2)$ for any $2t \geqslant 4$. Since by 3.4(c), $FT/T \cong Sp_{2n-2}(2)$ and by (2°) $O^2(F) \leqslant H$, we conclude that $2n - 2 = 2$ and $2n = 4$. Now 3.10(k) gives $T^* = T_0$. Since $L = HT^*$, the structure of $\widetilde{L}$ shows that $N_L(T_0 \cap H)$ is the unique maximal subgroup of $L$ containing $T^*$. Thus $FT \leqslant N_L(T_0 \cap H)$ and $D = N_L(T_0 \cap H)$. If $k = 1$, this implies that $D = T^*$, a contradiction to $F \leqslant D$.

4°. $k = 2, m = 1$, $\widetilde{L} \cong Sym(5)$, $D = FT$ *and* $O_2(D) = T$.

By 3.16(e), $L$ acts transitively on $\{H_1, \ldots, H_m\}$ and by (3°) $L = HT^*$. Hence $T^*$ acts transitively on $\{H_1, \ldots, H_m\}$.

Recall from 3.10(i) that $D = FTC_D(Y_M)$, so $D$ normalizes $Y$. From (3°) we get $D = N_L(T_0 \cap H)$ and $k > 1$, so $O^2(D \cap H_1) \neq 1$. Since $V_1^+$ is a natural $SL_2(q)$-module for $H_1$, we conclude that $C_{V_1^+}(O^2(D \cap H_1)) = 1$. As $Z_L^+ = V_1^+ \times \ldots \times V_m^+$, this gives $C_{Z_L^+}(O^2(D \cap H)) = 1$.

Put $X^+ := [Y^+, O^2(D \cap H_1)]$ that $X^+ = 1$. Then $Y^+ \leqslant C_{Z_L^+}(O^2(D \cap H_1))$ and so $Y^+ \leqslant C_{Z_L^+}(O^2(D \cap H)) = 1$, since $T^*$ normalizes $Y^+$ and acts transitively on $\{H_1, \ldots, H_m\}$. But $Y^+ \neq 1$ by (1°), a contradiction.

Thus $X^+ \neq 1$, and as $X^+ = [X^+, O^2(D \cap H_1)]$, $|X^+| \geqslant 4$. Since $Z_L^+ = V_1^+ \times \ldots \times V_m^+$, we have $X^+ \leqslant [Z_L^+, H_1] = V_1^+$, and since $D$ normalizes $Y$, $X^+ \leqslant Y^+$. Thus $X^+ \leqslant V_1^+ \cap Y^+$. By (3°) $2n = 4$ and so $|Y^+| \leqslant |Y/\langle y \rangle| = 4 \leqslant |X^+|$. Hence $Y^+ = X^+ \leqslant V_1^+$. Since $Y^+$ is $T^*$-invariant, we conclude that $T^*$ normalizes $V_1^+$ and so $m = 1$. Moreover, $Y^+$ is $D$-invariant and by (3°) $D = N_L(H \cap T_0)$. Since $Z_L^+ = V_1^+$ is a natural $SL_2(2^k)$-module for $H$, any non-trivial $N_H(H \cap T_0)$-submodule of $Z_L^+$ has order $2^k$ or $2^{2k}$. As $|Y^+| = 4$ and $k > 1$. we get $k = 2$. In particular, $\widetilde{H \cap D} \cong Alt(4)$. By (3°) $T^* = T_0$ and $L = HT^*$. Hence $O^2(\widetilde{D}) \leqslant \widetilde{H \cap D}$, and $F \leqslant D$ implies $O^2(F) = O^2(D)$. Since $T_0 = T^*$ and $T^* \in Syl_2(FT)$ this gives $D = O^2(D)T_0 = O^2(F)T^* = FT$. By 3.4(c), $T = O_2(FT)$ and $FT/T \cong Sym(3)$. Thus $\widetilde{T^*} \nleqslant \widetilde{H}$, $\widetilde{L} \cong Sym(5)$ and all parts of (4°) are proved.

By (3°) $T_0 = T^*$ and by 3.10(b) $[Q_z, P] \leqslant Q_z \cap P \leqslant T_0$. Hence $Q_z \leqslant N_G(T_0)$. Let $t \in Q_z \backslash L$. By 3.10(c) , $O_2(\langle L, L^t \rangle) = 1$. In particular, since $t$ normalizes $T_0$, no non-trivial characteristic subgroup of $T_0$ is normal in $L$. Since $L \cap M^\dagger$ is the unique maximal subgroup of $L$ containing $T_0$, we conclude that $N_L(X) \leqslant L \cap M^\dagger$ for every non-trivial characteristic subgroup $X$ of $T_0$. The main result of [**BHS**] now shows that $[O_2(L), O^2(L)] = [Z_L, H]$. By 3.10(d), $Z_L Z_L^t$ is not normal in $L$. Thus $Z_L^t \nleqslant O_2(L)$ and so $[Z_L, Z_L^t] \neq 1$. Observe that no element in $T_0$ acts as a transvection on $Z_L$ or $Z_L^t$. Thus $|\widetilde{Z_L^t}| = |Z_L^t / C_{Z_L^t}(Z_L)| \geqslant 4$. Since $Z_L^t \trianglelefteq T_0$, $Z_L^t$ acts quadratically on $Z_L$. Note that $\widetilde{H \cap T_0}$ is the unique subgroup of order at least four in $\widetilde{T_0}$ acting quadratically on $Z_L$, so

$$H \cap T_0 = C_{T_0}([Z_L, Z_L^t]) = Z_L^t O_2(L) = Z_L O_2(L^t).$$

Hence $O_2(L) = Z_L(O_2(L) \cap O_2(L)^t)$ and $\Phi(O_2(L)) = \Phi(O_2(L) \cap O_2(L^t))$. Since $[O_2(L), O^2(L)] \leqslant Z_L \leqslant \Omega_1 Z(O_2(L))$, 1.18 shows that $O^2(L)$ centralizes $\Phi(O_2(L))$ and so $\Phi(O_2(L) \cap O_2(L^t))$ is normalized by $O^2(L), T_0$ and $t$. This forces $\Phi(O_2(L) \cap O_2(L^t)) = 1$, whence $O_2(L)$ is elementary abelian. By (3°) $T_0 = T^* \leqslant D$ and by (4°) $\widetilde{L} \cong Sym(5)$. Since $\widetilde{D}$ is maximal subgroup of $\widetilde{L}$, this gives $D/O_2(L) \cong Sym(4)$ and so $D$ has no central composition factor on $O_2(D)/O_2(L)$. But $|Z_M/Y| = 2$ and $Y \leqslant O_2(L)$, so $Z_M \leqslant O_2(L)$ and $[Z_M, O_2(L)] = 1$. Since $T = O_2(D)$ by (4°) and $|T/O_2(M)| = 2$ by 3.1(c), a similar argument yields $T \leqslant O_2(L)O_2(M)$. But then $T$ centralizes $Z_M$, a contradiction. □

LEMMA 3.20. $C_{\widetilde{L}}(\widetilde{H}) = 1$.

PROOF. Put $N := C_L(\widetilde{H})$. Note that $Z(\widetilde{H_i}) = 1$ for all the groups listed in 3.16(d). Also $\widetilde{H} = \widetilde{H_1} \times \ldots \times \widetilde{H_m}$. Thus $Z(\widetilde{H}) = 1$ and so $N \cap H = O_2(L)$.

Suppose for a contradiction that $O^2(F) \leqslant N$. We claim that $D \cap HT^*$ is the unique maximal subgroup of $HT^*$ containing $T^*$. So let $T^* \leqslant U \leqslant HT^*$ and put $E := O^2(U)O_2(L)$. Then $U = ET^*$ and $E \leqslant H$. Thus $O^2(F) \leqslant N \leqslant C_L(\widetilde{E})$ and so $FT = O^2(F)T^*$ normalizes $E$. Hence $EFT$ is a subgroup of $L$ containing $FT$. Note that $ETF = ET^*F = UF$. By the minimal choice of $L$ either $UF \leqslant D$ or $L = UF$. In the first case $U \leqslant D \cap HT^*$. In the second case $L = ET^*F$ and $ET^* \leqslant HT^*$, so

$$HT^* = ET^*(HT^* \cap F) = ET^*(O^2(HT^* \cap F)) = UO^2(HT^* \cap F).$$

Since $O^2(HT^* \cap F) \leqslant O^2(HT^*) \cap O^2(F) \leqslant H \cap N = O_2(L) \leqslant T^* \leqslant U$ we conclude that $HT^* = U$. This completes the proof of the claim. It follows that $\widehat{HT^*}$ is 2-minimal. Hence C.13 shows that

$\widetilde{H}_i \cong SL_2(2^k)$ or $Sym(r)$, $r = 2^s + 1$, $s \geqslant 2$. The first case contradicts 3.19. In the second case $r \equiv 1 \pmod 4$, a contradiction to 3.16(d).

Thus $O^2(F) \nleqslant N$. Now 3.13 gives $N \cap T_0 = N \cap T \leqslant T \leqslant TH$. Since $N \cap H = O_2(L)$, $N \cap TH$ is 2-group. By 3.18 $L = HFT$. Since $F$ normalizes $T$, this gives $TH \trianglelefteq L$. So $N \cap T_0 \leqslant N \cap TH \leqslant O_2(N) \leqslant O_2(L)$. Thus $\widetilde{N}$ is a $2'$-group. Assume for a contradiction $\widetilde{N} \neq 1$. Since $N \cap TH = O_2(L)$,

$$\widetilde{N} \cong NTH/TH \trianglelefteq L/TH = FTH/TH.$$

On the other hand, as $FT/T \cong Sp_{2n-2}(2)$ and $\widetilde{N}$ is a non-trivial $2'$-group, we conclude that $2n-2 = 2$. Hence $F$ and $N$ are solvable.

Suppose that $N \nleqslant D$. Then the minimality of $L$ implies $L = NFT$, and so $L$ is solvable. The only solvable group listed in 3.16(d) is $\widetilde{H}_i \cong SL_2(2)$, a contradiction to 3.19. Hence $N \leqslant D$. Since $O^2(F) \nleqslant N$, 3.12(b) implies that $N \leqslant O_2(L)$. □

Lemma 3.21. *$m = 1$, $F \leqslant H_1 = H$, $L = HT$ and $P \cap H = C_H(z^+)$.*

Proof. Note that $C_H(z^+)$ centralizes $\langle z^+ \rangle$ and $C_{Y_L}(H)$. Thus $C_H(z^+)/C_H(z)$ is a 2-group. Since $T_0 \cap H$ centralizes $z$ and is a Sylow 2-subgroup of $H$ we conclude that $P \cap H = C_H(z) = C_H(z^+)$.

Recall from 3.16 that

$$\widetilde{H} = \widetilde{H_1} \times \dots \widetilde{H_m} \quad \text{and} \quad Z_L^+ = V_1^+ \times \dots V_m^+,$$

where $V_i^+ = [Z_L^+, \widetilde{H_i}]$. Let $z_i^+$ be the projection of $z^+$ onto $V_i^+$ and put $P_i := C_{H_i}(z_i^+)$. Then $P \cap H = \langle P_i \mid i = 1, \dots, m \rangle$. By 3.19 $\widetilde{H_i} \ncong SL_2(2^k)$ and so by 3.16(d)

$$(*) \qquad \widetilde{H_i} \cong SL_l(2^k),\ l \geqslant 3,\ Sp_{2l}(2^k),\ l \geqslant 2,\ G_2(2^k) \text{ or } Sym(l),\ l > 6,\ k \equiv 2,3 \pmod 4.$$

Moreover, $V_i^+$ is a corresponding natural module. In each of these cases we conclude that $\widetilde{P}_i = C_{\widetilde{H}_i}(z_i^+)$ is not a 2-group. On the other hand, $[P_i, O_2(P)]$ is a $p$-group, and so $O_2(P)$ normalizes $H_i$. Since $Q_z \cap L \leqslant O_2(P)$ by $Q!$, we get that

$$Q_z \cap L \leqslant O_2(P) \leqslant N_L(H_i),\ i = 1, \dots, m.$$

By 3.4(k) $F = \langle (Q_z \cap F)^F \rangle$ and we conclude that $F \leqslant N_L(H_i)$, $i = 1, \dots, m$. The structure of the groups in $(*)$ shows that no element of $O_2(P)$ induces an outer automorphism on $\widetilde{H}_i$. So $Q_z \cap F$ and thus also $F$ induces inner automorphisms on $\widetilde{H}$. Hence $F \leqslant C_L(\widetilde{H})H$, and 3.20 yields $F \leqslant H$. In particular, $L = HFT = HT$, and by 3.16(e) $L$ and so also $T$ acts transitively on $\{H_1, \dots, H_m\}$.

Let $O_2(L) \leqslant F_i \leqslant H_i$ such that $\widetilde{F}_i$ is the projection of $\widetilde{F}$ in $\widetilde{H}_i$, and put $N_0 := F_1 \cdots F_m$. Then $F \leqslant N_0$ and the minimality of $L$ shows that either $N_0 \leqslant D$ or $N_0 T = L$.

Assume first that $N_0 \leqslant D$. By 3.10(j) $F \trianglelefteq D$ and so $F \trianglelefteq N_0$. Since $F'$ is not a 2-group, also $[F_1, F]$ is not a 2-group. As $F/O_2(F) \cong Sp_{2n-2}(2)$ and $[F_1, F] \trianglelefteq F$, we conclude that $O^2(F) \leqslant [F_1, F] \leqslant F_1 \cap F$. Hence $T$ normalizes $H_1$ and the transitivity of $T$ gives $m = 1$. So the lemma holds in this case.

Assume now that $N_0 T = L$. Then $O^2(H) \leqslant N_0$. Note that none of the groups in $(*)$ is solvable. Hence also $H$, $N_0$ and $F$ are not solvable and thus $2n \geqslant 6$. By 3.4(d) $[Q_z \cap F, F \cap P] \nleqslant T$. Hence also $[Q_z \cap P, P] \nleqslant T$ and by the transitivity of $T$, $[Q_z \cap P, P_1] \nleqslant T$. Since by 3.4(a) $Q_z \nleqslant L$ and $|Q_z/Q_z \cap F| = 2$, we have $Q_z \cap H \leqslant F$. Thus $[Q_z \cap P, P_1] \leqslant Q_z \cap H_1 \leqslant F \cap H_1$ and $Q_z \cap F \cap H_1 \nleqslant T$. Since $F \cap H_1$ is normal in $F$ and $FT/T \cong Sp_{2n-2}(2)$ we get that $O^2(F) \leqslant H_1$, so $T$ normalizes $H_1$ and $m = 1$. □

Lemma 3.22. *$\widetilde{H} \ncong Sym(l)$, $l > 6$.*

Proof. By 3.16(d) $l \equiv 2,3 \pmod 4$, and $Z_L^+$ is the corresponding natural module. Since $Out(Sym(l)) = 1$, for $l > 6$, $L$ induces inner automorphism on $\widetilde{H}$. By 3.20 $C_{\widetilde{L}}(\widetilde{H}) = 1$ and so $L = H$.

By 3.10(f) $Z_0 := C_{Z_L}(T_0) = \langle y, z\rangle$ has order 4. Since $[y, L] = 1$ and $[z, H] \neq 1$, this gives $C_{Z_L}(H) = \Omega_1 Z(L) = \langle y\rangle$. Thus either $Z_L = \Omega_1 Z(L) \times [Z_L, L]$ or $[Z_L, L]$ is the even $Sym(l)$-permutation module of order $2^{l-1}$ for $\widetilde{H}$. As $|C_{Z_L}(T_0)| = 4$, the action of $T_0$ on $Z_L$ implies that $l = 2 + 2^k$; in particular, $l \equiv 2 \pmod 4$. Since $l > 6$, we have $k \geqslant 3$. Then $\widetilde{P} = C_{\widetilde{L}}(z^+) \cong C_2 \times Sym(l-2)$, $\widetilde{A} = O_2(\widetilde{P})$ is generated by a transposition and $[Z_L^+, A] = \langle z^+\rangle$. In particular $\widetilde{A} = \widetilde{Q_z \cap L}$.

Since by 3.4(k) $F = \langle (Q_z \cap F)^F\rangle$, $\widetilde{F}$ is generated by a conjugacy class of transpositions. Thus $\widetilde{F}$ is a naturally embedded symmetric subgroup of $\widetilde{H} \cong Sym(l)$. As $F/O_2(F) \cong Sp_{2n-2}(2)$, we get $\widetilde{F} \cong Sym(s)$ with $s = 3, 4$ or $6$, and since $s < l$, $\langle z^{+F}\rangle$ is the natural even permutation module of order $2^{s-1}$.

Suppose that $s$ is even. Then, as an $F$-module, $\langle z^{+F}\rangle$ is a non-split central extension of a simple module. On the other hand $Y^+ \cong Y/\langle y\rangle$ is simple for $F$ and by 3.1(c:c) $Y = \langle z^F\rangle$, so $\langle z^{+F}\rangle$ is simple $F$-module, a contradiction.

Thus $s = 3$, $\widetilde{F} \cong Sym(3)$ and $2n = 4$. By 3.10(k), $T_0 = T^*$, and so $T_0$ normalizes $F$. Hence $T_0$ has an orbit of length 1 on $\{1, \ldots, l\}$. But then $l \not\equiv 2 \mod 4$, a contradiction. □

Lemma 3.23. *The following hold:*

(a) $L = H$, $\widetilde{L} \cong SL_3(2)$, $Sp_4(2)$ *or* $G_2(2)$, $Z_L^+$ *is a corresponding natural module,* $2n = 4$, *and* $\widetilde{F}/O_2(\widetilde{F}) \cong SL_2(2)$.

(b) $\overline{M} \cong O_4^\epsilon(2)$, $Y_M$ *is a corresponding natural module,* $T_0 = T^*$ *and* $D = FT$.

(c) $D$ *and* $P$ *are the two maximal subgroups of* $L$ *containing* $T_0$. *Moreover,* $C_{Z_L^+}(T_0) = C_{Z_L^+}(O_2(P)) = \Omega_1 Z(T_0)^+ = \langle z^+\rangle$ *and* $P = C_L(z^+)$, *and* $C_{Z_L^+}(O_2(D)) = Y^+$ *is natural* $SL_2(2)$*-module for* $D$.

(d) $C_{Y_L}(H) = C_{Y_L}(O^2(L)) = \langle y\rangle = C_{Y_L}(O^2(F))$.

(e) *Either* $Z_L = Y_L$ *or* $\widetilde{L} \cong Sp_4(2)$ *and* $|Y_L/Z_L| = 2$.

Proof. (a) and (b): By 3.21 $L = HT$, $F \leqslant H$ and $m = 1$. By 3.19 $\widetilde{L} \not\cong SL_2(2^k)$ and by 3.22 $\widetilde{L} \not\cong Sym(l)$, $l > 6$. Thus, 3.16 shows that

$$(*) \qquad \widetilde{H} \cong SL_l(2^k), l \geqslant 3, \quad Sp_{2l}(2^k), l \geqslant 2, \quad \text{or} \quad G_2(2^k),$$

and $Z_L^+$ is a corresponding natural module. This implies that $\widetilde{J(T)} \leqslant \widetilde{H}$ and that no element of $\widetilde{L}$ induces a graph automorphism on $\widetilde{H}$. Moreover, by 3.14(b) either $\widetilde{J(T)} \neq 1$ or $(2n, \epsilon) = (4, +)$.

Suppose that $\widetilde{J(T)} \neq 1$, and choose a 2-subgroup $E$ of $H$ maximal with $FT \leqslant N_L(E)$ and $J(T) \leqslant E$. Then $E = O_2(N_H(E))$ and so by [**GLS3**, 3.1.5] (a corollary of the Borel-Tits Theorem) $\widetilde{D_0} := N_{\widetilde{H}}(\widetilde{E})$ is a proper Lie-parabolic subgroup of $\widetilde{H}$ normalized by $FT$. Observe that $F \leqslant FT \cap H \leqslant N_H(E)$, so $\widetilde{F} \leqslant \widetilde{FT \cap H} \leqslant \widetilde{D_0}$.

Suppose that $(2n, \epsilon) = (4, +)$, then by 3.10(k), $T^* \in Syl_2(L)$ and so $\widetilde{D_0} := N_{\widetilde{H}}(\widetilde{FT \cap H})$ is a proper Lie-parabolic subgroup of $\widetilde{H}$ normalized by $FT$. Moreover, $\widetilde{F} \leqslant \widetilde{FT \cap H} \leqslant \widetilde{D_0}$.

We have shown that in both cases $\widetilde{D_0}$ is a proper $FT$-invariant Lie-parabolic subgroup of $H$ with $\widetilde{F} \leqslant \widetilde{D_0}$. Let $\widetilde{T}_2 \in Syl_2(\widetilde{T}\widetilde{D_0})$ with $\widetilde{T} \leqslant \widetilde{T}_2$. Then $\widetilde{T}_2 \cap \widetilde{H}$ is a Sylow 2-subgroup of $\widetilde{H}$. Let $\Delta$ be the set of Lie-parabolic subgroups of $\widetilde{H}$ containing $\widetilde{T}_2 \cap \widetilde{H}$. Then $T$ acts on $\Delta$ and since no element of $L$ induces a graph automorphism on $\widetilde{H}$, $T$ acts trivially on $\Delta$. We conclude that $FT$ normalizes all Lie-parabolic subgroups of $\widetilde{H}$ containing $\widetilde{D_0}$. Thus, by the minimal choice of $L$, $\widetilde{D_0}$ is a maximal Lie-parabolic subgroup of $\widetilde{H}$ and $\widetilde{D_0} = \widetilde{H} \cap \widetilde{D}$. In particular, $O_2(\widetilde{D_0}) \neq 1$ and by Smith's Lemma A.63 $C_{Z_L^+}(O_2(\widetilde{D_0}))$ is a simple $\widetilde{D_0}$-module.

Since $T \leqslant D \leqslant L = HT$ we have $\widetilde{D} = (\widetilde{H} \cap \widetilde{D})\widetilde{T} = \widetilde{D_0}\widetilde{T} = \widetilde{D_0}C_{\widetilde{D}}(Y)$. By 3.10(i) $Y/\langle y\rangle$ is natural $Sp_{2n-2}(2)$-module for $D$. Hence $\langle y\rangle = C_Y(D) = C_Y(\widetilde{D_0}) = C_Y(H)$ and $Y^+$ is a natural $Sp_{2n-2}(2)$-module for $\widetilde{D_0}$. In particular, $1 \neq Y^+ \leqslant C_{Z_L^+}(O_2(\widetilde{D_0}))$. The simplicity of $C_{Z_L^+}(O_2(\widetilde{D_0}))$ as a $\widetilde{D_0}$-module now shows that $Y^+ = C_{Z_L^+}(O_2(\widetilde{D_0}))$. Thus $\widetilde{D_0}$ is a maximal Lie-parabolic subgroup in $\widetilde{H}$ such that

$$(**) \qquad Y^+ = C_{Z_L^+}(O_2(\widetilde{D_0})) \text{ is a natural } Sp_{2n-2}(2)\text{-module for } \widetilde{D_0}.$$

From the possible isomorphism types for $\widetilde{H}$ and $Z_L^+$ listed in $(*)$ we conclude that $Z_L^+$ is a natural $Sp_2(2)$-module for $\widetilde{D_0}$. Thus $2n-2=2$, $|Y^+|=4$ and $k=1$. Recall that $C_{\widetilde{L}}(\widetilde{H})=1$. Since $k=1$ and no element of $L$ induces a graph automorphism on $\widetilde{H}$ we get $L=H$.

Since $2n-2=2$ we have $2n=4$. So $\overline{M} \cong O_4^{\epsilon}(2)$. From 3.3(f) we conclude that $Y_M = Z_M$ and thus $Y_M$ is natural $O_4^{\epsilon}(2)$-module for $M$. Also 3.10(k) shows that $T^* = T_0$. So $T_0 = T^* \leqslant FT$ and $FT$ is parabolic subgroup of $H$. Since $k=1$ this implies that $\widetilde{FT}$ is a Lie-parabolic subgroup of $\widetilde{H}$. As $\widetilde{FT}/O_2(\widetilde{FT}) \cong FT/T \cong Sp_{2n-2}(2) \cong SL_2(2)$, $\widetilde{FT}$ has Lie rank 1. Since $FT$ is contained in a unique maximal subgroup of L, we conclude that $\widetilde{L}$ has Lie-rank two. Thus $\widetilde{L} \cong SL_3(2)$, $Sp_4(2)$ or $G_2(2)$ and $FT$ is maximal subgroup of $H$. Thus $D = FT$ and all parts of (a) and (b) are proved.

(c): By the choice of $L$, $D$ is a maximal subgroup of $L$. By (b) $T_0 = T^* \leqslant FT \leqslant D$. By (a) $2n=4$ and $H=L$. So $\widetilde{D} = \widetilde{D_0}$, and $(**)$ shows that $C_{Z_L+}(O_2(D)) = Y^+$ is a natural $SL_2(2)$-module for $D$ and $T_0 \leqslant D$. Hence, $D$ satisfies the statements of (c).

By 3.21 $P \cap H = C_H(z^+)$ and since $H = L$, $P = C_H(z^+)$. By Smith's Lemma A.63, $C_{Z_L^+}(O_2(\widetilde{P}))$ is a simple $P$-module and so $C_{Z_l^+}(O_2(\widetilde{P})) = \langle z^+ \rangle$. Since $O_2(\widetilde{P}) \leqslant \widetilde{T_0} \leqslant \widetilde{P}$, this gives $C_{Z_L^+}(T_0) = \langle z^+ \rangle$. By 3.10(d), $\Omega_1 Z(T_0) = \langle y, z \rangle$ and so $\Omega_1 Z(T_0)^+ = \langle z^+ \rangle$. Since $Z_L^+$ is a natural $SL_3(2)$-, $Sp_4(2)$- or $G_2(2)$-module for $L$ and $P = C_L(z^+)$, we conclude that $P$ is a maximal subgroup of $L$. As $L$ is a group of Lie-type of rank 2, $T_0$ is contained in exactly two maximal subgroups of $L$, namely $P$ and $D$. So (c) is proved.

(d): From (a) we get $H = L$, $C_{Z_L}(H) = C_{Z_L}(L) \leqslant \Omega_1 Z(F) = \langle y \rangle$. We now use $A \leqslant O_2(P)$ as chosen in 3.15. By C.9(e), $[C_{Y_L}(O^2(L)), A] = 1$. For $\widetilde{L} \cong Sp_4(2)$ or $G_2(2)$, C.8(c), shows that $A \nleqslant O^2(L)O_2(L)$. Thus $L = AO_2(L)O^2(L)$ and $C_{Y_L}(O^2(L)) = C_{Y_L}(L) = \langle y \rangle$.

For the equality $C_{Y_L}(O^2(F)) = \langle y \rangle$ it suffices to show that $C_{Y_L^+}(FT) = 1$. By C.10(b:b) $Y_L = Z_L C_{Y_L}(A)$. We conclude that $[Y_L, O^2(L)] \leqslant Z_L$, and by (b) $T_0 = T^* \leqslant FT$. Now Gaschütz's Theorem shows that $C_{Y_L^+}(FT) \leqslant C_{Y_L^+}(T_0) \leqslant C_{Y_L^+}(L) Z_L^+$, see C.17. But $C_{Y_L}(O^2(L)) = \langle y \rangle \leqslant Z_L$, and so $C_{Y_L^+}(FT) \leqslant Z_L^+$. As seen above $C_{Z_L^+}(O_2(\widetilde{D_0}))$ is a natural $Sp_2(2)$-module for $\widetilde{D_0}$. Since $\widetilde{FT} = \widetilde{D} = \widetilde{D_0}$ we conclude that $C_{Z_L^+}(FT) = 1$.

(e): Suppose that $Y_L \neq Z_L$. By (d) $C_{Y_L}(O^2(L)) = \langle y \rangle \leqslant Z_L$ and thus $Y_L$ does not split over $Z_L$ as an $L$-module. Since $A$ is an offender on $Y_L$, (a) and C.22 give $|Y_L/Z_L| = 2$ and $\widetilde{L} \cong SL_3(2)$ or $Sp_4(2)$. Moreover, in the $SL_3(2)$ case, $|[Z_L^+, A]| = 4$, which is a contradiction since $A \leqslant O_2(P)$ and $[Z_L^+, O_2(P)]| = 2$. $\square$

Notation 3.24. We fix $t \in Q_z \backslash F$ and set $G_0 := \langle L, L^t \rangle$.

Lemma 3.25. *The following hold:*

(a) $Q_z = \langle t \rangle (Q_z \cap F)$ *and* $t^2 \in F$.
(b) $Y_M \leqslant O_2(L)$.
(c) $O_2(G_0) = 1$, *and* $L \cap L^t = P = P^t$.
(d) $Y \nleqslant Y_L \cap Y_L^t$.

Proof. (a): By 3.4(a) $|Q_z / C_{Q_z}(y)| = 2$ and by 3.10(b) $C_{Q_z}(y) = Q_z \cap F$. Hence $Q_z = \langle t \rangle (Q_z \cap F)$ and $t^2 \in Q_z \cap F \leqslant F$.

(b): By 3.10(e), $O_2(L) \leqslant T$ and thus $Y_L \leqslant T$, and by 3.3(g) we have $[Y_M, T] = \langle y \rangle$. Thus $[Y_M, Y_L] \leqslant \langle y \rangle$. Since $L$ centralizes $y$, this gives $[\langle Y_M^L \rangle, Y_L] \leqslant \langle y \rangle$, and as $Y_L$ is $p$-reduced, $[\langle Y_M^L \rangle, Y_L] = 1$. By 3.11 $C_L(Y_L) = O_p(L)$ and so $Y_M \leqslant O_2(L)$. Hence (b) holds.

(c): By 3.10(c) $O_2(G_0) = 1 \neq O_2(L)$. So $G_0 \neq L$ and $L \neq L^t$. By 3.10(b) we have $[Q_z, P] \leqslant O_p(P)$, and since $t \in Q_z$ we get $P = P^t \leqslant L \cap L^t < L$. As $P$ is a maximal subgroup of $L$ by 3.23(c), this gives $P = L \cap L^t$ and (c) is proved.

(d): Note that by 3.23(b) $\overline{M} \cong O_4^\epsilon(2)$ and $Y_M$ is a natural $O_4^\epsilon(2)$-module. Also by 3.4(c) $\overline{FT} \cong C_2 \times Sp_2(2)$. It follows that $O^2(\overline{F}) \cong Sp_2(2)' \cong C_3$ and $C_{Y_M}(O^2(F))$ has order 4. By 3.23(d) $C_{Y_L}(O^2(F)) = \langle y \rangle$ has order 2. Hence $Y_M \nleqslant Y_L$. Since $Y_M = YY^t$ we get $Y^t \nleqslant Y_L$ and $Y \nleqslant Y_L^t$. □

LEMMA 3.26. *(a) $O_2(L)O_2(L^t) = O_2(P)$.*
*(b) $Z_L \cap Y_L^t = \langle y, z\rangle = Z(T_0)$.*
*(c) $Y_L Y_M$ is not normal in $L$.*
*(d) $[O_2(L), O^2(L)] \nleqslant Y_L$.*
*(e) $Y_L = Z_L$.*

PROOF. Put $R := O_2(L)O_2(L^t)$. By 3.14(c) $R \trianglelefteq P$ and $R \leqslant O_2(P)$, and by 3.14(a) $J(R) \nleqslant O_2(L)$. Thus, we can choose $B \in \mathcal{A}_R$ with $B \nleqslant O_2(L)$. By 3.11 $C_L(Z_L) = O_2(L)$ and so $[Z_L, B] \neq 1$. By A.40 $B$ is an offender on $Z_L$ and therefore , since $C_L(Z_L) = C_L(Z_L^+)$, $B$ is also an offender on $Z_L^+$.

Suppose for the moment that $\widetilde{L} \cong Sp_4(2) \cong Sym(6)$. Them $Z_L^+$ is a natural $Sym(6)$-module for $\widetilde{L}$, and since $P = C_L(z^+)$, $\widetilde{P} = C_L(t_0)$, where $t_0$ is the transposition in $\widetilde{L}$ with $[Z_L^+, t_0] = \langle z^+\rangle$. Note also that $t_0$ is the only transposition in $O_2(\widetilde{P})$. Part (h) of the Best Offender Theorem C.4 now shows that

$$(*) \qquad \widetilde{B} = \langle t_0\rangle, \quad \widetilde{B} = \langle t_1t_2, t_0\rangle \quad \text{or} \quad \widetilde{B} = \langle t_1t_2, s_1s_2, t_0\rangle$$

where $t_0, t_1, t_2$ are pairwise commuting transpositions and $s_1$ and $s_2$ are transpositions distinct from $t_1$ and $t_2$ and moving the same four symbols as $t_1t_2$.

(a): By 3.25(b),(d), $Y_M \leqslant O_2(L)$ and $Y \nleqslant Y_L^t$, so $Y_M \nleqslant Y_L$. By 3.11 $Y_L = \Omega_1 Z(O_2(L))$ and hence $[Y_M, O_2(L)] \neq 1$. Since $O_2(L) \leqslant T$ this gives $[Y_M, O_2(L)] = [Y_M, T] = \langle y\rangle$. Thus $[Y_M, R] = \langle y, y^t\rangle = \langle y, z\rangle$ and so $R \nleqslant T$. Since by 3.4(c) $T = O_2(FT)$ and by 3.23(b) $FT = D$, this gives $R \nleqslant O_2(D)$.

As $C_L(Z_L) = O_2(L) \leqslant R$, we get $\widetilde{R} \nleqslant O_2(\widetilde{D})$, and to prove (a) it suffices to show $O_2(\widetilde{P}) = \widetilde{R}$. We do this by discussing the cases for $\widetilde{L}$ given in 3.23. By 3.23(c) $\widetilde{P}$ and $\widetilde{D}$ are the two maximal parabolic subgroups of $\widetilde{L}$ containing $\widetilde{T_0}$ and, as seen above, $\widetilde{R} \trianglelefteq \widetilde{P}$ and $\widetilde{R} \nleqslant O_2(\widetilde{D})$.

Suppose first that $\widetilde{L} \cong SL_3(2)$. Then $O_2(\widetilde{P})$ is the unique non-trivial normal subgroup of $\widetilde{P}$. Since $1 \neq \widetilde{B} \leqslant \widetilde{R} \trianglelefteq \widetilde{P}$, we get $R = O_2(P)$.

Suppose next that $\widetilde{L} \cong G_2(q)$. Then by C.8, $\widetilde{B} \trianglelefteq \widetilde{P}$ and $|\widetilde{B}| = 8$. It follows that $P$ acts simply on $O_2(\widetilde{P})/\widetilde{B}$. Note that $\widetilde{B} \leqslant O_2(\widetilde{D})$. Since $\widetilde{R} \nleqslant O_2(\widetilde{D})$ and $\widetilde{B} \leqslant \widetilde{R} \trianglelefteq \widetilde{P}$, we conclude that $\widetilde{R} = O_2(\widetilde{P})$.

Suppose now that $\widetilde{L} \cong Sp_4(2)$. Choose notation as in $(*)$. Then $t_0 \in \widetilde{B} \leqslant \widetilde{R}$, $t_0 \in O_2(\widetilde{D})$ and $P$ acts simply on $O_2(\widetilde{P})/\langle t_0\rangle$. As $\widetilde{R} \nleqslant O_2(\widetilde{D})$, we again get that $\widetilde{R} = O_2(\widetilde{P})$. Thus (a) is proved.

(b): By 3.23(c) $C_{Z_L^+}(R) = C_{Z_L^+}(O_2(P)) = \langle z^+\rangle$ and so $C_{Z_L}(R) = \langle y, z\rangle$. Since $R = O_2(L)O_2(L^t)$ this gives $Z_L \cap Y_L^t = \langle y, z\rangle$, and (b) is proved.

(c): Assume for a contradiction that $Y_L Y_M$ is normal in $L$. By 3.10(h) $Y \leqslant Y_M \cap Z_L \leqslant Y_M \cap Y_L$ and so $|Y_M/Y_M \cap Y_L| \leqslant 2$. Hence $[Y_M, P] \leqslant Y_L$. On the other hand, $t$ normalizes $Y_M$ and $P$, so $[Y_M, P] \leqslant Y_L \cap Y_L^t$. Since $Y \leqslant Z_L$, this gives

$$[Y, P] \leqslant Z_L \cap Y_L^t \overset{\text{(b)}}{=} \langle y, z\rangle \leqslant Y.$$

Thus $P$ normalizes $Y$ and so by 3.8(b) $P \leqslant N_G(Y) \leqslant M^\dagger$, a contradiction.

(d): Suppose that $[O_2(L), O^2(L)] \leqslant Y_L$. Since $Y_M \leqslant O_2(L)$ we get $[Y_M, O^2(L)] \leqslant Y_L$ and $Y_M Y_L \trianglelefteq O^2(L)FT = L$, which contradicts (c).

(e): According to 3.23 we may assume that $\widetilde{L} \cong Sp_4(2)$ and $Z_L^+$ is a natural $Sp_4(2)$-module for $\widetilde{L}$. As we have seen already above $\widetilde{P}$ is a point stabilizer of $\widetilde{L}$ on $Z_L^+$.

Suppose for a contradiction that $\widetilde{J(R)} = \langle t_0\rangle$, $t_0$ as in (∗∗). Then $\widetilde{J(R))} = \widetilde{B}$, and it follows that $Z_0 := C_{Z_L}(J(R)) = C_{Z_L}(B)$. As $|\widetilde{B}| = 2$ and $B$ is an offender on $Z_L$, we have $|Z_L/Z_0| = 2$. Recall that $Z(T_0) = Z(T^*) = \langle y, z\rangle$. By the action of $P$ on $Z_L$

$$|Z_0/\Omega_1 Z(T_0)| = 4 \text{ and } [Z_0, O^2(P)]\Omega_1 Z(T_0) = Z_0.$$

By C.10(f) $[\Omega_1 Z(J(R)), \langle J(R)^L\rangle] \leqslant Z_L$ and so $[\Omega_1 Z(J(R)), O^2(L)] \leqslant Z_L$. Since $Z_0^t \leqslant \Omega_1 Z(J(R))$

$$Z_0^t = [Z_0^t, O^2(P)]\Omega_1 Z(T_0) \leqslant Z_L.$$

Thus $Z_0^t \leqslant Y^t \cap Z_L = \langle y, z\rangle = Z(T_0)$. Hence $Z_0 \leqslant Z(T_0)$ a contradiction.

Thus $\widetilde{J(R)} \neq \langle t_0\rangle$. Suppose that $Y_L^+ \neq Z_L^+$. Then Case (e:1) or (e:2) in C.22 holds, and so $\widetilde{B}$ is generated by transpositions in $\widetilde{L} \cong Sym(6)$. But then (∗∗) shows that $\widetilde{B} = \langle t_0\rangle$, so also $\widetilde{J(R)} = \langle t_0\rangle$, a contradiction. Hence (e) is proved. □

Lemma 3.27. $Y_L^t \leqslant O_2(L)$.

Proof. Assume for a contradiction that $Y_L^t \nleqslant O_2(L)$. Since $t^2 \in F \leqslant L \cap L^t$ we have $L^{t^2} = L$ and the situation is symmetric in $L$ and $L^t$. By 3.11, $C_L(Y_L) = O_2(L)$ and so $[Y_L^t, Y_L] \neq 1$. Since $\widetilde{Y_L^t} \leqslant O_2(\widetilde{P})$, C.9(f) shows that $\widetilde{Y_L^t}$ is not an over-offender [3] on $Y_L$, so $|Y_L/C_{Y_L}(Y_L^t)| \geqslant |Y_L^t/C_{Y_L^t}(Y_L)|$. Since the situation is symmetric in $L$ and $L^t$ equality holds in the preceding equation. Hence $\widetilde{Y_L^t}$ is an offender on $Y_L$ contained in $O_2(\widetilde{P})$. By 3.11 $C_L(Y_L) = C_L(Z_L)$, and as $[Y_L^t, Y_L] \neq 1$, we conclude that $[Z_L^t, Y_L] \neq 1$.

1°. $O^2(L) \leqslant \langle Y_L^{tL}\rangle$.

Observe that $O^2(\widetilde{L})$ is the unique minimal normal subgroup of $\widetilde{L}$ and so $O^2(\widetilde{L}) \leqslant \langle\widetilde{Y_L^{tL}}\rangle$. Hence $O^2(L) \leqslant \langle Y_L^{tL}\rangle O_2(L)$ and (1°) follows.

2°. $O_2(L^t) \nleqslant Y_L^t O_2(L)$ *and* $\widetilde{Y_L^t} \neq O_2(\widetilde{P})$.

Assume that $O_2(L^t) \leqslant Y_L^t O_2(L)$. Then $[Y_L, O_2(L^t)] \leqslant [Y_L, Y_L^t] \leqslant Y_L^t$ and after conjugation with $t$, $[O_2(L), Y_L^t] \leqslant Y_L$. Since $O^2(L) \leqslant \langle Y_L^{tL}\rangle$ by (1°), we conclude that $[O_2(L), O^2(L)] \leqslant Y_L$, which contradicts 3.26(c). Hence $O_2(L^t) \nleqslant Y_L^t O_2(L)$. Since $O_2(L^t) \leqslant O_2(P)$ this gives $Y_L^t O_2(L) \neq O_2(P)$ and so $\widetilde{Y_L^t} \neq O_2(\widetilde{P})$.

3°. $\widetilde{L} \cong Sp_4(2)$ *and* $|\widetilde{Y_L^t}| = |\widetilde{Z_L^t}| = 2$.

Recall that $\widetilde{Y_L^t}$ is a non-trivial offender on $Y_L$ in $O_2(\widetilde{P})$ and that $P$ normalizes $\widetilde{Y_L^t}$ since $P = P^t$. Also by 3.23(c), $P = C_L(z^+)$.

By 3.23(a) $Z_L^+$ is natural $SL_3(2)$, $Sp_4(2)$ or $G_2(2)$-module for $L$. We treat these three cases one by one.

Suppose that $Z_L^+$ is a natural $SL_3(2)$-module for $L$. Since $P = C_L(z^+)$ we conclude that $\widetilde{P}$ acts simply on $O_2(\widetilde{P})$ (see for example B.30). But then $\widetilde{Y_L^t} = O_2(\widetilde{P})$, contrary to (2°).

Suppose that $Z_L^+$ is a natural $Sp_4(2)$-module of $L$. Observe that $O_2(L^t)$ centralizes $[Z_L^+, Y_L^t]$. By 3.26(a) we have $O_2(P) = O_2(L)O_2(L^t)$, and by 3.23(d), $C_{Z_L^+}(O_2(P)) = \langle z^+\rangle$. It follows that $[Z_L^+, Y_L^t] = \langle z^+\rangle$ and so $|\widetilde{Y_L^t}| = 2 = |\widetilde{Z_L^t}|$.

Suppose that $Z_L^+$ is a natural $G_2(2)$-module of $L$. Note that $O_2(L^t)$ centralizes $Z_L^t$. By the Best Offender Theorem C.4(a), $C_{\widetilde{T_0}}(\widetilde{Z_L^t}) = \widetilde{Z_L^t}$ and so $O_2(L^t) \leqslant Y_L^t O_2(L)$, a contradiction to (2°).

4°. $\Phi(O_2(L)) \cap \Phi(O_2(L^t)) = 1$.

By (3°) $\widetilde{L} \cong Sp_4(2)$ and so $O_2(\widetilde{P})$ is elementary abelian. Thus $\Phi(O_2(L^t)) \leqslant O_2(L)$ and $[\Phi(O_2(L^t)), Y_L] = 1$. By (1°) we have $O^2(L^t) \leqslant \langle Y_L^{L^t}\rangle$, so $[\Phi(O_2(L^t)), O^2(L^t)] = 1$. This shows that $\Phi(O_2(L)) \cap \Phi(O_2(L^t))$ is centralized by $O^2(L^t)$ and normalized by $t$ and $P$. Since $L = O^2(L)P$

[3] for the definition of over-offender see A.7

we conclude that $\Phi(O_2(L)) \cap \Phi(O_2(L^t))$ is normalized by $\langle L, L^t\rangle$. By 3.10(c) $O_2(\langle L, L^t\rangle) = 1$ and, (4°) holds.

Put $U := Z_L^t \cap O_2(L)$ and $X := C_{O_2(L)}(U)$.

5°. $|O_2(L)/X| = 4$, *and* $X$ *is elementary abelian.*

By (3°) $U$ is a hyperplane of $Z_L^t$ centralized by $Y_L$. The action of $L^t$ on $Z_L^t$ shows that $O_2(P)/C_{O_2(P)}(U)$ and $C_{O_2(P)}(U)/O_2(L^t)$ have order 4 and 2, respectively. By 3.26(a) $O_2(P) = O_2(L)O_2(L^t)$ and so

$$X = Y_L(X \cap O_2(L^t)) \text{ and } |O_2(L)/X| = 4.$$

Moreover,

$$\Phi(X) = \Phi(X \cap O_2(L^t)) \leqslant \Phi(O_2(L)) \cap \Phi(O_2(L^t)).$$

Now (4°) yields $\Phi(X) = 1$.

6°. $|O_2(L)/Y_L| = 2^4$ *and* $[O_2(L), O^2(L)]Y_L = O_2(L)$.

Observe that the smallest $\mathbb{F}_2$-module $V$ for $Sp_4(2)$ with $[V, Sp_4(2)'] \neq 1$ has order $2^4$, while by (5°) $|O_2(L)/X| = 4$. By 3.26(c) $[O_2(L), O^2(L)] \nleqslant Y_L$ and so $|[O_2(L), O^2(L)]Y_L/Y_L| \geqslant 2^4$. Also by 3.11, $Y_L = \Omega_1 Z(O_2(L))$. Hence it suffices to show that $|O_2(L)/\Omega_1 Z(O_2(L))| \leqslant 2^4$.

Let $d \in L$ and put $B := XX^d$. Note that by (5°) $X$ is elementary abelian. Thus $X \cap X^d \leqslant \Omega_1 Z(B)$. So

$$(*) \qquad |B/\Omega_1 Z(B)| \leqslant |B/X \cap X^d| = |X/X \cap X^d||X^d/X \cap X^d| = |X/X \cap X^d|^2.$$

Suppose that $4 \leqslant |B/X|$. By (5°) $|O_2(L)/X| = 4$ and so $|B/X| = 4$ and $B = O_2(L)$. Since $|B/X| = |B/X^d| = |XX^d/X^d| = |X/X \cap X^d|$, also $|X/X \cap X^d| = 4$, and

$$|O_2(L)/\Omega_1 Z(O_2(L))| = |B/\Omega_1 Z(B)| \overset{(*)}{\leqslant} |X/X \cap X^d|^2 = 4^2.$$

Thus we may assume that $|B/X| = |X/X \cap X^d| \leqslant 2$ (for all $d \in L$). In particular, $|B/\Omega_1 Z(B)| \leqslant 4$ by $(*)$. Suppose that $B \trianglelefteq L$. Since $|O_2(L)/B| \leqslant |O_2(L)/X| \leqslant 4$ and $|B/\Omega_1 Z(B)| \leqslant 4$, it follows that $[O_2(L), O^2(L)] \leqslant \Omega_1 Z(B)$. From $U \leqslant X \leqslant B$, we conclude that $[O_2(L), O^2(P)] \leqslant C_P(U)$. This contradicts $O_2(L)O_2(L^t) = O_2(P)$ and $[O_2(P), O^2(P)] \nleqslant C_P(U)$.

Thus $XX^d \ntrianglelefteq L$ for all $d \in L$. In particular, $X \ntrianglelefteq L$ and $XX^d \neq O_2(L)$. Moreover, we can choose $d, h \in L$ such that $X \neq XX^d \neq XX^dX^h$. By (5°) $|O_2(L)/X| = 4$, so $O_2(L) = XX^dX^h$, $X \cap X^d \cap X^h \leqslant \Omega_1 Z(O_2(L))$ and $|X/\Omega_1 Z(O_2(L))| \leqslant 4$. Thus $|O_2(L)/\Omega_1 Z(O_2(L))| \leqslant 2^4$ and (6°) is proved.

We now are now able to derive a contradiction. By (6°) $|O_2(L)/Y_L| \leqslant 2^4 = 1 + 15$. Since the maximal parabolic subgroups of $Sp_4(2)$ have index 15, we conclude that $L$ is transitive on the non-trivial elements of $O_2(L)/Y_L$. Since $X$ is elementary abelian, $O_2(L)\backslash Y_L$ Hence all cosets of $Y_L$ in $O_2(L)$ contain involutions. As $Y_L \leqslant \Omega_1 Z(O_2(L))$ this implies that all non-trivial elements in $O_2(L)$ are involutions, so $O_2(L) = \Omega_1 Z(O_2(L)) = Y_L$. But this contradicts $|O_2(L)/Y_L| = 2^4$. □

3.28. **Proof of Theorem C:**

Put $R := O_2(L)O_2(L^t)$, and let $G^*$ be the free amalgamated product of $L$ and $L^t$ over $R$. Let $L_1$ and $L_2$ be the image of $L$ and $L^t$ in $G^*$, respectively, and identify $R$ with its image in $G^*$. An elementary property of free amalgamated products shows that $L_1 \cap L_2 = R$. We will now verify that Hypothesis 1 in [**P2**] is satisfied for $G^*$, $L_1$, $L_2$, $R$ and $p = 2$.

HYPOTHESIS 3.29 (Hypothesis 1 [**P2**]). Let $p$ be a prime and $G^*$ be a group generated by two finite subgroups $L_1$ and $L_2$ . For every $i \in \{1, 2\}$ put

$$R := L_1 \cap L_2, \qquad Z_i := \Omega_1 Z(O_p(L_i)), \qquad Z_i^+ := Z_i/C_{Z_i}(L_i), \qquad \widetilde{L}_i := L_i/O_p(L_i)$$

and suppose that the following hold:

(1) $R$ is a $p$-group with $C_{L_i}(Z_i) \leqslant R$.

(2) $\widetilde{L}_i \cong SL_{n_i}(q_i), Sp_{2n_i}(q_i)$ or $G_2(q_i)$, where $q_i$ is a power of $p$ and $p = 2$ in the last case; and $Z_i^+$ is a corresponding natural module for $\widetilde{L}_i$.

(3) There exists $S_i \in Syl_p(L_i)$ such that $R \trianglelefteq P_{L_i}(S_i)$ and either $R = O_p(P_{L_i}(S_i))$ or $\widetilde{L}_i \cong G_2(q_i)$ and $\widetilde{R}$ is elementary abelian of order $q_i^3$. (Here $P_{L_i}(S_i) := O^{p'}(C_L((\Omega_1 Z(S_i))))$

(4) $Z_1 Z_2 \leqslant O_p(L_i)$, and $Z_1 Z_2$ is not normal in $L_i$.

(5) No subgroup $U \neq 1$ of $R$ is normal in $G^*$.

(1): By 3.14(c) $R = O_p(L)O_p(L)^t) \trianglelefteq P$. In particular, $R$ is a $p$-group. By 3.11 $Y_L = \Omega_1 Z(O_p(L))$ and by 3.26 $Y_L = Z_L$, so $Z_L = \Omega_1 Z(O_p(L))$. By 3.11$C_L(Z_L) = O_p(L) \leqslant R$ and thus $C_{L_i}(Z_i) \leqslant R$ and (1) holds.

(2): By 3.23(a) $\widetilde{L} \cong SL_3(2)$, $Sp_4(2)$ or $G_2(2)$, and $Z_L^+$ is a corresponding natural module. Thus (2) holds.

(3): Recall that $R \trianglelefteq P$. By 3.10(g) $P = C_L(\Omega_1 Z(T_0))$ and so also $R \trianglelefteq P^* = O^{2'}(C_L(\Omega_1 Z(T_0)))$. By 3.26 $O_p(L)O_p(L^t) = O_p(P)$ and so also $R = O_p(P^*)$. Thus (3) holds.

(4): By 3.27 $Z_L^t \leqslant O_p(L)$ and so also $Z_L Z_L^t \leqslant O_p(L)$. By 3.10(d) $Z_L Z_L^t \ntrianglelefteq L$, and so (4) is proved.

(5) Let $U \leqslant R$ such that $U \trianglelefteq G^*$. Then $U$ is normal in $L_1$ and $L_2$ and so $U \leqslant O_2(\langle L, L^t \rangle)$. Since $O_2(\langle L, L^t \rangle) = 1$ by 3.10(c), this gives $U = 1$ and (5) holds.

So indeed Hypothesis 1 holds. According to the Main Theorem in [**P2**] this implies that $\widetilde{L}_i \cong SL_{n_i}(q_i)$ and either $p = 3$ and $n_i = 2$ or $q_i = 2$ and $n_i = 4$. Since in our case $p = 2$ and $\widetilde{L}_i$ is one of $SL_3(2), Sp_4(2)$ and $G_2(2)$, we finally have reached a contradiction.

CHAPTER 4

# The Symmetric Case

Recall from Section 2.1 that an abelian subgroup $Y$ of $G$ is called symmetric in $G$ if

$$(*) \qquad 1 \neq [Y, Y^g] \leqslant Y \cap Y^g \text{ for some } g \in G.$$

In this chapter we investigate the action of $M$ on $Y$ when $M \in \mathfrak{M}_G(S)$, $Y$ is a $p$-reduced elementary abelian normal $p$-subgroup of $M$, and $Y$ is symmetric in $G$. Note that for $Y = Y_M$ this is the symmetric case as defined in Section 2.1. Allowing $Y$ to be proper subgroup of $Y_M$ will turn out to be useful in Chapter 8.

It is immediate from $(*)$ that $Y$ is a quadratic offender on $Y^g$, or vice versa. So we are able to apply the General FF-module Theorem C.2 from Appendix C. But it is still a fairly general situation; for example, the General FF-module Theorem puts no restriction on number of components of $M/C_M(Y_M)$. This is one of the points where the existence of a large subgroup comes in handy, it allows us to apply the more restrictive Q!FF-Module Theorem C.24.

There is another point in the proof where large subgroups are essential. Assuming for a moment that $F^*(M/C_M(Y))$ is a classical group and $Y$ a corresponding natural module. Then again $(*)$ shows that $Y \cap Y^g$ is non-trivial and contains the commutator of a quadratic offender (either on $Y$ or $Y^g$). The structure of the natural module in question shows that, with very few exceptions, $[Y, Y^g]$ contains non-trivial elements that are centralized by conjugates of $Q$ in $N_G(Y)$ and in $N_G(Y^g)$. Then $Q!$ shows that $N_G(Y) \cap N_G(Y^g)$ contains these conjugates of $Q$ and so acts non-trivially on $Y$ and $Y^g$.

On the other hand

$$Y/C_Y(Y^g) \cong YC_G(Y^g)/C_G(Y^g),$$

and $N_G(Y) \cap N_G(Y^g)$ acts on the the left hand side as a subgroup of $N_G(Y)$ and on the right hand side as a subgroup of $N_G(Y^g)$. So these two actions must be isomorphic. But typically $Y/C_Y(Y^g)$ is a "natural" module for $N_G(Y) \cap N_G(Y^g)$, while $YC_G(Y^g)/C_G(Y^g)$ is the "square" of a natural module (cf. B.21). This simple observation poses a further restriction on the possible action of $M$ on $Y$.

We now state the main result of this chapter.

Theorem D. *Let $G$ be finite $\mathcal{K}_p$-group, $S \in Syl_p(G)$, and let $Q \leqslant S$ be a large subgroup of $G$. Suppose that $M \in \mathfrak{M}_G(S)$ and $Y$ is an elementary abelian normal $p$-subgroup of $M$ such that*

- (i) *$O_p(M/C_M(Y)) = 1$, and*
- (ii) *$Y$ is symmetric in $G$.*

*Then one of the following holds, where $q$ is some power of $p$ and $\overline{M} := M/C_M(Y)$:*

- (1) *$\overline{M^\circ} \cong SL_n(q)$, $n \geqslant 3$, and $Y$ is a corresponding natural module.*
- (2) (a) *$\overline{M^\circ} \cong Sp_{2n}(q)$, $n \geqslant 2$, or $Sp_4(q)'$ (and $q = 2$), and $[Y, M^\circ]$ is a corresponding natural module.*
  - (b) *If $Y \neq [Y, M^\circ]$, then $p = 2$ and $|Y/[Y, M^\circ]| \leqslant q$.*
  - (c) *If $Y \nleqslant Q^\bullet$, then $p = 2$ and $[Y, M^\circ] \nleqslant Q^\bullet$.*
- (3) *There exists a unique $\overline{M}$-invariant set $\mathcal{K}$ of subgroups of $\overline{M}$ such that $Y_M$ is a natural $SL_2(q)$-wreath product module for $\overline{M}$ with respect to $\mathcal{K}$. Moreover,*
  - (a) *$\overline{M^\circ} = O^p(\langle \mathcal{K} \rangle)\overline{Q}$,*
  - (b) *$Q$ acts transitively on $\mathcal{K}$,*
  - (c) *If $Y = Y_M$, then $Y_M = Y_{M^\circ S}$.*

(4) *$Y \nleq Q^\bullet$ and one of the following holds:*
  (1) *$\overline{M^\circ} \cong \Omega^+_{2n}(q)$ for $2n \geqslant 6$, $\Omega^-_{2n}(q)$ for $p = 2$ and $2n \geqslant 6$, $\Omega^-_{2n}(q)$ for $p$ odd and $2n \geqslant 8$, or $\Omega_{2n+1}(q)$ for $p$ odd and $2n+1 \geqslant 7$, and $Y$ is a corresponding natural-module.*
  (2) *$\overline{M^\circ} \cong SL_n(q)/\langle(-id)^{n-1}\rangle$, $n \geqslant 5$, and $Y$ is the exterior square of a corresponding natural module.*
  (3) *$\overline{M^\circ} \cong Spin^+_{10}(q)$, and $Y$ is a corresponding half-spin module.*
  (4) *$\overline{M^\circ} \cong SL_n(q) \circ SL_m(q)$, $n, m \geqslant 2, n + m \geqslant 5$, $p$ is odd, and $Y$ is the tensor product of corresponding natural modules.*

(5) (a) *$\overline{M} \cong O^\epsilon_{2n}(2)$, $\overline{M^\circ} \cong \Omega^\epsilon_{2n}(2)$, $2n \geqslant 4$ and $(2n, \epsilon) \neq (4, +)$[1] and $[Y, M]$ is a corresponding natural module.*
  (b) *If $Y \neq [Y, M]$, then $\overline{M} \cong O^+_6(2)$ and $|Y/[Y, M]| = 2$.*
  (c) *$C_G(y) \nleq M^\dagger$ for every non-singular element $y \in [Y, M]$.*
  (d) *If $Y = Y_M$, then $C_G(y)$ is not of characteristic 2 for every non-singular element $y \in [Y, M]$.*

Table 1 lists examples for $Y, M$ and $G$ fulfilling the hypothesis of Theorem D.

TABLE 1. Examples for Theorem D

| | Case | $[Y, M^\circ]$ for $M^\circ$ | c | Remarks | examples |
|---|---|---|---|---|---|
| | 1 | nat $SL_n(q)$ | 1 | $p$ odd | $L_{n+1}(q)\Phi_2$ |
| | 1 | nat $SL_n(q)$ | 1 | $n = 7, 8$ | $E_n(q)$ |
| | 1 | nat $SL_3(2)$ | 1 | - | $G_2(3)$, HS(.2), Ru, HN |
| | 1 | nat $SL_3(3)$ | 1 | - | $Fi'_{22,23,24}$, $F_4(2)$, ${}^2E_6(2)$, BM |
| | 1 | nat $SL_3(5)$ | 1 | - | Ly, BM, M |
| | 1 | nat $SL_5(2)$ | 1 | - | Th, BM |
| * | 2 | nat $Sp_8(2)$ | $\leqslant 2$ | - | BM |
| | 3 | nat $SL_2(q)$ | 1 | - | $L_3(q)$, $G_2(q)$ $p \neq 3$, $D_4(q)\Phi_3$ $p = 3$, ${}^3D_4(q)$ |
| | 3 | nat $SL_2(2)$ | 1 | - | $G_2(2)'$, $J_2$, $J_3$, $\Omega^-_6(3).X$, $\Omega^+_8(3).X$ |
| | 3 | nat $SL_2(3)$ | 1 | - | $Mat_{12}.2$, ${}^2F_4(2)'$, |
| | 3 | nat $SL_2(5)$ | 1 | - | Ru, HN, Th |
| | 3 | nat $SL_2(7)$ | 1 | - | O'N, M |
| | 3 | nat $SL_2(13)$ | 1 | - | M |
| | 4:1 | nat $\Omega_7(q)$ | 1 | $p$ odd | $F_4(q)$ |
| | 4:1 | nat $\Omega^-_6(q)$ | 1 | - | ${}^2E_6(q)$ |
| | 4:1 | nat $\Omega^+_8(q)$ | 1 | - | $E_6(q)\Phi_2$ |
| | 4:1 | nat $\Omega^+_{14}(q)$ | 1 | - | $E_8(q)$ |
| * | 4:1 | nat $\Omega^+_6(2)$ | 1 | - | $P\Omega^+_8(3).3(.2)$ |
| * | 4:1 | nat $\Omega^+_{10}(2)$ | 1 | - | M |
| | 1,4:1, 4:2 | $\Lambda^2$(nat) $SL_n(q)$, $n \geqslant 3$ | 1 | $p = 2$ | $O^+_{2n+2}(q)$ |
| | 4:3 | half-spin $Spin^+_{10}(q)$ | 1 | $p$ odd | $E_6(q)\Phi_2$ |
| | 4:4 | nat $SL_{t_1}(q) \otimes SL_{t_2}(q)$ | 1 | $p$ odd | $L_{t_1+t_2}(q)\Phi_2$,$t_1 \neq t_2$ |

Here $c := |Y_M/[Y_M, M^\circ]|$ and $\Phi_i$ denotes a group of graph automorphisms of order $i$. In the example $G = K.X$ with $K = \Omega^-_6(3)$ or $P\Omega^+_8(3)$, $X$ is a subgroup of $Out(K)$ such that $X$ acts transitively on $\mathcal{P}_{N_K(Q)}(K \cap S)$. Moreover, $*$ indicates that $(char\ Y_M)$ fails in $G$.

## 4.1. The Proof of Theorem D

In this section we assume the hypothesis of Theorem D and use the notation given there. We will prove this theorem in a sequence of lemmas.

[1] $O^+_4(2)$ appears as $SL_2(2) \wr C_2$ in Case (3)

LEMMA 4.1. *$Y \leqslant Y_M$ and $N_G(Y) = M^\dagger$.*

PROOF. By hypothesis, $O_p(M/C_M(Y)) = 1$ and so $Y$ is $p$-reduced for $M$. Hence $Y \leqslant Y_M$ and so $M^\dagger = MC_G(Y_M) \leqslant N_G(Y)$. As $Y \trianglelefteq M$, 2.2(c) gives $N_G(Y) \leqslant M^\dagger$, and 4.1 is proved. □

LEMMA 4.2. *There exists $u \in G$ such that $YY^u \leqslant S \cap S^u$ and $[Y, Y^u] \neq 1$.*

PROOF. As $Y$ is symmetric in $G$, there exists $u' \in G$ such that $1 \neq [Y^{u'}, Y] \leqslant Y^{u'} \cap Y$, so $Y^{u'} \leqslant N_G(Y) = M^\dagger$ and $Y \leqslant N_G(Y^{u'}) \leqslant M^{\dagger u'}$.

Since $S$ is a Sylow $p$-subgroup of $M^\dagger$ and $S^{u'}$ is a Sylow $p$-subgroup of $M^{\dagger u'}$, we can choose $m \in M^\dagger$ and $m' \in M^{\dagger u'}$ such that

$$Y^{u'} \leqslant S^m \qquad \text{and} \qquad Y \leqslant S^{u'm'}.$$

Set $u := u'm'm^{-1}$. Then $Y^{m^{-1}} = Y$, $Y^{u'm'} = Y^{u'}$, $Y^u = Y^{u'm'm^{-1}} = Y^{u'm^{-1}}$, and so

$$[Y, Y^u] = [Y^{m^{-1}}, Y^{u'm^{-1}}] = [Y, Y^{u'}]^{m^{-1}} \neq 1$$

and

$$Y^u = Y^{u'm^{-1}} \leqslant (S^m)^{m^{-1}} = S \qquad \text{and} \qquad Y = Y^{m^{-1}} \leqslant (S^{u'm'})^{m^{-1}} = S^u.$$

Also $Y \leqslant S$ and $Y^u \leqslant S^u$ and so $YY^u \leqslant S \cap S^u$. □

NOTATION 4.3. We fix $u$ as in 4.2. Let

$$M_1 := M, \quad S_1 := S, \quad Q_1 := Q, \quad Q_1^\bullet := Q^\bullet, \quad Y_1 := Y$$

and

$$M_2 := M^u, \quad S_2 := S^u, \quad Q_2 := Q^u, \quad Q_2^\bullet := (Q^\bullet)^u, \quad Y_2 := Y^u.$$

Note that $Y_1Y_2 \leqslant S_1 \cap S_2 \leqslant M_1 \cap M_2$ and $[Y_1, Y_2] \neq 1$.

For $i \in \{1, 2\}$ we further set $\overline{M_i} := M_i/C_{M_i}(Y_i)$, $A_i := C_{Y_i}(Q_i)$ and $\overline{L_i} := [F^*(\overline{M_i}), Q_i]$. Let $\overline{F_i}$ be the largest normal subgroup of $F^*(\overline{M_i})$ centralized by $Q_i$. $F_i$ and $L_i$ are the inverse images of $\overline{L_i}$ and $\overline{F_i}$ in $M_i$. If $U_i$ is a subgroup of $M_i$, then $\overline{U_i} := U_iC_{M_i}(Y_i)/C_{M_i}(Y_i)$. (So whether $\overline{U_i}$ denotes the image of $U_i$ in $\overline{M_1}$ or in $\overline{M_2}$ is determined by the subscript used to denoted $U_i$).

LEMMA 4.4. *$Y_1$ acts quadratically on $Y_2$ and vice versa.*

PROOF. Since $Y_1$ and $Y_2$ normalize each other, $[Y_1, Y_2] \leqslant Y_1 \cap Y_2$. Hence $[Y_2, Y_1, Y_i] \leqslant [Y_i, Y_i] = 1$ for $i = 1, 2$. □

LEMMA 4.5. (a) *$F_i \leqslant N_G(Q_i)$.*
(b) *$\overline{L_i}$ and $\overline{F_i}$ are normal in $F^*(\overline{M_i})\overline{S_i}$. In particular, $L_i$ and $F_i$ are normal in $L_iF_iS_i$.*
(c) *$\overline{F_i} = C_{F^*(\overline{M_i})}(L_iQ_i)$. In particular, $[\overline{L_i}, \overline{F_i}] = 1$.*
(d) *$\overline{L_i} = [\overline{L_i}, Q_i]$.*
(e) *$C_{\overline{M_i}}(\overline{L_iF_i})$ is a $p'$-group.*
(f) *If $B$ is a $p$-subgroup of $N_{M_i}(Q_i)$ with $[\overline{L_i}, B] \leqslant \overline{F_i}$, then $[\overline{L_i}, B] = 1$.*
(g) *$\overline{L_i} \cap \overline{F_i} \leqslant \Phi(\overline{L_i})$.*

PROOF. (a): Note that $Q_i \leqslant O_p(Q_iC_{M_i}(Y_i))$ since by $Q!$, $C_{M_i}(Y_i) \leqslant N_G(Q_i)$. Since $Q$ is weakly closed (or by 1.52(a)), $N_G(O_p(Q_iC_{M_i}(Y_i))) \leqslant N_G(Q_i)$. As $F_i$ normalizes $O_p(Q_iC_{M_i}(Y_i))$, we conclude that $F_i \leqslant N_G(Q_i)$.

(b): Since $\overline{L_i} = [F^*(\overline{M_i}), Q_i]$, $\overline{L_i} \trianglelefteq \overline{F^*(M_i)}$. By definition, $\overline{F_i} \trianglelefteq \overline{F^*(M_i)}$. As $S_i$ normalizes $Q_i$, it also normalizes $\overline{L_i}$ and $\overline{F_i}$.

The remaining claims follow from 1.17 applied to $\overline{M_i}$. □

LEMMA 4.6. *Either $Y_2$ centralizes $\overline{L_1}$ or $C_{Y_2}(\overline{L_1}) = C_{Y_2}(Y_1)$.*

PROOF. Recall from Hypothesis (i) of Theorem D that $O_p(\overline{M_1}) = 1$. As $\overline{L_1F_1}$ is subnormal in $\overline{M_1}$, this gives $O_p(\overline{L_1F_1}) = 1$. Put $H_1 := L_1F_1Q_1Y_2$ and $X := C_{Y_2}(\overline{L_1})$. By 4.5(b) both $F_1$ and $L_1$ are normal in $H_1$. Since $Q_1$ is weakly closed, 1.46(c) gives $H_1^\circ = \langle Q_1^{H_1}\rangle$. By 4.5(a) $F_i \leqslant N_G(Q_1)$, so $F_1Q_1Y_2$ normalizes $\overline{Q_1}$. As $\overline{L_1} = [\overline{L_1}, \overline{Q_1}]$ by 4.5(d), we get

$$\overline{H_1^\circ} = \langle \overline{Q_1}^{\overline{L_1F_1Q_1Y_2}}\rangle = \langle \overline{Q_1}^{\overline{L_1}}\rangle = [\overline{L_1}, \overline{Q_1}]\overline{Q_1} = \overline{L_1Q_1}.$$

Also $[O_p(\overline{H_1}), \overline{L_1F_1}] \leqslant O_p(\overline{L_1F_1}) = 1$. Since $X$ centralizes $\overline{L_1}$ and $Q_1$ centralizes $\overline{F_1}$ we have $[X, \overline{Q_1}] \leqslant C_{\overline{Q_1}}(\overline{L_1F_1})$. By 4.5(e) $C_{\overline{M_1}}(\overline{L_1F_1})$ is a $p'$-group, whence $[X, \overline{Q_1}] = 1$ and $O_p(\overline{H_1}) = 1$. The first property shows that

$$(*) \qquad X = C_{Y_2}(\overline{L_1Q_1}) = C_{Y_2}(\overline{H_1^\circ}).$$

We may assume that $Y_2$ does not centralize $\overline{L_1}$. Abusing our general convention, let $\overline{Y_2} := Y_2C_{M_1}(Y_1)/C_{M_1}(Y_1)$. Then $\overline{Y_2} \neq 1$, $\overline{L_1} \neq 1$ and $\overline{Q_1} \neq 1$. We will now show that the hypothesis of A.57, with $(Y_1, \overline{Q_1}, \overline{H_1}, \overline{Y_2})$ in place of $(V, Q, H, Y)$, is fulfilled.

We already have proved that $O_p(\overline{H_1}) = 1$. As $\overline{Q_1} \neq 1$, this gives that $\overline{Q_1} \ntrianglelefteq \overline{H_1}$. Hence by 1.57(b) $Y_1$ is a faithful $Q$!-module for $\overline{H_1}$ with respect to $\overline{Q_1}$. By 4.4 $Y_2$ acts quadratically on $Y_1$ and so $C_{\overline{Y_2}}([Y_1, \overline{Y_2}]) = \overline{Y_2} \neq 1$. Since $Y_2$ does not centralize $\overline{L_1}$ and $\overline{L_1} \leqslant \overline{H_1^\circ}$, we get $[\overline{H_1^\circ}, \overline{Y_2}] \neq 1$.

We have verified the hypothesis of A.57, and this result gives $C_{\overline{Y_2}}(\overline{H_1^\circ}) = 1$. Thus $C_{Y_2}(\overline{H_1^\circ}) = C_{Y_2}(Y_1)$ and so by $(*)$ $C_{Y_2}(Y_1) = X = C_{Y_2}(\overline{L_1})$. □

LEMMA 4.7. *Let $U \leqslant S_i$ with $[\overline{L_i}, U] = 1$ and $[Y_i, U] \neq 1$. Then $[A_i, U] \neq 1$.*

PROOF. Put $\overline{U} := UC_{M_i}(Y_i)/C_{M_i}(Y_i)$. Since $U \leqslant S_i$ and $[Y_i, U] \neq 1$, $\overline{U}$ is a non-trivial $p$-subgroup of $\overline{M_i}$. By 4.5(e) $C_{\overline{M_i}}(\overline{L_iF_i})$ is a $p'$-group. Thus $\overline{R_i} := [\overline{F_i}, \overline{U}] \neq 1$ and so $[Y_i, \overline{R_i}] \neq 1$. By 4.5(b) $S_i$ and so also $U$ normalizes $F_i$. So $\overline{R_i} \leqslant \overline{F_i}$, and we get $[\overline{R_i}, Q_i] = 1$. Since $\overline{R_i} \trianglelefteq\trianglelefteq F^*(\overline{M_i})$ and $O_p(\overline{M_i}) = 1$, we have $\overline{R_i} = O^p(\overline{R_i})$. Hence the $P \times Q$-Lemma gives $[A_i, \overline{R_i}] = [C_{Y_i}(Q_i), \overline{R_i}] \neq 1$. Since $\overline{F_i} \trianglelefteq F^*(\overline{M_i})$ we conclude that $(\overline{M_i}, \overline{F_i}, \overline{U})$ satisfy the hypothesis on $(H, L, Y)$ in 1.8. By 1.8(b), $[\overline{F_i}, \overline{U}] = [\overline{F_i}, \overline{U}, \overline{U}]$. Thus $\overline{R_i} = [\overline{R_i}, \overline{U}]$. Together with $[A_i, \overline{R_i}] \neq 1$ this implies $[A_i, U] \neq 1$. □

LEMMA 4.8. $[\overline{L_1}, Y_2] \neq 1$ *and* $[\overline{L_2}, Y_1] \neq 1$.

PROOF. By symmetry it suffices to show the claim for $[\overline{L_1}, Y_2]$. Therefore, we assume by way of contradiction:

1°. $\quad [\overline{L_1}, Y_2] = 1$.

By the choice of $u$, $[Y_1, Y_2] \neq 1$ (see 4.3). So we can apply 4.7 with $U = Y_2$ and $i = 1$, and conclude that $[A_1, Y_2] \neq 1$. Assume that also $[\overline{L_2}, Y_1] = 1$. Since $A_1 \leqslant Y_1$, also $[\overline{L_2}, A_1] = 1$. Thus 4.7 applied with $U = A_1$ and $i = 2$ gives $[A_2, A_1] \neq 1$. As $A_i \leqslant Z(Q_i)$, this is a contradiction to 2.3(a). Thus

2°. $\quad [\overline{L_2}, Y_1] \neq 1$.

Then 4.6 gives

3°. $\quad C_{Y_1}(\overline{L_2}) = C_{Y_1}(Y_2)$.

We now use the Fitting submodule $F_{Y_1}(\overline{M_1})$ defined in Appendix D. By D.6 $F_{Y_1}(\overline{M_1})$ is faithful for $\overline{M}$, and by D.8 $F_{Y_1}(\overline{M_1})$ is semisimple for $M_1^\circ$. Since $\overline{L_1} \trianglelefteq\trianglelefteq \overline{M_1^\circ}$, $F_{Y_1}(\overline{M_1})$ is also semisimple for $\overline{L_1}$, and since $F_{Y_1}(\overline{M_1})$ is faithful, $[F_{Y_1}(\overline{M_1}), Y_2] \neq 1$. Hence there exists a simple $L_1$-submodule $I_1$ of $F_{Y_1}(\overline{M_1})$ such that $[I_1, Y_2] \neq 1$; in particular, by (3°)

4°. $\quad [\overline{L_2}, I_1] \neq 1$.

Next we prove:

5°. *Put* $I_2 := I_1^u$. *Then there exists* $J_2 \in I_2^{F^*(\overline{M_2})\overline{Q_2}}$ *with* $[J_2,[L_2,I_1]] \neq 1$; *in particular* $[J_2,I_1] \neq 1$.

Put $U := \langle I_2^{F^*(\overline{M_2})\overline{Q_2}}\rangle$ and $\overline{F} := C_{F^*(\overline{M_2})}(U)$, and let $F$ be the inverse image of $\overline{F}$ in $M_2$. Note that $C_U(Q_2) \neq 1$ and so by $Q!$, $F \leqslant N_G(C_U(Q_2)) \leqslant N_G(Q_2)$. Also $\overline{F}$ is normal in $F^*(\overline{M_2})$. Hence

$$[\overline{F},\overline{Q_2}] \leqslant \overline{Q_2} \cap \overline{F} \leqslant O_p(\overline{F}) \leqslant O_p(\overline{M_2}) = 1,$$

and thus $\overline{F} \leqslant \overline{F_2}$.

Suppose that $[\overline{L_2},I_1] \leqslant \overline{F}$. Then $[\overline{L_2},I_1] \leqslant \overline{F_2}$ and 4.5(f) implies $[\overline{L_2},I_1] = 1$, a contradiction to (4°). Hence $[\overline{L_2},I_1] \nleqslant \overline{F}$, that is, $[U,[\overline{L_2},I_1]] \neq 1$, and (5°) holds.

Let $J_2$ be as in (5°). Observe that $|I_1| = |I_2| = |J_2|$. Let $x \in J_2$ with $[I_1,x] \neq 1$. Thus, $C_{I_1}(x)$ is a proper subgroup of $I_1$. By (1°) $[\overline{L_1},x] \leqslant [\overline{L_1},Y_2] = 1$. Hence $C_{I_1}(x)$ is a proper $L_1$-submodule of $I_1$. Since $I_1$ is a simple $L_1$-module we conclude:

6°. $C_{I_1}(x) = 1$. *In particular*, $|I_1| = |[I_1,x]|$.

Suppose that $C_{J_2}(I_1) \neq 1$. Let $y \in I_1$. Then $1 \neq C_{J_2}(I_1) \leqslant J_2 \cap J_2^y$. Since $J_2$ is a simple $L_2$-module and $y$ normalizes $L_2$, $J_2^y$ and $J_2^y$ are simple $L_2$-modules and $J_2 \cap J_2^y$ is a non-trivial $L_2$-submodule of $J_2$ and $J_2^y$. Thus $J_2 = J_2 \cap J_2^y = J_2^y$, and so $I_1$ normalizes $J_2$. But then $[I_1,J_2] < J_2$ and so $|[I_1,x]| < |J_2| = |I_1|$, a contradiction to (6°). We have proved:

7°. $C_{J_2}(I_1) = 1$.

By (5°) $[J_2,[L_2,I_1]] \neq 1$, and so there exists $y \in I_1$ such that $[J_2,[L_2,y]] \neq 1$. Put $W := J_2J_2^y$. By 4.4, $Y_1$ acts quadratically on $Y_2$. So $[J_2,y] \leqslant C_W(I_1)$ and $[W,I_1] \leqslant C_{Y_2}(I_1)$. By (7°) $C_{J_2}(I_1) = 1$, and we conclude that

8°. $[J_2,y] \leqslant C_W(I_1)$, $J_2 \cap [J_2,y] = 1$ *and* $J_2 \cap [W,I_1] = 1$.

In particular, $[J_2,N_{I_1}(J_2)] \leqslant J_2 \cap [W,I_1] = 1$ and so $N_{I_1}(J_2) = C_{I_1}(J_2) \leqslant C_{I_1}(x)$. By (6°) $C_{I_1}(x) = 1$ and thus

9°. $N_{I_1}(J_2) = 1$.

In particular, $J_2 \neq J_2^y$. Since $J_2$ and $J_2^y$ are simple $L_2$-modules, we conclude that $J_2 \cap J_2^y = 1$. By (8°), $J_2 \cap [J_2,y] = 1$ and so $W = J_2J_2^y = J_2[J_2,y] = J_2 \times [J_2,y]$. This gives

10°. $W = J_2 \times [J_2,y] = J_2^y \times [J_2,y] = J_2 \times J_2^y$.

Suppose for a contradiction that $[J_2,y]$ is $L_2$-invariant. Then (10°) shows that $J_2$ and $[J_2,y]$ are both isomorphic to $W/J_2^y$ as $L_2$-modules. Moreover, $y$ centralizes $[J_2,y]$ and so $[L_2,y]$ centralizes $[J_2,y]$. Hence $[L_2,y]$ also centralizes $J_2$, which contradicts the choice of $y$. Therefore,

11°. $[J_2,y]$ *is not* $L_2$-*invariant.*

By (8°) $[J_2,y] \leqslant C_W(I_1) \leqslant W \cap W^{y'}$ for every $y' \in I_1$. On the other hand, $J_2$ is a simple $L_2$-module and $W \cap W^{y'}$ is an $L_2$-submodule. By (10°) $W = J_2 \times J_2^y$. Hence every non-trivial $L_2$-submodule of $W$ has order $|J_2|$. Since $|W \cap W^{y'}| \geqslant |[J_2,y]| = |J_2|$, we conclude that either $W = W \cap W^{y'} = W^{y'}$ or $W \cap W^{y'} = [J_2,y]$. In the latter case, $[J_2,y]$ is $L_2$-invariant, a contradiction to (11°). Thus $W = W^{y'}$.

We have shown that $I_1$ normalizes $W$. By (9°) $N_{I_1}(J_2) = 1$, and so there are $|I_1|$ $I_1$-conjugates of $J_2$. Since $J_2 \cap [J_2,y] = 1$ and $I_1$ centralizes $[J_2,y]$, each of these conjugates intersects $[J_2,y]$ trivially and is $L_2$-invariant. Since $J_2$ is a simple $L_2$-module, the conjugates have pairwise trivial intersection. Note also that $|I_1| = |J_2|$ and by (10°) $|W| = |J_2||J_2^y| = |J_2|^2$ and $|[J_2,y] = |J_2|$. We conclude that these conjugates together with $[J_2,y]$ form a partition of $W$. Thus, $L_2$ also leaves invariant $[J_2,y]$, a contradiction to (11°). □

LEMMA 4.9. (a) $[\overline{M_1^\circ},Y_2] \neq 1$ *and* $[\overline{M_1^\circ},Y_1^{u^{-1}}] \neq 1$, *in particular* $\overline{M^\circ} \neq 1$.

(b) $Y_i$ *is a faithful* $Q!$-*module for* $\overline{M_i}$ *with respect to* $\overline{Q_i}$.

(c) $Y_2$ *or* $Y_1^{u^{-1}}$ *is a non-trivial quadratic offender on* $Y_1$.
(d) *The hypothesis of the Q!FF-Module Theorem C.24 is fulfilled for* $(\overline{M_i}, Y_i, \overline{Q_i})$ *in place of* $(H, V, Q)$.

Proof. (a): Recall from 4.3 that $\overline{L_i} = [F^*(\overline{M_i}), Q_i]$ and so $\overline{L_i} \leqslant \overline{M_i^\circ}$. By 4.8 $[\overline{L_1}, Y_2] \neq 1$ and $[\overline{L_2}, Y_1] \neq 1$ and so also $[\overline{M_2^\circ}, Y_1] \neq 1$ and $[\overline{M_1^\circ}, Y_2] \neq 1$. Conjugating the last equation by $u^{-1}$ gives $[\overline{M_1^\circ}, Y_1^{u^{-1}}] \neq 1$, and so (a) holds.

(b): Since $\overline{M_i^\circ} \neq 1$ we also have $\overline{Q_i} \neq 1$. As $O_p(\overline{M_i}) = 1$ this implies $Q_i \ntrianglelefteq M_i$. Hence by 1.57(b) $Y_i$ is a faithful $Q$!-module for $\overline{M}_i$ with respect to $\overline{Q_i}$.

(c): By 4.4 $Y_1$ acts quadratically on $Y_2$ and vice versa. If $Y_2$ is not an offender on $Y_1$, then $|Y_2/C_{Y_2}(Y_1)| \leqslant |Y_1/C_{Y_1}(Y_2)|$, and since $Y_2 = Y_1^u$, conjugation with $u^{-1}$ gives

$$|Y_1/C_{Y_1}(Y_1^{u^{-1}})| \leqslant |Y_1^{u^{-1}}/C_{Y_1^{u^{-1}}}(Y_1)|.$$

Hence $Y_1^{u^{-1}}$ is an offender on $Y_1$.

(d): According to (c) we can choose $Y_3 \in \{Y_2, Y_1^{u^{-1}}\}$ such that $Y_3$ is a non-trivial quadratic offender on $Y_1$. By (a) $[\overline{M_1^\circ}, Y_3] \neq 1$. Thus $Y_3$ fulfills the condition for $Y$ in the Q!FF-Module Theorem. By (b) $Y_1$ is a faithful $Q$!-module for $\overline{M}_1$ with respect to $\overline{Q_1}$. Also $O_p(\overline{M_1}) = 1$ and so the Hypothesis of the Q!FF-Module Theorem is fulfilled. □

Lemma 4.10. *Suppose that the following hold:*
(i) $M^\circ/C_{M^\circ}(Y) \cong \Omega_{2n}^\epsilon(2)$, $2n \geqslant 4$, *and* $M \nleqslant M^\circ C_M(Y)$.
(ii) $[Y, M^\circ]$ *is a natural* $\Omega_{2n}^\epsilon(2)$*-module for* $M^\circ$.
(iii) $[Y_1, Y_2]$ *contains a non-singular vector of* $[Y_1, M_1^\circ]$ *or* $[Y_2, M_2^\circ]$.

*Then Theorem D(5) holds if* $(2n, \epsilon) \neq (4, +)$, *and Theorem D(3) holds if* $(2n, \epsilon) = (4, +)$.

Proof. By B.35(d), $N_{Aut([Y,M^\circ])}(\overline{M^\circ}) \cong O_{2n}^\epsilon(2)$. Since $\Omega_{2n}^\epsilon(2)$ has index 2 in $O_{2n}^\epsilon(2)$ and $\overline{M} \neq \overline{M^\circ}$, we conclude that $\overline{M} \cong O_{2n}^\epsilon(2)$ and $[Y, M^\circ]$ is a corresponding natural module.

If $Y \neq [Y, M^\circ]$ then C.22 shows that $\overline{M} \cong O_6^+(2)$ and $|Y/[Y, M^\circ]| = 2$. In particular, $[Y, M^\circ] = [Y, M]$.

By (iii) and since the setup is symmetric in $M_1$ and $M_2$, we may assume that $[Y_1, Y_2]$ contains a non-singular vector $t$ of $[Y_1, M_1]$. As $M_1^\dagger = M_1 C_G(Y_1)$ fixes the $M_1$-invariant quadratic from on $[Y_1, M_1]$, we know that the non-singular elements of $[Y_1 . M_1]$ are precisely those elements that are not centralized by a Sylow $p$-subgroup of $M_1^\dagger$. In particular, $C_{M_1^\dagger}(t)$ does not contain a Sylow $p$-subgroup of $M_1^\dagger$. We claim that $C_G(t) \nleqslant M_1^\dagger$.

For this suppose first that $t$ is singular in $[Y_2, M_2]$. Then $C_{M_2}(t)$ contains a Sylow $p$-subgroup of $M_2$ and so also of $G$. As $C_{M_1^\dagger}(t)$ does not contain a Sylow $p$-subgroup of $M_1^\dagger$, we conclude that $C_{M_2}(t) \nleqslant C_{M_1^\dagger}(t)$ and so $C_G(t) \nleqslant M_1^\dagger$.

Suppose next that $t$ is non-singular in $[Y_2, M_2]$. Recall that $M_2 = M_1^u$, so as $t$ is non-singular in $[Y_1, M_1]$, $t^u$ is non-singular in $[Y_2, M_2] = [Y_1, M_1]^u$. Since $M_2$ is transitive on the non-singular vectors of $[Y_2, M_2]$, there exists $m \in M_2$ such that $t^{um} = t$. Since $Y_1^{um} = Y_2^m = Y_2 \neq Y_1$ we have $um \notin M_1^\dagger$ and so again $C_G(t) \nleqslant M^\dagger$.

We proved that $C_G(t) \nleqslant M_1^\dagger$. Since $M_1 = M$ and $M$ acts transitively on the non-singular vectors of $[Y, M]$, we conclude that $C_G(y) \nleqslant M^\dagger$ for all non-singular $y \in [Y, M]$.

Suppose that $\overline{M} \cong O_4^+(2)$. Then $\overline{M} \cong SL_2(2) \wr C_2$ and Theorem D(3) holds.

Suppose next that $\overline{M} \ncong O_4^+(2)$. By hypothesis $\overline{M^\circ} \cong \Omega_{2n}^\epsilon(2)$. Assume in addition that $Y = Y_M$. Then Theorem C shows that $C_G(y)$ is not of characteristic 2 for every non-singular element $y \in [Y, M]$. Thus Theorem D(5) holds in this case. □

By 4.9(d) the Hypothesis of the Q!FF-Module Theorem C.24 is fulfilled for $(\overline{M_i}, Y_i, \overline{Q_i})$. In the following we will discuss the various outcomes of the Q!FF-Module Theorem.

NOTATION 4.11. Let $\{i,j\} := \{1,2\}$. Put
$$J_i := J_{M_i}(Y_i) \qquad \text{and} \qquad \overline{R_i} := F^*(\overline{J_i}).$$
Let $R_i$ be the inverse image of $\overline{R_i}$ in $M_i$. Put
$$W_i := [Y_i, R_i] \qquad \text{and} \qquad T_i := Y_j R_i.$$

LEMMA 4.12. *Suppose that C.24(1) holds. Then Theorem D(3) holds.*

PROOF. By C.24(1) there exists an $\overline{M_1}$-invariant set $\mathcal{K}$ of subgroups of $\overline{M_1}$ such that $Y_1$ is a natural $SL_2(q)$-wreath product module for $\overline{M_1}$ with respect to $\mathcal{K}$, $\overline{M_1^\circ} = O^p(\langle\mathcal{K}\rangle)\overline{Q_1}$ and $Q_1$ acts transitively on $\mathcal{K}$. By A.27 this set is unique. By 1.24(f), $Y_{M^\circ S} \leqslant Y_M$ and since $Y_M$ is a simple $M^\circ S$-module, $Y_M = Y_{M^\circ S}$. So Theorem D(3) holds. □

LEMMA 4.13. *Suppose that C.24(2) holds and $W_1$ is not a simple $R_1$-module. Then Theorem D(4:4) holds.*

PROOF. Note that by 4.9(c) $Y_2 (= Y_1^u)$ or $Y_1^{u^{-1}}$ is an offender on $Y_1$. Also $u^{-1}$ in place of $u$ fulfills the conclusion of 4.2. So possibly after replacing $u$ by $u^{-1}$ we may assume that $Y_2$ is a offender on $Y_1$. Also $[Y_1, Y_2] \neq 1$, and so we can choose a minimal non-trivial offender $A$ on $Y_1$ with $A \leqslant Y_2$. By A.39 $A$ is a quadratic best offender on $Y_1$, so $A \leqslant J_1$.

Recall that $\{i,j\} = \{1,2\}$. For now let $I_i$ be any simple $R_i$-submodules of $W_i$. From C.24(2:a), (2:b) we conclude that

1°.

(a) *$\overline{R_i}$ is quasisimple and $\overline{R_i} \leqslant \overline{M_i^\circ}$.*

(b) *$C_{Y_i}(R_i) = 1$, $W_i$ is a semisimple $J_i$-module, and $\overline{M_i}$ acts faithfully on $W_i$.*

Next we prove:

2°. *Let $x \in M_j$ with $C_{I_j}(x) \neq 1$. Then $x$ normalizes $I_j$.*

Note that $1 \neq C_{I_j}(x) \leqslant I_j \cap I_j^x$. Since $R_j \trianglelefteq M_j$, $I_j$ and $I_j^x$ are simple $R_j$-modules, and so $I_j = I_j \cap I_j^x = I_j^x$.

3°. *Let $X_i \leqslant Y_i$. Suppose that $[X_i, Y_j] \neq 1$ and $X_i$ normalizes all the simple $R_j$-submodules of $W_j$. Then $[\overline{R_j}, X_i] = \overline{R_j} \neq 1$, $[I_j, X_i] \neq 1 \neq [I_j, Y_i]$, and $Y_i$ normalizes all simple $R_j$-submodules of $W_j$.*

Suppose for a contradiction that $[\overline{R_j}, X_i] = 1$. Since $X_i$ is a $p$-group and normalizes the simple $R_j$-submodule $I_j$, we conclude that $X_i$ centralizes $I_j$. By (1°)(b) $W_j$ is a semisimple $J_j$-module. Since $R_j \trianglelefteq J_j$, $W_j$ is also a semisimple $R_j$-module. It follows that $X_i$ centralizes $W_j$. As by (1°)(b) $W_j$ is a faithful $\overline{M_j}$-module, we conclude that $[X_i, Y_j] = 1$, a contradiction to the hypothesis of (3°).

Thus $[\overline{R_j}, X_i] \neq 1$. By (1°)(a) $\overline{R_j}$ is quasisimple, and we conclude that $[\overline{R_j}, X_i] = \overline{R_j}$. By (1°)(b) $C_{Y_j}(R_j) = 1$ and so $[I_j, R_j] \neq 1$. Together with $[\overline{R_j}, X_i] = \overline{R_j}$ this gives $[I_j, X_i] \neq 1$. Since $Y_i$ acts quadratically on $Y_j$, we conclude that $1 \neq [I_j, X_i] \leqslant C_{I_j}(Y_i)$ and so (2°) shows that $Y_j$ normalizes $I_j$, and (3°) is proved.

Recall from 4.11 that $T_i = Y_j R_i$.

4°. *$T_i$ normalizes all simple $R_i$-submodules of $W_i$. In particular, $W_i$ is a faithful semisimple $\overline{T_i}$-module and $O_p(\overline{T_i}) = 1$.*

We apply (1°). Since $\overline{R_1}$ is quasisimple, $\overline{R_1}$ is a $J_{\overline{M_1}}(Y_1)$-component of $\overline{M_1}$, and since $C_{Y_1}(R_1) = 1$ and $I_1$ is simple, $I_1$ is a perfect $R_1$-submodule of $Y_1$. Hence by A.44 $J_1$ and so also $A$ normalizes $I_1$. Since $I_1$ is any simple $R_1$-submodule of $W_1$, $A$ normalizes every simple $R_1$-submodule of $W_1$. Thus, we can apply (3°) with $X_2 = A$ and conclude that also $Y_2$ normalizes $I_1$ and that $[I_1, Y_2] \neq 1$. Therefore $T_2 = Y_2 R_1$ normalizes $I_1$.

In particular, $|[I_1, Y_2]| < |I_1| = |I_2|$. This implies that $C_{I_2}(y) \neq 1$ for all $y \in I_1$, and (2°) shows that $I_1$ normalizes $I_2$. Hence, $I_1$ normalizes all simple $R_2$-submodules of $W_2$. As $[Y_2, I_1] \neq 1$ we can apply (3°) with $X_1 = I_1$ and conclude that also $Y_1$ normalizes all simple $R_2$-submodules of $Y_2$. So the same holds for $T_2 = Y_1R_2$.

5°. *$C_{Y_1}(I_2) = C_{Y_1}(Y_2)$ and $C_{Y_2}(I_1) = C_{Y_2}(Y_1)$; in particular $[I_1, I_2] \neq 1$.*

By (4°) $X_i := C_{Y_i}(I_j)$ normalizes all simple $R_j$-submodules of $W_j$. If $[X_i, Y_j] \neq 1$, then (3°) shows that $[I_j, X_i] \neq 1$, a contradiction. Thus $[X_i, Y_j] = 1$ and $C_{Y_i}(I_j) = X_i \leqslant C_{Y_i}(Y_j)$. The other inclusion is obvious.

6°. *$W_i$ is not selfdual as a $T_i$-module.*

Suppose that $W_i$ is a selfdual $T_i$-module. Since $W_i \leqslant Y_i$, (5°) shows that $C_{W_i}(I_j) = C_{W_i}(Y_j)$. As $W_i$ is selfdual, this gives $[W_i, I_j] = [W_i, Y_j]$ (cf. B.6(c)). By (4°) $W_i$ normalizes $I_j$, thus $[W_i, Y_j] = [W_i, I_j] \leqslant I_j$. As $[Y_j, W_i] \neq 1$, (3°) gives $\overline{R_j} = [\overline{R_j}, W_i]$, and we conclude that $W_j = [Y_j, R_j] \leqslant I_j$, so $W_j = I_j$ is a simple $R_j$-module, a contradiction.

Put $\mathbb{K}_i := End_{R_i}(I_i)$.

7°. *$T_i$ acts $\mathbb{K}_i$-linearly on $I_i$.*

In this paragraph choose $I_j = I_i^u$ if $i = 1$ and $I_j = I_i^{u^{-1}}$ if $i = 2$. So $|\mathbb{K}_j| = |\mathbb{K}_i|$. By (4°) and (5°) $I_j$ normalizes and acts non-trivially on each of the simple $R_i$-submodules of $W_i$. Suppose that $I_j$ does not act $\mathbb{K}_i$-linearly on $I_i$. Then $p < |\mathbb{K}_i| = |\mathbb{K}_j|$, and 1.22 implies that $\dim_{\mathbb{K}_i} I_i = 1$, a contradiction, since $R_i$ is quasisimple (and so perfect) and $I_i$ is non-central $\mathbb{K}_iR_i$-module. Thus $I_j$ acts $\mathbb{K}_i$-linearly on $I_i$. As $Y_j$ acts quadratically on $I_i$, $Y_j$ centralizes the non-trivial $\mathbb{K}_i$-subspace $[I_i, I_j]$ of $I_i$, and so $Y_j$ acts $\mathbb{K}_i$-linearly on $I_i$. Since $T_i = R_iY_j$, (7°) follows.

8°. *One of the following holds.*

(1) (a) *$\overline{R_i} = \overline{J_i} \cap \overline{M_i^\circ} \cong SL_n(q)$, $n \geqslant 3$, $Sp_{2n}(q)$, $n \geqslant 3$, $SU_n(q)$, $n \geqslant 8$, or $\Omega_n^\pm(q)$, $n \geqslant 10$.*
(b) *$W_i$ is the direct sum of at least two isomorphic natural modules for $\overline{R_1}$.*
(c) *$\overline{M_i^\circ} = \overline{R_i}C_{\overline{M_i^\circ}}(\overline{R_i})$.*
(d) *If $Y_i \neq W_i$, then $\overline{R_i} \cong Sp_{2n}(q)$, $p = 2$, and $n \geqslant 4$.*

(2) *$p = 2$, $\overline{J_i} = \overline{R_i} \cong SL_4(q)$, and $Y_i$ is the direct sum of two non-isomorphic natural modules for $\overline{R_i}$.*

We consider the three cases of C.24(2:c).
In Case (1), (8°)(1) holds.
In Case (2), $W_1 = [Y_1, R_1]$ is a simple $R_1$-module, contrary to the hypothesis of this lemma.
In Case (3), (8°)(2) holds.

9°. *$\overline{T_i}$ acts faithfully on $I_i$.*

By (4°) $O_p(\overline{T_i}) = 1$, and by (8°) $\overline{R_i}$ acts faithfully on $I_i$. As $\overline{T_i}/\overline{R_i}$ is a $p$-group, we conclude $C_{\overline{T_i}}(I_i) \leqslant O_p(\overline{T_i}) = 1$.

10°. *$R_i = J_i$, $\overline{J_i} \cong SL_n(q)$, $n \geqslant 3$, $W_i = Y_i$, $\overline{M^\circ} = \overline{J_i}C_{\overline{M^\circ}}(\overline{J_i})$, and $Y_i$ is the direct sum of $m$ isomorphic natural modules for $\overline{J_i}$, $m \geqslant 2$.*

Suppose first that (8°)(1) holds and $\overline{R_i} \cong SL_n(q)$, $n \geqslant 3$. Then C.24(2:a) shows that $\overline{J_i} = \overline{R_i}$, so also $R_i = J_i$. The remaining assertion in (10°) now follows from (8°)(1).

Suppose next that (8°)(1) holds and $\overline{R_i} \not\cong SL_n(q)$, $n \geqslant 3$. Then $\overline{R_i} \cong Sp_{2n}(q)$, $n \geqslant 3$, $SU_n(q)$, $n \geqslant 8$, or $\Omega_n^\pm(q)$, $n \geqslant 10$, and $I_i$ is a corresponding natural module. Note that $T_i = Y_jR_i = O^{p'}(T_i)R_i$. Also $I_i$ is a selfdual as an $\mathbb{F}_pR_i$-module and $T_i$ acts $\mathbb{K}_i$-linearly on $I_i$. Thus, B.7(f) shows that there exists a $T_i$-invariant non-degenerate symmetric, symplectic or unitary $\mathbb{K}_i$-form on $I_i$. Hence $I_i$ is selfdual as an $\mathbb{F}_pT_i$-module. Since this holds for any simple $R_i$-submodule $I_i$ of $W_i$ and $W_i$ is a semisimple $R_i$-module, this shows that $W_i$ is a selfdual $T_i$-module, a contradiction to (6°).

Suppose now that (8°)(2) holds. Then $W_i = I_i \oplus I_i^\star$, where $I_i$ and $I_i^\star$ are non-isomorphic natural $SL_4(q)$-modules for $\overline{R_i}$. It follows that $I_i^\star$ is dual to $I_i$ as an $\mathbb{F}_pR_i$-module, and so $W_i$ is a selfdual $R_i$-module. By (7°) and (9°), $\overline{T_i}$ acts faithfully and $\mathbb{K}_i$-linearly on $I_i$. As $\overline{T_i}/\overline{R_i}$ is a $p$-group and $GL_4(q)/SL_4(q)$ is $p'$-group we conclude that $\overline{T_i} = \overline{R_i}$ and so again $W_i$ is a selfdual $T_i$-module, a contradiction.

$11^\circ$. $\quad Y_j \leqslant J_i$.

Just as in the previous paragraph, $I_i$ acts $\mathbb{K}_i$-linearly on $I_i$, $GL_n(q)/SL_n(q)$ is a $p'$-group and $\overline{T_i}/\overline{R_i}$ is a $p$-group. As there we conclude that $\overline{T_i} = \overline{R_i} = \overline{J_i}$, and ($11^\circ$) is established.

Since $Y_i$ is a direct sum of $m$ isomorphic simple $J_i$-modules, [**MS3**, 5.2(d)] implies that there exists an $S_i$-invariant simple $J_i$-submodule in $Y_i$. From now on $I_1$ and $I_2$ denote such $S_i$-invariant submodules with $I_1^u = I_2$.

Let $C_i$ be the inverse image of $C_{\overline{M_i^\circ}}(\overline{J_i})$ in $M_i^\circ$. By ($5^\circ$), $[I_1, I_2] \neq 1$. Pick $1 \neq x \in [I_1, I_2]$ and put $X_i := \mathbb{K}_i x$

We use the following simple facts about the action of $\overline{J_i}$ on the natural $SL_n(q)$-module $I_i$ and the structure of $\overline{J_i} \cong SL_n(q)$ and $J_iC_i/C_i \cong PSL_n(q)$.

(i) $\overline{J_i}$ is transitive on $I_i$.
(ii) $O^{p'}(N_{\overline{J_i}}(X_i)) = C_{\overline{J_i}}(x)$.
(iii) $O_p(C_{\overline{J_i}}(x))$ induces $Hom_{\mathbb{K}_i}(I_i/X_i, X_i)$ on $I_i$. In particular, $X_i = C_{I_i}(O_p(C_{\overline{J_i}}(x)))$ and $X_i = [I_i, O_p(C_{\overline{J_i}}(x))]$.
(iv) $O_p\big(C_{J_i}(x)C_i/C_i\big)$ is a natural $SL_{n-1}(q)$-module for $C_{J_i}(x)$. In particular, since $n-1 \geqslant 2$, $O_p\big(C_{J_i}(x)C_i/C_i\big)$ is a non-central simple $C_{J_i}(x)$-module.
(v) Let $W$ be an $\mathbb{K}_i$-subspace of $I_i$. Then $O^{p'}\big(N_{\overline{J_i}}(W)/C_{\overline{J_i}}(W)\big) \cong SL_{\mathbb{K}_i}(W)$ and $N_{\overline{J_i}}(W)$ acts transitively on $W$.

Note that $C_{I_i}(Q_i) \neq 1$. So by (i) there exists $y_i \in J_i$ with $x \in C_{I_i}(Q_i)^{y_i} \leqslant Z(Q_i^{y_i})$. By 1.52(e) $Z(Q)$ is a TI-set, so $Q_1^{y_1} = Q_2^{y_2} =: Q_0$. By ($10^\circ$), $\overline{M_i^\circ} = \overline{J_iC_i}$ and thus $Q_0 \leqslant J_iC_i$ and $Q_0 \nleqslant C_i$. By $Q!$, $C_{J_i}(x)$ normalizes $Q_0$. Observe that $C_{J_i}(x)C_i$ contains a Sylow $p$-subgroup of $J_iC_i$ and so $Q_0 \leqslant C_{J_i}(x)C_i$ and $Q_0C_i/C_i$ is a non-trivial normal $p$-subgroup of $C_{J_i}(x)C_i/C_i$. Now (iv) implies that

$$O_p\big(C_{J_i}(x)C_i/C_i\big) = Q_0C_i/C_i = [Q_0, C_{J_i}(x)]C_i/C_i \leqslant (Q_0 \cap J_i)C_i/C_i \leqslant O_p\big(C_{J_i}(x)C_i/C_i\big).$$

Thus

$12^\circ$. $\quad O_p\big(C_{J_i}(x)C_i/C_i\big) = Q_0C_i/C_i = (Q_0 \cap J_i)C_i/C_i$.

In particular, $Q_0C_i = (Q_0 \cap J_i)C_i$ and so $Q_0 = (Q_0 \cap J_i)(Q_0 \cap C_i)$. From $Q_0 = Q_i^{y_i} \leqslant J_iQ_i$ we get that $Q_0$ normalizes $I_i$. Since $J_i$ centralizes the $p$-group $\overline{Q_0 \cap C_i}$ and acts simply on $I_i$ we conclude that $Q_0 \cap C_i$ centralizes $I_i$. Thus

$13^\circ$. $\quad Q_0 = (Q_0 \cap J_i)(Q_0 \cap C_i) = (Q_0 \cap J_i)C_{Q_0}(I_i) \leqslant N_{M_i}(I_i)$.

As $\overline{J_i} \cap \overline{C_i}$ is a central $p'$-subgroup of $\overline{J_i}$, ($12^\circ$) implies

$14^\circ$. $\quad O_p(C_{\overline{J}_i}(x)) = \overline{Q_0 \cap J_i}$.

Hence using (iii), we get $X_i = C_{I_i}(O_p(C_{\overline{J}_i}(x))) = C_{I_i}(Q_0 \cap J_i)$ and $X_i = [I_i, O_p(C_{\overline{J}_i}(x))] = [I_i, Q_0 \cap J_i]$. As by ($13^\circ$) $Q_0 = (Q_0 \cap J_i)C_{Q_0}(I_i)$, this gives $X_i = C_{I_i}(Q_0)$ and $X_i = [I_i, Q_0]$.

Observe that $[I_1, I_2]$ is a $\mathbb{K}_i$-subspace of $I_i$. As $x \in [I_1, I_2]$ this gives $X_i \leqslant [I_1, I_2]$ and so $[I_i, Q_0] \leqslant [I_1, I_2]$. In particular, $Q_0$ normalizes $[I_1, I_2]$. Put $H := N_G([I_1, I_2])$. Then $H_i := N_{J_i}([I_1, I_2]))Q_0 \leqslant H$. Since $Q_0$ is weakly closed, 1.46(c) gives $H_i^\circ = \langle Q_0^{H_i^\circ} \rangle$, and since $[I_i, Q_0] \leqslant [I_1, I_2]$ and $H_i$ normalizes both, $I_i$ and $[I_1, I_2]$, we conclude that $[I_i, H_i^\circ] \leqslant [I_1, I_2]$.

By (v) $H_i$ acts transitively and so simply on $[I_1, I_2]$. Thus $[I_i, H_i^\circ] = [I_1, I_2]$. Moreover, the transitive action and 1.57(c) imply $H^\circ = H_i^\circ$. In particular, $[I_i, H^\circ] = [I_1, I_2]$ and $[I_i, H^\circ, I_j] = 1$. This holds for any $\{i, j\} = \{1, 2\}$. So $[I_1, H^\circ, I_2] = 1$ and $[I_2, H^\circ, I_1] = 1$. The Three Subgroups Lemma now gives $[I_1, I_2, H^\circ] = 1$. As $Q_0 \leqslant H^\circ$ and, as seen above, $C_{I_i}(Q_0) = X_i$, this gives $[I_1, I_2] = X_1 = X_2$. We have shown:

$15^\circ$. $\quad X_i = [I_1, I_2] = [I_i, Q_0] = C_{I_i}(Q_0)$. *In particular,* $[I_1, I_2]$ *is a* 1*-dimensional* $\mathbb{K}_i$*-subspace of* $I_i$ *and* $|[I_1, I_2]| = q$.

Put $Z_j := [I_i, Y_j]$ and $K_j := N_G(Z_j)$. We calculate the size of $Z_j$ by comparing the action of $Y_j$ on $I_i$ with the action of $I_i$ on $Y_j$. By ($11^\circ$) $Y_i \leqslant J_j$ and $Y_j \leqslant J_i$. Since $Y_i$ is a direct sum of $m$

copies of $I_j$, (15°) shows that $|[Y_j, I_i]| = |[I_1, I_2]|^m = q^m$. Since $Y_j$ acts $\mathbb{K}_i$-linearly on $I_i$, it follows that $Z_j$ is an $m$-dimensional $\mathbb{K}_i$-subspace of $I_i$. We have proved:

16°. *$Z_j$ is an $m$-dimensional $\mathbb{K}_i$-subspace of $I_i$.*

Assume that $Y_i \leqslant Q_i^\bullet$. Since $Q$ is weakly closed, $Q_i^\bullet$ and $Q_0^\bullet$ are conjugate in $M_i$, and as $Y_i \trianglelefteq M_i$, we get $Y_i \leqslant Q_0^\bullet$. Thus $Y_i \leqslant J_j \cap Q_0^\bullet \leqslant O_p\big(C_{J_j}(x)\big)$. Hence (iii) shows that $[I_j, Y_i] \leqslant X_j$. Thus $Z_i \leqslant X_j$, a contradiction since $|Z_i| = q^m > q = |X_j|$. We have proved:

17°. $Y_i \nleqslant Q_i^\bullet$.

By (13°) $Q_0$ normalizes $I_i$. As $Q_0 \leqslant M_j$, $Q_0$ normalizes $Y_j$ and so also normalizes $Z_i = [I_i, Y_j]$. Thus $Q_0 \leqslant K_j$.

By (11°) $I_i \leqslant Y_i \leqslant J_j \leqslant C_{M_j}(\overline{C_j})$. So $C_j$ normalizes $Z_j = [I_i, Y_j]$ and $C_j \leqslant K_j$. Since $I_j$ is a simple $J_j$-module, $\langle [I_i, I_j]^{J_j} \rangle = I_j$. As $Y_j$ is a direct sum of simple $J_j$-modules isomorphic to $I_j$ we conclude that $\langle [I_i, Y_j]^{J_j} \rangle = Y_j$. Thus $\langle Z_j^{J_j} \rangle = Y_j$, and since $J_j$ centralizes $\overline{C_j}$, we conclude that $C_{\overline{C}_j}(Z_j)$ centralizes $Y_j$. We record:

18°. *$Q_0 C_j \leqslant K_j$ and $\overline{C_j}$ acts faithfully on $Z_j$.*

By (13°) $Q_0 = (Q_0 \cap J_i) C_{Q_0}(I_i)$. Since $Q_0 \leqslant K_j$ and $Z_j \leqslant I_i$, this gives

$$Q_0 = (Q_0 \cap J_i) C_{Q_0}(Z_j) \qquad \text{and} \qquad Q_0 = (Q_0 \cap (J_i \cap K_j)) C_{Q_0}(Z_j) \leqslant (J_i \cap K_j) C_{M_i}(Z_j).$$

By (16°) $Z_j$ is an $\mathbb{K}_i$-subspace of $I_i$ and so (v) shows $J_i \cap K_j$ acts transitively on $Z_j$. Hence three applications of 1.57(c) give

19°. $\big\langle Q_0^{J_i \cap K_j} \big\rangle = K_j^\circ = \big((J_i \cap K_j) Q_0\big)^\circ = \big((J_i \cap K_j) C_{M_i}(Z_j)\big)^\circ$.

Put $\widetilde{K_j} := K_j / C_{K_j}(Z_j)$. By (v), $Z_j$ is a natural $SL_m(q)$-module for $O^{p'}(J_i \cap K_j)$. By (19°), $K_j^\circ = \big((J_i \cap K_j) C_{M_i}(Z_j)\big)^\circ$ and so $\widetilde{K_j^\circ} \leqslant O^{p'}(\widetilde{J_i \cap K_j}) \cong SL_m(q)$. As $SL_m(q)$ has no non-trivial proper normal subgroup generated by $p$-elements, we conclude that $\widetilde{K_j^\circ} = O^{p'}(\widetilde{J_i \cap K_j})$. Thus

20°. *$Z_j$ is a natural $SL_m(q)$-module for $K_j^\circ$, and $K_j^\circ$ acts $\mathbb{K}_i$-linearly on $Z_j$.*

By (15°) $Q_0$ centralizes $[I_1, I_2] = [I_i, I_j]$. Since $Z_j = [I_i, Y_j]$ and $Y_j$ is, as an $J_j$-module, the direct sum of copies of $I_j$, we conclude that $Q_0 \cap J_j$ centralizes $Z_j$. By (13°), $Q_0 = (Q_0 \cap J_j)(Q_0 \cap C_j)$ and thus $Q_0 = (Q_0 \cap C_j) C_{Q_0}(Z_j)$.

By (14°) $\overline{Q_0 \cap J_i} = O_p(C_{\overline{J_i}}(x))$. Hence, by (iii), $Q_0 \cap J_i$ induces $Hom_{\mathbb{K}_i}(I_i/X_i, X_i)$ on $I_i$. As by (15°) $X_i = [I_1, I_2] \leqslant Z_j$ and by (16°) $Z_j$ is a $\mathbb{K}_i$-subspace of $I_i$, we conclude that $Q_0 \cap J_i$ induces $Hom_{\mathbb{K}_i}(Z_j/X_i, X_i)$ on $Z_j$. Since

$$(Q_0 \cap J_i) C_{Q_0}(Z_j) = Q_0 = (Q_0 \cap C_j) C_{Q_0}(Z_j)$$

we infer:

21°. *$Q_0 \cap C_j$ induces $Hom_{\mathbb{K}_i}(Z_j/X_i, X_i)$ on $Z_j$.*

In this paragraph, $\overline{X} := X C_{M_j}(Y_j)/C_{M_j}(Y_j)$ for all $X \leqslant M_j$. Define

$$J_j^\star := \langle (Q_0 \cap J_j)^{J_j} \rangle \qquad \text{and} \qquad C_j^\star := \langle (Q_0 \cap C_j)^{C_j} \rangle.$$

Recall from (10°) that $R_i = J_i$ and $\overline{J_j} \cong SL_n(q)$, $n \geqslant 3$, and that by (14°) $\overline{Q_0 \cap J_j} = O_p(C_{\overline{J_j}}(x))$. Thus, we have $\overline{J_j^\star} = \overline{J_j}$, and by (8°)(1:c), $\overline{M_j^\circ} = \overline{J_j C_j}$. Also $[\overline{J_j}, \overline{C_j}] = 1$, and by (13°) $\overline{Q_0} = (\overline{Q_0 \cap J_j})(\overline{Q_0 \cap C_j})$. It follows that

$$\overline{M_j^\circ} = \langle \overline{Q_0}^{\overline{M_j^\circ}} \rangle = \overline{J_j^\star}\, \overline{C_j^\star} = \overline{J_j}\, \overline{C_j^\star} \qquad \text{and} \qquad [\overline{J_j}, \overline{C_j^\star}] = 1.$$

In addition, $O_p(\overline{C_j^\star}) \leqslant O_p(\overline{C_j}) \leqslant O_p(\overline{M_j}) = 1$, and by (18°) $\overline{C_j}$ is faithful on $Z_j$.

Recall that $C_j \leqslant K_j$ and $\widetilde{K_j} = K_j / C_{K_j}(Z_j)$. Hence $C_j^\star \leqslant K_j^\circ$, $\overline{C_j^\star} \cong \widetilde{C_j^\star}$ and $O_p(\widetilde{C_j^\star}) = 1$. By (20°) $Z_j$ is a natural $SL_m(q)$-module for $\widetilde{K_j}$ and by (21°) $Q_0 \cap C_j$ induces $Hom_{\mathbb{K}_i}(Z_j/X_i, X_i)$ on $Z_j$. Now [**MS3**, 7.2] implies that

22°. $\widetilde{C_j^\star} = \widetilde{K_j^\circ} \cong SL_m(q)$. *In particular, $Z_j$ a natural $SL_m(q)$-module for $\widetilde{C_j^\star}$.*

Since $Y_j$ is, as a $J_j$-module, the direct sum of natural $SL_n(q)$-modules isomorphic to $I_j$ and since $[\overline{J_j}, \overline{C_j^\star}] = 1$, $Y_j$ is, as a module for $\overline{M_j^\circ} = \overline{J_j}\,\overline{C_j^\star}$, isomorphic to $I_j \otimes_{\mathbb{K}_j} U_j$ for some $\mathbb{K}_j C_j^\star$-module $U_j$ (see for example [**MS3**, Lemma 5.2]).

Since $I_i \leqslant J_j$ by (11°) and $[I_1, I_2]$ is 1-dimensional in $I_j$ by (15°),

$$U_j \cong [I_1, I_2] \otimes U_j = [I_j \otimes U_j, I_i] \cong [Y_j, I_i] = Z_j$$

as a $C_j^\star$-module. Thus $U_j$ is a natural $SL_m(q)$-module for $C_j^\star$. Hence in order to establish Theorem D(4:4) it remains to prove that $p$ is odd.

By (22°) we have $\widetilde{C_j^\star} = \widetilde{K_j^\circ}$. Since $C_j^\star Q_0 \leqslant K_j^\circ$ we get $\widetilde{C_j^\star Q_0} = \widetilde{K_j^\circ}$. Hence 1.52(c) gives

23°. $K_j^\circ = (C_j^\star Q_0)^\circ$.

By (16°) $Z_2 = [I_1, Y_2]$ is an $m$-dimensional $\mathbb{K}_1$-subspace of $I_1$, so $Z_2^u$ is an $m$-dimensional $\mathbb{K}_2$-subspace of $I_1^u = I_2$ with $[I_1, I_2]^u \leqslant Z_2^u$. Also $[I_1, I_2]^u$ and $[I_1, I_2]$ are 1-dimensional $\mathbb{K}_2$-subspaces of $I_2$ by (15°), and again by (16°) $Z_1 = [I_2, Y_1]$ is an $m$-dimensional $\mathbb{K}_2$-subspace of $I_2$ with $[I_1, I_2] \leqslant Z_2$. As $I_2$ is a natural $SL_m(q)$-module for $J_2$, $J_2$ is transitive on the pairs of incident 1- and $m$-dimensional $\mathbb{K}_2$-subspaces of $I_2$. Hence, there exists $v \in J_2$ with $Z_2^{uv} = Z_1$ and $[I_1, I_2]^{uv} = [I_1, I_2]$. Put $g := uv$. Then

$$[I_1, I_2]^g = [I_1, I_2], \quad I_1^g = I_2^v = I_2, \quad Y_1^g = Y_2^v = Y_2, \quad Z_2^g = Z_1$$

and

$$Z_1^g = [I_2^g, Y_1^g] = [I_2^g, Y_2], \quad [I_2^g, I_2] = [I_2^g, I_1^g] = [I_1, I_2]^g = [I_1, I_2].$$

Since $I_2 \leqslant Y_2 \leqslant J_1$, $I_2^g \leqslant J_1^g = J_2$. Also $I_1 \leqslant J_2$, and since $Y_2$ is the direct sum of copies of the $J_2$-module $I_2$, we conclude from $[I_2^g, I_2] = [I_1, I_2]$ that $[I_2^g, Y_2] = [I_1, Y_2] = Z_2$. Thus $Z_1^g = Z_2$ and so $g$ acts non-trivially on the sets $\{Z_1, Z_2\}$. Thus $g$ also acts non-trivially on $\{K_1, K_2\}$ and $\{K_1^\circ, K_2^\circ\}$. By (23°) $K_1^\circ = (C_1^\star Q_0)^\circ$ and by (19°) $K_2^\circ = \big((J_1 \cap K_2)Q_0\big)^\circ$. Thus

$$\Big\{K_1^\circ, K_2^\circ\Big\} = \Big\{\big(C_1^\star Q_0\big)^\circ, \big((J_1 \cap K_2)Q_0\big)^\circ\Big\}.$$

Recall that $S_1$ normalizes $I_1$ and $1 \neq x \in [I_1, I_2]$. Note that the number of pairs $(x_0, Z_0)$, where $Z_0$ is an $m$-dimensional $\mathbb{K}_1$-subspaces of $I_1$ and $1 \neq x_0 \in Z_0$, is not divisible by $p$ and that $J_1$ acts transitively on such pairs. Hence every such pair is normalized by a Sylow $p$-subgroup of $M_1$. Since $Z_2$ is an $m$-dimensional subspace of $I_1$ and $x \in Z_2$, we conclude that $N_{M_1}(Z_2) \cap C_{M_1}(x)$ contains a Sylow $p$-subgroup $S_0$ of $M_1$. Then $S_0 \leqslant M_1 \cap K_2$ and by $Q!$, $S_0$ normalizes $Q_0$. It follows that $S_0$ acts trivially on $\Big\{\big(C_1^\star Q_0\big)^\circ, \big((J_1 \cap K_2)Q_0\big)^\circ\Big\}$, that is on $\{K_1^\circ, K_2^\circ\}$. Since $S_0$ is a Sylow $p$-subgroup of $G$ and $g$ acts non-trivially on $\{K_1^\circ, K_2^\circ\}$, this shows $p \neq 2$. □

LEMMA 4.14. *Suppose that C.24(2) holds and $W_1$ is a simple $R_1$-module. Then Theorem D holds.*

PROOF. Since C.24(2) holds and $W_1$ is a simple $R_1$-module we are in case (2:c:2) of C.24. Thus

1°.
- (a) *$\overline{R_i}$ is quasisimple, $\overline{R_i} \leqslant \overline{M_i^\circ}$, and either $\overline{J_i} = \overline{R_i}$ or $p = 2$ and $\overline{J_i} \cong O_{2n}^\pm(q)$, $Sp_4(2)$ or $G_2(2)$.*
- (b) *$C_{Y_i}(R_i) = 1$ and $\overline{M_i}$ acts faithfully on $W_i$. In particular, $C_{M_i}(W_i) = C_{M_i}(Y_i)$.*
- (c) *Either $\overline{M_i^\circ} = \overline{R_i} = \overline{M_i^\circ} \cap \overline{J_i}$ or $\overline{M_i^\circ} \cong Sp_4(2)$, $3{\cdot}Sym(6)$, $SU_4(q).2$ ($\cong O_6^-(q)$ and $W_i$ is the natural $SU_4(q)$-module), or $G_2(2)$.*
- (d) *One of the cases C.3 (1) - (9), (12) applies to $(\overline{J_i}, W_i)$, with $n \geqslant 3$ in case (1), $n \geqslant 2$ in case (2), and $n = 6$ in case (12).*

Recall that $M_{i\circ} = O^p(M_i^\circ)$. Next we show:

2°. $\overline{R_i} = F^*(\overline{M_i^\circ}) = \overline{M_{i\circ}}$. *In particular, $W_i = [Y_i, M_{i\circ}]$.*

Suppose first that $\overline{R_i} = \overline{M_i^\circ}$. As by (1°)(a) $\overline{R_i}$ is quasisimple, we conclude that (2°) holds.

Suppose next that $\overline{R_i} \neq \overline{M_i^\circ}$. Then by (1°)(c), $\overline{M_i^\circ} \cong Sp_4(2)$, $3{\cdot}Sym(6)$, $SU_4(q).2$ or $G_2(2)$. In each case $F^*(\overline{M_i^\circ})$ is quasisimple and has index 2 in $\overline{M_i^\circ}$. Thus $F^*(\overline{M_i^\circ}) = O^2(\overline{M_i^\circ}) = \overline{M_{i^\circ}}$. As $\overline{R_i} \leqslant \overline{M_i^\circ}$ and $\overline{R_i}$ is a quasisimple normal subgroup of $\overline{M_i}$ we conclude that $\overline{R_i} = F^*(\overline{M_i^\circ})$ and again (2°) holds. Hence (2°) is proved.

Note that

$$|W_1/C_{W_1}(W_2)| \leqslant |W_2/C_{W_2}(W_1)| \qquad \text{or} \qquad |W_2/C_{W_2}(W_1)| \leqslant |W_1/C_{W_1}(W_2)|.$$

In the second case, conjugation by $u^{-1}$ shows that $|W_1/C_{W_1}(W_1^{u^{-1}})| \leqslant ||W_1^{u^{-1}}/C_{W_1^{u^{-1}}}(W_1)|$. Also $u^{-1}$ in place of $u$ fulfills the conclusion of 4.2. So possibly after replacing $u$ by $u^{-1}$, we may assume $W_2$ is an offender in $W_1$.

Put $Z := [W_1, W_2]$. Abusing our general convention, define

$$\overline{W_i} := W_iC_{M_j}(Y_j)/C_{M_j}(Y_j) \qquad (\text{and not } \overline{W_i} = W_iC_{M_i}(W_i)/C_{M_i}(W_i)\,).$$

By (1°)(b) $C_{M_i}(Y_j) = C_{M_j}(W_j)$ and so $\overline{W_i} \cong C_{W_i}/C_{W_i}(W_j)$ as an $M_1 \cap M_2$-module.

3°. *$W_2$ is a non-trivial quadratic offender on $W_1$; in particular $Z \neq 1$.*

Recall from 4.4 that $Y_2$ acts quadratically on $Y_1$ and from 4.3 that $[Y_1, Y_2] \neq 1$. In particular $W_2$ is a quadratic offender on $W_1$. It remains to prove that $W_1$ acts non-trivially on $W_2$.

By (1°)(b) $C_{M_i}(Y_i) = C_{M_i}(W_i)$. For $i = 1$ this shows that $[Y_1, Y_2] \neq 1$ implies $[W_1, Y_2] \neq 1$, and then for $i = 2$ that $[W_1, Y_2] \neq 1$ implies $[W_1, W_2] \neq 1$.

Let $v \in G^\sharp$ and suppose that $v$ is centralized by a conjugate $Q^g$ of $Q$ in $G$. Since $C_G(Q) \leqslant Q$ we get $v \in Z(Q^g)$. By 1.52(e) $Z(Q)$ is a TI-set. Thus $Q^g$ is unique determined by $v$ and we define $Q_v := Q^g$.

Let $\mathcal{V}$ be the set of all $1 \neq v \in Z$ such that for each $i \in \{1, 2\}$ there exists $Q_{v,i} \in Q^G$ with $Q_{v,i} \leqslant M_i$ and $[v, Q_{v,i}] = 1$. Note that $Q_v = Q_{v,1} = Q_{v,2} \leqslant M_1 \cap M_2$.

Let $L := \langle Q_v \mid v \in \mathcal{V}\rangle$. Then $L \leqslant M_1 \cap M_2$.

4°.

(a) *$M_1 \cap M_2$ normalizes $W_1, W_2$ and $Z$; in particular $L \leqslant N_{M_i}(Z)$.*
(b) *$L = (M_1 \cap M_2)^\circ \leqslant N_{M_1}(Z)^\circ \cap N_{M_2}(Z)^\circ$.*
(c) *Suppose $\mathcal{V} = Z^\sharp$. Then $L = N_{M_1}(Z)^\circ = N_{M_2}(Z)^\circ$.*

(a): $M_1 \cap M_2$ normalizes $W_1$ and $W_2$, so also $Z = [W_1, W_2]$. Since $L \leqslant M_1 \cap M_2$, (a) follows.

(b): As $L \leqslant M_1 \cap M_2$ and $L$ is generated by conjugates of $Q$, $L \leqslant (M_1 \cap M_2)^\circ$. Let $g \in G$ with $Q^g \leqslant M_1 \cap M_2$. Then $Q^g$ normalizes $Z$, and since $Z \neq 1$ by (3°), there exists $1 \neq v \in C_Z(Q^g)$. Thus $Q_v = Q^g \leqslant M_1 \cap M_2$. Hence $v \in \mathcal{V}$ and $Q^g = Q_v \leqslant L$. Thus $(M_1 \cap M_2)^\circ \leqslant L$ and so $(M_1 \cap M_2)^\circ = L$. As $M_1 \cap M_2 \leqslant N_{M_i}(Z)$, we have $L = (M_1 \cap M_2)^\circ \leqslant M_{M_i}(Z)^\circ$.

(c): Suppose that $\mathcal{V} = Z^\sharp$ and let $g \in G$ with $Q^g \leqslant N_{M_i}(Z)$. Again there exists $1 \neq z \in C_Z(Q^g)$ and so $z \in \mathcal{V}$ and $Q^g = Q_v \leqslant L$. Hence $N_{M_i}(Z)^\circ \leqslant L$ and so $N_{M_i}(Z)^\circ = L$.

5°. *Suppose $M_i$ acts transitively on $W_i$. Then $\mathcal{V} = Z^\sharp$.*

Since $M_i$ acts transitively on $W_i$ and $C_{W_i}(Q_i) \neq 1$ each elements of $W_i$ (and so also of $Z$) is centralized by a conjugate of $Q_i$ in $M_i$. Thus $Z^\sharp = \mathcal{V}$.

6°. *Suppose $1 \neq z \in C_Z(L)$ and $K_i \leqslant M_i$ acts transitively on $W_i$. Then $L = Q_z$ and $Z^\sharp \subseteq z^{N_{K_i}(L)}$.*

Let $v \in Z^\sharp$. By (5°), $\mathcal{V} = Z^\sharp$ and so $Q_v \leqslant L$ and $[z, Q_v] = 1$. Thus $Q_v = Q_z$, and we conclude that $L = Q_z$. Since $K_i$ acts transitively on $W_i$, there exists $k \in K_i$ with $z^k = v$. Then $Q_z^k = Q_v = Q_z$ and $k \in N_{K_i}(L)$. Hence, (6°) holds.

Let $\{i, j\} := \{1, 2\}$ and put $\mathbb{K}_i := End_{R_i}(W_i)$.

7°. *$T_j$ acts $\mathbb{K}_j$-linearly on $W_j$. In particular, $Z$ is a $\mathbb{K}_j$-subspace of $W_j$.*

Suppose for a contradiction, that $W_i$ does not act $\mathbb{K}_j$-linearly on $W_j$. Then $p < |\mathbb{K}_j| = |\mathbb{K}_i|$, and 1.22 shows that $\dim_{\mathbb{K}_i} W_i = 1$, a contradiction. Hence $W_i$ acts $\mathbb{K}_j$-linearly on $W_j$. Recall that $Y_i$ acts quadratically on $V_j$, so $Y_i$ centralizes the non-trivial $\mathbb{K}_i$-subspace $[W_j, W_i]$ of $W_j$. Thus $Y_i$ and so also $T_j = Y_i R_j$ acts $\mathbb{K}_j$-linearly on $W_j$.

We now discuss the cases of C.3 listed in $(1^\circ)$(d). Observe that a natural $Sym(6)$- or $Alt(6)$-module (Case 12) of C.3 for $n = 6$), is also a natural $Sp_4(2)$- or $Sp_4(2)'$-module, respectively. We will treat this case together with the symplectic groups in (Case 2).

Case 1. *Case (1) of C.3 holds with $n \geqslant 3$, that is, $\overline{J_1} \cong SL_n(q)$ and $W_1$ is a corresponding natural module.*

By $(1^\circ)$(a),(c) $\overline{J_1} = \overline{R_1} = \overline{M_1^\circ}$. Also C.22 shows that either $Y_1 = [Y_1, R_1] = W_1$ or $\overline{J_1} \cong SL_3(2)$ and $|Y_1| = 2^4$. In the first case Theorem D (1) holds. So we need to rule out the second case.

Assume that $\overline{J_1} \cong SL_3(2)$ and $|Y_1| = 2^4$, so $\overline{J_i} = \overline{M_i}$. Put $Z_0 := [Y_1, Y_2]$ and note that $Z_0 \leqslant W_1 \cap W_2 \leqslant C_{Y_i}(Y_j)$. Since $J_i$ acts transitively on $W_i$ each $v \in Z_0^\sharp$ is centralized by some $Q_{vi} \in Q_i^{M_i}$. Thus $Q_v = Q_{v1} = Q_{v2} \leqslant M_1 \cap M_2$ and $L_0 := \langle Q_v \mid v \in Z_0^\sharp \rangle \leqslant M_1 \cap M_2$. Choose $\{i, j\}$ such that $Y_j$ is an offender on $Y_i$. Then $Y_j$ contains a non-trivial best offender $A$ on $Y_i$. From C.22 we conclude that $C_{Y_i}(A) = [Y_i, A]$ has order 4. Since $Y_j$ acts quadratically on $Y_i$, this implies that $C_{Y_i}(Y_j) = C_{Y_i}(A) = [Y_i, A] = [Y_i, Y_j] = Z_0$. Thus $Z_0$ has order 4. Note that $L_0 = \langle Q_i^g \mid g \in J_i, C_{W_i}(Q_i^g) \neq 1 \rangle$, and so B.38(c) shows that $Z_0$ is natural $SL_2(2)$-module for $L_0$. As $Z_0 \leqslant Y_j$ this implies that $O^2(L_0) \nleqslant C_{M_j}(Y_j)$. Since $|Y_i/W_i| = 2 = |W_i/Z_0|$, $O^2(L_0)$ centralizes $Y_i/Z_0$ and so $[Y_i, O^2(L_0)] \leqslant Z_0$. As $Z_0 \leqslant Y_j \leqslant C_{M_j}(Y_j)$, we conclude that $[Y_i, O^2(L_0)] \leqslant C_{M_j}(Y_j)$. On the other hand, $L_0 \leqslant M_1 \cap M_2$ and $M_j/C_{M_j}(Y_j) = \overline{J_j} \cong SL_3(2)$. Hence, the centralizer of an involution in $M_j/C_{M_j}(Y_j)$ is a 2-group, so $O^2(L_0) \leqslant C_{M_i}(Y_i)$, a contradiction.

Case 2. *Case (2) of C.3 holds with $n \geqslant 2$ or Case(12) holds with $n = 6$, that is, $\overline{J_1} \cong Sp_{2n}(q)$, $n \geqslant 2$, or $Sp_4(q)'$ (and $q = 2$), and $W_1$ is a corresponding natural module.*

Suppose that $p$ is odd. Then by $(1^\circ)$(a),(c) $\overline{J_1} = \overline{R_1} = \overline{M_1^\circ}$ and so by 2.26 $W_1 \leqslant Q^\bullet$. Since $p$ is odd, $|Z(\overline{J_1})| = 2$, and coprime action gives

$$Y_1 = [Y_1, Z(\overline{J_1})] \times C_{Y_1}\big(Z(\overline{J_1})\big) = W_1 \times C_{Y_1}(J_1).$$

Moreover, by $(1^\circ)$(b), $C_{Y_1}(J_1) = 1$ and so $Y_1 = W_1$. Thus Theorem D(2) holds.

Suppose that $p = 2$. Then $(1^\circ)$(a),(c) show that also $\overline{M_1^\circ} \cong Sp_{2n}(q)$, $n \geqslant 2$, or $Sp_4(q)'$ (note that $\overline{J_1}$ and $\overline{M_1^\circ}$ do not need to be equal if one of them is isomorphic to $Sp_4(q)'$). Since $C_{Y_1}(J_1) = 1$, C.22 shows $W_1 = [Y_1, R_1] = [Y_1, J_1]$ and $|Y_1/W_1| \leqslant q$. Since either $\overline{M_1^\circ} = \overline{R_1}$ or $q = 2$ and $|\overline{M_1^\circ}/\overline{R_1}| = 2$, this gives $W_1 = [Y_1, M_1^\circ]$. If $W_1 \leqslant Q^\bullet$, then 2.25(b) shows that $Y_1 \leqslant Q^\bullet$. So again Theorem D(2) holds.

Case 3. *Case (3) of C.3 holds, that is, $\overline{J_1} \cong SU_n(q)$, $n \geqslant 4$, and $W_1$ is a corresponding natural module.*

Note that $\mathbb{K}_j \cong \mathbb{F}_{q^2}$. By $(7^\circ)$ $W_i$ is $p$-group acting $\mathbb{K}_j$-linearly on $W_j$. As $W_i$ normalizes $\overline{J_j}$, we conclude from B.35(d) that $W_i \leqslant J_i$. Since $W_i$ acts quadratically on $W_j$, $Z$ is an isotropic and so also a singular subspace of $W_j$, see B.6(b) and B.5. It follows that each element of $Z$ is $p$-central in $M_i$ and so centralized by a conjugate of $Q_i$. Thus $Z^\sharp = \mathcal{V}$. Put $m := \dim_{\mathbb{F}_{q^2}} Z$ and $E := C_{\overline{J_1}}(C_{W_1}(W_2))$. Note that $1 \neq W_2/C_{W_2}(W_1) \cong \overline{W_2} \leqslant E$. Let $H_1$ be an $\mathbb{K}_1$-hyperplane of $W_1$ with $C_{W_1}(W_2) \leqslant H_1$. Then $|C_{\overline{J}_1}(H_1)| = q$ and so

$$|C_{W_2}(H_1)/C_{W_2}(W_1)| = |\overline{W_2} \cap C_{\overline{J}_1}(H_1)| \leqslant q.$$

As $W_1$ acts $\mathbb{K}_2$-linearly on $W_2$ this gives $C_{W_2}(H_1) = C_{W_2}(W_1)$. In particular, $H_1 \neq C_{W_2}(W_1)$ and so $m \geqslant 2$. Moreover, $\overline{W_2} \cap C_{\overline{J}_1}(H_1) = 1$, and as $1 \neq C_{\overline{J}_1}(H_1) \leqslant E$, we get

$$1 < \overline{W_2} < E.$$

Since $L$ normalizes this series, $E$ is not a simple $L$-module. As $Z^\sharp = \mathcal{V}$, $(4^\circ)$(c) shows that $L = N_{M_1}(Z)^\circ$. Now B.38(c) implies that there exists a subgroup $F \leqslant L \cap J_1$ such that $Z$ is a natural

$SL_m(q^2)$-module for $F$. Since $GL_m(q^2)/SL_m(q^2)$ is a $p'$-group this implies that $O^{p'}(N_{J_1}(Z)) \leqslant FC_{J_1}(Z)$. Note that $C_{J_1}(Z)$ centralizes $W_1/Z^\perp$ and so (for example by the Three Subgroups Lemma) also $E$. By B.21(b) $E = C_{\overline{J_1}}(W_1/Z) \cap C_{\overline{J_1}}(Z)$. Hence, by B.22(a) $E$ is a simple $O^{p'}(N_{J_1}(Z))$-module and we infer that $E$ is a simple $F$- and a simple $L$-module, a contradiction.

Case 4. *Case (4) of C.3 holds, that is,* $\overline{J_1} \cong \Omega^+_{2n}(q)$ *for* $2n \geqslant 6$, $\Omega^-_{2n}(q)$ *for* $p = 2$ *and* $2n \geqslant 6$, $\Omega^-_{2n}(q)$ *for* $p$ *odd and* $2n \geqslant 8$, $\Omega_{2n+1}(q)$ *for* $p$ *odd and* $2n+1 \geqslant 7$, $O^-_4(2)$, *or* $O^\epsilon_{2n}(q)$ *for* $p = 2$ *and* $2n \geqslant 6$, *and* $W_1$ *is a corresponding natural module.*

Note that in all these cases $\overline{R_i} = F^*(\overline{J_i}) \cong \Omega^\epsilon_m(q)$ for appropriate $\epsilon$ and $m$. Moreover, by (1°)(c) $\overline{M_i^\circ} = \overline{R_i}$ and so $W_i = [Y_i, M_i^\circ]$.

Recall that $T_j = Y_i R_j$. Since $Y_i$ is $p$-subgroup acting $\mathbb{K}_j$-linearly on $W_j$ and since $Y_i$ normalizes $\overline{R_1}$, we conclude from B.35(d) that either $\overline{T_j} = \overline{R_j} \cong \Omega^\epsilon_m(q)$ or $p = 2$ and $\overline{T_j} \cong O^\epsilon_m(q)$. Moreover, $W_j$ is the corresponding natural module.

Assume first that $|Z| \leqslant q$. Then by B.9(c) $p = 2$ and $Z$ is not singular in $W_1$, and $|\overline{W_2}| = 2$. Since $W_2$ is an offender on $W_1$, we get $q = 2$ and $\overline{T_1} \cong O^\epsilon_{2n}(2)$. Hence 4.10 shows that Theorem D(5) holds.

Assume next that $Y_1 \neq W_1$. Then C.22 shows that $\overline{J_1} \cong O^+_6(2) \cong Sym(8)$. Hence by C.4(h) every offender in $\overline{T_i}$ on $Y_i$ is a best offender. Choose $i$ and $j$ such that $Y_i$ is an offender (and so a best offender) on $Y_j$. Then C.22 shows that image of $Y_i$ in $\overline{M_j}$ is generated by transpositions and thus $[Y_i, Y_j]$ contains a non-singular vector of $Y_j$. So using 4.10 a second time, this shows that Theorem D(5) holds.

Assume finally that $Y_1 = W_1$ and $|Z| > q$. Suppose that $\overline{J_1} \cong O^-_4(2) \cong Sym(5)$. Then $\overline{T_1} \cong Alt(5)$ or $Sym(5)$. Since $W_2$ is an offender on $W_1$, C.4(g) shows that $\overline{W_2}$ is generated by transpositions in $\overline{T_1}$. Thus $Z$ contains a non-singular element of $W_1$, and so using 4.10 a third time, this shows that Theorem D (5) holds. If $\overline{J_1} \not\cong O^-_4(2)$, then $2n \geqslant 6$, and Theorem D (4:1) holds, except that we still need to show that $Y \not\leqslant Q^\bullet$.

Suppose that $Y \leqslant Q^\bullet$, so $Y_i \leqslant Q_i^\bullet$, $i = 1, 2$. Since $W_1$ acts quadratically on $W_2$, an isotropic subspace of $W_2$, see B.6(b). By B.5 the singular vectors of $W_2$ contained in $Z$ form a $\mathbb{K}_2$-subspace of $Z$ of codimension at most 1. Thus, as $|Z| > q$, there exists $1 \neq v \in Z$ such that $v$ is singular in $W_2$. Hence there exists $x \in M_2$ such that $[v, Q_2^x] = 1$. By $Q!$, $C_G(v) \leqslant N_G(Q_2^x)$, and since $Y_2 \leqslant (Q_2^\bullet)^x$, we get $Y_2 \leqslant O_p(C_G(v))$. In particular, $W_2 \leqslant O_p(C_{M_1}(v))$.

Suppose that $v$ is singular in $W_1$. Then $v$ is centralized by a Sylow $p$-subgroup of $M_1$, and since $W_2$ is a non-trivial offender on $W_1$, we obtain a contradiction to the Point-Stabilizer Theorem C.8. Thus $v$ is non-singular. It follows that $|O_p(C_{\overline{M_1}}(v))| = 1$ if $p$ is odd and $|O_p(C_{\overline{M_1}}(v))| \leqslant 2$ if $p = 2$. Hence $|W_2/C_{W_2}(W_1)| \leqslant 2$ and then $|Z| = |C_{W_2}(W_1)^\perp| = 2 \leqslant q$, a contradiction.

Case 5. *Case (5) of C.3 holds, that is,* $p = 2$, $\overline{J_1} \cong G_2(q)$, *and* $W_1$ *is a corresponding natural module.*

Put $L_i := N_{J_i}(W_j C_{M_i}(W_i))$, so $\overline{L_i} = N_{\overline{J_i}}(\overline{W_i})$. Since $W_2$ is a non-trivial offender on $W_1$, we conclude from the Best Offender Theorem C.4(a) that $Z = C_{W_1}(W_2)$, $|Z| = |W_1/Z| = |\overline{W_2}| = q^3$, and $L_1$ is a maximal parabolic subgroup of $J_1$. Note also that $L_1$ normalizes $Z$ and by the action of $J_1$ on the natural $G_2(q)$-module $W_1$, $L_1 = N_{J_1}(Z_1)$ for some 1-dimensional $\mathbb{K}_1$-subspace $Z_1$ of $W_1$. Observe that $[Z_1, O^{p'}(L_1)] = 1$.

By (1°)(c) $\overline{M_1^\circ} = \overline{R_1}$ or $\overline{M_1^\circ} = \overline{J_1}$. Thus $\overline{M_1^\circ} \leqslant \overline{J_1}$. In particular, $M_1^\circ \leqslant J_1$, $M_1^\circ$ acts $\mathbb{K}_1$-linearly on $W_1$ and $L \leqslant N_{M_1^\circ}(Z) \leqslant L_1$. Since $O^{p'}(L_1)$ centralizes $Z_1$ and $L$ is generated by $p$-elements, we get that $L \leqslant C_{J_1}(Z_1)$. Note that $J_1$ acts transitively on $W_1$. Thus by (5°) $\mathcal{V} = Z^\sharp$ and by (6°), $Z^\sharp \subseteq z^{N_{J_1}(L)}$, where $1 \neq z \in Z_1$. As $\mathcal{V} = Z^\sharp$, $L_1$ normalizes $L$ and so, since $L_1$ is maximal subgroup of $J_1$, we get $N_{J_1}(L) = L_1$. But then $Z^\sharp \subseteq z^{L_1} \subseteq Z_1$, a contradiction.

Case 6. *Case (6) of C.3 holds, that is,* $\overline{J_1} \cong SL_n(q)/\langle(-id)^{n-1}\rangle$, $n \geqslant 5$, *and* $W_1$ *is the corresponding exterior square of a natural module.*

Then $Y_1 = W_1$ by C.22. Since $W_i$ is the exterior square of a natural $SL_n(q)$-module, there exists a central $p'$-extension $\widehat{L_i}$ of $\overline{L_i}$ and a natural $SL_n(q)$-module $N_i$ for $\widehat{L_i}$ such that $Y_i \cong \Lambda^2 N_i$

as $\widehat{L_i}$-module. By C.4 $W_2$ is not an over-offender on $W_1$ and so $W_1$ is an offender on $W_2$. This also shows that $W_i$ is a best offender on $W_j$ and so $W_i \leqslant J_j$. Let $\widehat{W_i}$ be the unique Sylow $p$-subgroup of the inverse image of $W_j C_{J_i}(Y_i)/C_{J_i}(Y_i)$ in $\widehat{J}_i$. By C.4 there exists a $\mathbb{K}_i$-hyperplane $H_i$ of $N_i$ with $\widehat{W_j} = C_{\widehat{J_i}}(H_i)$. Put $L_i := C_{\widehat{J_i}}(N_i/H_i)$. The action of $\widehat{J}_i$ on $N_i$ shows that $\widehat{W_j} = C_{L_i}(H_i) = O_p(L_i)$ is a natural $SL_{n-1}(q)$-module for $L_i$ isomorphic to $H_i$, $Z = C_{W_i}(W_j)$, $W_i/Z \cong H_i$ and $Z \cong \Lambda^2 H_i$ as $L_i$-modules. Let $X \leqslant W_1$ such that $Z \leqslant X$ and $|X/Z| = p$.

Consider the action of $L_2$ on $N_2$ and $W_2$. Note that $|X/C_X(W_2)| = p$ and so $X$ acts as a subgroup of the transvection group with axis $H_2$ and center say $P_2$ on $N_2$. It follows that $[W_2, X] \cong P_2 \wedge H_2$ and so $[W_2, X]$ is a natural $SL_{n-2}(q)$-module isomorphic to $H_2/P_2$ for $C_{L_2}(P_2)$. Thus each element of $[W_2, X]$ is centralized by a Sylow $p$-subgroup of $C_{L_2}(P_2)$ and so also by a Sylow $p$-subgroup of $J_2$, since $C_{L_2}(P_2)$ is a parabolic subgroup of $L_2$ and $\widehat{J_2}$.

Next consider the action of $L_1$ on $N_1$ and $W_1$. Identify $W_1$ with $\Lambda^2 N_1$. Then $X = \langle n \wedge x\rangle Z$, where $n \in N_1 \backslash H_1$ and $1 \neq x \in H_1$. If $T$ is the transvection group with axis $H_1$ and center say $P_1$, then $[X, T] = P_1 \wedge x$ and so $[X, W_2] = [X, C_{L_1}(H_1)] = P_1 \wedge H_1$. So as above each element of $[X, W_2]$ is centralized by a Sylow $p$-subgroups of $J_1$.

Let $1 \neq v \in [X, W_2]$. We have proved that $v$ is centralized by a Sylow $p$-subgroup $S_i^*$ of $J_i$. By $(1^\circ)$(c), $M_i^\circ \leqslant J_i$ so $S_i^*$ contains a $J_i$- conjugate of $Q_i$ and thus $v \in \mathcal{V}$. Since $v \in C_{W_2}(S_2^*)$, $C_{J_2}(v)$ contains the point-stabilizer of $J_2$ on $W_2$ with respect to $S_2$. Since the exterior square of a natural module does not appear in the conclusion of the Point-Stabilizer Theorem C.8 and since $W_1$ is an quadratic offender on $W_2$, we conclude that $W_1 \nleqslant O_p(C_{J_2}(v))$ and so also $Y_1 = W_1 \nleqslant Q_v^\bullet$. Hence Theorem D (4:2) holds.

Case 7. *Case (7) of C.3 holds, that is,* $\overline{J_1} \cong Spin_7(q)$ *and* $W_1$ *is the corresponding spinmodule.*

Observe that $\overline{J_i}$ is quasisimple and so $(1^\circ)$(c) gives $\overline{M_i^\circ} = \overline{R_i} = \overline{J_i}$. Hence $M_i^\circ \leqslant R_i = J_i$. Put $L_i := O^{p'}(N_{M_i^\circ}(Z))$.

Note that $W_i$ is a selfdual $J_i$-module (see for example A.65). Since by $(7^\circ)$ $T_i$ acts $\mathbb{K}_i$-linearly on $W_i$ and $T_i/J_i$ is $p$-group we conclude from B.7(f) that $W_i$ is also a self-dual $T_i$-module. Hence $C_{W_i}(W_j) = Z^\perp$ (in $W_i$) and so $|W_1/C_{W_1}(W_2)| = |Z| = |W_2/C_{W_2}(W_1|$. Thus $W_j$ is non-trivial quadratic offender on $W_i$ and we can apply C.4(c).

Let $A_i$ be maximal offender in $\overline{J_i}$ on $W_i$ with $\overline{W_j} \leqslant A_i$. We conclude from C.4(c) that $Z = C_{W_i}(W_j)$, $|Z| = q^4 = |W_i/Z|$, $|[W_i, A_i]| = q^4$ and $O^{p'}(N_{\overline{J_i}}(A_i)/A_i) \cong Sp_4(q)$. It follows that $Z = [W_i, A_i]$, $N_{\overline{J}_i}(A_i) \leqslant N_{J_i}(Z)$, and $N_{\overline{J}_i}(A_i)$ is maximal parabolic subgroup of $\overline{J_i}$. Therefore $\overline{L_i} = O^{p'}(N_{\overline{J}_i}(A_i))$, and $Z$ is natural $Sp_4(q)$-module for $L_i$. Hence $L_i$ is transitive on $Z$. In particular, each element of $Z$ is $p$-central in $L_i$ and so also in $J_i$. As $M_i^\circ \leqslant J_i$, this shows that each element of $Z$ is centralized by a conjugate of $Q_i$ in $J_i$, and so $Z^\sharp = \mathcal{V}$. Thus $(4^\circ)$(c) shows that $L = L_i^\circ \trianglelefteq L_i$.

Let $g \in J_i$ with $Q_i^g \leqslant L_i$. Suppose for a contradiction that $[Z, Q_i^g] = 1$, and let $Z_i$ be a 1-dimensional $\mathbb{K}_i$ subspace of $Z$. Then $Q!$ implies that $Q_i^g \trianglelefteq L_i$ and $Q_i^g \trianglelefteq N_{J_i}(Z_i)$; in particular $\langle L_i, N_{J_i}(Z_i)\rangle \leqslant N_{J_i}(Q_i^g)$. On the other hand, by the action of $J_i$ on the spin module $W_i$, $N_{J_1}(Z_i)$ is a maximal parabolic of $J_i$. We conclude that $N_{J_i}(Q^g) = N_{J_i}(Z_i)$ and $L_i \leqslant N_{J_i}(Z_i)$, a contradiction since $L_i$ is transitive on $Z$. Thus $[Z, L] \neq 1$. As $Sp_4(q)$ is quasisimple, except for $q = 2$, we conclude that $L/C_L(Z) \cong Sp_4(q)$ or $Sp_4(2)'$. Put $E := C_{\overline{J_1}}(Z)$. In $\overline{J_1}$ we see that $E$ is natural $\Omega_5(q)$- respectively $\Omega_5(2)'$-module for $L$ and so by B.29 $E$ has no $L$- submodule of order $q^4$, Put $E := C_{\overline{J_1}}(Z)$. On the other hand, $\overline{W_2} \leqslant E$ and $|\overline{W_2}| = |W_2/Z| = q^4$, so $\overline{W_2}$ is an $L$-submodule of $E$ order $q^4$, a contradiction.

Case 8. *Case (8) of C.3 holds, that is,* $\overline{J_1} \cong Spin_{10}^+(q)$, *and* $W_1$ *is the corresponding half-spinmodule.*

Just as in the previous case, the fact that $\overline{J_i}$ is quasisimple implies that $\overline{M_i^\circ} = \overline{R_i} = \overline{J_i}$, and $M_i^\circ \leqslant R_i = J_i$. Put $L_i := O^{p'}(N_{M_i^\circ}(Z))$.

Since $W_2$ is a non-trivial offender on $W_1$, C.4(d) shows that $|\overline{W_2}| = q^8 = |W_1/C_{W_1}(W_2)|$. Hence also $W_1$ is a non-trivial offender on $W_2$, so $W_i$ is a best offender on $W_j$, $W_i \leqslant J_j$, and we can apply

C.4(d) to $J_1$ and $J_2$. It follows that $Z = C_{W_i}(W_j)$, $|Z| = q^8$, and $O^{p'}(N_{\overline{J_i}}(\overline{W_j}))/\overline{W_j} \cong Spin_8^+(q)$. In particular, $N_{J_i}(\overline{W_j})$ contains a Sylow $p$-subgroup of $\overline{J_i}$ and $O_p(N_{\overline{J_i}}(\overline{W_i})) = \overline{W_i}$. The structure of $\overline{J_i}$ now implies that $N_{\overline{J_i}}(\overline{W_j})$ is maximal parabolic subgroup of $\overline{J_i}$. As $N_{J_i}(\overline{W_j})$ normalizes $Z = [W_i, W_j]$, we conclude that $\overline{L_i} = O^{p'}(N_{J_i}(\overline{W_j}))$, and $Z$ is a natural $\Omega_8^+(q)$-module for $L_i$ (note here that a half-spin $Spin_8^+(q)$-module is also a natural $\Omega_8^+(q)$-module). Thus $L_i$ preserves a non-degenerate quadratic form $q_i$ of $+$-type on $Z$. Note that the $q_i$-singular elements in $Z$ are $p$-central in $L_i$ and so also in $J_i$. Hence each of these singular elements is centralized by a $J_i$-conjugate of $Q_i$. Observe that more than half of the non-trivial elements in $Z$ are $q_i$-singular and so there exists $1 \neq v \in Z$ such that $z$ is singular with respect to $q_1$ and $q_2$. Thus, $v \in \mathcal{V}$ and $Q_v \leqslant M_1 \cap M_2$. Let $Z_1$ be the 1-dimensional $\mathbb{K}_1$-subspace of $Z$ with $v \in Z_1$. From $Q_v \leqslant M_1^\circ \leqslant J_1$ we conclude that $Q_v$ acts $\mathbb{K}_1$-linearly on $W_1$, and so $[Z_1, Q_v] = 1$. Thus by $Q!$, $N_{J_1}(Z_1) \leqslant N_G(Q_v)$. By the action of $J_1$ on the half-spin module $W_1$, $N_{J_1}(Z_1)$ is a maximal parabolic subgroups of $J_1$ distinct from the maximal parabolic subgroup $N_{J_1}(Z)$. Hence $O_p(N_{\overline{J_1}}(Z)) \nleqslant O_p(N_{\overline{J_1}}(Z_1))$. As seen above $\overline{W_2} = O_p(N_{\overline{J_1}}(Z))$ and so $W_2 \nleqslant O_p(N_{J_1}(Z_1))$ and $Y_2 \nleqslant O_p(N_G(Q_v))$. Thus $Y_2 \nleqslant Q_2^\bullet$ and $Y \nleqslant Q^\bullet$. Moreover C.22 shows that $Y_1 = W_1$. Therefore Theorem D (4:3) holds.

Case 9. *Case (9) of C.3 holds, that is,* $\overline{J_1} \cong 3{}^{\cdot}Alt(6)$ *and* $|W_1| = 2^6$.

As in the $Sp_{2n}(q)$-case for odd $q$, the action of $Z(\overline{J_1})$ on $Y_1$ and $C_{Y_1}(J_1) = 1$ give $W_1 = Y_1$ and thus also $W_2 = Y_2$. This action also shows that $\mathbb{K}_1 \cong \mathbb{F}_4$. Since $W_i = Y_i$, $Y_2$ is an non-trivial offender on $Y_1$. Hence C.4(e) shows that $|Y_2/C_{Y_2}(Y_1)| = 4 = |Y_1/C_{Y_1}(Y_2)|$. In particular, $Y_1$ is a non-trivial offender on $Y_2$. Now C.4(e) shows that the non-trivial offenders in $\overline{J_i}$ on $W_i$ are conjugate in $\overline{J_i}$, $|Z| = 2^4$, and $Z = C_{Y_i}(Y_j)$. Since also $Y_2^u$ is an offender on $Y_1^u = Y_2$ we see in $\overline{M_2}$ that $\overline{Y_2^{uh}} = \overline{Y_1}$ for some $h \in M_2$. Put $g := uh$. Then $Y_1^g = Y_2^h = Y_2$, $\overline{Y_2^g} = \overline{Y_1}$ (in $\overline{M_2}$) and $Z^g = [Y_1^g, Y_2^g] = [Y_2, Y_2^g] = [Y_2, Y_1] = Z$. Define

$$\Delta := \{[y_1, y_2] \mid y_1 \in Y_1 \backslash Z, y_2 \in Y_2 \backslash Z\}.$$

For $y_j \in Y_j \backslash Z$, $[Y_i, y_j]$ is a 1-dimensional $\mathbb{K}_i$-subspace of $Y_i$. It follows that

$$\Delta_i := \left\{ [Y_i, y_j]^\sharp \mid y_j \in Y_j \backslash Z \right\}$$

is a partition of $\Delta$ into three subsets of size three. From $(Y_1, Y_2)^g = (Y_2, Y_2^g)$ and $\overline{Y_2^g} = \overline{Y_1}$ (in $\overline{M_2}$) we conclude that $\Delta^g = \Delta$, $\Delta_1^g = \Delta_2$ and $\Delta_2^g = \Delta_1$. Thus $g \in N_G(\{\Delta_1, \Delta_2\})$. On the other hand, in $\overline{M_1}$, $\overline{Y_2}$ is normalized by a Sylow 2-subgroup of $\overline{M_1}$. It follows that $C_G(\{\Delta_1, \Delta_2\})$ contains a Sylow 2-subgroup of $G$. Thus $N_G(\{\Delta_1, \Delta_2\}) = C_G(\{\Delta_1, \Delta_2\})$ and $\Delta_1 = \Delta_2$, and so $[y_1, Y_2]^\sharp \in \Delta_1$ for $y_1 \in Y_1 \backslash Z$. But $[y_1, Y_2]^\sharp$ has an element in common with each $[Y_1, y_2]$, $y_2 \in Y_2 \backslash Z$ of $\Delta_1$ (namely $[y_1, y_2]$), a contradiction since $\Delta_1$ is a partition of $\Delta$. □

4.15. **Proof of Theorem D:**

By 4.9(d) the hypothesis of the Q!FF-Module Theorem C.24 is fulfilled for $(\overline{M_i}, Y_i, \overline{Q_i})$ in place of $(H, V, Q)$. Hence Theorem D follows from 4.12 if C.24(1) holds, from 4.13 if C.24(2) holds and $W_1$ is not a simple $R_1$-module, and from 4.14 if C.24(2) holds and $W_1$ is a simple $R_1$-module.

CHAPTER 5

# The Short Asymmetric Case

In this chapter we begin to investigate the action of $M \in \mathfrak{M}_G(S)$ on $Y_M$, when $Y_M$ is asymmetric. This investigation will occupy the next five chapters. In this chapter we treat the short asymmetric case, that is, in addition,

$$Y_M \leqslant O_p(L) \quad \text{for all} \quad L \leqslant G \quad \text{with } O_p(M) \leqslant L \text{ and } O_p(L) \neq 1.$$

For all such $L$ asymmetry shows that $L \cap M^{\dagger}$ is a parabolic subgroup of $L$ and then shortness that $\langle Y_M^L \rangle$ is an elementary abelian normal subgroup of $L$ (see 2.6).

The proof of Theorem E is carried out using particular choices for $L$, namely the $Y_i$-indicators $L_i$ of a symmetric pair $(V_1, V_2)$. It is here where for the first time $p$-minimal subgroups enter the stage. Apart from technical details, $Y_i$ is a conjugate of $Y_M$, $V_i = \langle Y_i^{L_i} \rangle$ is elementary abelian, and

$$V_1 V_2 \leqslant L_1 \cap L_2 \text{ and } 1 \neq [V_1, V_2] \leqslant V_1 \cap V_2.$$

From a formal point of view the last property is very similar to the one discussed at the beginning of the previous chapter. But in contrast to the situation there neither is $V_i$ a $p$-reduced normal subgroup of $L_i$ nor are we really interested in the structure of $L_i$ but in the structure of $N_G(Y_i)/C_G(Y_i)$. So we use the action of $L_i$ on non-central $L_i$-chief factors of $V_i$ to get information about the action of $N_G(Y_i)$ on $Y_i$. This is carried out be a rather technical argument. A maybe easier way to understand how the action of $L_i$ on $V_i$ influences the action of $N_G(Y_i)$ on $Y_i$ is by studying the more transparent situation of the *qrc*-Lemma in [**MS4**], from where some of our arguments are borrowed.

Here is the main result of this chapter.

THEOREM E. *Let $G$ be finite $\mathcal{K}_p$-group, $S \in Syl_p(G)$, and let $Q \leqslant S$ be a large subgroup of $G$. Suppose that $M \in \mathfrak{M}_G(S)$ such that*

(i) *$Q \ntrianglelefteq M$ and $\mathcal{M}_G(S) \neq \{M^{\dagger}\}$, and*
(ii) *$Y_M$ is short and asymmetric in $G$.*

*Then one of the following holds, where $q$ is a power of $p$ and $\overline{M} := M/C_M(Y_M)$:*

(1) (a) *$\overline{M^{\circ}} \cong SL_n(q)$, $n \geqslant 3$, and $[Y, M^{\circ}]$ is a corresponding natural module for $\overline{M^{\circ}}$.*
 (b) *If $Y \neq [Y, M^{\circ}]$ then $\overline{M^{\circ}} \cong SL_3(2)$ and $|Y/[Y, M^{\circ}]| = 2$.*

(2) (a) *$\overline{M^{\circ}} \cong Sp_{2n}(q)$, $n \geqslant 2$, or $Sp_4(q)'$ (and $q = 2$), and $[Y, M^{\circ}]$ is the corresponding natural module for $\overline{M^{\circ}}$.*
 (b) *If $Y \neq [Y, M^{\circ}]$, then $p = 2$ and $|Y/[Y, M^{\circ}]| \leqslant q$.*

(3) *There exists a unique $\overline{M}$-invariant set $\mathcal{K}$ of subgroups of $\overline{M}$ such that $Y_M$ is a natural $SL_2(q)$-wreath product module for $\overline{M}$ with respect to $\mathcal{K}$. Moreover, $\overline{M^{\circ}} = O^p(\langle \mathcal{K} \rangle)\overline{Q}$ and $Q$ acts transitively on $\mathcal{K}$.*

(4) (a) *$\overline{M} \cong O^{\epsilon}_{2n}(2)$, $\overline{M^{\circ}} \cong \Omega^{\epsilon}_{2n}(2)$, $2n \geqslant 4$ and $(2n, \epsilon) \neq (4, +)$,[1] and $[Y, M]$ is a corresponding natural module.*
 (b) *If $Y_M \neq [Y_M, M]$, then $\overline{M} \cong O^+_6(2)$ and $|Y_M/[Y_M, M]| = 2$.*
 (c) *$C_G(y) \nleqslant M^{\dagger}$ and $C_G(y)$ is not of characteristic 2 for every non-singular element $y \in [Y, M]$.*

[1] $O^+_4(2)$ appears as $SL_2(2) \wr C_2$ in Case (3)

Table 1 lists examples for $Y_M, M$ and $G$ fulfilling the hypothesis of Theorem E.

TABLE 1. Examples for Theorem E

| Case | $[Y_M, M^\circ]$ for $M^\circ$ | c | examples for $G$ |
|---|---|---|---|
| 3 | nat $SL_2(q)$ | 1 | ${}^2F_4(q)$ |
| 3 | nat $SL_2(2)$ | 1 | $Mat_{12}(.2)$, ${}^2F_4(2)'(.2)$ |
| 3 | nat $SL_2(3)$ | 1 | Th |

Here $c = |Y_M/[Y_M, M^\circ]|$.

We fix the following hypothesis and notation for the remainder of this chapter. For the definition of a symmetric pair and a $Y$-indicator see Definition 2.19.

HYPOTHESIS AND NOTATION 5.1. The groups $G$, $S$, $Q$, $M^\dagger$, and $M$ have the properties given in the hypothesis of Theorem E. In particular $Q \ntrianglelefteq M^\dagger$, $\mathcal{M}_G(M) = \{M^\dagger\}$, and $Y_M = Y_{M^\dagger}$ is asymmetric and short in $G$

By 2.23 there exist conjugates $Y_1$ and $Y_2$ of $Y_M$ such that $(Y_1, Y_2)$ is a symmetric pair; i.e., there exist $Y_i$-indicators $L_i$ for $i = 1, 2$ such that for $V_i := \langle Y_i^{L_i}\rangle$

$$V_1V_2 \leqslant L_1 \cap L_2 \text{ and } [V_1, V_2] \neq 1.$$

Recall from 2.20 that $V_1$ and $V_2$ are elementary abelian $p$-subgroups. We choose such $Y_1, Y_2, L_1$ and $L_2$ with the additional property that $|L_1||L_2|$ is minimal. We further fix:

(a) $\{i, j\} = \{1, 2\}$.

(b) (1) If case 2.19(2:i) holds for $(Y_i, L_i)$ then $Q_i \in Q^G$ such that $Q_i^\bullet \leqslant N_G(Y_i)$ and $L_i \leqslant N_G(Q_i)$.

(2) If case 2.19(2:ii) holds for $(Y_i, L_i)$ then $S_i \in Syl_p(N_G(Y_i))$ such that $S_i \cap L_i \in Syl_p(N_{L_i}(Y_i))$, and $[Y_i, O^p(L_i)] \nleqslant [\Omega_1 Z(S_i), O^p(L_i))] \neq 1$.

(c) $R_i := O_p(L_i)$.

(d) $g_i \in G$ such that $Y_M^{g_i} = Y_i$ and $M^{g_i} \cap L_i$ is a parabolic subgroup of $L_i$. Note that that such a $g_i$ exists since $N_{L_i}(Y_i)$ is a parabolic subgroup of $L_i$ and $M$ a parabolic subgroup of $M^\dagger = N_G(Y_M)$.

(e) $M_i := M^{g_i}$ and $M_i^\dagger := M^{\dagger g_i}$. In particular, $M_i^\dagger = M_iC_G(Y_i) = N_G(Y_i)$, see 2.2(d).

LEMMA 5.2. (a) *$V_i \leqslant R_i \leqslant N_{L_i}(Y_i) \leqslant M_i^\dagger$. In particular, $[Y_i^t, V_i] \leqslant Y_i^t \cap V_i$ and $[Y_i^t, R_i] \leqslant Y_i^t \cap R_i$ for all $t \in L_i$.*

(b) *Suppose 2.19(2) holds for $(L_i, Y_i)$. Then $O^p(L_i) \nleqslant M_i^\dagger$ and $[Y_i, O^p(L_i)] \neq 1$.*

PROOF. Since by definition $V_i$ is normal $p$-subgroup of $L_i$, $V_i \leqslant R_i$. Also $N_{L_i}(Y_i) \leqslant N_G(Y_i) = M_i^\dagger$.

Suppose that Case 2.19(1) holds. Then $Y_i \trianglelefteq L_i$ and (a) holds.

Suppose that 2.19(2) holds. Then $N_{L_i}(Y_i)$ is a maximal and parabolic subgroup of $L_i$. In particular, $N_{L_i}(Y_i) \neq L_i$ and $N_{L_i}(Y_i)$ contains a Sylow $p$-subgroup $T_i$ of $L_i$. Hence $R_i = O_p(L_i) \leqslant T_i \leqslant N_{L_i}(Y_i)$, $L_i = T_iO^p(L_i) = N_{L_i}(Y_i)O^p(L_i)$ and $O^p(L_i) \nleqslant N_{L_i}(Y_i)$; in particular, $O^p(L_i) \nleqslant C_{L_i}(Y_i)$.

Thus (a) and (b) hold. □

LEMMA 5.3. *Suppose that one of the following holds:*

(i) *There exists $Y \in Y_i^{L_i}$ with $1 \neq [Y, V_j] \leqslant Y$, or*

(ii) *$V_j \leqslant R_i$.*

*Then*

(a) *Case 1 of 2.19 holds for $(L_i, Y_i)$.*

(b) *$L_i = Y_iV_j = R_i$. In particular, $V_j \leqslant R_i$ and $L_i$ is a $p$-group.*

(c) *$Y_i \trianglelefteq L_i$. In particular, $V_i = Y_i$.*

PROOF. Suppose first that (i) holds. Then $YV_j$ is a $p$-group with $Y \trianglelefteq YV_j$. Thus $YV_j$ fulfills Case 1 in the Definition 2.19 of $Y$-indicator, so $YV_j$ is a $Y$-indicator. Moreover

$$YV_j \leqslant V_iV_j \leqslant L_j \qquad \text{and} \qquad 1 \neq [Y, V_j].$$

Hence $(Y, Y_j)$ is a symmetric pair and $|YV_j||L_j| \leqslant |L_i||L_j| = |L_1||L_2|$. The minimal choice of $|L_1||L_2|$ now implies $L_i = YV_j$. Then $Y \trianglelefteq L_i$ and so $Y = V_i = Y_i$. If Case 2 of 2.19 holds for $(L_i, Y_i)$ then $Y_i \ntrianglelefteq L_i$, which is not the case. So Case 1 holds for $(L_i, Y_i)$, and the Lemma is proved in this case.

Suppose next that (ii) holds. Since $V_i = \langle Y_i^{L_i} \rangle$ and $[V_i, V_j] \neq 1$, we can choose $Y \in Y_i^{L_i}$ with $[Y, V_j] \neq 1$. By 5.2(a) $[Y, R_i] \leqslant Y$ and by assumption $V_j \leqslant R_i$. Hence $1 \neq [Y, V_j] \leqslant Y$. Thus (i) holds, and we are done by the previous case. □

LEMMA 5.4. *Suppose that $V_j \nleqslant R_i$.*

(a) *Case 2 of 2.19 holds for $(L_i, Y_i)$. In particular, $Y_i \ntrianglelefteq L_i$ and $L_i \nleqslant M_i^\dagger$.*
(b) *$C_{V_j}(V_i) \leqslant R_i$.*
(c) *$L_i$ is $V_iV_j$-minimal.*
(d) *There exists $X_i \in Y_i^{L_i}$ such that $[V_j, X_i] = 1$ and $N_{L_i}(X_i)$ is the unique maximal subgroup of $L_i$ containing $V_iV_j$. In particular, $Y_i \ntrianglelefteq L_i$ and $L_i = \langle V_j, V_j^x \rangle V_i$ for every $x \in L_i \backslash N_{L_i}(X_i)$.*

PROOF. Since $V_j \nleqslant R_i$ we know that $L_i$ is a not a $p$-group, so Case 2 of 2.19 holds for $(L_i, Y_i)$. Then $N_{L_i}(Y_i)$ is a maximal and parabolic subgroup of $L_i$. In particular $Y_i \ntrianglelefteq L_i$, and as $V_j$ is a $p$-subgroup of $L_i$, $V_j^g \leqslant N_{L_i}(Y_i)$ for some $g \in L_i$. By 5.2(a) also $V_i^g = V_i \leqslant N_{L_i}(Y_i)$. Put $X_i := Y_i^{g^{-1}}$. Then $V_iV_j \leqslant N_{L_i}(X_i)$ and $V_i = \langle X_i^{L_i} \rangle$.

1°. *There exist $L_i^* \leqslant L_i$ and $h \in L_i$ such that for $Y_i^* := X_i^h$:*

(a) *$L_i^*$ is $V_iV_j$-minimal and $N_{L_i^*}(Y_i^*)$ is the unique maximal subgroup of $L_i^*$ containing $V_iV_j$. In particular, $V_j \nleqslant O_p(L_i^*)$.*
(b) *$\langle V_j^{L_i^*} \rangle V_i = L_i^*$ and $\langle V_j, V_j^x \rangle V_i = L_i^*$ for all $x \in L_i^* \backslash N_{L_i^*}(Y_i^*)$.*

Observe that the $L$-Lemma 1.41 applies with $(L_i, V_j, N_{L_i}(X_i))$ in place of $(H, A, M)$. Hence, there exist $L \leqslant L_i$ and $h \in L_i$ such that for $Y_i^* := X_i^h$

$$(*) \qquad L = \langle V_j, V_j^x \rangle O_p(L) \quad \text{for all } x \in L \backslash N_L(Y_i^*),$$

and $N_L(Y_i^*)$ is the unique maximal subgroup of $L$ containing $V_jO_p(L)$.

Pick $t \in L \backslash N_L(Y_i^*)$ such that $L_i^* := \langle V_j, V_j^t \rangle V_i$ is minimal. Let $x \in L_i^* \backslash N_{L_i^*}(Y_i^*)$. Then $\langle V_j, V_j^x \rangle V_i \leqslant L_i^*$, and the minimal choice of $L_i^*$ shows $\langle V_j, V_j^x \rangle V_i = L_i^*$. By $(*)$, $L = L_i^* O_p(L)$. Since $V_jO_p(L) \leqslant N_L(Y_i^*) < L$ we conclude that $V_iV_j \leqslant N_{L_i^*}(Y_i^*) < L_i^*$. In particular, $N_{L_i^*}(Y_i^*)$ is the unique maximal subgroup of $L_i^*$ containing $V_iV_j$. Thus, there exists $x \in L_i^* \backslash N_{L_i^*}(Y_i^*)$, and so $L_i^* = \langle V_j, V_j^x \rangle V_i = \langle V_j^{L_i^*} \rangle$. Hence (1°) holds.

We fix the groups $L_i^*$ and $Y_i^*$ given in (1°); in particular, $Y_i^* = X_i^h = Y_i^{g^{-1}h}$ for certain $g, h \in L_i$. Furthermore we set $V_i^* := \langle Y_i^{*L_i^*} \rangle$. Note that $V_i \leqslant C_{L_i^*}(V_i^*) \leqslant N_{L_i^*}(Y_i^*)$. Since $N_{L_i^*}(Y_i^*)$ is the unique maximal subgroup of $L_i^*$ containing $V_iV_j$, the assumptions of 1.42(e) are fulfilled with $\big(C_{L_i^*}(V_i^*), N_{L_i^*}(Y_i^*)\big)$ in place of $(N, L_0)$. Thus $C_{L_i^*}(V_i^*)$ is $p$-closed. Also by (1°)(a) $V_j \nleqslant O_p(L_i^*)$, and it follows that

2°. *$C_{V_j}(V_i^*) \leqslant O_p(L_i^*)$ and $[V_j, V_i^*] \neq 1$.*

Next we show:

3°. *$L_i^*$ is an $Y_i^*$-indicator.*

By (1°)(a) $L_i^*$ is $V_iV_j$-minimal and so also $p$-minimal. Moreover, $N_{L_i^*}(Y_i^*)$ is a maximal and parabolic subgroup of $L_i^*$. Recall that $(V_i, V_j)$ is a symmetric pair with $V_j \nleqslant R_i$. Thus, $L_i$ is not a $p$-group and one of of the cases 2.19(2:i) or (2:ii) holds for $L_i$ and $Y_i$.

Suppose that 2.19(2:i) holds for $L_i$ and $Y_i$. Then $L_i^* \leqslant L_i \leqslant N_G(Q_i) \leqslant N_G(Q_i^\bullet)$ and $Q_i^\bullet \leqslant N_G(Y_i)$. Since $Y_i^* = X_i^h = Y_i^{g^{-1}h}$ and $g^{-1}h \in L_i \leqslant N_G(Q_i^\bullet)$, this implies $Q_i^\bullet \leqslant N_G(Y_i^*)$ and so $L_i^*$ is an $Y_i^*$-indicator.

Suppose next that 2.19(2:ii) holds for $L_i$ and $Y_i$. Let $T_i^* \in Syl_p(N_{L_i^*}(Y_i^*))$ with $V_iV_j \leqslant T_i^*$ and let $T_i \in Syl_p(N_{L_i}(Y_i^*))$ with $T_i^* \leqslant T_i$. Since $S_i \cap L_i \in Syl_p(N_{L_i}(Y_i))$ and $Y_i^* \in Y_i^{L_i}$, there exists $t \in L_i$ with $Y_i^t = Y_i^*$ and $T_i = S_i^t \cap L_i$. Put $S_i^* := S_i^t$. Then $T_i = S_i^* \cap L_i$, in particular $T_i^* \leqslant S_i^* \cap L_i^*$. Since $S_i \in Syl_p(N_G(Y_i))$ we have $S_i^* \in Syl_p(N_G(Y_i^*))$. As $N_{L_i^*}(Y_i)$ is a parabolic subgroup of $L_i^*$, $T_i^* \in Syl_p(L_i^*)$, and $T_i^* \leqslant S_i^* \cap L_i^*$ gives $T_i^* = S_i^* \cap L_i^*$. We collect:

$$S_i^* \in Syl_p(N_G(Y_i^*)),\ V_iV_j \leqslant T_i^* = S_i^* \cap L_i^* \in Syl_p(N_{L_i^*}(Y_i^*)) \text{ and } S_i^* \cap L_i = T_i \in Syl_p(N_{L_i}(Y_i^*)).$$

Also $[\Omega_1 Z(S_i), O^p(L_i)] \neq 1$ implies $[\Omega_1 Z(S_i^*), O^p(L_i)] \neq 1$.

Note that $L_i$ is $p$-minimal, $N_{L_i}(Y_i^*)$ is the unique maximal subgroup of $L_i$ containing $S_i^* \cap L_i$, $L_i^* \nleqslant N_{L_i}(Y_i^*)$ and $L_i^* = O^p(L_i)(S_i^* \cap L_i^*)$. Hence

$$L_i = \langle S_i^* \cap L_i, L_i^* \rangle = \langle S_i^* \cap L_i, O^p(L_i^*) \rangle.$$

Thus $[\Omega_1 Z(S_i^*), O^p(L_i^*)] \neq 1$. Moreover,

$$L_i = \langle O^p(L_i^*)^{S_i^* \cap L_i} \rangle (S_i^* \cap L_i) \qquad \text{and} \qquad O^p(L_i) = \langle O^p(L_i^*)^{S_i^* \cap L_i} \rangle.$$

Suppose that $[Y_i^*, O^p(L_i^*)] \leqslant [\Omega_1 Z(S_i^*), O^p(L_i^*)]$. Then

$$\begin{aligned}[Y_i^*, O^p(L_i)] = {} & [Y_i^*, \langle O^p(L_i^*)^{S_i^* \cap L_i} \rangle] = \Big\langle [Y_i^*, O^p(L_i^*)]^{S_i^* \cap L_i} \Big\rangle \leqslant \Big\langle [\Omega_1 Z(S_i^*), O^p(L_i^*)]^{S_i^* \cap L_i} \Big\rangle \\ = {} & [\Omega_1 Z(S_i^*), \big\langle O^p(L_i^*)^{S_i^* \cap L_i} \big\rangle] = [\Omega_1 Z(S_i^*), O^p(L_i)].\end{aligned}$$

Conjugation by $t^{-1}$ shows $[Y_i, O^p(L_i)] \leqslant [\Omega_1 Z(S_i), O^p(L_i)]$, a contradiction to 2.19(2:ii) .

Hence $[Y_i^*, O^p(L_i^*)] \nleqslant [\Omega_1 Z(S_i^*), O^p(L_i^*)]$ and so also in this case $L_i^*$ is a $Y_i^*$ indicator.

By (2°) and (3°) we know that $[V_i^*, V_j] =\neq 1$ and that $L_i^*$ is a $Y_i^*$-indicator. So $(Y_i^*, Y_j)$ is a symmetric pair, and the minimality of $|L_i||L_j|$ yields $L_i = L_i^*$. Hence $V_i^* = V_i$, and (2°) gives $C_{V_j}(V_i) \leqslant O_p(L_i^*) = R_i$. Since $V_j$ normalizes $Y_i^*$ we have $[V_j, Y_i^*] \leqslant Y_i^*$. If $[V_j, Y_i^*] \neq 1$ then hypothesis 5.3(i) is satisfied, and 5.3(b) shows that $L_i = R_i$. But then $V_j \leqslant R_i$, contrary to the hypothesis of the lemma. If $[V_j, Y_i^*] = 1$ then (1°) shows that (b) holds with $Y_i^*$ in place of $X_i$. □

Recall from Definition A.7 that a strong dual offender $A$ on a module $V$ satisfies $[V, A] = [v, A]$ for every $v \in V \backslash C_V(A)$.

LEMMA 5.5. *Suppose that there exists $A \leqslant M$ such that the following hold:*

(i) *$A$ is a non-trivial strong dual offender on $Y_M$.*
(ii) *If $|A/C_A(Y_M)| = 2$, then $C_G([Y_M, A]) \nleqslant M^\dagger$.*

*Then Theorem E holds.*

PROOF. By 1.57(b) $Y_M$ is a faithful $p$-reduced $Q!$-module for $\overline{M}$ with respect to $\overline{Q}$. Since $\overline{A}$ is a non-trivial strong dual offender on $Y_M$, we can apply C.27. This shows that Theorem E holds, except that, in Case C.27(4) ($[Y_M, M]$ a natural $O_{2n}^\epsilon(2)$-module for $M$), we still have to verify that $C_G(y)$ is not of characteristic 2 for every non-singular element $y \in [Y_M, M]$.

By C.27(4:c) $|\overline{A}| = 2$. Since $\overline{A}$ is a strong dual offender, this gives $|Y_M/C_Y(A)| = 2$ and $|[Y_M, A]| = 2$. Let $1 \neq y \in [Y_M, A]$. Then, for example by B.9(c), $y$ is non-singular, and by 3.1(a) every non-singular element of $[Y_M, M]$ is conjugate to $y$. By Hypothesis (ii) $C_G([Y_M, A]) \nleqslant M^\dagger$ and so also $C_G(y) \nleqslant M^\dagger$. Hence the hypothesis of Theorem C is fulfilled, and we conclude that $C_G(y)$ is not of characteristic 2. □

LEMMA 5.6. *Suppose that $V_j \nleqslant R_i$ and let $D \leqslant V_j$.*

(a) $[Y_i, O^p(L_i)] \neq 1$. *In particular,* $[V_i, O^p(L_i)] \neq 1$ *and there exists non-central chief factor for* $L_i$ *on* $V_i$.
(b) $L_i$ *is p-irreducible.*
(c) *Let* $X$ *be any* $L_i$*-section of* $V_j$ *with* $[X, O^p(L_i)] \neq 1$. *Then* $C_D(X) \leqslant D \cap R_j$. *In particular, if* $D \nleqslant R_j$ *then* $[X, D] \neq 1$.
(d) *Let* $X$ *be any* $L_i$*-section of* $V_j$ *with* $[X, O^p(L_i)] \neq 1$ *and* $[X, O_p(L_i)] = 1$[2]. *Then* $C_D(X) = D \cap R$ *and*

$$|X/C_X(D)| \geqslant |D/C_D(X)| = |D/D \cap R_i| = |DR_i/R_i|$$

PROOF. Note first that by 5.4(a) Case 2 of 2.19 holds for $(L_i, Y_i)$. In particular, $L_i$ is $p$-minimal.

(a): This holds by 5.2(b).

(b): Since $L_i$ is $p$-minimal, $L_i$ is also $p$-irreducible, see 1.37.

(c): As $V_i$ is an elementary abelian $p$-group, $X$ is an $\mathbb{F}_p L_i$-module. Since $L_i$ is $p$-irreducible and $D$ is $p$-subgroup of $L_i$, 1.33(b) shows that $C_D(X) \leqslant O_p(L_i) = R_i$. Thus (c) holds.

(d): By (c) $C_D(X) \leqslant D \cap R_i$, and since by hypothesis $R_i$ centralizes $X$, we get $C_D(X) = D \cap R_i$. Since $L_i$ is $p$-minimal, C.13(e) shows that no subgroup of $L_i$ is an over-offender on $X$. As $D \leqslant V_i$, $D$ is an elementary abelian $p$-group, and we conclude that $|X/C_X(D)| \geqslant |D/C_D(X)|$. Together with $C_D(X) = D \cap X$ this gives (d). $\square$

LEMMA 5.7. *Suppose that* $Y_j \nleqslant R_i$.
(a) $[V_i, V_j \cap R_i] \leqslant Y_i \cap Z(L_i)$.
(b) $Y_j \cap Z(L_i) = 1$.
(c) $[V_i \cap R_j, V_j] \cap Z(L_i) = 1$.
(d) $[V_i \cap R_j, V_j \cap R_i] = 1$.
(e) $C_{V_i}(V_j) = [V_i, V_j]C_{V_i}(L_i) = [V_i, v]C_{V_i}(L_i) = C_{V_i}(v) = C_{V_i}(Y_j)$ *for every* $v \in V_j \backslash R_i$.

PROOF. Since $V_j \nleqslant R_i$ we can apply 5.4. By 5.4(d) there exists $X_i \in Y_i^{L_i}$ with $[X_i, V_j] = 1$ and

$$(+) \qquad L_i = \langle V_j, V_j^x \rangle V_i \quad \text{for every } x \in L_i \backslash N_{L_i}(X_i).$$

Recall from 2.20 that $V_i$ and $V_j$ are elementary abelian $p$-groups.

(a): Let $t \in L_i$. Since $[X_i, V_j] = 1$ we have $[X_i^t, V_j^t] = 1$. By 5.2(a) $[X_i^t, R_i] \leqslant X_i^t \cap R_i$. As $V_j$ is abelian, it follows that

$$[X_i^t, V_j \cap R_i] \leqslant X_i^t \cap V_j \leqslant C_{X_i^t}(\langle V_j, V_j^t \rangle V_i).$$

If $t \in N_{L_i}(X_i)$ then $[X_i^t, V_j] = 1$, and if $t \notin N_{L_i}(X_i)$ then by (+) and the previous line $[X_i^t, V_j \cap R_i] \leqslant X_i^t \cap Z(L_i)$. Since $X_i^t \cap Z(L_i) = Y_i \cap Z(L_i)$ for every $t \in L_i$, (a) holds.

(b): Suppose that $Y_j \cap Z(L_i) \neq 1$. Then $N := N_G(Y_j \cap Z(L_i))$ is a $p$-local subgroup of $G$. Also $O_p(M_j) \leqslant N$ since $Y_j \leqslant Z(O_p(M_j))$. Hence $Y_j \leqslant O_p(N)$ since $Y_j$ is short.[3] But this contradicts $L_i \leqslant N$ and $Y_j \nleqslant R_i$.

(c): According to (b) it suffices to show that

$$(*) \qquad [V_j, V_i \cap R_j] \leqslant Y_j.$$

If $V_i \leqslant R_j$, then 5.3 gives $V_j = Y_j$ and so $(*)$ holds. If $V_i \nleqslant R_j$, then the hypothesis of this lemma is satisfied with $i$ and $j$ interchanged, and (a) yields $(*)$.

(d): By (a) and (c), $[V_i \cap R_j, V_j \cap R_i] \leqslant Z(L_i) \cap [V_i \cap R_j, V_j] = 1$.

(e): Let $v \in V_j \backslash R_i$. By 5.4(c),(d) $L_i$ is $V_iV_j$-minimal and $N_{L_i}(X_i)$ is a maximal subgroup of $L_i$ containing $V_iV_j$. So by 1.42(d) $\bigcap N_{L_i}(X_i)^{L_i}$ is $p$-closed. Hence there exists $t \in L_i$ with $v \notin N_{L_i}(X_i^t)$. Thus by (+)

$$L_i = \langle V_j^t, V_j^{tv} \rangle V_i = \langle v, V_j^t \rangle V_i.$$

[2]Observe that condition holds for any non-central chief-factor of $L_i$ on $V_i$

[3]Apart from the existence of symmetric pairs, this is the only place in this chapter where one needs shortness and not only *char* $p$-shortness

Since $V_i$ normalizes $V_j$ and $V_j$ is abelian, $[V_i, V_j] \leqslant V_i \cap V_j \leqslant C_{V_i}(V_j)$. As $[X_i^t, V_j^t] = 1$, we get

$$C_{V_i}(v) \cap [V_i, V_j^t]X_i^t \leqslant C_{V_i}(v) \cap C_{V_i}(V_j^t) = C_{V_i}(\langle v, V_j^t \rangle V_i) = C_{V_i}(L_i)$$

and

$$V_i = \langle Y_i^{L_i} \rangle = \langle X_i^{tL_i} \rangle = [V_i, L_i]X_i^t = [V_i, v][V_i, V_j^t]X_i^t.$$

Therefore,

$$C_{V_i}(v) = [V_i, v](C_{V_i}(v) \cap [V_i, V_j^t]X_i^t) = [V_i, v]C_{V_i}(L_i) \leqslant [V_i, V_j]C_{V_i}(L_j) \leqslant C_{V_i}(V_j) \leqslant C_{V_i}(v),$$

so equality holds everywhere in the preceding chain of inclusions; in particular $C_{V_i}(V_j) = C_{V_i}(v)$. Since $Y_j \nleqslant R_i$ we can choose $v \in Y_j \backslash R_i$. Then $v \in Y_j \leqslant V_j$, and we conclude that also $C_{V_i}(Y_j) = C_{V_i}(V_j)$. Thus (e) holds. □

LEMMA 5.8. *Suppose that $Y_j \nleqslant R_i$ and $L_i$ has a unique non-central chief factor on $V_i$. Then Theorem E holds.*

PROOF. As $Y_j \nleqslant R_i$, 5.4(a) shows that we are in case 2.19(2). Suppose that 2.19(2:ii) holds for the $Y_i$-indicator $L_i$. Then $[\Omega_1 Z(S_i), O^p(L_i)] \neq 1$, and $[Y_i, O^p(L_i)] \nleqslant [\Omega_1 Z(S_i), O^p(L_i)]$ (see 5.1(b)). Hence $L_i$ has a non-trivial chief factor on both, $[\Omega_1 Z(S_i), O^p(L_i)]$ and $V_i/[\Omega_1 Z(S_i), O^p(L_i)]$, a contradiction.

Thus 2.19(2:i) holds for $L_i$, so $L_i \leqslant N_G(Q_i) \leqslant N_G(Q_i^\bullet)$ and $Q_i^\bullet \leqslant N_G(Y_i)$. Since $V_i = \langle Y_i^{L_i} \rangle$, we conclude that $Q_i^\bullet \leqslant N_G(V_i)$. Hence $L_i Q_i^\bullet$ acts on $V_i$, $Q_i^\bullet \leqslant O_p(L_i Q_i^\bullet)$ and $Q_i^\bullet$ centralizes any chief factor of $L_i Q_i^\bullet$ on $V_i$. It follows that $L_i Q_i^\bullet$ has a unique non-central chief factor on $V_i$. Set $A_i := [O^p(L_i Q_i^\bullet), O_p(L_i Q_i^\bullet)]$.

1°. *Suppose that $[C_{V_i}(A_i), O^p(L_i)] = 1$. Then Theorem E holds.*

Note that we can apply A.45 with $(L_i Q_i^\bullet, Y_i, A_i, S_i, V_i)$ in place of $(H, Y, R, T, V)$. We conclude that one of the following holds:

(A) $[V_i, A_i] = 1$,
(B) $A_i$ is a non-trivial strong dual offender on $Y_i$,
(C) There exist $A_i O^p(L_i Q_i^\bullet)$-invariant subgroups $Z_1 \leqslant X_1 \leqslant Z_2 \leqslant X_2$ of $V_i$ such that for $l = 1, 2$, $X_l/Z_l$ is a non-central simple $O^p(L_i Q_i^\bullet)$-module and $X_l \cap Y_i \nleqslant Z_l$.

Suppose that (A) holds. Then $C_{V_i}(A_i) = V_i$, a contradiction since $[C_{V_i}(A_i), O^p(L_i)] = 1$ in the current case while $L_i$ has a non-central chief factor on $V_i$.

Suppose that (B) holds. By A.32(a) any strong dual offender is quadratic and so $[Y_i, A_i] \leqslant C_{V_i}(A_i) \leqslant C_{V_i}(O^p(L_i))$. Since $O^p(L_i) \nleqslant M_i^\dagger$ by 5.2(b), this gives $C_G([Y_i, A_i]) \nleqslant M_i^\dagger$. Thus the hypothesis of 5.5 is fulfilled, and we conclude that Theorem E holds.

Suppose that (C) holds. Let $l \in \{1, 2\}$ and put $X_l^* := \langle (X_l \cap Y_i)^{O^p(L_i)} \rangle$. Then $X_1^* \leqslant X_1 \leqslant Z_2$ and $X_1^* \leqslant X_2^*$. Since $X_l/Z_l$ is a non-central simple $O^p(L_i)$-module and $X_l \cap Y_i \nleqslant Z_l$, we have $[X_l^*, O^p(L_i)] \nleqslant Z_l$. Thus $[X_1^*, O^p(L_i)] \neq 1$, and since $X_1^* \leqslant Z_2$, $[X_2^*, O^p(L_i)] \nleqslant X_1^*$. By 5.4(d) $V_j$ centralizes an $L_i$-conjugate of $Y_i$. Thus there exists $t \in L_i$ with $[Y_i, V_j^t] = 1$. Also by 5.4(c), $L_i$ is $V_i V_j$ minimal and so $L_i = O^p(L_i) V_i V_j = O^p(L_i) V_i V_j^t$. As $V_i V_j^t$ centralizes $Y_i$ and so also $X_l \cap Y_i$, this implies that $X_l^* = \langle (X_l \cap Y_i)^{L_i} \rangle$. Hence $X_l^*$ is $L_i$-invariant for $l = 1, 2$, and $L_i$ has at least two non-central chief factors on $V_i$, a contradiction.

2°. *Suppose that $[C_{V_i}(A_i), O^p(L_i)] \neq 1$. Then Theorem E holds.*

Put $D_i := C_{V_i}(O_p(L_i Q_i^\bullet))$. Since $Q_i$ is large, $C_G(Q_i) \leqslant Q_i \leqslant Q_i^\bullet$, so $D_i \leqslant Z(Q_i^\bullet) \leqslant Z(Q_i)$. Also as $Q_i^\bullet \leqslant O_p(L_i Q_i^\bullet)$, we have $D_i \leqslant C_{V_i}(A_i)$ and

$$[O_p(L_i Q_i^\bullet), O^p(L_i)] \leqslant A_i \leqslant C_{L_i Q_i^\bullet}(C_{V_i}(A_i)),$$

so the $P \times Q$-Lemma implies

$$(*) \qquad [D_i, O^p(L_i)] \neq 1.$$

Since $V_j \nleqslant R_i$, 5.6(c) applied with $(V_j, D_i)$ in place of $(D, X)$ gives $[V_j, D_i] \neq 1$. Moreover, as $[D_i, R_i] = 1$ we can also apply 5.6(d) and conclude that

$$(**) \qquad |V_j/V_j \cap R_i| = |V_j/C_{V_j}(D_i)| \leqslant |D_i/C_{D_i}(V_j)|.$$

Suppose for a contradiction that $[V_j, D_i \cap R_j] \neq 1$ and choose $Y_j^* \in Y_j^{L_j}$ with $[Y_j^*, D_i \cap R_j] \neq 1$. By 5.2(a) $[Y_j^*, R_j] \leqslant Y_j^*$ and so $[Y_j^*, D_i \cap R_j] \leqslant D_i \cap Y_j^*$. Thus $D_i \cap Y_j^* \neq 1$. Since $D_i \leqslant Z(Q_i^\bullet)$ and $Y_j^*$ is short and so also $Q$-short, we conclude from 2.3(c) that $[Y_j^*, D_i] = 1$, a contradiction.

We have shown that $[V_j, D_i] \neq 1$ and $[V_j, D_i \cap R_j] = 1$. Hence $D_i \nleqslant R_j$ and so also $V_i \nleqslant R_j$. Thus we can apply 5.6 with the roles of $i$ and $j$ interchanged. In particular, there exists a non-central chief factor $W$ for $L_j$ on $V_j$. Moreover, 5.6(d) shows that $C_{D_i}(W) = D_i \cap R_j = C_{D_i}(V_j)$ and

$$|V_j/C_{V_j}(D_i)| \geqslant |W/C_W(D_i)| \geqslant |D_i/C_{D_i}(W)| = |D_i/C_{D_i}(V_j)|.$$

Combined with $(**)$ this gives

$$|W/C_W(D_i)| = |V_j/C_{V_j}(D_i)| = |D_i/C_{D_i}(V_j)|.$$

In particular, there exists a unique non-central chief factor of $L_j$ in $V_j$, so also $L_j$ satisfies the hypothesis of this lemma (for some $L_i$-conjugate of $Y_i$). Put $A_j := [O^p(L_jQ_j^\bullet), O_p(L_jQ_j^\bullet)]$ and $D_j := C_{V_j}(O_p(L_jQ_j^\bullet))$.

If $[C_{V_j}(A_j), O^p(L_j)] = 1$, then $(1^\circ)$, with $j$ in place of $i$, shows that we are done. Otherwise $(*)$, again with $j$ in place of $i$, gives $[D_j, O^p(L_j)] \neq 1$. Since $D_i \nleqslant R_j$, we conclude from 5.6(c) that $1 \neq [D_i, D_j] \leqslant D_i \cap D_j$. As $D_i \leqslant Z(Q_i)$, this contradicts 2.3(a). $\square$

Lemma 5.9. *Suppose that $V_j \leqslant R_i$. Then Theorem E holds.*

Proof. By 5.3 $Y_i = V_i$. Assume that also $V_i \leqslant R_j$. Then $V_i = Y_i$ and $V_j = Y_j$ and so $1 \neq [Y_i, Y_j] \leqslant V_i \cap V_j = Y_i \cap Y_j$. Hence $Y_M$ is not asymmetric in $G$, a contradiction.

Thus $Y_i = V_i \nleqslant R_j$, and we can apply 5.7 with the roles of $i$ and $j$ interchanged. By 5.7(d)

$$[V_j, V_i \cap R_j] = [V_j \cap R_i, V_i \cap R_j] = 1.$$

Since $Y_i = V_j$ and by 5.4(b), again with $i$ and $j$ interchanged, $C_{V_i}(V_j) \leqslant R_j$, this gives

$$Y_i \cap R_j = V_i \cap R_j = C_{V_i}(V_j) = C_{Y_i}(V_j).$$

By 5.7(b) $V_i \cap Z(L_j) = Y_i \cap Z(L_j) = 1$, in particular $[Y_i, V_j] \cap C_{V_j}(L_j) = 1$. Let $v \in Y_i \backslash C_{Y_i}(V_j)$. Then $v \in V_i \backslash R_i$, and 5.7(e) shows

$$[Y_i, V_j] \leqslant C_{V_j}(V_i) = [v, V_j]C_{V_j}(L_j).$$

Thus $[Y_i, V_j] = [v, V_j]\big([Y_i, V_j] \cap C_{V_j}(L_j)\big) = [v, V_j]$. We conclude that $V_j$ is a non-trivial strong dual offender on $Y_i$.

If $|V_j/C_{V_j}(Y_i)| > 2$, we are done by 5.5. If $|V_j/C_{V_j}(Y_i)| = 2$, then $L_j$ has a unique non-central chief factor on $V_j$ since $Y_i \nleqslant R_j$ and $L_j$ is 2-minimal. So we are done by 5.8. $\square$

Lemma 5.10. *Let $q$ be a power of $p$, $H \cong SL_2(q)$, $W$ a natural $SL_2(q)$-module for $H$ and $V$ an $\mathbb{F}_pH$-module isomorphic to $W^n$, $n \geqslant 1$, the direct sum of $n$ copies of $W$. Let $B_1, B_2 \leqslant H$ with $B_1B_2 \in Syl_p(H)$ and $B_1 \neq 1 \neq B_2$. Suppose that there exists $A \leqslant V$ with $C_V(B_1B_2) \leqslant A$, $[A, B_1] \cap [A, B_2] = 0$ and $|V/A| \leqslant |A/C_V(B_1B_2)|$. Then*

(a) *There exist a subfield $\mathbb{F}$ of $\mathbb{K} := End_H(W)$ with $\dim_{\mathbb{F}} \mathbb{K} = 2$, a 3-dimensional $\mathbb{F}$-subspace $D$ of $W$ with $C_W(B_1B_2) \leqslant D$ and $\mathbb{F}_pH$-monomorphisms $\alpha_i : W \to V$, $1 \leqslant i \leqslant n$, such that*

$$V = \bigoplus_{i=1}^{n} V_i \qquad \text{and} \qquad A = \bigoplus_{i=1}^{n} A_i, \qquad \text{where } V_i := \alpha_i(W) \text{ and } A_i := \alpha_i(D).$$

(b) *$|V/A| = |A/C_V(B_1B_2)|$ and $|B_1| = |B_2|$.*

(c) *There exists $h \in H$ with $[A, B_1] \leqslant A^h$ and $[A, B_2] \cap A^h = 0$.*

Proof. Let $\mathcal{I}$ be the set of simple $\mathbb{F}_pH$-submodules of $V$ and put $Z := C_V(B_1B_2)$. Since $W$ is a natural $SL_2(q)$-module for $H$, $C_W(B_1B_2) = C_W(B_i)$ is a one dimensional $\mathbb{F}_q$-submodule of $W$. So $Z = C_V(B_i)$, $i = 1, 2$, since $V \cong W^n$. Observe that $V = \bigcup_{I \in \mathcal{I}}(I + Z)$ and so, since $Z \leqslant A$, $A = \bigcup_{I \in \mathcal{I}} \big((A \cap I) + Z\big)$. Put

$$\mathcal{J} := \{I \in \mathcal{I} \mid A \cap I \nleqslant Z\} \qquad \text{and} \qquad X := \sum \mathcal{J}.$$

Then $A = (X \cap A) + Z$ and $C_X(B_1B_2) = X \cap Z \leqslant X \cap A$. By assumption $|V/A| \leqslant |A/C_V(B_1B_2)| = |A/Z|$. Thus

$$(*)\quad \begin{array}{ccccccc} |X \cap A/X \cap Z| & = & |X \cap A/(X \cap A) \cap Z| & = & |(X \cap A) + Z/Z| & = & |A/Z| \\ & \geqslant & |V/A| & \geqslant & |X + A/A| & = & |X/X \cap A|. \end{array}$$

So $(X, X \cap A)$ in place of $(V, A)$ fulfills the assumption of the lemma. Suppose that $X \neq V$. Then induction on $|V|$ gives $|X \cap A/X \cap Z| = |X/X \cap A|$. Thus equality holds in $(*)$ and so $V = X + A = X + (A \cap X) + Z = X + Z$. But then $[V, B_1B_2] \leqslant X$, a contradiction.

Thus $X = V$ and so there exist $V_1, \dots, V_n \in \mathcal{J}$ with $V = \bigoplus_{i=1}^n V_i$. Pick $a \in W \backslash C_W(B_1B_2)$ and choose an $\mathbb{F}_pH$-isomorphism $\alpha_i : W \to V_i$ for each $1 \leqslant i \leqslant n$. By definition of $\mathcal{J}$, $V_i \cap Z < V_i \cap A$. Also $W = C_W(B_1B_2) + \mathbb{K}a$ and so there exists $k_i \in \mathbb{K}$ with $\alpha_i(k_ia) \in V_i \cap A \backslash V_i \cap Z$. Replacing $\alpha_i$ by $\alpha_i \circ k_i$ we may assume that $a_i := \alpha_i(a) \in V_i \cap A \backslash V_i \cap Z$. View $V$ as a $\mathbb{K}$-module such that each $\alpha_i$ is a $\mathbb{K}H$-isomorphism.

If $d \in A$ then $d + Z = (\sum_{i=1}^n f_i(d)a_i) + Z$ for some $f_i(d) \in \mathbb{K}$. Put $\mathbb{F}_i := \{f_i(d) \mid d \in A\}$. Then $\mathbb{F}_i$ is an additive subgroup of $\mathbb{K}$ and $A \leqslant Z + \sum_{i=1}^n \mathbb{F}_ia_i$.

For $l = 1, 2$ fix $1 \neq b_l \in B_l$ and put $x_l := [a, b_l]$ and $x_{il} := \alpha_i(x_l)$. Define $\mathbb{K}_l \subseteq \mathbb{K}$ by $[a, B_l] = \mathbb{K}_lx_l$. Since $[a, b_l] = 1x_l$, $1 \in \mathbb{K}_l$. Also $\mathbb{K}_l$ is an additive subgroup of $\mathbb{K}$ and $|\mathbb{K}_l| = |B_l|$. Thus $Z_l := \sum_{i=1}^n \mathbb{K}_lx_{il}$ has order $|B_l|^n$. Since $[a_i, B_l] = \mathbb{K}_lx_{il}$ we have $Z_l \leqslant [A, B_l]$. From $[A, B_1] \cap [A, B_2] = 0$ we get $Z_1 \cap Z_2 = 0$ and $B_1 \cap B_2 = 1$. We conclude that

$$|Z_1 + Z_2| = |Z_1||Z_2| = |B_1|^n|B_2|^n = |B_1B_2|^n = q^n = |Z|.$$

Thus $Z = Z_1 \oplus Z_2$ and $[A, B_l] = Z_l$.

Fix $m$ and $l$ with $1 \leqslant m \leqslant n$ and $l \in \{1, 2\}$. Let $g_m \in \mathbb{F}_m$ and $k_l \in \mathbb{K}_l$. Then there exists $d \in A$ with $g_m = f_m(d)$ and $e \in B_l$ with $k_lx_l = [a, e]$. Since $\alpha_i$ is an $H$-monomorphism we get $k_lx_{il} = [a_i, e]$ for all $1 \leqslant i \leqslant n$. Thus

$$[d, e] = [\sum_{i=1}^n f_i(d)a_i, e] = \sum_{i=1}^n f_i(d)[a_i, e] = \sum_{i=1}^n f_i(d)k_lx_{il}.$$

As $[d, e] \in [A, B_l] = Z_l$ we get that $f_i(d)k_lx_{il} \in \mathbb{K}_lx_{il}$ and so $f_i(d)k_l \in \mathbb{K}_l$ for all $1 \leqslant i \leqslant n$. For $i = m$ we infer $g_mk_l \in \mathbb{K}_l$ and so

$$(**)\qquad \mathbb{F}_m\mathbb{K}_l \subseteq \mathbb{K}_l.$$

Since $1 \in \mathbb{K}_l$, we conclude $\mathbb{F}_m \leqslant \mathbb{K}_l$ and so $|\mathbb{F}_m| \leqslant \min(|\mathbb{K}_1|, |\mathbb{K}_2|)$. From $|\mathbb{K}_1||\mathbb{K}_2| = |B_1||B_2| = q$ we get $|\mathbb{F}_m| \leqslant \sqrt{q}$ for all $1 \leqslant m \leqslant n$. Recall that $A \leqslant Z + \sum_{i=1}^n \mathbb{F}_ia_i$, so $|A/Z| \leqslant \prod_{i=1}^n |\mathbb{F}_i| \leqslant \sqrt{q^n}$. As $|V/Z| = q^n$ and $|V/A| \leqslant |A/Z|$, this gives $|V/A| = |A/Z|$, and equality holds in all of the preceding inequalities. So $|\mathbb{F}_m| = |\mathbb{K}_l| = \sqrt{q}$, $\mathbb{F}_m = \mathbb{K}_l$, and $A = Z + \sum_{i=1}^n \mathbb{F}_ia_i$. In particular, $|B_l| = |\mathbb{K}_l| = \sqrt{q}$ and $|B_1| = |B_2|$.

Hence $\mathbb{F} := \mathbb{F}_m = \mathbb{K}_l$ for all $1 \leqslant m \leqslant n$ and $1 \leqslant l \leqslant 2$, and $A = Z + \sum_{i=1}^n \mathbb{F}a_i$. By (**) $\mathbb{F}\mathbb{F} \subseteq \mathbb{F}$ and so $\mathbb{F}$ is a subring of $\mathbb{K}$. Thus $\mathbb{F}$ is a finite integral domain and so a field. Since $|\mathbb{K}| = q = |\mathbb{F}|^2$, $\dim_{\mathbb{F}} \mathbb{K} = 2$. Put $E := C_W(B_1B_2)$ and $D := E + \mathbb{F}a$. Then $A = Z + \sum_{i=1}^n \mathbb{F}a_i = \sum_{i=1}^n \alpha_i(D)$. So (a) and (b) hold.

Let $h \in H \backslash N_H(E)$. Note that $W = E \oplus E^h$. So $D^h = (D^h \cap E) \oplus E^h$ and thus $D^h \cap E$ is a 1-dimensional $\mathbb{F}$-subspace of $E$. Since $N_H(E)$ acts transitively on $E$, we can choose $h$ such that $x_1 \in D^h \cap E$. Then $D^h \cap E = \mathbb{F}x_1$. Applying the $\alpha_i$'s gives $A_i^h \cap Z = \mathbb{F}x_{i1}$. As $A = \bigoplus_{i=1}^m A_i$ and $Z = \bigoplus_{i=1}^m V_i \cap Z$, this yields $A^h \cap Z = \sum_{i=1}^n \mathbb{F}x_{i1} = [A, B_1]$. In particular, $[A, B_1] \leqslant A^h$ and, since $[A, B_2] \leqslant Z$,

$$[A, B_2] \cap A^h = [A, B_2] \cap (A^h \cap Z) \leqslant [A, B_2] \cap [A, B_1] = 0.$$

So (c) is proved. □

LEMMA 5.11. *Suppose that $Y_1 \nleqslant R_2$ and $Y_2 \nleqslant R_1$. Then Theorem E holds.*

PROOF. Since $Y_1 \nleqslant R_2$ and $Y_2 \nleqslant R_2$, we can apply 5.4 with $(i, j) = (1, 2)$ and $(i, j) = (2, 1)$. As the hypothesis is symmetric in $i$ and $j$ we choose our notation such that

1°. $\quad |V_1R_2/R_2| \geqslant |V_2R_1/R_1|$.

By 5.7(e) $C_{V_i}(V_j) = C_{V_i}(Y_j)$. Also 5.4(b) (applied to $(j,i)$ in place of $(i,j)$) gives $C_{V_i}(V_j) \leqslant R_j$. Thus

2°. $C_{V_i}(Y_j) = C_{V_i}(V_j) \leqslant V_i \cap R_j$.

Let $r_i$ be the number of non-central chief factors for $L_i$ on $V_i$. By 5.6 we have $[V_i, O^p(L_i)] \neq 1$. So $r_i \geqslant 1$. If $r_i = 1$ then 5.8 shows that Theorem E holds. So we may assume that $r_i \geqslant 2$ for $i = 1, 2$. By 5.6(d) we have

$$|X/C_X(V_j)| \geqslant |V_j/C_{V_j}(X)| = |V_jR_i/R_i|$$

for any non-central chief factor $X$ of $L_i$ on $V_i$. Thus

3°. *$|V_i/C_{V_i}(V_j)| \geqslant |V_jR_i/R_i|^{r_i} \geqslant |V_jR_i/R_i|^2$. Moreover, if $|V_i/C_{V_i}(V_j)| = |V_jR_i/R_i|^2$, then $r_i = 2$ and $V_j$ is a non-trivial offender on each non-central chief factor of $L_i$ on $V_i$.*

As $C_{V_i}(Y_j) = C_{V_i}(V_j)$ by (2°), this gives

4°. $|V_i/C_{V_i}(Y_j)| \geqslant |V_jR_i/R_i|^2$.

Hence

5°. $|V_2/C_{V_2}(Y_1)| \overset{(4^\circ)}{\geqslant} |V_1R_2/R_2|^2 \overset{(1^\circ)}{\geqslant} |V_2R_1/R_1||V_1R_2/R_2| = |V_2/V_2 \cap R_1||V_1R_2/R_2|$.

Since $V_1 \nleqslant R_2$, this gives $|V_2/C_{V_2}(Y_1)| > |V_2/V_2 \cap R_1|$, so

6°. $[Y_1, V_2 \cap R_1] \neq 1$.

By 5.7(d) $[Y_1 \cap R_2, V_2 \cap R_1] = 1$. Let $x \in V_2 \cap R_1 \backslash C_{V_2}(V_1)$ and $y \in Y_1 \backslash R_2$. By 5.7(e) $C_{V_2}(V_1) = C_{V_2}(y)$. Thus $[x,y] \neq 1$, so $C_{Y_1}(x) \leqslant Y_1 \cap R_2$, and

$$C_{Y_1}(x) \leqslant Y_1 \cap R_2 \leqslant C_{Y_1}(V_2 \cap R_1) \leqslant C_{Y_1}(x).$$

Hence

7°. *$C_{Y_1}(x) = Y_1 \cap R_2 = C_{Y_1}(V_2 \cap R_1)$ for $x \in V_2 \cap R_1 \backslash C_{V_2}(Y_1)$.*

Recall from (2°) that $C_{V_2}(Y_1) \leqslant V_2 \cap R_1$. So $C_{V_2}(Y_1) = C_{V_2 \cap R_1}(Y_1)$ and

$$|V_2/C_{V_2}(Y_1)| = |V_2/V_2 \cap R_1||V_2 \cap R_1/C_{V_2 \cap R_1}(Y_1)|.$$

By (5°)

$$|V_2/C_{V_2}(Y_1)| \geqslant |V_2/V_2 \cap R_1||V_1R_2/R_2|.$$

Comparing the last two displayed statements gives

8°. $|V_2 \cap R_1/C_{V_2 \cap R_1}(Y_1)| \geqslant |V_1R_2/R_2| \geqslant |Y_1R_2/R_2| = |Y_1/Y_1 \cap R_2|$,

and so, since $Y_1 \cap R_2 = C_{Y_1}(V_2 \cap R_1)$ by (7°),

9°. $|V_2 \cap R_1/C_{V_2 \cap R_1}(Y_1)| \geqslant |Y_1/C_{Y_1}(V_2 \cap R_1)|$.

Combining (6°), (7°) and (9°) we get:

10°. *$A := V_2 \cap R_1$ is a non-trivial strong offender on $Y_1$.*

By A.34 all strong offenders are best offenders, so

11°. *$A$ is a non-trivial best offender on $Y_1$.*

By 5.7(a), $[V_1, V_2 \cap R_1] \leqslant Z(L_1)$, so $L_1 \leqslant C_G([Y_1, A])$. By 5.4(a) $L_1 \nleqslant M_1^\dagger$. We record:

12°. *$L_1 \nleqslant M_1^\dagger$ and $L_1 \leqslant C_G([Y_1, A]) \nleqslant M_1^\dagger$.*

Next we prove:

13°. *Let $N \leqslant M_1$ with $N = N^\circ$ and $1 \neq O^p(N) \trianglelefteq M_1$, then $N$ does not normalize any non-trivial subgroup of $[Y_1, A]$.*

Suppose that there exists $1 \neq U \leqslant [Y_1, A]$ with $N \leqslant N_G(U)$. Pick $Q_0 \in Q^G$ with $Q_0 \leqslant N$. Then $C_U(Q_0) \neq 1$ and thus by $Q!$, $C_G(U) \leqslant N_G(Q_0)$. Now 1.52 gives $(NC_G(U))^\circ = N^\circ = N$, so $N$ is normalized by $C_G(U)$. Hence

$$C_G([Y_1, A]) \leqslant C_G(U) \leqslant N_G(N) \leqslant N_G(O^p(N)).$$

By hypothesis, $1 \neq O^p(N) \trianglelefteq M_1$, and so 2.2(c) gives $N_G(O^p(N)) \leqslant M_1^\dagger$. Thus $C_G([Y_1, A]) \leqslant M_1^\dagger$, a contradiction to (12°).

14°. $[M_1^\circ, A] \nleqslant C_{M_1}(Y_1)$.

Otherwise, $M_1^\circ$ normalizes $[Y_1, A]$, a contradiction to (13°) applied to $N = M_1^\circ$.

Define

$$\overline{M_1} := M_1/C_{M_1}(Y_1), \qquad J := J_{M_1}(Y_1), \qquad \overline{J} := J/C_{M_1}(Y_1) \qquad \overline{K} := F^*(\overline{J}).$$

Since $Q_1$ is large and $Q_1 \ntrianglelefteq M_1$, 1.57(b) shows that $Y_1$ is a $Q!$-module for $\overline{M}_1$ with respect to $\overline{Q_1}$. Since $Y_1$ is $p$-reduced for $M_1$, $O_p(\overline{M_1}) = 1$. By (11°) $A$ is a best offender on $Y_1$. By (14°), $[\overline{M_1^\circ}, A] \neq 1$. Thus the assumption of the Q!FF-Module Theorem C.24 are fulfilled for $(\overline{M_1}, \overline{Q_1}, \overline{A}, Y_1)$ in place of $(H, Q, Y, V)$.

Suppose that C.24(1) holds. Then there exists an $\overline{M_1}$-invariant set $\mathcal{K}$ of subgroups of $\overline{M_1}$ such that $Y_1$ is a natural $SL_2(q)$-wreath product module for $\overline{M_1}$ with respect to $\mathcal{K}$, $\overline{M_1^\circ} = O^p(\langle\mathcal{K}\rangle)\overline{Q_1}$ and $\overline{Q_1}$ acts transitively on $\mathcal{K}$. By A.27(c) $\mathcal{K}$ is unique. So Case (3) of Theorem E holds.

Thus, we may assume from now:

15°. *C.24(2) holds for $\overline{M_1}$ and $Y_1$.*

In particular, by C.24(2:a) and (2:b)

16°.

(a) *$\overline{K}$ is quasisimple.*
(b) *$C_{Y_1}(\overline{K}) = 0$ and $[Y_1, \overline{K}]$ is a semisimple $\overline{J}$-module.*

Note that by (16°) all non-trivial $\overline{J}$-submodules of $[Y_1, \overline{K}]$ are perfect. Thus A.44 shows that all $\overline{K}$-submodules of $[Y_1, \overline{K}]$ are $\overline{J}$-invariant. In particular, the simple $\overline{K}$-submodules of $[Y_1, \overline{K}]$ are exactly the simple $\overline{J}$-submodules of $[Y_1, \overline{K}]$.

By (11°) $A$ is a best offender on $Y_1$. Thus $\overline{A} \leqslant \overline{J}$. Put $T := KA$ and let $I$ be a simple $T$-submodule of $[Y_1, \overline{K}]$.

Suppose that there exists a simple $T$-submodule $I_0$ in $Y_1$ such that $I^* \cong I_0$ as a $T$-module, where $I^*$ is the dual of the $\mathbb{F}_p\overline{J}$-module $I$. (Note that we can choose $I = I_0$ if $I \cong I^*$). By (10°) $A$ is a strong offender on $Y_1$, so $A$ is also a strong offender on the submodules $I$ and $I_0$. It follows that $A$ is strong offender on $I^*$, and so by A.35 $A$ is a root offender on $I$. Hence A.37 shows that $|I/C_I(A)| = |A/C_A(I)|$ and $A$ is strong dual offender on $I$. As $A$ is strong offender on $Y_1$, $C_A(Y_1) = C_A(I)$. Thus

$$|A/C_A(Y_1)| = |A/C_A(I)| = |I/C_I(A)| = |IC_{Y_1}(A)/C_{Y_1}(A)| \leqslant |Y_1/C_{Y_1}(A)| \leqslant |A/C_A(Y_1)|.$$

Hence equality holds everywhere, $Y_1 = IC_{Y_1}(A)$, and $A$ is a strong dual offender on $Y_1$. By (12°), $C_G([Y_1, A]) \nleqslant M_1^\dagger$ and so $M_1$ and $A$ satisfy the hypothesis of 5.5, and Theorem E follows. So we may assume from now on:

17°. *$I^*$ is not isomorphic to any $T$-submodule of $Y_M$; in particular $I$ is not selfdual as an $\mathbb{F}_pT$-module.*

Since $\overline{K} = F^*(\overline{J})$ is quasisimple and $\overline{A} \leqslant \overline{J}$, we get $\overline{T} = \overline{AK} = \langle \overline{A}^{\overline{T}} \rangle$ and $\overline{K} = F^*(\overline{T})$. As seen above, $\overline{A}$ is a strong offender on $I$, so we can apply the Strong Offender Theorem C.6 to $(\overline{T}, \overline{K}, I, \overline{A})$ in place of $(M, K, V, A)$. Hence one of the following holds:

(A) $\overline{T} \cong SL_n(\tilde{q})$ or $Sp_{2n}(\tilde{q})$ and $I$ is a corresponding natural module.
(B) $p = 2$, $\overline{T} \cong Alt(6)$, $3{\cdot}Alt(6)$ or $Alt(7)$, $|V| = 2^4$, $2^6$ or $2^4$, respectively, and $|\overline{A}| = 4$.
(C) $p = 2$, $\overline{T} \cong O_{2n}^\epsilon(2)$ or $Sym(n)$, $V$ is a corresponding natural module, and $|\overline{A}| = 2$.

Note that the natural $SL_2(\tilde{q})$-, $Sp_{2n}(\tilde{q})$, $Alt(6)$-, $O^{\epsilon}_{2n}(2)$- and $Sym(n)$-modules all are selfdual and so are ruled out by (17°). Moreover, the module of order $2^4$ for $Alt(7)$ is rule out since it does not appear as a conclusion of the Q!FF-module Theorem (in fact this module is not a $Q!$-module).

We have proved:

18°. *$\overline{T} \cong SL_n(\tilde{q})$, $n \geqslant 3$, or $3^{\cdot}Alt(6)$, and $I$ is a corresponding natural module for $\overline{T}$.*

Next we prove

19°. *$\overline{J} = \overline{T} = \overline{K} \leqslant \overline{M^\circ}$ and one of the following holds:*

(1) *$\overline{K} \cong SL_n(\tilde{q})$, $n \geqslant 3$, $\overline{M^\circ} = \overline{K}C_{\overline{M^\circ}}(\overline{K})$, and $Y_1 = \bigoplus_{l=1}^{k} Y_{1l}$, where $k \geqslant 2$ and the modules $Y_{1l}$ are isomorphic natural $SL_n(\tilde{q})$-modules for $\overline{K}$.*
(2) *$\overline{K} \cong 3^{\cdot}Alt(6)$, $\overline{M^\circ} \cong 3^{\cdot}Alt(6)$ or $3^{\cdot}Sym(6)$ and $Y_1 = [Y_1, \overline{K}]$ has order $2^6$.*
(3) *$\overline{\overline{K}} \cong SL_n(\tilde{q})$, $n \geqslant 3$, $\overline{M^\circ} = \overline{K}$, and $[Y_1, K]$ is natural $SL_n(\tilde{q})$-modules for $\overline{K}$. Moreover, either $Y_1 = [Y_1, \overline{K}]$ or $\overline{K} \cong SL_3(2)$ and $|Y_1/[Y_1, K]| = 2$.*

Since $\overline{T} \cong SL_n(\tilde{q})$, $n \geqslant 3$, or $3^{\cdot}Alt(6)$, we have $\overline{K} = F^*(\overline{T}) = \overline{T}$. Recall that C.24(2) holds. By C.24(2:a) $\overline{K} \leqslant \overline{M_1^\circ}$ and either $\overline{J} = \overline{K}$ or $\overline{J} \cong O^{\epsilon}_{2n}(2), Sp_4(2)$ or $G_2(2)$. As $\overline{K} \cong SL_n(\tilde{q})$, $n \geqslant 3$, or $3^{\cdot}Alt(6)$, we get $\overline{J} = \overline{K}$ or $\overline{K} \cong SL_4(2)$ and $\overline{J} \cong O_6^+(2)$. In the later case, recall that $I$ is $J$-invariant, which contradicts the fact that $\overline{J} \cong O_6^+(2)$ induces graph automorphisms on $\overline{K} \cong SL_4(2)$ and so does not act on that natural $SL_4(2)$-module $I$. Thus $\overline{J} = \overline{K}$ and the initial statement in (19°) is proved. We now consider the three cases of C.24(2:c).

Suppose that C.24(2:c:1) holds. Since $3^{\cdot}Alt(6)$ does not appear in C.24(2:c:1:a) we conclude that $\overline{K} \cong SL_n(\tilde{q})$. Moreover, $[Y_1, \overline{K}]$ is a direct sum of at least two isomorphic natural modules and $\overline{M^\circ} = \overline{K}C_{\overline{M^\circ}}(\overline{K})$. Since $SL_n(\tilde{q})$ does not appear in C.24(2:c:1:d), we have $Y_1 = [Y_1, \overline{K}]$ and so (19°)(1) holds.

Suppose that C.24(2:c:2) holds. Then $[Y_1, \overline{K}]$ is a simple $\overline{K}$-module and either $\overline{M^\circ} = \overline{K}$ or $\overline{M^\circ} \cong Sp_4(2)$, $3^{\cdot}Sym(6)$, $SU_4(q).2$ or $G_2(2)$. Thus $I = [Y_1, \overline{K}]$.

Assume that $I$ is natural $SL_n(\tilde{q})$-module for $\overline{K}$. Then $\overline{M^\circ} = \overline{K}$. Moreover, by (16°)(b) $C_{Y_1}(\overline{K}) = 1$, and C.22 shows that either $Y_1 = [Y_1, \overline{K}]$ or $\overline{K} \cong SL_3(2)$ and $|Y_1/[Y_1, \overline{K}]| = 2$. Thus (19°)(3) holds.

Assume that $I$ is a natural $3^{\cdot}Alt(6)$-module for $\overline{K}$. Then $\overline{M^\circ} = \overline{K} \cong 3^{\cdot}Alt(6)$ or $\overline{M^\circ} \cong 3^{\cdot}Sym(6)$. As $C_{Y_1}(\overline{K}) = 1$, the fixed-point free action of $Z(\overline{K})$ on $I$ shows that $I = Y_1$. Thus (19°)(2) holds.

Suppose that C.24(2:c:3) holds. Then $Y_1$ is the direct sum of two non-isomorphic natural $SL_4(\tilde{q})$-modules for $\overline{K}$. Since non-isomorphic natural $SL_4(\tilde{q})$-modules are dual to each other, this contradicts (17°). This completes the proof of (19°).

Observe that

20°. *If (19°)(3) holds, then Case 1 of Theorem E holds.*

So we may assume from now on that (19°)(1) or (2) holds. The next statement will allow us to derive a contradiction in these two cases, simultaneously.

21°.

(a) *$N_{M_1}([Y_1, A])$ is a parabolic subgroup of $M_1$. In particular, there exists an $M_1$-conjugate $Q_3$ of $Q_1$ with $Q_2 \leqslant N_{M_1}([Y_1, A])$.*
(b) *Put $E_1 := O^{p'}(N_J([Y_1, A]))$. There exist isomorphic $E_1$-submodules $Y_{1l}, 1 \leqslant l \leqslant k$, with $Y_1 = \bigoplus_{l=1}^{k} Y_{1l}$ and $k \geqslant 2$.*

Suppose first that (19°)(1) holds. Then $Y_1 = \bigoplus_{l=1}^{k} Y_{1l}$ as an $\mathbb{F}_p J$-module and so also as an $E_1$-module. Since $[I, A]$ is an $\mathbb{F}_{\tilde{q}}$-subspace of $I$ and $I$ is natural $SL_n(\tilde{q})$-module, $N_J([I, A])$ is a parabolic subgroup of $J$ and $[I, A] = [I, O_p(N_{\overline{J}}([I, A]))]$. Since each $Y_{1l}$ is isomorphic to $I$, this implies that $E_1 = O^{p'}(N_J([I, A]))$, $E_1$ is parabolic subgroup of $J$ and $[Y_1, A] = [Y_1, O_p(\overline{E_1})]$. Since $I^*$ is not isomorphic to any $J$-submodule of $Y_1$, no element of $M_1$ induces a non-trivial graph automorphism on $\overline{J} \cong SL_n(\tilde{q})$. It follows that

$$\overline{M_1} = N_{\overline{M_1}}(\overline{E_1})\overline{J} = N_{\overline{M_1}}(O_p(\overline{E_1}))\overline{J} = N_{\overline{M_1}}([Y_1, A])\overline{J} = \overline{N_{M_1}([Y_1, A])J}.$$

As $E_1 \leqslant N_{M_1}([Y_1, A])$ and $E_1$ is parabolic subgroup of $J$, this shows that $N_{M_1}([Y_1, A])$ is parabolic subgroup of $M_1$.

Suppose next that (19°)(2) holds. By C.16(b), $C_{Y_1}(A) = [Y_1, A]$, so by C.16(c), $N_{M_1}([Y_1, A])$ is parabolic subgroup of $M_1$. Put $\overline{K_2} := C_{\overline{M_1}}(Y_1/C_{Y_1}(A))$ and let $\mathcal{V}$ be the set of 3-dimensional $\overline{K_2}$-submodules of $Y_1$. Then by C.16(e) $\mathcal{V} = \{Y_{11}, Y_{12}, Y_{13}\}$ and $Z(\overline{K})$ acts transitively on $\mathcal{V}$. By C.16(d), $\overline{K_2} = \overline{E_1}$, and we conclude that $Y_{11}$ and $Y_{12}$ are isomorphic $E_1$-submodules of $Y_1$. By C.16(f), $Y = Y_{11} \times Y_{12}$, and so (21°)(b) holds with $k = 2$.

22°. *$\overline{M_1^\circ} = \overline{J}$ and $C_{[Y_{1l}, A]}(Q_3) \neq 1$ for all $1 \leqslant l \leqslant k$.*

Put $F := C_{M_1^\circ}(\overline{J})$ and $F_0 := O^p((FQ_3)^\circ)$. Note that $F_0$ is normalized by $FN_{M_1}(Q_3)$. We claim that $F_0$ is normal in $M_1$.

Define $J_0 := (J \cap M_1^\circ)^\infty$, so $J_0$ is the largest perfect subgroup of $J \cap M_1^\circ$. By (19°) $\overline{J} = \overline{K} \leqslant \overline{M_1^\circ}$ and so $J = (J \cap M_1^\circ)C_{M_1}(Y_1)$. As $\overline{J}$ is perfect, we conclude that $J = J_0 C_{M_1}(Y_1)$. By 1.52(c) $[C_{M_1}(Y_1), M_1^\circ] \leqslant O_p(M_1^\circ)$. Since $[F, J_0] \leqslant C_{M_1}(Y_1)$ and $J_0 \leqslant M_1^\circ$, this gives $[F, J_0, J_0] \leqslant O_p(M_1^\circ)$. As $J_0$ is perfect, the Three Subgroups Lemma implies $[F, J_0] \leqslant O_p(M_1^\circ)$. In particular, $J_0$ normalizes $F_0 O_p(M_1^\circ)$. Since $O_p(M_1^\circ) \leqslant F$, $O_p(M_1^\circ)$ normalizes $F_0$. Hence $O^p(F_0 O_p(M_1^\circ)) = O^p(F_0) = F_0$, and $F_0$ is normalized by $J_0$. As seen above, also $FN_{M_1}(Q_3)$ normalizes $F_0$. Since $C_{M_1}(Y_1) \leqslant N_{M_1}(Q_3)$ by $Q$! and $J = J_0 C_{M_1}(Y_1)$, this shows that $F_0 \trianglelefteq JFN_{M_1}(Q_3)$.

By (19°) either $\overline{J} \cong SL_n(q)$ and $\overline{M_1^\circ} = \overline{FJ}$ or $\overline{J} \sim 3^{\cdot}Alt(6)$ and $|\overline{M^\circ}/\overline{J}| \leqslant 2$, thus in any case $\overline{M_1^\circ} = \overline{FJQ_3}$. Moreover, since $Q_3$ is a weakly closed subgroup of $G$, a Frattini argument shows $M_1 = FJN_{M_1}(Q_3)$. As proved above $F_0 \trianglelefteq JFN_{M_1}(Q_3)$ and thus $F_0 \trianglelefteq M_1$, as claimed.

Suppose that $F_0 \neq 1$. Then we can apply (13°) with $N = (FQ_3)^\circ$ and conclude that $(FQ_3)^\circ$ does not normalizes any non-trivial subgroup of $[Y_1, A]$. But $\overline{A} \leqslant \overline{J}$, so $[\overline{A}, \overline{F}] = 1$, and $F$ normalizes $[Y_1, A]$. By the choice of $Q_3$, also $Q_3$ normalizes $[Y_1, A]$, a contradiction.

Thus $F_0 = 1$, $(FQ_3)^\circ$ is a $p$-group and $(FQ_3)^\circ = Q_3$. Hence

$$[\overline{F}, \overline{Q_3}] \leqslant \overline{F} \cap \overline{Q_3} \leqslant O_p(\overline{F}) \leqslant O_p(\overline{M_1}) = 1.$$

If $\overline{J} \sim 3^{\cdot}Alt(6)$ we get $[Z(\overline{J}), Q_3] = 1$ and so $\overline{M_1^\circ} \not\sim 3^{\cdot}Sym(6)$ and $\overline{J} = \overline{M_1^\circ}$.

Suppose now that $\overline{J} \cong SL_n(q)$. We have

$$M_1^\circ = \langle Q_3^{M_1} \rangle = \langle Q_3^{FJN_{M_1}(Q_3)} \rangle = \langle Q_3^J \rangle \leqslant Q_3 J,$$

and so $\overline{M_1^\circ} = \overline{Q_3 J}$. Hence $\overline{FJ}/\overline{J}$ is a $p$-group. Since $\overline{F} \cap \overline{J} \leqslant Z(\overline{F})$ this implies that $\overline{F}$ is nilpotent. As $O_p(\overline{F}) = 1$ we conclude that $\overline{F}$ is a $p'$-group. Since $\overline{FJ}/\overline{J}$ is a $p$-group, we get $\overline{F} = \overline{F} \cap \overline{J} \leqslant \overline{J}$ and again $\overline{M_1^\circ} = \overline{J}$. So the first statement in (22°) holds. In particular, $Q_3 \leqslant O^{p'}(N_J([Y_1, A]))$ and $Q_3$ normalizes each $Y_{1l}$. Hence also the second statement holds.

Recall that $A = V_2 \cap R_1$.

23°. *Put $q := |V_1 R_2/R_2|$ and $\widehat{V_2} := V_2/V_2 \cap Z(L_2)$. Then the following hold:*

(a) $q = |V_1 R_2/R_2| = |V_2 R_1/R_1| = |V_2/A|$.
(b) $k = 2 = r_2$.
(c) $|\widehat{V_2}| = q^4$, *and every composition factor for $L_2$ on $\widehat{V_2}$ is a natural $SL_2(q)$-module for $L_2$. In particular, every non-trivial proper $L_2$-submodule of $\widehat{V_2}$ is a natural $SL_2(q)$-module for $L_2$.*

By (2°) $C_{V_2}(V_1) = C_{V_2}(Y_1)$. Since $A = V_2 \cap R_1$, also $C_A(V_1) = C_A(Y_1)$ and

$$\text{(I)} \qquad |A/C_A(V_1)| = |A/C_A(Y_1)| \overset{(8^\circ)}{\geqslant} |V_1 R_2/R_2| \overset{(1^\circ)}{\geqslant} |V_2 R_1/R_1| = |V_2/V_2 \cap R_1| = |V_2/A|.$$

From 5.4(d) we get $L_2 = \langle V_1, V_1^x \rangle V_2$ for a suitable $x \in L_2$, and $[X_2, V_1] = 1$ for suitable $X_2 \in Y_2^{L_2}$. Note that $[V_2, V_1] \leqslant V_1 \cap V_2 \leqslant C_{V_2}(V_1)$ and so $X_2[V_2, V_1] \leqslant C_{V_2}(V_1)$ and recall that $V_2$ is abelian. It follows that

$$V_2 = \langle Y_2^{L_2} \rangle = \langle X_2^{L_2} \rangle = X_2[V_2, L_2] = X_2[V_2, \langle V_1, V_1^x \rangle V_2] = X_2[V_2, V_1][V_2, V_1^x] = C_{V_2}(V_1)C_{V_2}(V_1^x)$$

and

$$C_{V_2}(V_1) \cap C_{V_2}(V_1^x) = C_{V_2}(\langle V_1, V_1^x \rangle V_2) = C_{V_2}(L_2) = V_2 \cap Z(L_2).$$

Thus

$$\text{(II)} \qquad \widehat{V_2} = \widehat{C_{V_2}(V_1)} \times \widehat{C_{V_2}(V_1^x)}.$$

and so

$$\text{(III)} \qquad |\widehat{C_{V_2}(V_1)}| = |\widehat{V_2}/\widehat{C_{V_2}(V_1^x)}| = |V_2/C_{V_2}(V_1^x)| = |V_2/C_{V_2}(V_1)| = |V_2/A||A/C_A(V_1)|.$$

As by (I) $|V_2/A| \leqslant |A/C_A(V_1)|$, this gives

$$\text{(IV)} \qquad |\widehat{C_{V_2}(V_1)}| = |V_2/A||A/C_A(V_1) \leqslant |A/C_A(V_1)|^2.$$

By 5.7(c) $[V_1, A] \cap Z(L_2) = 1$ and so

$$\text{(V)} \qquad |[V_1, A]| \ = |\widehat{[V_1, A]}| \leqslant |\widehat{C_{V_2}(V_1)}| \leqslant |A/C_A(V_1)|^2.$$

Let $y \in Y_{1l} \backslash C_{Y_1}(A)$ and $a \in A \backslash C_A(Y_1)$. Since by (10°) $A$ is a strong offender on $Y_1$, $C_{Y_1}(A) = C_{Y_1}(a)$ and so $[y, a] \neq 1$. Thus $C_A(y) = C_A(Y_1)$. Hence

$$|[Y_{1l}, A]| \geqslant |[y, A]| \geqslant |A/C_A(y)| = |A/C_A(Y_1)|.$$

Since this holds for all $1 \leqslant l \leqslant k$,

$$|[Y_1, A]| \geqslant |A/C_A(Y_1)|^k.$$

Now (V) implies

$$|A/C_A/V_1)|^2 \geqslant |[V_1, A]| \geqslant |[Y_1, A]| \geqslant |A/C_A(Y_1)|^k \geqslant |A/C_A(V_1)|^k.$$

Hence $k = 2$ since $k > 1$, and $|[V_1, A]| = |A/C_A(V_1)|^2$. From this we conclude that equality holds in (V), so

$$\text{(VI)} \qquad |\widehat{C_{V_2}(V_1)}| = |A/C_A(V_1)|^2.$$

As a consequence equality holds in (IV) so $|V_2/A| = |A/C_A(V_1)|$, and then equality holds in (I), so

$$\text{(VII)} \qquad q = |V_1R_2/R_2| = |V_2R_1/R_1| = |A/C_A(V_1)| = |V_2/A|.$$

In particular, (23°)(a) is proved. Moreover,

$$\text{(VIII)} \qquad |\widehat{C_{V_2}(V_1)}| \overset{\text{(VI)}}{=} |A/C_A(V_1)|^2 \overset{\text{(VII)}}{=} q^2 \quad \text{and} \quad |V_2/C_{V_2}(V_1)| \overset{\text{(III)}}{=} |\widehat{C_{V_2}(V_1)}| = q^2.$$

Hence

$$|\widehat{V_2}| \overset{\text{(II)}}{=} |\widehat{C_{V_2}(V_1)} \times \widehat{C_{V_2}(V_1^x)}| = |\widehat{C_{V_2}(V_1)}||\widehat{C_{V_2}(V_1^x)}| = |\widehat{C_{V_2}(V_1)}|^2 = (q^2)^2 = q^4.$$

Also $|V_2/C_{V_2}(V_1)| = q^2 = |V_1R_2/R_2|^2$, and so (3°) shows that $r_2 = 2$ and $V_1$ is a non-trivial offender on each non-central chief factor $X$ of $L_2$ on $V_2$. Since $L_2$ is $V_1V_2$-minimal we can apply C.11 and conclude that $X$ is natural $SL_2(q)$-module for $L_2$. In particular, $|X| = q^2$. Since $r_2 = 2$ and $|\widehat{V_2}| = q^4$ this show that all composition factors of $L_2$ on $\widehat{V_2}$ are non-central. Thus (23°)(c) holds.

As proved above $k = 2$ and $r_2 = 2$. So also (23°)(b) holds, and (23°) is proved.

Define $J_2 := J_{M_2}(Y_2)$. By (23°)(a) $|V_1R_2/R_2| = |V_2R_1/R_1|$, so our initial choice of notation given in (1°) holds with 1 and 2 interchanged. Hence also all the results proven are also valid with 1 and 2 interchanged. In particular, (21°) shows that there exist isomorphic $O^{p'}(N_{J_2}([Y_2, V_1 \cap R_2]))$-submodules $Y_{2l}, 1 \leqslant l \leqslant 2$, such that $Y_2 = Y_{21} \times Y_{22}$.

Put $V_{2l} := \langle Y_{2l}^{L_2} \rangle$ and $E := \langle (V_1 \cap R_2)^{L_2} \rangle$. By 5.7(a), $[V_2, V_1 \cap R_2] \leqslant Y_2 \cap Z(L_2)$, and so conjugation in $L_2$ gives $[V_2, E] \leqslant Y_2 \cap Z(L_2)$. Note that $Y_{2l} \leqslant V_2$. So $[Y_{2l}, E] \leqslant Y_2 \cap Z(L_2)$ and again by conjugation in $L_2$, $[Y_{2l}, E] = [V_{2l}, E]$. Hence

$$[V_{2l}, E] = [Y_{2l}, E] \leqslant Y_2 \cap Z(L_2).$$

Moreover, since $[Y_2, V_1 \cap R_2] \leqslant Y_2 \cap Z(L_2)$ and $E \leqslant L_2$, $E$ centralizes $[Y_2, V_1 \cap R_2]$.

We first show that $E \leqslant J_2$. Let $x \in L_2$. Note that $(Y_2, Y_1^x)$ is a symmetric pair with indicators $L_2$ and $L_1^x$. Moreover, $Y_1^x \nleqslant R_2$.

Suppose that $Y_2 \nleqslant R_1^x$. Then $(Y_2, Y_1^x)$ fulfills the hypothesis of the lemma and so by (11°), applied to the symmetric pair $(Y_1^x, Y_2)$ in place of $(Y_2, Y_1)$, $V_1^x \cap R_2$ is a best offender on $Y_2$. Thus $V_1^x \cap R_2 \leqslant J_2$. Suppose that $Y_2 \leqslant R_1^x$. By 5.7(d) applied with $(Y_1^x, Y_2)$ in place of $(Y_j, Y_i)$ we have $[V_2 \cap R_1^x, V_1^x \cap R_2] = 1$. In particular, $[Y_2, V_1^x \cap R_2] = 1$ since $Y_2 \leqslant R_1^x$. So again $V_1^x \cap R_2 \leqslant J_2$.

We have shown that all $L_2$-conjugates of $V_1 \cap R_2$ are in $J_2$, and so $E = \langle (V_2 \cap R_1)^{L_2} \rangle \leqslant J_2$. Therefore, $E \leqslant O^{p'}(C_{J_2}([Y_2, V_1 \cap R_2]))$. Thus $Y_{21}$ and $Y_{22}$ are isomorphic $E$-submodules of $Y_2$. Hence

$$\text{(IX)} \qquad [V_{2l}, E] = [Y_{2l}, E] \leqslant Y_{2l} \cap Z(L_2).$$

Note that $[Y_2, V_1 \cap R_2] \neq 1$, $Y_2 = Y_{21} \times Y_{22}$ and $Y_{21}$ and $Y_{22}$ are isomorphic $V_1 \cap R_2$-modules. Thus $[Y_{21}, V_1 \cap R_2] \neq 1$. Suppose that $\widehat{V_{21}} \leqslant \widehat{V_{22}}$. Then

$$1 \neq [Y_{21}, V_1 \cap R_2] \leqslant [V_{22}, V_1 \cap R_2] \leqslant [V_{22}, E] \overset{\text{(IX)}}{\leqslant} Y_{22} \cap Z(L_2),$$

which contradicts $[Y_{21}, V_1 \cap R_2] \leqslant Y_{21}$ and $Y_{21} \cap Y_{22} = 1$.

Thus $\widehat{V_{21}} \nleqslant \widehat{V_{22}}$ and by symmetry $\widehat{V_{22}} \nleqslant \widehat{V_{21}}$. By (23°)(c) every non-trivial proper $L_2$-submodule of $\widehat{V_2}$ is natural $SL_2(q)$-module. It follows that $\widehat{V_2} = \widehat{V_{21}} \times \widehat{V_{22}}$, and $\widehat{V_{2l}}$ is a natural $SL_2(q)$-module for $L_2$.

Put $\widetilde{L_2} := L_2/C_{L_2}(\widehat{V_2})$. By C.14 $\widehat{V_{21}}$ and $\widehat{V_{22}}$ are isomorphic $L_2$-modules and $\widetilde{L_2} \cong SL_2(q)$. Since by (23°)(a) $|V_1R_2/R_2| = q$, this gives $\widetilde{V_1} \in Syl_p(\widetilde{L_2})$. By 5.4(d) there exists $X_2 \in Y_2^{L_2}$ with $[X_2, V_1] = 1$. Since $N_{L_2}(X_2)$ is a maximal parabolic subgroup of $L_2$ containing $V_1$, we conclude that $N_{\widetilde{L_2}}(X_2) = N_{\widetilde{L_2}}(\widetilde{V_1})$. As $\widehat{V_2}$ is the direct sum of isomorphic natural $SL_2(q)$-modules, $C_{\widehat{V_2}}(V_1)$ is a direct sum of simple $N_{L_2}(X_2)$ submodules (of order $q$), and any simple $N_{L_2}(X_2)$-submodule of $C_{\widehat{V_2}}(V_1)$ is contained in a simple $L_2$-submodule of $\widehat{V_2}$. Since $[\widehat{X_2}, V_1] = 1$ and $\widehat{V_2} = \langle \widehat{Y_2}^{L_2} \rangle = \langle \widehat{X_2}^{L_2} \rangle$, this implies that $\widehat{X_2} = C_{\widehat{V_2}}(V_1)$. In particular, either $\widehat{X_2} = \widehat{Y_2}$ or $\widehat{V_2} = \widehat{X_2}\widehat{Y_2}$.

By (VIII) $|\widehat{C_{V_2}(V_1)}| = q^2$. Since also $|C_{\widehat{V_2}}(V_1)| = q^2$, we conclude that $\widehat{C_{V_2}(V_1)} = C_{\widehat{V_2}}(V_1)$. Together with $[Y_2, V_1] \neq 1$ this gives $\widehat{Y_2} \nleqslant C_{\widehat{V_2}}(V_1)$. Thus $\widehat{V_2} = \widehat{X_2}\widehat{Y_2}$ and $V_2 = C_{V_2}(V_1)Y_2$. In particular, since by (2°) $C_{V_2}(V_1) \leqslant V_2 \cap R_1 = A$, $Y_2R_1 = V_2R_1$. By symmetry, also $Y_1R_2 = V_1R_2$ and so

$$\widetilde{Y_{11}}\widetilde{Y_{12}} = \widetilde{Y_1} = \widetilde{V_1} \in Syl_p(\widetilde{L_2}).$$

By (23°)(a), $|\widehat{V_2}/\widehat{A}| = |V_2/A| = q$. Also $|C_{\widehat{V_2}}(Y_1) = |C_{\widehat{V_2}}(V_1)| = q^2$ and therefore $|\widehat{A}/C_{\widehat{V_2}}(Y_1)| = q = |\widehat{V_2}/\widehat{A}|$.

By (IX) $[Y_{2l}, E] \leqslant Y_{2l}$, so $[Y_{2l}, V_1 \cap R_2] \leqslant Y_{2l}$. By symmetry also $[Y_{1l}, V_2 \cap R_1] = [Y_{1l}, A] \leqslant Y_{1l}$. Since $Y_{11} \cap Y_{12} = 1$, we get $[Y_{11}, A] \cap [Y_{12}, A] = 1$. By 5.7(c) $[V_2, V_2 \cap R_1] \cap Z(L_2) = 1$, and since $A = V_2 \cap R_1$, we conclude that $[\widehat{A}, Y_{11}] \cap [\widehat{A}, Y_{12}] = 1$.

Thus we can apply 5.10 with $(\widetilde{L_2}, \widetilde{Y_{11}}, \widetilde{Y_{12}}, \widehat{V_2}, \widehat{A})$ in place of $(H, B_1, B_2, V, A)$. Hence, there exists $h \in L_2$ with $[Y_{11}, \widehat{A}] \leqslant \widehat{A}^h$ and $[Y_{12}, \widehat{A}] \cap \widehat{A}^h = 1$, so $[Y_{11}, A] \leqslant A^h$ and $[Y_{12}, A] \cap A^h \leqslant Y_1 \cap Z(L_2)$. By 5.7(b), $Y_1 \cap Z(L_2) = 1$. Thus

$$[Y_{12}, A] \cap A^h \leqslant Y_1 \cap Z(L_2) = 1.$$

On the other hand, (22°) gives $C_l := C_{[Y_{1l}, A]}(Q_3) \neq 1$. We conclude that $1 \neq C_1 \leqslant A^h$, $C_2 \cap A^h = 1$ and $C_2 \nleqslant A^h$.

Put $U := (Y_1 \cap R_2)^h$. Recall that $V_2 = C_{V_2}(V_1)Y_2$, so

$$A = V_2 \cap R_1 = C_{V_2}(V_1)(Y_2 \cap R_1).$$

Since $[Y_1, A] \neq 1$, this gives $[Y_1, Y_2 \cap R_1] \neq 1$. By symmetry $[Y_2, Y_1 \cap R_2] \neq 1$. By (7°) applied with 1 and 2 interchanged, $C_{Y_2}(x) = Y_2 \cap R_1$ for all $x \in V_1 \cap R_2 \backslash C_{V_1}(Y_2)$ and so $C_{Y_2}(Y_1 \cap R_2) = Y_2 \cap R_1$. Thus

$$C_{V_2}(Y_1 \cap R_2) = C_{C_{V_2}(V_1)Y_2}(Y_1 \cap R_2) = C_{V_2}(V_1)C_{Y_2}(Y_1 \cap R_2) = C_{V_2}(V_1)(Y_2 \cap R_1) = A.$$

Conjugation by $h$ gives $C_{V_2}(U) = A^h$. As $C_1 \leqslant A^h$ and $C_2 \nleqslant A^h$, this shows that $[C_1, U] = 1$ while $[C_2, U] \neq 1$.

By 5.7(a), $[V_2, V_1 \cap R_2] \leqslant Z(L_2)$. Since $C_2 \leqslant V_2$ and $U \leqslant V_1 \cap R_2$, we get $[C_2, U] \leqslant Z(L_2)$. Since $C_1 \leqslant C_G(Q_3)$, $Q!$ implies $U \leqslant C_G(C_1) \leqslant N_G(Q_3)$, and since $C_2 \leqslant C_G(Q_3)$, also $1 \neq [C_2, U] \leqslant C_G(Q_3)$. We conclude, again by $Q!$, that $N_G([C_2, U]) \leqslant N_G(Q_3)$. As seen above, $[C_2, U] \leqslant Z(L_2)$, so

$$Y_1 \leqslant L_2 \leqslant N_G([C_2, U]) \leqslant N_G(Q_3).$$

Since $Y_1 \nleqslant R_2 = O_p(L_2)$ this gives $Y_2 \nleqslant O_p(N_G(Q_3))$, a contradiction since $Y_1$ is short and so also $Q$-short.

This contradiction completes the proof of 5.11. $\square$

5.12. **Proof of Theorem E:**

If $V_1 \leqslant R_2$ or $V_2 \leqslant R_1$, then Theorem E follows from 5.9.

Suppose that $V_1 \nleqslant R_2$ and $V_2 \nleqslant R_1$. Since $V_i = \langle Y_i^{L_i} \rangle$ there exist $h_i \in L_i$ with $Y_1^{h_1} \nleqslant R_2$ and $Y_2^{h_2} \nleqslant R_1$. As also $(Y_1^{h_1}, Y_2^{h_2})$ is a symmetric pair for every $h_1 \in L_1, h_2 \in L_2$, we may assume that $Y_1 \nleqslant R_2$ and $Y_2 \nleqslant R_1$. Now Theorem E follows from 5.11.

CHAPTER 6

# The Tall *char* $p$-Short Asymmetric Case

In this short chapter we will show that $Y_M$ is *char* $p$-tall in $G$ provided that $Y_M$ is tall and asymmetric in $G$ and the centralizers of the non-trivial elements of $Y_M$ are of characteristic $p$. In other words we show that the tall *char* $p$-short asymmetric case does not occur if the centralizers of the non-trivial elements of $Y_M$ are of characteristic $p$.

THEOREM F. *Let $G$ be finite $\mathcal{K}_p$-group, $S \in Syl_p(G)$, and let $Q \leqslant S$ be a large subgroup of $G$. Suppose that $M \in \mathfrak{M}_G(S)$ such that*

(i) *$Y_M$ is tall and asymmetric in $G$.*

(ii) *$C_G(y)$ is of characteristic $p$ for all $1 \neq y \in Y_M$.*

*Then $Y_M$ is char $p$-tall.*

PROOF. By 2.2(f) $O_p(M) \in Syl_p(C_G(Y_M))$. Since $Y_M$ is tall we conclude that there exits a subgroup $L$ of $G$ with $O_p(M) \leqslant L$, $O_p(L) \neq 1$ and $Y_M \nleqslant O_p(L)$. By 2.2(a) $C_G(O_p(M)) \leqslant O_p(M)$. Since $Y_M$ is asymmetric in $G$, 2.6(b) shows that $O_p(M)$ is a weakly closed subgroup of $G$. By 2.2(e) $Y_M = \Omega_1 Z(O_p(M))$ and so by Hypothesis (ii) of Theorem F $C_G(y)$ is of characteristic $p$ for all $1 \neq y \in \Omega_1 Z(O_p(M))$. Thus the hypothesis of 1.50 is fulfilled and we conclude that $L$ is of characteristic $p$. Hence $Y_M$ is *char* $p$-tall. $\square$

We remark that $G = Sym(9)$ and $M = Sym(3) \wr Sym(3)$ provides an example for $p = 3$ where $Y_M$ is tall and asymmetric in $G$, but not *char* $p$-tall. Similar examples occur in $Alt(9)$, $Alt(10)$, $Sym(10)$ and $Alt(11)$.

CHAPTER 7

# The *char* $p$-Tall $Q$-Short Asymmetric Case

In this chapter we treat the *char* $p$-tall $Q$-short asymmetric case. That is, $M \in \mathfrak{M}_G(S)$, $Y_M$ is asymmetric in $G$, and there exists a subgroup $L$ such that

$$(*) \qquad L \text{ has characteristic } p, \quad O_p(M) \leqslant L \quad \text{and} \quad Y_M \nleqslant O_p(L),$$

but $Y_M \leqslant O_p(N_G(Q))$. Here and in the next two chapters the subgroups in $\mathfrak{H}_G(O_p(M))$ introduced in Chapter 2 play a prominent role. These subgroups can be seen as being minimal satisfying $(*)$. But the crucial trick is to choose even smaller subgroups by looking at subgroups $L \leqslant H \in \mathfrak{H}_G(O_p(M))$ such that $L$ is minimal satisfying $Y_M \leqslant L$ and $Y_M \nleqslant O_p(L)$. According to the Asymmetric L-Lemma these subgroups $L$ are in $\mathfrak{L}_G(Y_M)$, see 2.16, so they have a very transparent structure. For example, $O_p(L) = \langle (Y_M \cap O_p(L))^L \rangle$ and

$$L/O_p(L) \cong SL_2(q), Sz(q),\ q := |Y_M O_p(L)/O_p(L)|, \text{ or } D_{2r}.$$

Since $Y_M$ is $Q$-short we have $[\Omega_1 Z(O_p(H)), H] \neq 1$, see 7.1(e), and one can investigate the action of $L$ on quasisimple $H$-submodules $U$ of $\Omega_1 Z(O_p(H))$. By 2.17, $W := [U, L]$ is a strong offender on $Y_M$, so the action of $\langle W^M \rangle$ on $Y_M$ can be investigated via some of the FF-module results from Appendix C.

Here is the main result of this chapter.

Theorem G. *Let $p$ be a prime, $G$ be finite $\mathcal{K}_p$-group, $S \in Syl_p(G)$, and let $Q \leqslant S$ be a large $p$-subgroup of $G$. Suppose that $M \in \mathfrak{M}_G(S)$ such that*

(i) *$Y_M$ is $Q$-short* [1] *and $Q \ntrianglelefteq M$,*
(ii) *$Y_M$ is char $p$-tall and asymmetric in $G$.*

*Then one of the following holds, where $q$ is some power of $p$ and $\overline{M^\dagger} := M^\dagger/C_{M^\dagger}(Y_M)$:*

(1) *$\overline{M^\circ} \cong SL_n(q)$, $n \geqslant 3$, and $Y_M$ is a corresponding natural module.*
(2) *$p = 2$, $\overline{M} \cong O_4^-(2)$, $Sp_4(2)'$ or $Sp_4(2)$, $Y_M$ is a corresponding natural module, $Y_M = O_2(M)$, and $N_G(Q) \leqslant M^\dagger$. Moreover, (in the $O_4^-(2)$-case) for all non-singular $x \in Y_M$, $C_G(x)$ is not of characteristic 2.*
(3) *There exists a unique $\overline{M}$-invariant set $\mathcal{K}$ of subgroups of $\overline{M}$ such that $Y_M$ is a natural $SL_2(q)$-wreath product module for $\overline{M}$ with respect to $\mathcal{K}$. Moreover,*
   (a) *$Y_M = O_p(M)$.*
   (b) *$N_G(Q) \leqslant M^\dagger$.*
   (c) *$\overline{M^\circ} = O^p(\langle \mathcal{K} \rangle)\overline{Q}$.*
   (d) *$Q$ acts transitively on $\mathcal{K}$.*
   (e) *If $|\mathcal{K}| \geqslant 2$ then $q = 2$ or $4$ and, for all $K \in \mathcal{K}$, $C_G(\langle [V, A] \mid A \in \mathcal{K} \backslash \{K\} \rangle)$ is not of characteristic $2$.*

Table 1 lists examples for $Y_M, M$ and $G$ fulfilling the hypothesis of Theorem G.

[1]Note that by 2.6(e) this is equivalent to $Y_M \leqslant O_p(N_G(Q))$.

TABLE 1. Examples for Theorem G

| | Case | $[Y_M, M^\circ]$ for $M^\circ$ | c | Remarks | examples for $G$ |
|---|---|---|---|---|---|
| | 1 | nat $SL_n(q)$ | 1 | $N_G(Q) \leqslant M$ | $L_{n+1}(q)$ |
| | 1 | nat $SL_3(2)$ | 1 | $N_G(Q) \leqslant M$ | $Alt(9)$ |
| | 1 | nat $SL_4(2)$ | 1 | $N_G(Q) \leqslant M$ | $Mat_{24}$ |
| * | 2 | nat $\Omega_4^-(2)$ | 1 | $\overline{M} \cong O_4^-(2)$ | $Alt(10)$ |
| | 2 | nat $Sp_4(2)'$ | 1 | - | $Mat_{22}(.2)$ |
| | 2 | nat $Sp_4(2)$ | 1 | - | $Mat_{22}.2$ |
| | 3 | nat $SL_2(q)$ | 1 | - | $L_3(q)$ |
| | 3 | nat $SL_2(2)$ | 1 | - | $Sp_4(2)'$ |
| | 3 | nat $SL_2(3)$ | 1 | - | $Mat_{12}$ |
| | 3 | nat $\Gamma SL_2(4)$ | 1 | - | $\Gamma L_3(4)$ |
| * | 3 | nat $SL_2(q)$ wreath | 1 | $\lvert\mathcal{K}\rvert > 1$ | $(\Gamma)L_3(q) \wr$ 2-group, $q = 2, 4$ |

Here $c = |Y_M/[Y_M, M^\circ]|$, and $*$ indicates that $(char\ Y_M)$ fails in $G$.

## 7.1. The Proof of Theorem G

Throughout this section we assume the hypothesis of Theorem G and use the notation introduced there. Note that by 2.10 $\mathfrak{H}_G(O_p(M)) \neq \emptyset$ (for the definition of $\mathfrak{H}_G(O_p(M))$ see 2.1).

Choose $H \in \mathfrak{H}_G(O_p(M))$. By definition of $\mathfrak{H}_G(O_p(M))$, $O_p(M) \leqslant H$ and so we can choose $T \in Syl_p(H \cap M^\dagger)$ with $O_p(M) \leqslant T$. By 2.6(b) $O_p(M)$ is a weakly closed subgroup of $G$, and so $T \leqslant N_G(O_p(M)) \leqslant N_G(Y_M)$. Thus there exists $g \in N_G(O_p(M))$ with $T^g \leqslant S$. Since $g$ normalizes $O_p(M)$ and $Y_M$, $H^g \in \mathfrak{H}_G(O_p(M))$, and replacing $H$ by $H^g$ we may assume that $T \leqslant S$.

LEMMA 7.1. (a) *$T \in Syl_p(H)$ and $O_p(H) \leqslant T \leqslant S \leqslant M$.*

(b) *$Y_M \leqslant O_p(M) \leqslant T$ and $Y_H \nleqslant O_p(H)$.*

(c) *$\Omega_1 Z(S) \leqslant \Omega_1 Z(T) \leqslant Y_M \cap Y_H$.*

(d) *$[\Omega_1 Z(S), H] \neq 1$, $[Y_H, H] \neq 1$, $[\Omega_1 Z(O_p(H)), O^p(H)] \neq 1$ and $Y_H = \Omega_1 Z(O_p(H))$.*

(e) *$Y_M \leqslant O_p(C_H(C_{Y_H}(T)))$.*

(f) *$Y_M \cap Y_H = C_{Y_H}(O_p(M)) = C_{Y_M}(O_p(H))$.*

PROOF. (a): By 2.6(c) $H \cap M^\dagger$ is a parabolic subgroup of $H$ and so $T \in Syl_p(H)$. In particular, $O_p(H) \leqslant T$. By the above choice $T \leqslant S \leqslant M$ and so (a) holds.

(b): The first statement is true by choice of $T$ and the second by definition of $\mathfrak{H}_G(O_p(M))$.

(c): By 2.2(a) and (e), $C_G(O_p(M)) \leqslant O_p(M)$ and $\Omega_1 Z(O_p(M)) = Y_M$. Since $O_p(M) \leqslant T \leqslant S$ this gives $\Omega_1 Z(T)\Omega_1 Z(S) \leqslant Y_M$ and

$$\Omega_1 Z(S) = C_{Y_M}(S) \leqslant C_{Y_M}(T) = \Omega_1 Z(T).$$

Thus $\Omega_1 Z(S) \leqslant \Omega_1 Z(T) \leqslant Y_M$, and by 1.24(g), $\Omega_1 Z(T) \leqslant Y_H$, and (c) holds.

(d): Suppose that $[\Omega_1 Z(S), H] = 1$. Then $Q!$ shows that $H \leqslant C_G(\Omega_1 Z(S)) \leqslant N_G(Q)$. But then by 2.12(a) $Y_M$ is $Q$-tall, a contradiction.

Hence $[\Omega_1 Z(S), H] \neq 1$. By (c), $\Omega_1 Z(S) \leqslant Y_H$ and so $[Y_H, H] \neq 1$. Since by 2.11(e) $H$ is $p$-irreducible, 1.35 implies $[\Omega_1 Z(O_p(H)), O^p(H)] \neq 1$ and $Y_H = \Omega_1 Z(O_p(H))$. Hence (d) holds.

(e): By (c) $\Omega_1 Z(S) \leqslant Y_H$ and so $\Omega_1 Z(S) \leqslant C_{Y_H}(T)$. Put $C := C_H(C_{Y_H}(T))$. Then

$$Y_M \leqslant O_p(M) \leqslant C \leqslant C_G(\Omega_1 Z(S)).$$

By $Q!$, $C_G(\Omega_1 Z(S)) \leqslant N_G(Q)$, and by Hypothesis (i) of Theorem G (and its footnote) $Y_M \leqslant O_p(N_G(Q))$. Hence $Y_M \leqslant C \cap O_p(N_G(Q)) \leqslant O_p(C)$, and so (e) holds.

(f): Both groups, $H$ and $M$, are of characteristic $p$, and by (d) and 2.2(e), respectively, $Y_H = \Omega_1 Z(O_p(H))$ and $Y_M = \Omega_1 Z(O_p(M))$. Hence $C_H(O_p(H)) \leqslant O_p(H)$ and so

$$Y_M \cap Y_H \leqslant C_{Y_M}(O_p(H)) \leqslant \Omega_1 Z(O_p(H)) = Y_H,$$

and with a symmetric argument $Y_M \cap Y_H \leqslant C_{Y_H}(O_p(M)) \leqslant Y_M$. Now (f) follows. □

According to 7.1(d) $[\Omega_1 Z(O_p(H)), O^p(H)] \neq 1$. Hence $H$ satisfies the hypothesis of 2.17. In particular, $\mathfrak{L}_H(Y_M) \neq \varnothing$ and there exists a quasisimple $H$-submodule of $Y_M$. We fix the following notation:

NOTATION 7.2. (a) $U$ is a quasisimple $H$-submodule of $Y_H$, $\widehat{U} = U/C_U(O^p(H))$, $\widetilde{H} = H/C_H(U)$ and $\widetilde{q} = |\widetilde{Y_M}|$.
(b) $L \in \mathfrak{L}_H(Y_M)$, $W := [U, L]$, $R := C_{Y_M}(L)$, $A = O_p(L)$ and $l \in L \backslash N_L(Y_M)$.
(c) $K$ is the subnormal closure of $W$ in $M$, $K^* := \langle W^M \rangle = \langle K^M \rangle$ and $Y := [Y_M, K]$.

REMARK 7.3. *Note that we can apply 2.17 with $(H, L, U, W)$ in place of $(H, L, V, W)$. In particular, $W$ is a strong offender on $Y_M$.*

*By definition of $\mathfrak{L}_G(Y_M)$, $L$ is $Y_M$-minimal with $L \cap M^\dagger$ the unique maximal subgroup of $L$ containing $Y_M$. In particular, $O_p(L)Y_M \leqslant L \cap M^\dagger$. So $O_p(L)$ normalizes $Y_M$, and we can apply 1.43 with $Y_M$ in place of $Y$.*

*We will use these two results, 2.17 and 1.43, frequently.*

LEMMA 7.4. (a) *If $[O_p(H), Y_M] \leqslant [W, Y_M]$ then $[O_p(H), O^p(H)Y_M] = [O_p(H), O^p(H)] = U$.*
(b) *$K = \langle W^K \rangle = O^p(K)W$.*
(c) *$W \leqslant Z(A) \leqslant A \leqslant O_p(H)$.*
(d) *$A = L \cap O_p(H)$ and $Y_M \cap A = Y_M \cap O_p(H)$.*
(e) *$C_{Y_M}(A) \leqslant C_{Y_M}(W) = C_A(Y_M) = Y_M \cap A = [y, A]R = C_A(y)$ for every $y \in Y_M \backslash A$.*
(f) *$O_p(H)$ normalizes $K$, $Q$ and any perfect $K$-submodule of $Y_M$.*

PROOF. (a): Suppose that $[O_p(H), Y_M] \leqslant [W, Y_M]$. As $W \leqslant U \trianglelefteq H$ this gives $[O_p(H), Y_M] \leqslant U$. By 2.11(d) $\langle Y_M^H \rangle = O^p(H)Y_M$, and since $U$ is $H$-quasisimple, $U = [U, H] = [U, O^p(H)]$. So

$$U = [U, O^p(H)] \leqslant [O_p(H), O^p(H)Y_M] = [O_p(H), \langle Y_M^H \rangle] \leqslant U,$$

and (a) holds.

(b): Since $K$ is the subnormal closure of $W$, this follows from 1.13.

(c): By 2.17(b) $W \leqslant Z(A)$ and $A \leqslant O_p(H)$.

(d): Note that $A \leqslant L \cap O_p(H) \leqslant O_p(L) = A$ and so $L \cap O_p(H) = A$. Since $Y_M \leqslant L$ we also get $Y_M \cap A = Y_M \cap O_p(H)$.

(e): By 1.43(g) applied with $Y = Y_M$ and $B = A$,

$$Y_M \cap A = C_A(Y_M) = C_A(y) = [A, y]C_{Y_M}(L) = [A, y]R$$

for $y \in Y_M \backslash A$. Since $L$ is $p$-minimal, $L$ is $p$-irreducible. Also $[W, O^p(L)] = W \neq 1$, and 1.34(a) gives $C_{Y_M}(W) = Y_M \cap O_p(L) = Y_M \cap A$. Since by (c) $W \leqslant A$, $C_{Y_M}(A) \leqslant C_{Y_M}(W)$.

(f): Since $W \leqslant U \leqslant Y_H$, $O_p(H)$ centralizes $W$. As $O_p(H) \leqslant T \leqslant S \leqslant M$, we get $O_p(H) \leqslant N_M(W)$. Hence $O_p(H)$ normalizes the subnormal closure $K$ of $W$ in $M$. Since $O_p(H) \leqslant S$, $O_p(H)$ also normalizes $Q$.

Let $X$ be a perfect $K$-submodule of $Y_M$, and let $h \in O_p(H)$. Since $X \leqslant L$, $X$ normalizes $W$ and since $W \leqslant K$, $W$ normalizes $X$. So $[X, W] \leqslant X \cap W \leqslant C_X(h) \leqslant X^h$. Since $h$ normalizes $K$, $K$ normalizes $X^h$. Also $K = \langle W^K \rangle$ and thus $X = [X, K] = [X, \langle W^K \rangle] \leqslant X^h$ and so $X = X^h$. □

LEMMA 7.5. (a) *$R = C_{Y_H}(H)$. In particular, $R \cap U = C_U(H)$.*
(b) *$U \cap Y_M = [W, Y_M](U \cap R) = [U, Y_M](U \cap R)$.*
(c) *$W \cap Y_M = [W, Y_M]$ and $W \cap R = [W, Y_M] \cap R$.*
(d) *$C_U(Y_M) = U \cap O_p(M)$ and $C_U(O_p(M)) = U \cap Y_M$.*
(e) *$C_R(Q^g) = 1$ for all $g \in G$.*
(f) *$C_G(M^\circ) = 1$. In particular, $C_{Y_M}(M^\circ) = 1$.*
(g) *$C_T(U) = C_T(\widehat{U}) = O_p(H)$.*

PROOF. (a): By 7.1(f), $C_{Y_H}(O_p(M)) = Y_M \cap Y_H = C_{Y_M}(O_p(H))$. Since $O_p(M)O_p(H) \leqslant H$ this gives $C_{Y_H}(H) = C_{Y_H \cap Y_M}(H) = C_{Y_M}(H)$.

By 2.17(a), $H = \langle O_p(M), L\rangle$. Recall that $R = C_{Y_M}(L)$ and both $O_p(M)$ and $L$ centralize $R$. Thus $R = C_{Y_M}(H)$, and (a) holds.

(b): Since $B := U \cap A$ is an $L$-invariant subgroup of $A$, 1.43(g) gives

$$B \cap Y_M = [B, Y_M]C_{B\cap Y_M}(L) = [B, Y_M](B \cap R).$$

Note that $U \cap Y_M \leqslant O_p(H) \cap Y_M$ and by 7.4(d), $O_p(H) \cap Y_M = A \cap Y_M$. So $U \cap Y_M = U \cap (A \cap Y_M) = B \cap Y_M$. By 2.17(c) $U = WC_U(Y_M)$. Since $W \leqslant U \cap A = B \leqslant U$ this gives $[W, Y_M] = [B, Y_M] = [U, Y_M]$, and so (b) holds.

(c): Recall from Notation 7.2(b) that $l \in L\backslash N_L(Y_M)$ and so by 1.42(f) $L = \langle Y_M, Y_M^l\rangle$. Since $W = [U, L]$, 1.40 shows $C_W(Y_M) = [W, Y_M]$. Thus $W \cap R = C_W(L) \leqslant C_W(Y_M) = [W, Y_M]$ and so $W \cap R = [W, Y_M] \cap R$.

(d): Note that $U \leqslant O_p(H) \leqslant S$ and by 2.2(f) $C_S(Y_M) = O_p(M)$. Thus $C_U(Y_M) = U \cap O_p(M)$. Also $U \leqslant Y_H$, and by 7.1(f), $C_{Y_H}(O_p(M)) = Y_H \cap Y_M$. So $C_U(O_p(M)) = U \cap Y_M$, and (d) holds.

(e): Assume that there exists $g \in G$ such that $C_R(Q^g) \neq 1$. By (a) $H$ centralizes $R$ and so also $C_R(Q^g)$. Thus by $Q!$, $H \leqslant N_G(Q^g)$, and by 2.12(a) $Y_M$ is $Q$-tall, a contradiction, since $Y_M$ is $Q$-short by Hypothesis (i) of Theorem G.

(f): By Hypothesis (i) of Theorem G $Q \ntrianglelefteq M$. Thus $M^\circ \neq Q$ and 1.55(d) implies $C_G(M^\circ) = 1$.

(g): Since $U$ is quasisimple, $\widehat{U}$ is a non-central simple $H$-module. Thus $[\widehat{U}, O^p(H)] \neq 1$. By 2.11(e) $H$ is $p$-irreducible, and so 1.34(a) gives (g). □

LEMMA 7.6. *Put* $H_0 := \langle Y_M^H\rangle$.

(a) $\widehat{U}$ *is a non-central simple* $H_0$*-module, and* $U$ *is a quasisimple* $H_0$*-module.*
(b) *Put* $\mathbb{K} := End_{H_0}(\widehat{U})$*. Then* $\mathbb{K}$ *is a finite field and* $O_p(M)$ *and* $H$ *act* $\mathbb{K}$*-linearly on* $\widehat{U}$*.*
(c) $C_H(U) = C_H(\widehat{U})$.
(d) *Suppose that* $\widetilde{O_p(M)} \leqslant \widetilde{H_0}$*. Then* $H = H_0C_H(\widehat{U}) = H_0C_H(U) = H_0O_p(H)$ *and* $U \cap R = C_U(O^p(H))$.
(e) $C_{\widetilde{H}}(\widetilde{H_0}) \leqslant \widetilde{H_0}$.

PROOF. By 2.11(c),(d), $H = O^p(H)O_p(M)$ and $Y_MO^p(H) = \langle Y_M^{O^p(H)}\rangle = \langle Y_M^H\rangle = H_0$. So $H_0 = \langle Y_M^{H_0}\rangle$, and since $\widehat{U}$ is a non-central simple $H$-module, $C_{\widehat{U}}(H_0) = 1$.

Note also that $[U, Y_M, O_p(M)] \leqslant [Y_M, O_p(M)] = 1$ and so $[\widehat{U}, Y_M] \leqslant C_{\widehat{U}}(O_p(M))$.

(a) : Let $\widehat{I}$ be a simple $H_0$-submodule of $\widehat{U}$. Since $\widehat{U}$ is a simple $H$-module with $[\widehat{U}, H] \neq 1$ and $H_0 \trianglelefteq H$, also $[\widehat{I}, H_0] \neq 1$, and since $H_0 = \langle Y_M^{H_0}\rangle$, $[\widehat{I}, Y_M] \neq 1$. Hence also $C_{\widehat{I}}(O_p(M)) \neq 1$, and since distinct simple $H_0$-submodules have trivial intersection, $O_p(M)$ normalizes $\widehat{I}$. Thus $\widehat{I}$ is invariant under $H_0O_p(M) = H$, and since $\widehat{U}$ is a simple $H$-module, $\widehat{I} = \widehat{U}$. Since $U$ is a perfect $H$-module and $O^p(H) \leqslant H_0$, $U$ is a perfect $H_0$-module. As $H_0 \trianglelefteq H$ and $U$ is a $p$-reduced $H$-module, $U$ is a $p$-reduced $H_0$-module. Hence $U$ is a quasisimple $H_0$-module, and (a) holds.

(b): Since by (a) $\widehat{U}$ is a simple $H_0$-module, Schur's Lemma shows that $\mathbb{K}$ is a finite division ring and so by Wedderburn's Theorem a field. Since $H$ normalizes $H_0$, $H$ acts $\mathbb{K}$-semilinearly on $\widehat{U}$. Note that $[\widehat{U}, Y_M]$ is a non-trivial $\mathbb{K}$-subspace of $\widehat{U}$ centralized by $O_p(M)$. Thus $O_p(M)$ acts $\mathbb{K}$-linearly on $\widehat{U}$, and since $H = H_0O_p(M)$, also $H$ acts $\mathbb{K}$-linearly on $\widehat{U}$.

(c): By 7.5(g) $C_T(\widehat{U}) = O_p(H) \leqslant C_H(U)$. Also $[U, C_H(\widehat{U})] \leqslant C_U(O^p(H))$ and therefore $[U, O^p(C_H(\widehat{U}))] = 1$. Thus $C_H(\widehat{U}) = O^p(C_H(\widehat{U}))C_T(\widehat{U}) \leqslant C_H(U) \leqslant C_H(\widehat{U})$.

(d): Suppose that $O_p(M) \leqslant H_0C_H(\widehat{U})$. Then $H = H_0O_p(M) = H_0C_H(\widehat{U})$, and by (c) also $H = H_0C_H(U)$. Hence $O_p(M) \leqslant T \leqslant (T \cap H_0)C_T(\widehat{U})$. By 7.5(g) $C_T(\widehat{U}) = O_p(H)$ and so $O_p(M) \leqslant (T \cap H_0)O_p(H)$. This shows that $H = H_0O_p(M) = H_0O_p(H)$, and the first part of (d) is proved. By 2.17(d), $C_U(O^p(H)) = C_U(H_0)$. Since $H = H_0C_H(U)$, $C_U(H) = C_U(H_0)$ and by 7.5(a), $C_U(H) = U \cap R$. Thus $C_H(O^p(H)) = U \cap R$.

(e): By (c) $\widetilde{H} = H/C_H(\widehat{U})$. Since $\widehat{U}$ is a simple $H_0$-module we conclude that $C_{\widetilde{H}}(\widetilde{H_0})$ is $p'$-group. As $H = H_0O_p(M)$, $H/H_0$ is a $p$-group and so $C_{\widetilde{H}}(\widetilde{H_0}) \leqslant \widetilde{H_0}$. □

LEMMA 7.7. (a) $C_{Y_M}(K) \cap C_{Y_M}(Q) = 1$.
(b) $C_{O_p(M)}(\langle K^Q \rangle) = 1$.

PROOF. (a): Suppose for a contradiction that $C_{Y_M}(K) \cap C_{Y_M}(Q) \neq 1$. Then by A.54(c) $\overline{K} \leqslant N_{\overline{M}}(\overline{Q})$, and by A.54(e) $\overline{K}$ acts faithfully on $X := C_{Y_M}(Q)$. In particular, $[X, K] \neq 1$ and since $K = \langle W^K \rangle$, also $[X, W] \neq 1$.

Suppose first that $|X/C_X(W)| > 2$. Then 2.17(f) shows that $[W, Y_M] = [W, X] \leqslant X$. Using $K = \langle W^K \rangle$ this gives $[K, Y_M] \leqslant X = C_{Y_M}(Q)$ and $[Y_M, K, Q] = 1$. Since $\overline{K} \neq 1$, this contradicts A.54(d).

Hence $|X/C_X(W)| = 2$. By 7.4(c), (f) $A \leqslant O_p(H)$, and $O_p(H)$ normalizes $Q$ and $K$. In particular, $A$ normalizes $Q$ and $K$, and so

$$\text{(I)} \qquad [K, A] \leqslant K \quad \text{and} \quad [X, A] \leqslant X.$$

Choose $y \in X \backslash C_X(W)$. By 7.4(c), $W \leqslant Z(A)$. So $y \notin A$, and 7.4(e) gives $Y_M \cap A = C_{Y_M}(W) = [y, A]R$. Also by 7.5(e), $R \cap X \leqslant C_R(Q) = 1$. Note that $C_X(K) = C_{Y_M}(K) \cap C_{Y_M}(Q) \neq 1$, and by (I) $[y, A] \leqslant [X, A] \leqslant X$, so

$$1 \neq C_X(K) \leqslant C_{Y_M}(W) \cap X \leqslant [y, A]R \cap X = [y, A](R \cap X) = [y, A].$$

By 1.43(a)

$$A' = [Y_M \cap A, A] \leqslant C_{Y_M}(L) = R.$$

On the other hand $[Y_M \cap A, A] \leqslant [XR, A] = [X, A] \leqslant X$ and so $A' = [Y_M \cap A, A] \leqslant R \cap X = 1$. Thus $A$ is abelian and so $[y, A] = \{[y, a] \mid a \in A\}$. As $1 \neq C_X(K) \leqslant [y, A]$ we can choose $a \in A$ with $1 \neq [y, a] \in C_X(K)$. From $C_X(W) \leqslant C_{Y_M}(W) = Y_M \cap A$ we also get $[C_X(W), A] = 1$. Since $|X/C_X(W)| = 2$, $X = \langle y \rangle C_X(W)$, and it follows that $[X, a] = \langle [y, a] \rangle \leqslant C_X(K)$ and $C_X(a) = C_X(W)$. By (I) $[K, A] \leqslant K$, and so $[K, a]$ centralizes the factors of the $K$-invariant series $1 \leqslant C_X(K) \leqslant X$. As $X$ is a faithful $\overline{K}$-module we get $[\overline{K}, \overline{a}] \leqslant O_p(\overline{K}) \leqslant O_p(\overline{M}) = 1$. The Three Subgroups Lemma now shows that $[X, K, a] = 1$ and $[X, K] \leqslant C_X(a) = C_X(W)$. But then $[X, K, W] = 1$, and since $K = \langle W^K \rangle$, $[X, K, K] = 1$, a contradiction since $\overline{K}$ is not a $p$-group and acts faithfully on $X$. This completes the proof of (a).

(b): Put $K_0 := O^p(\langle K^Q \rangle)$ and $C := C_{O_p(M)}(K_0)$. Since $K$ is subnormal in $M$, $O_p(M)$ normalizes $O^p(K)$ and thus also $K_0$ and $C$; in particular $C \trianglelefteq O_p(M)$. Assume that $C \neq 1$. Then $C \cap Z(O_p(M)) \neq 1$, and since $\Omega_1 Z(O_p(M)) = Y_M$, also $C \cap Y_M \neq 1$. On the other hand, $\langle Q, K \rangle / K_0$ is a $p$-group, and so $C \cap Y_M \neq 1$ implies $C_{C \cap Y_M}(\langle Q, K \rangle) \neq 1$. This contradicts (a). Hence $C = 1$, and (b) holds. □

LEMMA 7.8. *Let $1 \neq X \leqslant R$ and suppose that*

$$O_p(C_{\overline{M}}(X)) = 1 \quad \text{or} \quad [C_{Y_M}(O_p(C_{\overline{M}}(X))), W] \neq 1.$$

*Then $C_G(X)$ is not of characteristic $p$.*

PROOF. Note that $O_p(C_{\overline{M}}(X)) = 1$ implies $Y_M = C_{Y_M}(O_p(C_{\overline{M}}(X)))$. Thus, also in this case

$$(*) \qquad [C_{Y_M}(O_p(C_{\overline{M}}(X))), W] \neq 1.$$

Put $P := O_p(C_G(X))$. Since $R \leqslant Y_M$, $X \leqslant Y_M$ and $O_p(M) \leqslant C_G(X)$. Hence 2.6(c) shows that $M^\dagger \cap C_G(X)$ is a parabolic subgroup of $C_G(X)$, and so $P \leqslant M^\dagger$. Thus $\overline{P} \leqslant O_p(C_{\overline{M^\dagger}}(X))$. As $M^\dagger = MC_G(Y_M)$, $\overline{M} = \overline{M^\dagger}$ and so $\overline{P} \leqslant O_p(C_{\overline{M}}(X))$. Hence $C_{Y_M}(O_p(C_{\overline{M}}(X))) \leqslant C_{Y_M}(P)$. Now $(*)$ implies $[C_{Y_M}(P), W] \neq 1$. By 7.4(e) $C_{Y_M}(W) = Y_M \cap A$, and so $C_{Y_M}(P) \nleqslant A = O_p(L)$.

As $X \leqslant R = C_{Y_M}(L)$, $L \leqslant C_G(X)$, and since $C_{Y_M}(P) \nleqslant O_p(L)$, $C_{Y_M}(P) \nleqslant O_p(C_G(X)) = P$. Thus $C_G(X)$ is not of characteristic $p$. □

LEMMA 7.9. *Suppose that $N_G(Q) \leqslant N_G(Y_M)$. Then there exists $t \in A \backslash C_A(Y_M)$ such that $[C_D(t), L] \leqslant A$ for all $p$-subgroups $D$ of $M$ with $[Y_M, D] \leqslant A$.*

PROOF. By 7.5(e), $\Omega_1 Z(S) \leqslant Y_M \cap Y_H \leqslant Y_M \cap O_p(H)$, and by 7.4(d), $Y_M \cap O_p(H) = Y_M \cap A$, so $\Omega_1 Z(S) \leqslant C_{Y_M \cap A}(Q) \neq 1$. Let $l \in L \backslash N_L(Y_M)$ and choose $1 \neq t \in C_{Y_M \cap A}(Q)^l$. By 7.5(e) $C_R(Q^l) = 1$, so $t \notin R$. Since $L = \langle Y_M^l, Y_M \rangle$, this gives $[t, Y_M] \neq 1$. By $Q!$, $C_G(t) \leqslant N_G(Q^l)$, and as $N_G(Q^l) \leqslant N_G(Y_M^l)$, $C_G(t)$ normalizes $Y_M^l$. Since $D$ normalizes $Y_M$, $C_D(t) \leqslant N_G(\langle Y_M, Y_M^l \rangle) = N_G(L)$. In particular, $C_D(t)$ acts on $Y_M^l A/A$, and since $C_D(t)$ is a $p$-group, we can choose $h \in Y_M^l \backslash A$ with $[h, C_D(t)] \leqslant A$. By 1.43(k) $N_L(Y_M) \cap Y_M^l \leqslant A$. So $h \notin N_L(Y_M)$ and $L = \langle h, Y_M \rangle$. Since $[Y_M, C_D(t)] \leqslant Y_M \cap A \leqslant A$ and $[h, C_D(t)] \leqslant A$ this gives $[L, C_D(t)] \leqslant A$. □

LEMMA 7.10. *Suppose that $Y_M$ is an offender on $W$. Then $\widetilde{Y_M} \in Syl_p(\widetilde{L})$, and both, $W/W \cap R$ and $\widehat{W}$, are natural $SL_2(\widetilde{q})$-modules for $\widetilde{L}$.*

PROOF. By 1.43(h) $C_{W/W \cap R}(L) = 1$. Also $W = [W, O^p(L)] \nleqslant W \cap R$ and $[W, O_p(L)] = 1$. Hence 1.34(b) shows that $W$ and $W/W \cap R$ are $p$-reduced for $L$ and $C_{Y_M}(W/W \cap R) = Y_M \cap A = C_{Y_M}(W)$. So $Y_M$ is an offender on $W/W \cap R$. Since $L$ is $Y_M$-minimal, C.13 shows that $W/W \cap R$ is a natural $SL_2(\widetilde{q})$-module for $L/A$ and $Y_M A/A \in Syl_p(L/A)$. By 7.4(d) $A = L \cap O_p(H)$, so $\widetilde{L} = LO_p(H)/O_p(H) \cong L/A$ and $\widetilde{Y_M} \in Syl_p(\widetilde{L})$.

By 2.17(e) $W \cap R = W \cap C_{Y_M}(L) = C_W(O^p(H))$. Hence $\widehat{W} \cong W/W \cap R$ and so also $\widehat{W}$ is a natural $SL_2(\widetilde{q})$-module for $\widetilde{L}$, and the lemma is proved. □

LEMMA 7.11. *Suppose that $Y_M$ is an offender on $W$. Then one of the following holds:*

(1) *$\widehat{U}$ is natural $SL_2(\widetilde{q})$-module for $H$, $Y_M = O_p(M) = C_G(Y_M)$, $M = M^\dagger$, $N_G(Q) \leqslant M$, $H = L$ and $U = W$.*
(2) *$U$ is a natural $SL_m(\widetilde{q})$-module for $H$, $m \geqslant 3$, $U \cap R = 1$ and $\widetilde{Y_M} = Z(\widetilde{T})$ is a transvection group on $U$.*

PROOF. Since $Y_M$ is an offender on $W$, 7.10 shows $\widetilde{Y_M} \in Syl_p(\widetilde{L})$ and $\widehat{W}$ is a natural $SL_2(\widetilde{q})$-module for $\widetilde{L}$. It follows that

$$C_{\widehat{W}}(y) = C_{\widehat{W}}(Y_M) \qquad \text{and} \qquad [\widehat{W}, y] = [\widehat{W}, Y_M]$$

for all $y \in Y_M \backslash C_{Y_M}(W)$. Also $|\widetilde{Y_M}| = |\widetilde{q}| = |\widehat{W}/C_{\widehat{W}}(Y_M)|$ and so $Y_M$ is a root offender[2] on $\widehat{W}$. By 2.17(c), $U = WC_U(Y_M)$. Hence $\widehat{U} = \widehat{W} C_{\widehat{U}}(Y_M)$. It follows that $Y_M$ is a root offender on $\widehat{U}$. By A.37(b) any root offender is a strong dual offender. Thus

1°. *$Y_M$ is a strong dual offender and a root offender on $\widehat{U}$.*

Put $H_0 := \langle Y_M^H \rangle$ and $\mathbb{K} := End_{H_0}(\widehat{U})$. By 7.6(a) $\widehat{U}$ is a non-central simple $H_0$-module. Hence we can apply the Strong Dual FF-Module Theorem C.5, and get:

2°. *One of the following cases holds:*

(A) *$\widetilde{H_0} \cong Alt(7)$, $p = 2$, and $\widehat{U}$ is a spin module of order $2^4$ for $\widetilde{H_0}$.*
(B) *$\widetilde{H_0} \cong O_{2m}^\epsilon(2)$, $m \geqslant 2$ and $p = 2$, $|\widetilde{Y_M}| = 2$, and $\widehat{U}$ is a natural $O_{2m}^\epsilon(2)$-module for $\widetilde{H_0}$.*
(C) *$\widetilde{H_0} \cong SL_m(q_1)$, $m \geqslant 3$, and $\widehat{U}$ is a natural $SL_m(q_1)$-module for $\widetilde{H_0}$.*
(D) *$\widetilde{H_0} \cong Sp_{2m}(q_1)$, $m \geqslant 1$, or $Sp_4(2)'$ (and $p = 2$), and $\widehat{U}$ is a corresponding natural module for $\widetilde{H_0}$.*
(E) *$\widetilde{H_0} \cong Sym(m)$, $m \geqslant 5$, $m \neq 6$ and $p = 2$, and $\widehat{U}$ is a natural $Sym(m)$-module for $\widetilde{H_0}$.*

[2] For the definitions of a root offender and a strong dual offender see A.7(5),(6)

Note here that $Alt(6) \cong Sp_4(2)'$ and a natural $Alt(6)$-module is also a natural $Sp_4(2)'$-module. Similarly, $SL_2(q_1) \cong Sp_2(q_1)$, and a natural $SL_2(q_1)$-module is also a natural $Sp_2(q_1)$-module. So these two cases are included in Case (D).

Suppose that Case (A) holds. Then $\widetilde{H_0} \cong Alt(7)$ and $|\widehat{U}| = 2^4$. Since $Alt(7)$ is a maximal subgroup of $Alt(8) \cong GL_4(2)$ and $H_0 \trianglelefteq H$, we conclude that $\widetilde{H} = \widetilde{H_0} \cong Alt(7)$. It follows that there exists a proper subgroup $P$ of $H$ with $O_2(M) \leqslant P$ and $\widetilde{P} \cong Alt(6)$. Note that $O_2(\widetilde{P}) = 1$ and so $\widetilde{Y_M} \nleqslant O_2(\widetilde{P})$. Hence also $Y_M \nleqslant O_2(P)$. Since $H \in \mathfrak{H}_G(O_2(M))$ this contradicts the definition of $\mathfrak{H}_G(O_2(M))$.

Suppose that Case (B) holds. Then $\widetilde{U}$ is a natural $O^{\epsilon}_{2m}(2)$-module for $H_0$ and $|\widetilde{Y_M}| = 2$. Since $Y_M \trianglelefteq T$ and $[\widehat{U}, Y_M] = 2$ we conclude that $[\widehat{U}, Y_M] \leqslant C_{\widehat{U}}(T)$, a contradiction since $[\widehat{U}, Y_M]$ is non-singular and $C_{\widehat{U}}(T)$ is singular by B.9(c) and B.23(g) respectively.

Suppose that Case (C) holds. Then $\widetilde{U}$ is a natural $SL_m(q_1)$-module for $H_0$, $m \geqslant 3$. Recall that $\mathbb{K} = End_{H_0}(\widehat{U})$. Hence $\mathbb{K}$ is a finite field of order $q_1$, and by 7.6(b), $O_p(M)$ acts $\mathbb{K}$-linearly on $\widehat{U}$. Since $GL_m(q_1)/SL_m(q_1)$ is a $p'$-group this gives $\widetilde{O_p(M)} \leqslant \widetilde{H_0}$. So 7.6(d) implies

$$H = H_0 C_H(\widehat{U}) = H_0 C_H(U) = H_0 O_p(H).$$

Since $Y_M \trianglelefteq T$ we can choose $y \in Y_M \backslash C_{Y_M}(U)$ with $\widetilde{y} \in Z(\widetilde{T})$. Note that $Z(\widetilde{T})$ is a transvection group. So $[\widehat{U}, y]$ and $\widehat{U}/C_{\widehat{U}}(y)$ are 1-dimensional over $\mathbb{K}$ and

$$Z(\widetilde{T}) = C_{\widetilde{H}}([\widetilde{U}, y]) \cap C_{\widetilde{H}}(C_{\widehat{U}}(y)).$$

By (1°) $Y_M$ is a root offender on $\widehat{U}$. Thus $[\widehat{U}, Y_M] = [\widehat{U}, y]$ and $C_{\widehat{U}}(Y_M) = C_{\widehat{U}}(y)$. It follows that $\widetilde{Y_M} \leqslant Z(\widetilde{T})$, and since $Y_M$ is an offender on $\widehat{U}$, $\widetilde{Y_M} = Z(\widetilde{T})$, $\widetilde{Y_M}$ is a transvection group, and $q_1 = |Z(\widetilde{T})| = |\widetilde{Y_M}| = \widetilde{q}$.

Suppose that $C_U(H) \neq 1$. Since $U = [U, H]$ and $Y_M$ is a offender on $W$ and so on $U$, C.22 shows that $\widehat{U}$ is a natural $SL_3(2)$-module for $H$ and $|\widetilde{Y_M}| = 4$, a contradiction to $2 = q_1 = \widetilde{q} = |\widetilde{Y_M}|$. Thus $C_U(H) = 1$, $U$ is a natural $SL_n(\widetilde{q})$-module and $U \cap R = 1$. So (2) holds in this case.

For the remainder of the proof we can assume now that Case (D) or (E) holds. We show next:

3°. *$H = H_0 O_p(H) = HC_H(\widehat{U}) = HC_H(U)$, and one of the following holds:*

(i) *$\widehat{U}$ is a natural $Sp_{2m}(\widetilde{q})$-module for $H$, $m \geqslant 1$, and $\widetilde{Y_M}$ acts as a transvection group on $\widehat{U}$.*

(ii) *$p = 2$, $\widehat{U}$ is a natural $Sym(m)$ module for $H$, $m \geqslant 5$ and $m \neq 6$, and $\widetilde{Y_M}$ is generated by a transposition of $\widetilde{H}$.*

Suppose that Case (D) holds, so $\widehat{U}$ is a natural $Sp_{2m}(q_1)$-module, $m \geqslant 1$, or $Sp_4(2)'$-module for $H_0$. By 7.6(b), $H$ acts $\mathbb{K}$-linearly on $\widehat{U}$. Note hat $\mathbb{K}$ is a field of order $q_1$ and the set of $H_0$-invariant symplectic forms on $\widehat{U}$ form 1-dimensional $\mathbb{K}$-space. Since $O_p(M)$ acts $\mathbb{K}$-linearly on $\widehat{U}$ and is a $p$-group, we conclude that $O_p(M)$ acts trivially on this $\mathbb{K}$-space. So any $H_0$-invariant non-degenerate symplectic form on $\widehat{U}$ is also $O_p(M)$-invariant.

Let $X = C_{\widehat{U}}(T)$ and $P = C_H(X)$. Note that $X$ is a 1-dimensional singular $\mathbb{K}$-subspace of $\widehat{U}$ and $[X^{\perp}, O_p(P)] \leqslant X$, cf. B.23(g) and B.28(b:b). Since $O_p(M) \leqslant T \leqslant P < H$ and $H \in \mathfrak{H}_G(O_p(M))$ we have $Y_M \leqslant O_p(P)$. Suppose that $[X^{\perp}, Y_M] \neq 1$. By (1°) $Y_M$ is a strong dual offender on $\widehat{U}$ and so $[\widehat{U}, Y_M] = [X^{\perp}, Y_M] = X$. But then $C_{\widehat{U}}(Y_M) = [\widehat{U}, Y_M]^{\perp} = X^{\perp}$ contrary to $[X^{\perp}, Y_M] \neq 1$. Thus $[X^{\perp}, Y_M] = 1$. Hence

$$q_1 = |\widehat{U}/X^{\perp}| = |U/C_U(Y_M)| \leqslant |\widetilde{Y_M}| \leqslant |C_{\widetilde{H_0}}(X^{\perp})| \leqslant q_1.$$

Thus $\widetilde{Y_M}$ is a transvection group on $\widehat{U}$, and $q_1 = |\widetilde{Y_M}| = \widetilde{q}$. Moreover, since $Sp_4(2)'$ does not contain a transvection, $\widetilde{H_0} \cong Sp_{2n}(\widetilde{q})$. As $O_p(M)$ fixes the $\widetilde{H_0}$-invariant symplectic forms we get $\widetilde{O_p(M)}) \leqslant \widetilde{H_0}$. Now 7.6(d) shows that the first statement of (3°) holds. In particular, $\widetilde{H} = \widetilde{H_0}$ and (3°)(i) holds.

Suppose that Case (E) holds, so $\widehat{U}$ is a natural $Sym(m)$ module for $H_0$, $m \geqslant 5$ and $m \neq 6$, and $|\widetilde{Y_M}| = 2$. Since $|\widetilde{Y_M}| = 2$ and $Y_M$ is an offender, $\widetilde{Y_M}$ is generated by a transposition. Note that $Out(Sym(m)) = 1$ since $m \neq 6$. Hence $O_p(M)$ induces inner automorphisms of $\widetilde{H_0}$. By 7.6(e), $C_{\widetilde{H}}(\widetilde{H_0}) \leqslant \widetilde{H_0}$ and thus $\widetilde{O_p(M)} \leqslant \widetilde{H_0}$. Now 7.6(d) shows that the first statement of (3°) holds. Thus $\widetilde{H} = \widetilde{H_0}$, and (3°)(ii) follows. This completes the proof of (3°).

4°. $\quad$ *$U \cap R = C_U(O^p(H))$. In particular, $\widehat{U} = U/U \cap R$.*

By 7.5(a) $U \cap R = C_U(H)$, by (3°) $H = H_0 C_H(U)$, and by 2.17(d) $C_U(H_0) = C_H(O^p(H))$. Hence $U \cap R = C_U(H_0) = C_U(O^p(H))$.

Let $Z_2$ be maximal in $U \cap O_p(M)$ with $[Z_2, O_p(M)] \leqslant U \cap Y_M$ and put $E := [Z_2, O_p(M)]$.

5°. $\quad$ *$\widehat{Z_2}$ and $\widehat{E}$ are $\mathbb{K}$-subspaces of $\widehat{U}$, $\widehat{E} \leqslant [\widehat{U}, Y_M]$, $[\widehat{U}, Y_M]$ is 1-dimensional, and $\widehat{E}$ is at most 1-dimensional over $\mathbb{K}$.*

By 7.5(b), $U \cap Y_M = [U, Y_M](U \cap R)$. Since by (4°) $\widehat{U} = U/U \cap R$, it follows that $\widehat{U \cap Y_M} = [\widehat{U}, Y_M]$ is a $\mathbb{K}$-subspace of $\widehat{U}$, and as $U \cap R \leqslant U \cap Y_M$, $\widehat{Z_2}$ is maximal in $\widehat{U}$ with $[\widehat{Z_2}, O_p(M)] \leqslant \widehat{U \cap Y_M}$. By 7.6(b), $O_p(M)$ acts $\mathbb{K}$-linearly on $\widehat{U}$ and so $\widehat{Z_2}$ is a $\mathbb{K}$-subspace of $\widehat{U}$. Hence also $\widehat{E} = [\widehat{Z_2}, O_p(M)]$ is a $\mathbb{K}$-subspace of $\widehat{U}$. By definition of $E$ and $Z_2$, $\widehat{E} = [\widehat{Z_2}, O_p(M)] \leqslant \widehat{U \cap Y_M} = [\widehat{U}, Y_M]$.

Since by (3°) $\widetilde{Y_M}$ is a transvection group (in the $Sp_{2n}(\widetilde{q})$-case) or generated by a transposition (in the $Sym(m)$-case), $[\widehat{U}, Y_M]$ is 1-dimensional over $\mathbb{K}$.

6°. $\quad$ *$E \cap R = 1$.*

If $U \cap R = 1$ then also $E \cap R = 1$. So we may assume that $U \cap R \neq 1$. Suppose first that Case (3°)(i) holds, that is, $\widehat{U}$ is a natural $Sp_{2m}(\widetilde{q})$-module. Note that $C_U(H) = U \cap R \neq 1$ and $U = [U, H]$. Thus C.22 shows that $p = 2$, and $U$ is a central quotient of a natural $\Omega_{2m+1}(\widetilde{q})$-module $\check{U}$ for $H$. For $X \subseteq U$, let $\check{X}$ be the inverse image of $X$ in $\check{U}$. Since $Z_2 \leqslant U \cap O_2(M) = C_U(Y_M)$,

$$\widehat{Z_2} \leqslant \widehat{U \cap O_2(M)} \leqslant C_{\widehat{U}}(Y_M) = [\widehat{U}, Y_M]^{\perp}.$$

As $[\widehat{Z_2}, O_2(M)] \leqslant [\widehat{U}, Y_M]$ this gives $[\check{Z_2}, O_2(M)] \leqslant \check{Z_2}^{\perp}$. Hence by B.9(d) $[\check{Z_2}, O_2(M)]$ is a singular subspace in $\check{U}$. Since all the non-trivial vectors in $\check{U}^{\perp}$ are non-singular, this gives $[\check{Z_2}, O_p(M)] \cap \check{U}^{\perp} = 1$. Taking images in $U$ gives $E \cap R = E \cap (U \cap R) = 1$.

Suppose next that Case (3°)(ii) holds, that is, $\widehat{U}$ is a natural $Sym(m)$-module. Since $U \cap R \neq 1$, C.22 shows that $m$ is even and $U$ is the even permutation module for $Sym(m)$. Let $\check{U}$ be the permutation module for $H$ with $H$-invariant basis $v_1, \dots, v_m$. Identify $U$ with $[\check{U}, H] = \langle v_i + v_j \mid 1 \leqslant i < j \leqslant m\rangle$ such that $\widetilde{Y_M}$ acts as $\langle (1,2)\rangle$. Put $P := N_H(\widetilde{Y_M})$. Then $\widetilde{P} \cong C_2 \times Sym(m-2)$,

$$U \cap O_2(M) = C_U(Y_M) = \langle v_1 + v_2, v_i + v_j \mid 3 \leqslant i < j \leqslant m\rangle$$

and

$$[U \cap O_2(M), P] \leqslant \langle v_i + v_j \mid 3 \leqslant i < j \leqslant m\rangle.$$

Thus $[U \cap O_2(M), P] \cap R = 1$. Since $Z_2 \leqslant U \cap O_2(M)$ and $O_2(M) \leqslant P$ we have $E = [Z_2, O_2(M)] \leqslant [U \cap O_2(M), P]$ and so $E \cap R = 1$. Thus (6°) is proved.

7°. $\quad$ *$E = 1$.*

Suppose that $E \neq 1$. Note that $E \leqslant Y_M$, $W \leqslant U$, $\overline{K^*} = \langle \overline{W}^{\overline{M}} \rangle = \langle K^M \rangle$, and by 7.7(b) $C_{O_p(M)}(\langle K^Q \rangle) = 1$. Hence $C_{Y_M}(\overline{K^*}) = 1$. It follows that $[E, U^g] \neq 1$ for some $g \in M$. By definition of $Z_2$ and $E$, $[Z_2, U^g \cap O_p(M)] \leqslant [Z_2, O_p(M)] = E$. On the other hand $Z_2 \leqslant U \cap O_p(M) \leqslant O_p(M) = O_p(M)^g \leqslant H^g$ and so $Z_2$ normalizes $U^g$. Since $U^g$ is abelian we have

$$[U^g \cap O_p(M), Z_2] \leqslant U^g \cap E \leqslant C_E(U^g) < E.$$

As by (6°) $R \cap E = 1$, this gives
$$[U^g \cap O_p(M), \widehat{Z_2}] < \hat{E}.$$

Since $U^g \cap O_p(M)$ acts $\mathbb{K}$-linearly on $\hat{U}$ and $\widehat{Z_2}$ is a $\mathbb{K}$-subspace of $\hat{U}$, also $[U^g \cap O_p(M), \widehat{Z_2}]$ is a $\mathbb{K}$-subspace of $\hat{U}$. As by (5°) $\hat{E}$ is at most 1-dimensional over $\mathbb{K}$, this gives $[U^g \cap O_p(M), \widehat{Z_2}] = 1$ and $[U^g \cap O_p(M), Z_2] \leqslant R \cap E = 1$.

We now shift attention to $H^g$ and the $H^g$-modules $U^g$ and $\widehat{U^g} := U^g/C_{U^g}(O^p(H^g))$. Observe that $O_p(M) \leqslant H^g$ since $g \in M$. From $[U^g, O_p(M)] \leqslant U^g \cap O_p(M)$ we conclude that $[U^g, O_p(M), Z_2] = 1$ and so also $[\widehat{U^g}, O_p(M), Z_2] = 1$. Since $\widehat{U^g}$ is selfdual as an $H^g$ module, B.8 shows $[\widehat{U^g}, Z_2, O_p(M)] = 1$, and the Three Subgroup Lemma gives $[E, \widehat{U^g}] = [Z_2, O_p(M), \widehat{U^g}] = 1$. By 7.5(g) $C_T(U) = C_T(\hat{U})$, and thus also $C_{T^g}(U^g) = C_{T^g}(\widehat{U^g})$, so $[E, U^g] = 1$. This contradicts the choice of $g$. Hence (7°) holds.

8°. $U \cap O_p(M) = U \cap Y_M$.

By (7°), $[Z_2, O_p(M)] = E = 1$. By the definition of $Z_2$ this means
$$C_{U \cap O_p(M)/U \cap Y_M}(O_p(M)) = Z_2/U \cap Y_M = 1,$$
and so $U \cap O_p(M) = U \cap Y_M$.

We are now able to prove the Lemma. From (8°) we have, $[U, O_p(M)] \leqslant U \cap Y_M \leqslant Y_M$, and since $W \leqslant U$ and $K = \langle W^K \rangle$, $[K, O_p(M)] \leqslant Y_M = \Omega_1 Z(O_p(M))$. Thus, 1.18 gives $[\Phi(O_p(M)), K] = 1$ and so also $[\Phi(O_p(M)), \langle K^Q \rangle] = 1$. By 7.7(b), $C_{O_p(M)}(\langle K^Q \rangle) = 1$ and so $\Phi(O_p(H)) = 1$. It follows that $O_p(M)$ is elementary abelian. Hence $O_p(M) = \Omega_1 Z(O_p(M)) = Y_M$. Since $M \in \mathcal{L}_G(S)$ we have $Y_M \leqslant C_G(O_p(M)) \leqslant O_p(M)$, and so $C_G(Y_M) = Y_M$ and $M^\dagger = MC_G(Y_M) = MY_M = M$. Since $Y_M$ is $Q$-short, $O_p(M) = Y_M \leqslant Q$, and since by 2.6(b) $O_p(M)$ is a weakly closed subgroup of $G$, $N_G(Q) \leqslant N_G(O_p(M))) \leqslant M^\dagger = M$.

Also by 2.17(a) $H = \langle L, O_p(M) \rangle = \langle L, Y_M \rangle = L$ and so $U = W$. By 7.10 $\widehat{W}$ is a natural $SL_2(\tilde{q})$-module for $L$, and so Case (1) of the lemma holds. □

Lemma 7.12. *Suppose that there exists a non-degenerate $\overline{K^*S}$-invariant symplectic form on $V := [Y_M, \overline{K^*}]$. Put $H_0 := \langle Y_M^H \rangle$.*

(a) *$Y_M = VC_{Y_M}(W)$ and $C_{Y_M}(W) = Y_M \cap A$; in particular $[V, W] = [Y_M, W]$ and $C_W(V) = C_W(Y_M)$.*
(b) *$W$ is a root offender on $V$ and $Y_M$.*
(c) *$V$ and $Y_M$ are root offenders on $W$.*
(d) *$|\overline{W}| = |W/C_W(V)| = |[V, W]| = |V/C_V(W)|$.*
(e) *$A = W \times R$, $C_{Y_M}(W) = [V, W] \times R$, $[Y_M, O_p(H)] = [V, W]$, $[O_p(H), O^p(H)] = U$, and $W$ is a natural $SL_2(\tilde{q})$-module for $\widetilde{L}$.*
(f) *$C_V(W) = [V, W]^\perp = [V, W] \times (V \cap R)$, and $[V, W]$ is a singular subspace of $V$.*
(g) *$|V| = |\overline{W}|^2 |V \cap R|$.*
(h) *$C_G(Y_M) = O_p(M) = Y_M = VR$, $M = M^\dagger$ and $N_G(Q) \leqslant M$.*
(i) *$H = L$ and $O_p(H) = Y_H = A = W \times R = C_L(W)$.*

Proof. Since $V$ carries a $\overline{K^*S}$-invariant non-degenerate symplectic form, $V$ is selfdual as an $\mathbb{F}_p\overline{K^*S}$-module. By 2.17(c), $W$ is a strong offender on $Y_M$ and so $W$ is also a strong offender on the submodule $V$ of $Y_M$. Since $V$ is selfdual, A.38 shows that $W$ is a root offender on $V$. In particular, by A.37,

$$\text{(I)} \qquad |[W, V]| = |V/C_V(W)| = |W/C_W(V)|.$$

(a): Since $W$ is an offender on $Y_M$, $|Y_M/C_{Y_M}(W)| \leqslant |\overline{W}|$, and (I) yields
$$|Y_M/C_{Y_M}(W)| \leqslant |\overline{W}| = |W/C_W(V)| = |V/C_V(W)| = |VC_{Y_M}(W)/C_{Y_M}(W)| \leqslant |Y_M/C_{Y_M}(W)|.$$
Thus $Y_M = VC_{Y_M}(W)$, and 7.4(e) gives $C_{Y_M}(W) = Y_M \cap A$.

(b): We already know that $W$ is a root offender on $V$. Since $Y_M = VC_{Y_M}(W)$ by (a), $W$ is also a root offender on $Y_M$.

(c): Since $W$ is a root offender on $V$, 1.21 shows that $V$ is a root offender on $W$. Since $Y_M = VC_{Y_M}(W)$, also $Y_M$ is a root offender on $W$.

(d): By (a) $C_W(V) = C_W(Y_M)$, and so $|W/C_W(V)| = |\overline{W}|$. Now (d) follows from (I).

(e) and (f): By (a) $[Y_M, W] = [V, W]$. Let $w \in W \backslash C_W(V)$. Then by 1.43(i), $[w, Y_M] \cap C_{Y_M}(L) = 1$. By (c) $V$ is a root offender on $W$ and so by A.37(b), $V$ is a strong dual offender on $W$. Thus $[w, V] = [W, V]$, and we conclude that $[W, V] \cap R = 1$. By 7.5(c), $W \cap R = [Y_M, W] \cap R$. Since $[Y_M, W] = [V, W]$, this gives $W \cap R = [V, W] \cap R = 1$. By (c) $Y_M$ is an offender on $W$, so 7.10 shows that $W \cong W/W \cap R$ is a natural $SL_2(\tilde{q})$-module for $L$.

By (a) $C_{Y_M}(W) = Y_M \cap A$, and we conclude that

$$V \cap A = C_V(W) = [V, W]^{\perp}.$$

As $[V, W] \leqslant V \cap A$, this implies that $[V, W]$ is singular.

By 1.43(a), $A' \leqslant C_{Y_M}(L) = R$ and so

$$[A, V \cap A] \cap [W, V] \leqslant R \cap [W, V] = 1.$$

Hence there exists a subgroup $V_0$ of $V \cap A$ with $[A, V \cap A] \leqslant V_0$ and $V \cap A = [V, W] \times V_0$. Note that $A$ normalizes $V_0$ and so also $V_0^{\perp}$. From $V_0 \cap [V, W] = 1$ we get $V = V_0^{\perp}[V, W]^{\perp}$. Since $V \cap A = [V, W]^{\perp}$ this gives $V = V_0^{\perp}(V \cap A)$. Also

$$\begin{aligned}[A, V_0^{\perp}] \leqslant V_0^{\perp} \cap [A, V] \leqslant V_0^{\perp} \cap (V \cap A) = V_0^{\perp} \cap [V, W]^{\perp} &= (V_0 + [V, W])^{\perp} \\ = (V \cap A)^{\perp} \quad = [V, W]^{\perp\perp} \quad = [V, W] \quad &\leqslant W.\end{aligned}$$

Since $V = V_0^{\perp}(V \cap A)$ and $Y_M = VC_{Y_M}(W) = V(Y_M \cap A)$ we have $Y_M \leqslant VA \leqslant V_0^{\perp}A$, and so

$$[A, Y_M] \leqslant [A, V_0^{\perp}A] = [A, V_0^{\perp}][A, A] \leqslant WA' \leqslant WR.$$

From $L = \langle Y_M^L \rangle$ we conclude $[A, L] \leqslant WR$. By 1.43(p) $L$ has no central chief factors on $A/R$ and so $A = WR$, and since $W \cap R = 1$, $A = W \times R$. In particular, $A$ is abelian.

Note that $[W, V] \leqslant W \cap V \leqslant W \cap Y_M \leqslant C_W(V)$. Since $W$ is a natural $SL_2(\tilde{q})$ module for $L$ we have $[W, V] = C_W(V)$ and so $[W, V] = W \cap V = W \cap Y_M$. Recall that $C_{Y_M}(A) = Y_M \cap A$, $A = W \times R$ and $R \leqslant Y_M \cap A$. Hence

$$C_{Y_M}(W) = Y_M \cap A = (Y_M \cap W)R = [V, W] \times R \text{ and } C_V(W) = [V, W] \cap (R \cap W).$$

Since $A \leqslant WR \leqslant UR$ we have $[A, O_p(H)] = 1$. In particular, $V \cap A \leqslant C_V(O_p(H))$ and so

$$[V, O_p(H)] = C_V(O_p(H))^{\perp} \leqslant (V \cap A)^{\perp} = [V, W].$$

Since by (a) $Y_M = V(Y_M \cap A)$ we get that $[Y_M, O_p(H)] = [V, W] = [Y_M, W]$. Hence, (e) and (f) are proved.

(g): By (f) $C_V(W) = [V, W] \times (V \cap R)$ and by (d) $|\overline{W}| = |[V, W]| = |V/C_V(W)|$. Thus $|V| = |V/C_V(W)||C_V(W)| = |\overline{W}|^2|V \cap R|$.

(h) and (i): Since $Y_M$ is an offender on $W$, we can apply 7.11. We now treat the two cases arising there separately.

Case 1. *Suppose that 7.11(1) holds ($\widehat{U}$ is a natural $SL_2(\tilde{q})$-module for $H$).*

According to 7.11(1) we have

$$Y_M = O_p(M) = C_G(Y_M), \quad M = M^{\dagger}, \quad N_G(Q) \leqslant M, \quad H = L, \quad U = W.$$

Then $O_p(H) = O_p(L) = A$, and by (e) $A \cap Y_M = [W, Y_M]R$ and $A = W \times R \leqslant Y_H$, so $A = Y_H$ follows. By (e) $C_{Y_M}(W) = [V, W]R$, so (a) implies

$$Y_M = VC_{Y_M}(W) = V[V, W]R = VR.$$

Since $H$ is of characteristic $p$ and $A = O_p(H)$, $C_H(A) \leqslant A$. As $A$ is abelian, $C_H(A) = A$. By 7.5(a), $R = C_{Y_H}(R)$ and so $H$ centralizes $R$. Thus

$$C_L(W) = C_H(W) = C_H(WR) = C_H(A) = A = WR.$$

Hence (h) and (i) hold in this case.

Case 2. *Suppose that 7.11(2) holds.*

According to 7.11(2) $U$ is a natural $SL_m(\widetilde{q})$-module for $H$, $m \geqslant 3$, and $\widetilde{Y_M}$ is a transvection group on $U$. Our goal is to derive a contradiction in this situation.

Put $H_1 := O^p(C_H(Y_M \cap A))$ and $B := \langle V^{H_1} \rangle$. We show:

1°. $C_{Y_M \cap A}(Q) \neq 1$. *In particular,* $H_1 \leqslant N_H(Q)$.

By (a) $Y_M \cap A = C_{Y_M}(W)$. Note that $W \leqslant S$ and $Q \leqslant S$. Thus $1 \neq C_{Y_M}(S) \leqslant Y_M \cap A$ and so $C_{Y_M \cap A}(Q) \neq 1$. Then $Q!$ implies $H_1 \leqslant C_G(C_{Y_M \cap A}(Q)) \leqslant N_G(Q)$.

2°. $B/B \cap O_p(H)$ *is a non-central simple* $H_1$*-module and* $B \cap O_p(H) = C_B(H_1)$.

By (e) $Y_M \cap A = [W, Y_M] \times R = [U, Y_M] \times R$ and by 7.5(a) $[R, H] = 1$. Hence $H_1 = O^p(C_H([U, Y_M]))$. Since $\widetilde{Y_M}$ acts as a transvection group on $U$, $[W, Y_M] = [U, Y_M]$ is a 1-dimensional $\mathbb{K}$-space, where $\mathbb{K} := End_H(U)$. Note that $U/[U, Y_M]$ and $C_{\widetilde{H}}(U/[U, Y_M])$ are natural $SL_{m-1}(q)'$-modules for $H_1$ dual to each other. In particular, $H_1$ acts simply on $C_{\widetilde{H}}([U, Y_M])$ and $C_{\widetilde{H}}([U, Y_M]) = O_p(\widetilde{H}_1)$. As

$$1 \neq \widetilde{Y_M} = \widetilde{V} \leqslant C_{\widetilde{H}}([U, Y_M]) = O_p(\widetilde{H_1}),$$

the simple action of $H_1$ gives $\widetilde{B} = O_p(\widetilde{H_1})$. Since $B/B \cap O_p(H) \cong \widetilde{B}$ the first statement in (2°) holds.

As $H_1$ acts simply on $U/[U, Y_M]$ and centralizes $[U, Y_M]$, we have

$$C_U(B) = [U, Y_M] = C_U(H_1).$$

Since $Y_M$ is $Q$-short, $Y_M \leqslant O_p(N_G(Q))$. Also $S \leqslant Syl_p(N_G(Q))$ and so $O_p(N_G(Q)) \leqslant S$ and $Y_M \trianglelefteq O_p(N_G(Q))$. Now 2.5 shows that $\langle Y_M^{N_G(O_p(Q))} \rangle$ is abelian. Since $V \leqslant Y_M$ and by (1°) $H_1 \leqslant N_G(Q)$, we conclude that $B$ is abelian. Moreover, by (a) $Y_M = V(Y_M \cap A)$ and so $[U, Y_M] = [U, V]$. Hence

$$[U, Y_M] = [U, V] \leqslant U \cap B \leqslant C_U(B) = [U, Y_M] = C_U(H_1)$$

and so $U \cap B = C_U(H_1)$.

By (e) $[O_p(H), O^p(H)] = U$. Since $H_1 \leqslant O^p(H)$ this gives $[O_p(H), H_1] \leqslant U$. Thus

$$[B \cap O_p(H), H_1] \leqslant U \cap B \leqslant C_U(H_1),$$

and since $H_1 = O^p(H_1)$, $B \cap O_p(H) \leqslant C_B(H_1)$. Since $B/B \cap O_p(H)$ is a non-central simple $H_1$ module, $C_B(H_1) \leqslant B \cap O_p(H)$ and so (2°) holds.

3°. $[V, W, Q] = 1$. *Moreover,* $Q = Q^g$ *for all* $g \in M$ *with* $C_{Y_M \cap A}(Q^g) \neq 1$.

Let $g \in M$ with $C_{Y_M \cap A}(Q^g) \neq 1$. Suppose for contradiction that $[V, W, Q^g] \neq 1$. Then B.8 gives $[V^*, Q^g, W] \neq 1$, where $V^*$ is the $\mathbb{F}_p$-dual of $V$. Note that $V$ is a selfdual $\overline{K^*Q}$-module and so also a selfdual $\overline{K^*Q^g}$-module. Since $\overline{WQ^g} \leqslant \overline{K^*Q^g}$ we conclude that $[V, Q^g, W] \neq 1$. Hence $[V, Q^g] \nleqslant O_p(H)$ and so also $[B, Q^g] \neq O_p(H)$.

Note that $[H_1, C_{Y_M \cap A}(Q^g)] = 1$ and $Q!$ show that $H_1$ normalizes $Q^g$. Since $Q^g$ normalizes $V$, we conclude that $Q^g$ normalizes $V^h$ for any $h \in H_1$. It follows that $Q^g$ normalizes $B$, and $H_1$ normalizes $[B, Q^g]$. By (2°) $H_1$ acts simply on $B/B \cap O_p(H)$ and $C_B(H_1) = B \cap O_p(H)$. As $[B, Q^g] \nleqslant B \cap O_p(H)$ this gives $B = [B, Q^g]C_B(H_1)$. Hence $[B, H_1] \leqslant [B, Q^g]$ and $H_1$ normalizes $[B, Q^g]V$. Thus

$$B = \langle V^{H_1} \rangle = [V, H_1]V \leqslant [B, Q^g]$$

and $B/V = [B/V, Q^g]$, so $B = V$. But since $m \geqslant 3$, $\widetilde{V} \leqslant \widetilde{Y_M} \neq O_p(\widetilde{H_1}) = \widetilde{B}$ and so $B \neq V$, a contradiction.

We have proved that $[V, W, Q^g] = 1$. But then also $[V, W, Q] = 1$, since by (1°) $C_{Y_M \cap A}(Q) \neq 1$, and so $1 \neq [V, W] \leqslant C_G(Q) \cap C_G(Q^g)$. Hence 1.52(e) gives $Q = Q^g$ and (3°) is proved.

4°. *Put* $q := |\overline{W}|$. *Then* $q = \widetilde{q}$, $\overline{W} = \overline{Q} = \overline{U}$, $\overline{K} = \overline{K^*} = \overline{M^\circ} \cong SL_2(q)$, *and* $V$ *is a natural* $SL_2(q)$*-module for* $\overline{K}$.

Let $g \in M$ with $Q \neq Q^g$. Then (3°) shows that $C_{Y_M \cap A}(Q^g) = 1$ and $[V, W, Q] = 1$. So also $[V, W^g, Q^g] = 1$. Since $[V, W] \leqslant C_V(W) = V \cap A$, this gives

(II) $$C_V(W) \cap [V, W]^g \leqslant C_V(W) \cap C_V(Q^g) = V \cap A \cap C_V(Q^g) = 1 \text{ for all } g \in M \backslash N_M(Q).$$

In particular, $|C_V(Q^g)| \leqslant |V/C_V(W)|$. By (d), $|V/C_V(W)| = |[V, W]|$, and so $|C_V(Q^g)| \leqslant |[V, W]|$. Since by (3°) $[V, W] \leqslant C_V(Q)$. we conclude that $|[V, W]| \leqslant |C_V(Q)| = |C_V(Q^g)|$ and so $[V, W] = C_V(Q)$. Now $Q!$ shows that $N_M([V, W]) = N_M(Q)$. Hence (II) gives $C_V(W) \cap [V, W]^g = 1$ for all $g \in M \backslash N_M([V, W])$.

Since $[Y_M, W] = [V, W]$ we have $V = [V, K^*]$. Moreover, by (b) $W$ is a root offender on $V$. Hence $\overline{M}$, $\mathcal{D} := \overline{W}^{\overline{M}}$ and $V$ satisfy the hypothesis of C.12. We conclude that $V$ is a natural $SL_2(q)$-module for $K^*$. Moreover, $|\overline{W}| = q = |[W, V]| = \widetilde{q}$. As $[V, W, Q] = 1$ and $N_M([V, W])$ normalizes $\overline{Q}$, $\overline{Q} = \overline{W}$. Hence
$$\overline{M^\circ} = \langle \overline{Q}^{\overline{K^*}} \rangle = \langle \overline{W}^{\overline{K^*}} \rangle = \overline{K^*} = \overline{K}.$$
By 2.17(c) $U = WC_U(Y_M)$. Hence $\overline{U} = \overline{W}$, and (4°) holds.

5°. $O_p(H) = Y_H = U$, $C_H(U) = U$ *and* $H/U \cong SL_m(q)$.

As $R \leqslant C_{Y_M}(W)$ and by (4°) $\overline{W} = \overline{Q}$, $[R, Q] = 1$. By 7.5(e), $C_R(Q) = 1$, and so $R = 1$. Hence by 7.5(a) $C_{Y_H}(H) = R = 1$. Since $Y_H = \Omega_1 Z(O_p(H))$ by 7.1(d), this gives $C_{O_p(H)}(H) = C_{O_p(H)}(O^p(H)) = 1$. By (h),
$$[O_p(H), O^p(H)] = U \leqslant Y_H = \Omega_1 Z(O_p(H)),$$
and 1.18 yields $[\Phi(O_p(H)), O^p(H)] = 1$. Hence $\Phi(O_p(H)) = 1$ and $O_p(H) = \Omega_1 Z(O_p(H)) = Y_H$.

By (4°) $V$ is a natural $SL_2(q)$-module for $\overline{K^*}$ and $q = |\overline{W}|$. Since $Y_H$ centralizes the $\mathbb{F}_q$-subspace $[V, W]$ of $Y_H$ we conclude that $Y_H$ acts $\mathbb{F}_q$-linearly on $V$. Hence $\overline{Y_H} \leqslant \overline{K^*}$, $\overline{Y_H} = \overline{W} = \overline{U}$ and $Y_H \leqslant UC_M(V)$. Thus $Y_H = C_{Y_H}(V)U$ and $V$ is an offender on $Y_H$. Hence C.22 shows that $Y_H = UC_U(H) = U$. Thus $O_p(H) = U$, and since $H$ is of characteristic $p$, $C_H(U) = U$ and $H/U \cong SL_m(q)$, and (5°) is proved.

We are now able to show that (Case 2) leads to a contradiction. By (4°), $\overline{M^\circ} \cong SL_2(q)$ and $\overline{Q} = \overline{U}$. So we can choose $M_1$ minimal in $M^\circ U$ with $U \leqslant M_1$ and $[V, U] \ntrianglelefteq M_1$. It follows $\overline{M_1} = \overline{M^\circ}$ and $C_S(Y_M)U = O_p(M)U = O_p(M)Q$. Thus $U \leqslant M^\circ O_p(M)$ and $M_1 \leqslant M^\circ U \leqslant M^\circ O_p(M)$. Also $M_1$ acts transitively on $V$, and so by 1.57(c), $M^\circ = \langle Q^{M_1} \rangle \leqslant M_1 O_p(M)$. Thus $M_1 O_p(M) = M^\circ O_p(M)$. The minimal choice of $M_1$ shows that $M_1 = \langle U^{M_1} \rangle$. Thus, since $O_p(M)$ normalizes $U$, it also normalizes $M_1$. Therefore
$$O^p(M_1) = O^p(M_1 O_p(M)) = O^p(M^\circ O_p(M)) = O^p(M^\circ) = M_\circ.$$
Since by 1.55(d) $C_{O_p(M)}(M^\circ) = 1$, this gives $C_{O_p(M)}(M_1) = 1$ and thus $C_U(M_1) = 1$. Note that $U \ntrianglelefteq M_1$ and $N_{M_1}([V, U])$ is the unique maximal subgroup of $M_1$ containing $U$. Hence $M_1$ is $U$-minimal, and we can apply 1.43.

Put $D := \langle U \cap O_p(M_1)^{M_1} \rangle$ and let $m \in M_1 \backslash N_{M_1}([V, U])$. Then by 1.43(e), $D = (U \cap D) \times (U^m \cap D)$, by 1.43(a), $\Phi(D) \leqslant C_U(M_1) = 1$, and by 1.43(p) $M_1$ has no central chief factor on $D/C_D(M_1) \cong D$. Hence $D = [D, M_1]$. Note that
$$[U, O_p(M)] \leqslant U \cap O_p(M) \leqslant U \cap O_p(M_1) \leqslant C_U(V) \leqslant U \cap O_p(M),$$
and so
$$[U, O_p(M)] \leqslant U \cap O_p(M) = C_U(V) = U \cap O_p(M_1) = U \cap D.$$
Hence $D = [O_p(M), O^p(M_1)] = [O_p(M), M_\circ] \trianglelefteq M$.

Recall that $U$ is a natural $SL_m(\widetilde{q})$ module and $\widetilde{Y_M}$ is a transvection group on $U$. By (4°) $q = \widetilde{q}$, and by (a) $\widetilde{V} = \widetilde{Y_M}$. Hence $U \cap D = C_U(V)$ is an $\mathbb{F}_q$-hyperplane of $U$. In particular, $U \cap D$ has order $q^{m-1}$. As $D = (U \cap D) \times (U^m \cap D)$, $D$ has order $q^{2(m-1)}$ and $UD$ has order $q^{2m-1}$.

Put $H_2 := N_H(C_U(V)) = N_H(U \cap D)$. By (5°) $C_H(U) = U$ and thus $|C_H(U)| = q^m$. Since $U$ is a natural $SL_m(q)$-module for $U$, $C_H(C_U(V))/C_H(U)$ is a natural $SL_{m-1}(q)$-module for $O^{p'}(H_2)$ (isomorphic to $C_U(V)$), and so has order $q^{m-1}$. Thus, $|C_H(C_U(V))| = q^m q^{m-1} = q^{2m-1} = |UD|$. Note that $D$ and $V$ are abelian. Hence $UD \leqslant C_H(U \cap D) = C_H(UV)$ and $UD = C_H(C_U(V)) \trianglelefteq H_2$.

As $U \cap D$ is an $\mathbb{F}_q$-hyperplane of $U$ and the elements of $D$ act $\mathbb{F}_q$-linearly on $U$, for every $d \in D \backslash C_D(U)$

$$C_{DU}(d) = DC_U(d) = D(U \cap D) = D.$$

In particular, for every elementary abelian subgroup $E \leqslant DU$ either $E \leqslant D$ or $E \cap D = E \cap C_D(U)$. In the latter case $|E/E \cap C_D(U)| \leqslant q$ since $|DU/D| = q$, while $|D/C_D(U)| = q^{m-1}$. As $m \geqslant 3$ we conclude that $D$ is the only maximal elementary abelian subgroup of order $q^{2(m-1)}$ in $DU$. Since $UD \trianglelefteq H_2$ we get $H_2 \leqslant N_G(D)$.

As we have seen above, $D \trianglelefteq M$ and so $M \leqslant N_G(D)$. The basic property of $M$ gives $H_2 \leqslant N_G(D) \leqslant M^\dagger$ and $Y_M \trianglelefteq H_2$. But $\widetilde{Y_M}$ is a transvection group on $U$ and since $m \geqslant 3$ we get $\widetilde{Y_M} \ntrianglelefteq \widetilde{H_2}$, a contradiction. □

Lemma 7.13. *Put* $\mathcal{K} := \overline{K}^{\overline{M}}$. *Then*

$$(*) \qquad \overline{K^*} = \bigtimes_{F \in \mathcal{K}} F \quad \text{and } [Y_M, \overline{K^*}] = \bigtimes_{F \in \mathcal{K}} [V, F].$$

*Moreover, one of the following holds, where $q$ is a power of $p$:*

(A) $\overline{K} \trianglelefteq \overline{M}$, $\overline{K} = \overline{M^\circ} \cong SL_n(q)$, $n \geqslant 3$, *and $Y$ is a natural $SL_n(q)$-module for $K$.*

(B) $\overline{M^\circ} = O^p(\overline{K^*})\overline{Q}$ *and there exists a non-degenerate $\overline{K^*}\,\overline{S}$-invariant symplectic form on* $[Y_M, \overline{K^*}]$. *In addition, one of the following holds:*
  (1) $\overline{K} \trianglelefteq \overline{M}$, $\overline{K} \cong Sp_{2n}(q)$, $n \geqslant 1$, *or* $Sp_4(2)'$ *(and $p = 2$), and $Y$ is a corresponding natural module for* $\overline{K}$,
  (2) $\overline{K} \trianglelefteq \overline{M}$, $p = 2$, $\overline{K} \cong O^\epsilon_{2n}(2)$, $n \geqslant 2$ *and* $(n, \epsilon) \neq (2, +)$, *and $Y$ is a corresponding natural module for* $\overline{K}$. *Moreover,* $\overline{M^\circ} = \overline{K}' \cong \Omega^\epsilon_{2n}(2)$ *and* $|\overline{W}| = |Y_M/C_{Y_M}(W)| = 2$.
  (3) $\overline{K} \ntrianglelefteq \overline{M}$, $Y_M$ *is a natural $SL_2(q)$-wreath product module for $\overline{M}$ with respect to $\mathcal{K}$, and* $\overline{Q}$ *acts transitively on* $\mathcal{K}$.

(C) (a) $\overline{K} \trianglelefteq M$, $Y = Y_M$ *and* $|Y/C_Y(W)| = 4$.
  (b) *Put* $M_2 := N_M(C_Y(W))$ *and* $K_2 := C_{M_2}(Y/C_Y(W))$. *Then* $\overline{K_2} \leqslant \overline{K}$, *and there exists an $M_2$-invariant set* $\{V_1, V_2, V_3\}$ *of $K_2$-submodules of $Y$ such that* $Y = V_i \times V_j$ *for all* $1 \leqslant i < j \leqslant 3$.
  (c) *For all* $1 \leqslant i \leqslant 3$ *and* $1 \neq x \in C_{V_i}(W)$ *there exists* $g \in M$ *with* $[x, Q^g] = 1$.
  (d) *One of the following holds:*
    (1) $p = 2$, $\overline{K} = \overline{K}' \cong SL_n(2)$, $n \geqslant 3$, $\{V_1, V_2, V_3\}$ *is the set of proper $K$-submodules of $Y_M$, and the $V_i$'s are isomorphic natural $SL_n(2)$-modules for $K$. Moreover,* $\overline{M^\circ} \cong SL_n(2)$, $SL_n(2) \times SL_2(2)$ *or* $SL_2(2)$, *with* $\overline{K} \leqslant \overline{M^\circ}$ *in the first two cases and* $[\overline{K}, \overline{M^\circ}] = 1$ *in the last case.*
    (2) $p = 2$, $\overline{K} = \overline{K}' \leqslant \overline{M^\circ}$, $\overline{K} \cong 3{\cdot}Alt(6)$ *and* $\overline{M^\circ} \cong 3{\cdot}Alt(6)$ *or* $3{\cdot}Sym(6)$, *and $Y_M$ is corresponding natural module for $K$.*

Proof. Recall that $Y_M$ is a $p$-reduced $Q$!-module for $\overline{M}$. By 2.17(c), $W$ is a non-trivial strong offender on $Y_M$, and by 2.17(a), $[W, Y_M] = [W, X]$ for all $X \leqslant Y_M$ with $|X/C_X(W)| > 2$. Thus we can apply C.25. Hence $(*)$ holds. Also most of the other statements follow directly from C.25, but we still need to show:

(Task 1) In cases C.25(1:b:2), (1:b:3),(1:b:5),(2) ($Y$ is a natural $Sp_{2n}(q)$-, $Sp_4(2)$-, $Sp_4(2)'$-, $O^\epsilon_{2n}(2)$- or $SL_2(q)$-module for $K$) show that there exists an $\overline{K^*}\,\overline{S}$-invariant non-degenerate symplectic form on $[Y_M, \overline{K^*}]$ over $\mathbb{F}_p$ (to prove (B)).

(Task 2) In case C.25(1:b:4) ($Y$ is natural $3{\cdot}Alt(6)$-module for $K$) show that $Y_M = Y$, $|Y/C_Y(W)| = 4$, $\overline{K_2} \leqslant \overline{K}$, and prove the existence of $\{V_1, V_2, V_3\}$ fulfilling (C:b) and (C:c).

(Task 3) In case C.25(4) ($Y$ is a direct sum of two isomorphic natural $SL_n(q)$-module and $[\overline{K}, \overline{M^\circ}] = 1$) show that $\overline{K} \trianglelefteq \overline{M}$ and $Y = Y_M$ (to prove (C)).

(Task 4) In cases C.25(3) and (4) ($Y$ is a direct sum of two isomorphic natural $SL_n(q)$-modules) prove $\overline{K_2} \leqslant \overline{K}$ and the existence of $\{V_1, V_2, V_3\}$ fulfilling (C:b), (C:c) and (C:d:1).

(Task 1): Put $\mathbb{K} := End_K(Y)$. Then $\mathbb{K}$ is a finite field (of order $q$ or 2 depending on the case). Also in each case there exists a $K$-invariant non-degenerate symplectic form $s$ on $Y$ over $\mathbb{K}$. Note here that $SL_2(q) \cong Sp_2(q)$ and a natural $SL_2(q)$-module is also a natural $Sp_2(q)$-module. Moreover, $s$ is unique up to multiplication by a non-zero $k \in \mathbb{K}$. Since $|\mathbb{K}| - 1$ is not divisible by $p$, we can choose $s$ to be $N_{\overline{S}}(\overline{K})$-invariant. If $\overline{K} \trianglelefteq \overline{M}$ we are done.

Assume that $\overline{K} \ntrianglelefteq \overline{M}$. Then we are in Case (2) of Theorem C.25, so $\overline{K} \cong SL_2(q)$, $Y$ is a natural $SL_2(q)$-module for $\overline{K}$, and $Q$ and so also $S$ acts transitively on $\mathcal{K}$.

Let $F \in \mathcal{K}$ with $F \neq \overline{K}$. Then $(*)$ shows that $[F, \overline{K}] = 1$ and $[Y_M, K] \cap [Y_M, F] = 1$. So $F \leqslant C_{\overline{K^*}}(Y)$ and

$$\overline{K^*} = KC_{\overline{K^*}}(Y).$$

For any $F \in \mathcal{K}$ choose $g \in \overline{K^*}\,\overline{S}$ with $F = \overline{K}^g$. Define a symplectic form $s_F$ on $[Y_M, F] = Y^g$ via $s_F(v^g, w^g) := s(v, w)$ for all $v, w \in Y$. If also $F = \overline{K}^h$ for some $h \in \overline{K^*S}$, then

$$h^{-1}g \in N_{\overline{K^*S}}(K) = N_{\overline{S}}(\overline{K})\overline{K^*} = N_{\overline{S}}(\overline{K})\overline{K}C_{\overline{K^*}}(Y),$$

and we conclude that the definition of $s_F$ is independent of the choice of $g$.

By $(*)$, $Y_M = \times_{F \in \mathcal{K}} [Y_M, F]$, and so there exists a unique symplectic form $t$ on $Y_M$ such that the restriction of $t$ to $[Y_M, F]$ is $s_F$ for all $F \in \mathcal{K}$, and $[Y_M, F] \perp [Y_M, F^*]$ for distinct $F, F^* \in \mathcal{K}$. Then $t$ is $\overline{K^*S}$-invariant, and (Task 1) is accomplished

(Task 2): By coprime action $Y_M = C_{Y_M}(Z(\overline{K})) \times [Y_M, Z(\overline{K})]$, and since $Z(\overline{K})$ acts fixed-point freely on $Y = [Y_M, K]$, $Y_M = C_{Y_M}(K) \times Y$. Since $\overline{M_\circ} \leqslant \overline{K}$ and by 1.55(d) $C_{Y_M}(M^\circ) = 1$, this gives $Y_M = Y$.

As $W$ is a nontrivial (strong) offender on $Y_M$, the Offender Theorem C.4(e) gives

$$|Y/C_Y(W)| = 4 = |\overline{W}| \qquad \text{and} \qquad C_Y(W) = [W, Y].$$

Let $\mathcal{V}$ be the set of 3-dimensional $K_2$-submodules of $Y$. By C.16 $M_2$ is a parabolic subgroup of $M$, $M_2 = N_M(\overline{W})$, $\overline{K_2} = O^{2'}(N_{\overline{K}}(\overline{W}))$, $\mathcal{V} = \{V_1, V_2, V_3\}$, $Z(\overline{K})$ acts transitively on $\mathcal{V}$, $Y = V_i \times V_j$ for all $1 \leqslant i < j \leqslant 3$, and $C_{V_i}(W)$ is a natural $SL_2(2)$-module for $K_2$. In particular, $\overline{K_2} \leqslant \overline{K}$. Let $1 \neq x \in C_{V_i}(W)$. Since $Z(\overline{K}) \leqslant \overline{M_2}$, $M_2$ acts transitively on the three elements of $\mathcal{V}$ and, since $K_2 \leqslant N_{M_2}(V_i)$, $N_{M_2}(V_1)$ acts transitively the three elements of $C_{V_i}(W)^\sharp$. Thus $C_{M_2}(x)$ has index 9 in $M_2$, so $C_M(x)$ contains a Sylow 2-subgroup of $M_2$ and of $M$. Hence $C_M(x)$ also contains a conjugate of $Q$ in $M$.

(Task 3): Since $[\overline{K}, \overline{M^\circ}] = 1$, $\langle \overline{K}^{\overline{Q}} \rangle = \overline{K}$. Thus 7.7 shows that

$$C_{Y_M}(K) = C_{Y_M}(\langle K^Q \rangle) \leqslant C_{O_p(M)}\langle K^Q \rangle) = 1.$$

Hence $(*)$ implies $Y_M = [Y_M, K] \times C_{Y_M}(K) = [Y_M, K] = Y$ and $\mathcal{K} = \overline{K}$. Thus $\overline{K} \trianglelefteq \overline{M}$ and (Task 3) is accomplished.

(Task 4): Since $Y$ is the direct sum of two isomorphic natural $SL_n(2)$-modules for $K$, there exist exactly three simple $K$-submodules $V_1, V_2$ and $V_3$ in $Y$. Moreover, $Y = V_i \times V_j$ for any $1 \leqslant i < j \leqslant 3$. Since $K$ induces $Aut(V_i)$ on $V_i$ and $\overline{K} \trianglelefteq \overline{M}$, $\overline{M} = \overline{K} \times C_{\overline{M}}(\overline{K})$. Also $C_{\overline{M}}(K)$ is isomorphic to a subgroup of $SL_2(2)$ and $O_2(\overline{M}) = 1$. Thus $C_{\overline{M}}(K)$ is isomorphic to one of $1, C_3$ or $SL_2(2)$. So $M$ acts either trivially or transitively on $\{V_1, V_2, V_3\}$. In either case $V_i$ is normalized by a Sylow 2-subgroup of $M$, and since $K$ acts transitively on $V_i$ each $1 \neq x \in V_i$ is centralized by a Sylow 2 subgroup of $V$. So again $C_M(x)$ contains a conjugate of $Q$ in $M$. Note that $C_Y(W) = C_{Y_1}(W) \times C_{Y_2}(W)$ and $C_{\overline{M}}(K)$ normalizes $C_Y(W)$. It follows that $\overline{M_2} = (\overline{M_2} \cap \overline{K})C_{\overline{M}}(K)$, $C_{\overline{M}}(K)$ acts faithfully on $Y/C_Y(W)$, and $\overline{M_2} \cap \overline{K}$ centralizes $Y/C_Y(W)$. Thus $\overline{K_2} = \overline{M_2} \cap \overline{K} \leqslant \overline{K}$, and all assertions in (Task 4) hold. □

LEMMA 7.14. *Suppose that Case 7.13(A) holds. Then $Y_M = Y$ and Theorem G(1) holds.*

Proof. In this case $Y$ is a natural $SL_n(q)$-module for $\overline{K} = \overline{M^\circ}$ with $n \geqslant 3$, and by 7.5(f), $C_{Y_M}(M^\circ) = 1$. If $Y_M = Y$ we conclude that Theorem G(1) holds.

Suppose that $Y_M \neq Y$. Then $Y_M$ is a non-trivial non-split central extension of $Y$. Since, by 2.17(c), $W$ is a (strong) offender on $Y_M$, C.22 shows that $p = 2$, and

$$\overline{K} \cong SL_3(2),\ |Y_M| = 2^4,\ C_{Y_M}(W) = C_Y(W) \text{ and } \widetilde{q} = |Y_M/C_{Y_M}(W)| = |\overline{W}| = 4.$$

In particular, $[Y_M, M^\dagger] = [Y, M] = Y$, and $Y_M$ is an offender on $W$. Now 7.10 implies $\widetilde{L} \cong SL_2(4)$ and $\widetilde{Y_M} \in Syl_2(\widetilde{L})$. By definition of $\mathfrak{L}_H(Y_M)$, $N_L(Y_M)(= L \cap M^\dagger)$ is unique maximal subgroup of $L$ containing $Y_M$, and the structure of $SL_2(4)$ shows that $[\widetilde{Y_M}, L \cap M^\dagger] = \widetilde{Y_M}$. It follows that

$$Y_M = [Y_M, L \cap M^\dagger] C_{Y_M}(U) = Y C_{Y_M}(W) = Y C_Y(W) = Y,$$

which contradicts $Y_M \neq Y$. □

Lemma 7.15. *Suppose that Case 7.13(B) holds. Then Theorem G(2) or Theorem G(3) holds.*

Proof. Put $H_0 := \langle Y_M^H \rangle$. Note that in Case 7.13(B) there exists a $\overline{K^*}\,\overline{S}$-invariant non-degenerate symplectic form on $V := [Y_M, \overline{K^*}]$. Thus we can apply 7.12. We will now treat each of the three subcases of 7.13(B) separately.

Case 1. *Suppose that 7.13(B:1) holds, that is, $\overline{K} \trianglelefteq \overline{M}$ and $Y = V$ is a natural $Sp_{2n}(q)$-module ($n \geqslant 1$) or a natural $Sp_4(2)'$-module ($p = 2$) for $K$.*

Put $n := 2$ and $q := 2$ in the $Sp_4(2)'$-case. Note that $K'$ acts transitively on the natural $Sp_{2n}(q)'$-module $V$, and so each non-trivial element of $V$ is centralized by a conjugate $Q^g$ of $Q$ under $K$. Since by 7.5(e) $C_R(Q^g) = 1$ for all such $Q^g$, this gives $V \cap R = 1$.

Suppose for a contradiction that $Y \neq Y_M$. By 7.7(a) $C_{Y_M}(K) \cap C_{Y_M}(Q) = 1$. Since $\overline{K} \trianglelefteq \overline{M}$, this gives $C_{Y_M}(K) = 1$. Hence, $Y_M$ is a non-split central extension of $Y$. Also by 2.17(c) $W$ is a strong offender on $Y_M$. Since strong offenders are best offenders, C.22 shows that $Y_M$ is a submodule of the dual of a natural $O_{2n+1}(q)$-module, $n \geqslant 2$, or a natural $O_5(2)'$-module for $\overline{K}$.

By 7.12(h) $Y_M = VR$, and so there exists $y \in R \backslash V$. Since $Y_M$ is a submodule of the dual of the orthogonal module for $\overline{K}$, $C_{\overline{K}}(y) \cong O^\epsilon_{2n}(q)$ or $\Omega^\epsilon_4(2)$. Since by 7.12(b), $W$ is a root offender on $V$, and since $\overline{W} \leqslant C_{\overline{K}}(y)$, C.6 shows that $|\overline{W}| = 2$ Hence by 7.12(g), $|V| = |\overline{W}|^2 |V \cap R| = 2^2 \cdot 1 = 4$, a contradiction since $|V| = q^{2n}$ and $n \geqslant 2$.

We have shown that

$$Y = Y_M = V \quad \text{and} \quad R = R \cap V = 1.$$

By 7.12(h) $O_p(M) = C_G(Y_M) = Y_M = V$ and $N_G(Q) \leqslant M$. So if $\overline{K} \cong Sp_4(2)'$, then Theorem G(2) holds. We therefore may assume that $\overline{K} \cong Sp_{2n}(q)$.

Since $R = 1$, 7.12(e) gives $A = W \times R = W$, and $A$ is a natural $SL_2(\widetilde{q})$ module. Put $D := C_K(V/[V, W) \cap C_K([V, W])$. Then $D$ acts nilpotenly on $V$ and so $D/C_D(V)$ is a $p$-group. As $C_G(V) = C_G(V) = Y_M$, $D$ is a $p$-group. Since $V = Y_M$ we have $[Y_M, D] = [V, D] \leqslant [V, W] \leqslant A$. Also by 7.12(h) $N_G(Q) \leqslant M$. Thus, by 7.9 there exists $t \in A$ with $[t, Y_M] \neq 1$ and $[C_D(t), L] \leqslant A = W$. Put $B := C_D(t)W$. Then $B$ and $W$ are normal in $LB$, and since $W$ is a simple $L$-module, $[B, W] = 1$. Hence $\Phi(B) = \Phi(C_D(t))$ is centralized by $L = \langle Y_M^L \rangle$. From $C_G(Y_M) = Y_M$ we get $C_G(L) \leqslant C_{Y_M}(L) = R = 1$. In particular, $\Phi(B) = 1$, and $B$ is elementary abelian with $C_B(L) = 1$. It follows that $B$ is isomorphic to a submodule of the dual of the natural $\Omega_3(\widetilde{q})$-module for $\widetilde{L}$. Let $d \in C_D(t) \leqslant B$. Then $C_{\widetilde{L}}(d)$ is isomorphic to $\mathbb{F}_{\widetilde{q}}$ or $O_2^\pm(q)$,

$$\text{(I)} \qquad |Y_M/C_{Y_M}(d)| \in \{1, \frac{\widetilde{q}}{2}, \widetilde{q}\}.$$

Since $t \in A = W$ and $W$ is the natural $SL_2(\widetilde{q})$-module, $\{[t, y] \mid y \in Y_M\} = [t, Y_M] = [W, Y_M]$. Let $d \in D$. Using the definition of $D$ we have $[t, d] \in [D, V] \leqslant [W, V] = [W, Y_M]$. Thus $[t, d] = [t, y]$ for some $y \in Y_M$. Hence $t^d = t^y$, $dy^{-1} \in C_D(t)$ and $D = C_D(t) Y_M$. By 7.12(f), $[V, W]$ is singular subspace of $V$ and $[V, W]^\perp = [V, W] \times (V \cap R) = [V, W]$. Hence $[V, W]$ is a maximal singular subspace of $V$ and $|V| = q^n$. The action of $D$ on the natural $Sp_{2n}(q)$-module $Y_M$ now shows $\{|Y_M/C_{Y_M}(d)| \mid d \in D\} = \{q^i \mid 0 \leqslant i \leqslant n\}$, and so also

$$\text{(II)} \qquad \{|Y_M/C_{Y_M}(d)| \mid d \in C_D(t)\} = \{q^i \mid 0 \leqslant i \leqslant n\}.$$

A comparison of (I) and (II) shows that either $n = 1$ and $\tilde{q} = q$ or $n = 2$, $\tilde{q} = 4$ and $q = 2$. We already know that $Y_M = O_p(M)$ and $N_G(Q) \leqslant M$. If $n = 1$ and $q = \tilde{q}$, then $Y_M$ is a natural $SL_2(q)$-module for $\overline{K}$, and Theorem G(3) holds with $r = 1$. If $n = 2$ and $q = 2$, then $Y_M$ is a natural $Sp_4(2)$-module, and Theorem G(2) holds.

Case 2. *Suppose that 7.13(B:2) holds,that is, $\overline{K} \trianglelefteq \overline{M}$, $p = 2$, $\overline{K} \cong O^{\epsilon}_{2n}(2)$, $n \geqslant 2$ and $(n, \epsilon) \neq (2, +)$, $Y$ is a corresponding natural module for $\overline{K}$, $\overline{M^{\circ}} = \overline{K}' \cong \Omega^{\epsilon}_{2n}(2)$, and $|\overline{W}| = |Y_M/C_{Y_M}(W)| = 2$.*

Since $\overline{K} \trianglelefteq \overline{M}$, $\overline{K^*} = \overline{K}$ and so $Y = [Y_M, K] = [Y_M, K^*] = V$. Moreover, $\overline{M}$ fixes the unique $\overline{K}$- invariant quadratic form $h$ on $Y$ and so $\overline{M} = \overline{K}$. Note also that the $\overline{K}$-invariant symplectic form on $V$ given by 7.13(B) is exactly the symmetric form associated with $h$.

Note that each singular vector in $V$ is centralized by a Sylow 2-subgroup of $M$ and so also by a conjugate of $Q$. By 7.5(e) $C_R(Q^g) = 1$ for all $g \in R$, so this implies that $R$ contains no non-trivial singular vectors. Thus $R \cap V$ has dimension at most 2 and so $|R \cap V| \leqslant 2^2$. Hence, by 7.12(g), $|V| = |\overline{W}|^2|V \cap R| \leqslant 2^2 \cdot 2^2 = 2^4$. Thus $n = 4$. Since $(2n, \epsilon) \neq (4, +)$, $V$ is a natural $O_4^-(2)$-module for $\overline{M}$.

As above, since $\overline{K} \trianglelefteq \overline{M}$, 7.7(a) shows that $C_{Y_M}(K) = 1$. Thus C.18 implies that $Y_M = V$. Hence $R = R \cap V$, $R$ has order 4, and all non-trivial elements in $R$ are non-singular vectors of $V$.

Pick $1 \neq x \in R$ and put $\overline{B} := O_2(C_{\overline{M}}(x))$. Then $C_{\overline{M}}(x) \cong C_2 \times Sp_2(2)$ and $[Y_M, \overline{B}] = \langle x \rangle$. Since $[Y_M, W] \nleqslant R$ this means $[C_{Y_M}(\overline{B}), W] \neq 1$. Thus by 7.8 $C_G(x)$ is not of characteristic 2. Since by 7.12(h) $O_2(M) = Y_M = V$ and $N_G(Q) \leqslant Y_M$, and since $\overline{M} = \overline{K} \cong O_4^-(2)$, Theorem G(2) holds.

Case 3. *Suppose that 7.13(B:3) holds, that is, $\overline{K} \ntrianglelefteq \overline{M}$, $Y_M$ is a natural $SL_2(q)$-wreath product module for $\overline{M}$ with respect to $\mathcal{K} := \overline{K}^{\overline{M}}$, $\overline{M^{\circ}} = O^p(\overline{K^*})\overline{Q}$, and $\overline{Q}$ acts transitively on $\mathcal{K}$.*

Put $\mathcal{K} =: \{\overline{K_1}, \dots, \overline{K_r}\}$ and $V_i := [Y, \overline{K_i}]$ with $\overline{K} = \overline{K_1}$, so $Y = V_1$. Since $Y_M$ is a natural $SL_2(q)$-wreath product module, $Y$ is a natural $SL_2(q)$-module for $K$, and

$$Y_M = V = V_1 \times V_2 \times \ldots \times V_r.$$

Since $\overline{K} \ntrianglelefteq \overline{M}$, $r \geqslant 2$. Put

$$\mathcal{S} := \{v \in V \mid [v, F] \neq 1 \text{ for all } F \in \mathcal{K}\}.$$

In the following we apply A.28 to $\overline{K^*S}$ in place of $H$. Since $\overline{Q}$ acts transitively on $\mathcal{K}$, A.28(e) shows that $\overline{K^*}$ acts transitively on $\mathcal{S}$ and $C_V(Q)^{\sharp} \subseteq \mathcal{S}$. Thus $C_{\mathcal{S}}(Q) \neq \varnothing$, and every element of $\mathcal{S}$ is centralized by a conjugate of $Q$. As by 7.5(e) $C_R(Q^g) = 1$, we get

$$R \cap \mathcal{S} = \varnothing.$$

Since $\overline{W} \leqslant \overline{K} = \overline{K_1}$ we get

$$C_V(W) = C_{V_1}(W) \times V_2 \times \ldots \times V_r.$$

Since $V_i$ is 2-dimensional over $\mathbb{F}_q$, $[V, W] = [V_1, W] = C_{V_1}(W))$. Thus by 7.12(e)

$$C_V(W) = [V, W] \times R = C_{V_1}(W) \times R.$$

As $|C_{V_1}(W)| = q$ this gives $|C_V(W)/R| = q$. Let $2 \leqslant i \leqslant r$. Then $V_i \leqslant C_V(W)$, and since $|V_i| = q^2$ and $|C_V(W)/R| = q$, we get $|V_i \cap R| \geqslant q$. In particular, there exists $1 \neq t_i \in V_i \cap R$.

Suppose that $V_j \nleqslant R$ for some $2 \leqslant j \leqslant r$. Say $j = 2$. Since $V_2 \leqslant C_V(A) = C_{V_1}(W) \times R$ there exist $1 \neq s_2 \in V_2$ and $1 \neq s_1 \in C_{V_1}(W)$ with $s_1 s_2 \in R$. Put $t = s_1 s_2 t_3 \ldots t_r$. Then $t \in R \cap \mathcal{S} = \varnothing$, a contradiction. Thus $V_j \leqslant R$ and so $V_2 \ldots V_r \leqslant R$. Together with

$$C_{V_1}(W) \times V_2 \times \cdots \times V_r = C_V(W) = C_{V_1}(W) \times R$$

this gives $R = V_2 \times \cdots \times V_r$. In particular, $\overline{K_1} \leqslant C_{\overline{M}}(R)$ and so $[V_1, O_p(C_{\overline{M}}(R))] = 1$. Since $[V_1, W] \neq 1$, 7.8 shows that $C_G(R)$ is not of characteristic $p$.

We will now show that $q \in \{2,4\}$. For this put $M_1 := C_M(R) \cap N_M([V,W])$ and let $1 \neq x \in C_V(\langle W^Q \rangle) \cap C_V(Q)$ and $x_1$ be the projection of $x$ onto $V_1$. As already seen above, A.28(e) gives $C_V(Q)^\sharp \subseteq \mathcal{S}$. Thus $x \in \mathcal{S}$ and so $x_1 \neq 1$. Moreover, $x \in x_1 V_2 \cdots V_r = x_1 R$, and so $Q!$ implies $C_{M_1}(x_1) \leqslant C_G(x) \leqslant N_G(Q)$. Thus $[Q, C_{M_1}(x_1)] \leqslant Q$.

Let $m \in C_{M_1}(x_1)$ and $q \in Q$ with $V_1 = V_2^q$. Since $m$ centralizes $V_2$, $m^q$ centralizes $V_1$. Hence

$$m = m^q[q^{-1}, m] \in m^q Q \subseteq C_M(V_1)Q,$$

and so $m$ acts a $p$-element on $V_1$. It follows that $C_{M_1}(x_1)/C_{M_1}(V_1)$ is a $p$-group. Since $C_{M_1}(V_1) = C_M(V_1 R) = C_M(V) = C_M(Y_M)$ and by 7.12(h) $C_G(Y_M) = Y_M$, $C_{M_1}(x_1)$ is a $p$-group.

Put $B_1 := M_1 \cap KV$. Then $B_1 \trianglelefteq M_1$, $VW \in Syl_p(B_1)$, $B_1/VW \cong C_{q-1}$, and $B_1$ acts transitively on $[V,W]$. It follows that $M_1 = C_{M_1}(x_1)B_1$ and $M_1/B_1$ is a $p$-group. Thus $O^p(M_1) \leqslant B_1$. Since $[R,L] = 1$ and $[V,W] \trianglelefteq N_L(V)$, $N_L(V) \leqslant M_1$, and since $L/A = L/WR \cong SL_2(q)$ and $VW \in Syl_p(L)$, $N_L(V)/VW$ is cyclic of order $q-1$. Let $H_1$ be a complement to $VW$ in $N_L(V)$. Then $H_1 \leqslant O^p(M_1) \leqslant B_1$. As $B_1/VW$ has order $q-1$, we get $B_1 = H_1 VW = N_L(V)$.

Suppose that $p$ is odd and let $i$ be the involution in $H_1$. In $L$ we see that $[VW, i] = W$ and in $M$ we see that $[VW, i] = V_1$, a contradiction.

Thus $p = 2$. In $L$ we see that the $\mathbb{F}_2 H_1$-module $W/C_W(V)$ is isomorphic to the dual of $[V,W]$ and in $M$ that the $\mathbb{F}_2 H_1$-module $V/C_V(W)$ is isomorphic to the dual of $[V,W]$. It follows that $W/C_W(V)$ and $V/C_V(W)$ are isomorphic $\mathbb{F}_2 H_1$-module. Let $H_1 =: \langle h_1 \rangle$. In $L$ we see that there exists $\xi \in \mathbb{F}_q$ and $\mathbb{F}_q H_1$-module structures on $[V,W]$, $W/C_W(V)$ and $V/C_V(W)$ such that $h_1$ acts as multiplication by $\xi, \xi^{-1}$ and $\xi^2$, respectively. It follows that there exists $\sigma \in Aut(\mathbb{F}_q)$ with $(\xi^2)^\sigma = \xi^{-1}$. Since $|\xi| = |h_1| = q-1 = |\mathbb{F}_q^\sharp|$ and also squaring is an field automorphism of $\mathbb{F}_q$, we conclude that $\mu : \mathbb{F}_q \mapsto \mathbb{F}_q, \lambda \to (\lambda^2)^\sigma$, is a field automorphism and $\lambda^\mu = \lambda^{-1}$ for all $\lambda \in \mathbb{F}_q^\sharp$. It follows that $\mathbb{F}_2$ is the fixed field of $\mu$, and $\mu$ as order 1 or 2; so $\mathbb{F}_q = \mathbb{F}_2$ or $\mathbb{F}_q = \mathbb{F}_4$.

Thus indeed $q \in \{2,4\}$. We already know that $C_G(R) = C_G(V_2 \ldots V_r)$ is not of characteristic 2. By 7.12(h) we have $N_G(Q) \leqslant M$ and $O_2(M) = Y_M$. Hence, Theorem G(3) holds with $\mathcal{K} := \overline{K}^M$, where the uniqueness of $\mathcal{K}$ follows from A.27(c). □

LEMMA 7.16. *Case 7.13(C) does not hold.*

PROOF. Let $\{i,j,k\} = \{1,2,3\}$. Recall from 7.13(C) that $p = 2$, $Y = Y_M$, $M_2 = N_M(C_Y(W))$, $K_2 = C_{M_2}(Y/C_Y(M))$, $\overline{K_2} \leqslant \overline{K}$ and that there exists an $M_2$-invariant set $\{V_1, V_2, V_3\}$ of $K_2$-submodules of $Y$ with $Y = V_i \times V_j$. Note that the projection of $V_k$ onto $V_i$ and $V_j$ shows that $V_k$ is isomorphic to $V_i$ and $V_j$ as an $K_2$-module. In particular, $\overline{K_2}$ acts faithfully on $V_i$.

Define $n$ by $2^n := |V_i|$. Then by 7.13(C) either $n = 3$ and $Y$ is a natural $3^{\cdot}Alt(6)$-module for $K$, or $n \geqslant 3$ and each $V_i$ is a natural $SL_n(2)$-module for $K$.

1°. *$V_i \cap R = 1$.*

Since $[R,W] = 1$, $V_i \cap R = C_{V_i}(W) \cap R$. Let $1 \neq x \in C_{V_i}(W)$. According to 7.13(C:c) for all $1 \neq x \in C_{V_i}(W)$ there exists $g \in M$ with $[x, Q^g] = 1$. By 7.5(g) $C_R(Q^g) = 1$ for all $g \in G$ and so $x \notin R$. Hence $V_i \cap R = 1$.

2°. *$A \cap Y = C_Y(W)$ and $A \leqslant K_2$. In particular, $W$ and $A$ normalize $V_i$.*

By 7.4(e) $A \cap Y_M = C_{Y_M}(W)$. Since $Y = Y_M$ this gives $A \cap Y = C_Y(W)$. As $[A,Y] \leqslant A \cap Y$, we conclude that $A$ normalizes $C_Y(W)$ and centralizes $Y/C_Y(W)$, so $A \leqslant K_2$. As $W \leqslant A$ and $K_2$ normalizes $V_i$, (2°) holds.

3°. *$A \cap Y = (A \cap V_i) \times (A \cap V_j)$.*

By (2°) $A \cap Y = C_Y(W)$, and $W$ normalizes $V_i$. As $Y = V_i \times V_j$, this implies

$$A \cap Y = C_Y(W) = C_{V_i}(W) \times C_{V_j}(W) = (A \cap V_i) \times (A \cap V_j),$$

and (3°) is proved.

4°. *$A$ is elementary abelian.*

By (2°) $A$ normalizes $V_i$, and by 1.43(a),

$$\Phi(A) = [A \cap Y, A] \leqslant C_Y(L) = R,$$

so

$$[A \cap V_i, A] \leqslant V_i \cap R \overset{(1^\circ)}{=} 1.$$

By (3°) $A \cap Y = (A \cap V_i) \times (A \cap V_j)$ and so $[A \cap Y, A] = 1$. It follows that $\Phi(A) = 1$ and $A$ is elementary abelian.

5°.

(a) $|A| = 2^{3(n-1)}$ *and* $|\overline{A}| = |A/A \cap Y| = |R| = 2^{n-1}$.
(b) $A \cap V_i$ *is a hyperplane of* $V_i$ *and* $\overline{A} = C_{\overline{K_2}}(A \cap V_i)$.
(c) *Let* $B$ *be any* $L$*-invariant subgroup of* $A$. *Then* $|Y \cap B/R \cap B| \leqslant |R \cap B|$.

By 7.13(C:a) $|Y/C_Y(W)| = 4$. Since $Y = V_i \times V_j$, this gives $|V_i/C_{V_i}(W)| = 2$, and since by (2°) $C_Y(W) = Y \cap W$, $V_i \cap A = C_{V_i}(W)$. Hence $V_i \cap A$ is a hyperplane of $V_i$. As by (4°) $A$ is abelian, $A$ centralizes $V_i \cap A$ and so $V_i \cap A = C_{V_i}(A)$.

Let $B$ be any $L$-invariant subgroup of $A$. Pick $v_i \in V_i \backslash A$. By 1.43(g), $Y \cap B = [v_i, B](R \cap B)$, and so $Y \cap B = [v_i, B](R \cap B)$. Since by (1°) $V_i \cap R = 1$, we have $[v_i, B] \cap (R \cap B) = 1$. This gives

$$(*) \qquad |Y \cap B| = |[v_i, B]||R \cap B|.$$

Also $|Y \cap B| \geqslant |[Y, B]| = |[v_i, B] \times [v_j, B]| = |[v_i, B]|^2$, and we conclude with $(*)$ that

$$|R \cap B| \geqslant |[v_i, B]| = |Y \cap B/R \cap B|.$$

Thus (c) holds.

Using $A = B$ in $(*)$, $|Y \cap A| = |[v_i, A]||R|$ and so $|R| = |Y \cap A||[v_i, A]|^{-1}$. On the other hand, by (1°) $V_i \cap A \cap R = 1$ and so

$$|R| = |R(V_i \cap A)/V_i \cap A| \leqslant |Y \cap A/V_i \cap A|.$$

Since $[v_i, A] \leqslant V_i \cap A$, we get

$$|Y \cap A/V_i \cap A| \leqslant |Y \cap A/[v_i, A]| = |R| \leqslant |Y \cap A/V_i \cap A|.$$

It follows that equality holds in the preceding inequalities. In particular, $[v_i, A] = V_i \cap A$ and so

$$|[v_i, A]| = |V_i \cap A| = 2^{n-1}.$$

Thus

$$\overline{A} = |A/C_A(V_i)| = |A/C_A(v_i)| = |[v_i, A]| = 2^{n-1}.$$

Since $\overline{A} \leqslant C_{\overline{K_2}}(A \cap V_i)$ and $|C_{\overline{K_2}}(A \cap V_i)| \leqslant |A \cap V_i| = 2^{n-1}$ this gives $\overline{A} = C_{\overline{K_2}}(A \cap V_i)$. So all parts of (5°) are proved.

6°.

(a) $Y = Y_M = O_2(M)$, $M = M^\dagger$ *and* $N_G(Q) \leqslant M$.
(b) $H = L$, $U = W$, $\widehat{U}$ *is natural* $SL_2(4)$*-module for* $H$, *and* $U$ *is a natural* $\Omega_3(4)$*-module for* $H$.

Recall that $L \in \mathfrak{L}_G(Y_M)$ and so $L/A \cong SL_2(\tilde{q})$, $Sz(\tilde{q})$ or $Dih_{2r}$. In the $Sz(\tilde{q})$-case $\tilde{q}$ is an odd power of 2 and in the $Dih_{2r}$- case $\widetilde{q} = 2$. Since $\widetilde{q} = |Y_M/C_{Y_M}(W)| = |Y_M/Y_M \cap A| = 4$ we get $\widetilde{L} \cong SL_2(4)$.

By 2.13 $\widehat{U}$ is a faithful simple minimal asymmetric $\mathbb{F}_2\widetilde{H}$-module, so we can apply the Minimal Asymmetric Modules Theorems C.28 and C.29. Put $H_0 := \langle Y_M^H \rangle$. Since $\widetilde{L} \cong SL_2(4)$, $\widetilde{H_0}$ is not solvable. Thus we are in Case (1) of C.29. In particular, $\widetilde{H_0}$ is a group of Lie-type defined over $\mathbb{F}_4$ and $\widetilde{Y_M}$ is a long root subgroup of $\widetilde{H_0}$. Note that $U \cap Y = (U \cap A) \cap Y$ and $U \cap R = (U \cap A) \cap R$. Thus by (5°)(c) applied to $B = U \cap A$

$$(**) \qquad |U \cap Y/U \cap R| \leqslant |U \cap R|.$$

In particular, $U \cap R \neq 1$. So by 7.5(a) $C_U(H) = U \cap R \neq 1$ and $C_U(H_0) \neq 1$. By 7.6(a) $U$ is a quasisimple $H_0$-module. A comparison of C.29(1) with C.18 shows that $p = 2$ and either

$\widetilde{H_0} \cong Sp_{2m}(4)$ and $U$ is a quotient of the natural $\Omega_{2m+1}(4)$-module for $\widetilde{H_0}$, or $\widetilde{H_0} \cong G_2(4)$ and $\widehat{U}$ is the corresponding natural module of order $4^6$. In the first case $|[\widehat{U}, Y_M]| = 4$ and in the second case $|[\widehat{U}, Y_M]| = 16$, and in both cases $|U \cap R| \leqslant |C_U(H_0)| \leqslant 4$.

By 2.17(e)

$$W \cap R = C_W(O^2(H)) = W \cap C_U(O^2(H)).$$

It follows that

$$\widehat{W \cap Y} = (W \cap Y)C_U(O^2(H))/C_U(O^2(H)) \cong W \cap Y/W \cap R \cong (W \cap Y)(U \cap R)/U \cap R \leqslant U \cap Y/U \cap R.$$

Hence

$$|\widehat{W \cap Y}| \leqslant |U \cap Y/U \cap R| \overset{(**)}{\leqslant} |U \cap R| \leqslant |C_U(H_0)| \leqslant 4.$$

On the other hand, by 2.17(c) $U = WC_U(Y_M)$. Thus $[\widehat{U}, Y] = [\widehat{W}, Y] \leqslant \widehat{W \cap Y}$ and so $|[\widehat{U}, Y]| \leqslant 4$. This excludes the $G_2(q)$-case and shows that $|C_U(H_0)| = 4$, so $U$ is a natural $\Omega_{2m+1}(4)$ for $\widetilde{H_0}$ and $|\widehat{W \cap Y}| = 4$. Moreover, by 1.43(e) $|W/C_W(Y)| = |W \cap Y/C_{W \cap Y}(L)| = |\widehat{W \cap Y}| = 4$. Hence $Y_M$ is an offender on $W$, and so also an offender on $U$ since $U = WC_U(Y)$. Thus we can apply 7.11. In the second case of 7.11 $U \cap R = 1$, a contradiction. So the first case holds. Hence $\widehat{U}$ is natural $SL_2(\widetilde{q})$-module for $H$ and

$$Y_M = O_2(M),\ M = M^\dagger,\ N_G(Q) \leqslant M,\ H = L \text{ and } U = W.$$

Since $\widetilde{q} = 4$ and $U$ is a natural $\Omega_{2m+1}(4)$-module, this gives ($6^\circ$).

$7^\circ$. *$C_M(Y) = Y$, $M = N_G(Y)$, $AY = C_{K_2}(Y \cap A)$ and $A \trianglelefteq M_2$.*

By ($6^\circ$)(a) $Y = O_2(M)$ and $N_G(Y) = M^\dagger = M$. By ($5^\circ$)(b) $\overline{A} = C_{\overline{K_2}}(Y \cap A)$ and so $AY = AC_M(Y) = C_{K_2}(Y \cap A)$. In particular, $AY \trianglelefteq M_2$.

Let $v \in Y \backslash Y \cap A$. Then $v \in V_i(Y \cap A)$ for some $i$. Since $V_i$ is a faithful $K_2$-module and $|V_i/V_i \cap A| = 2$ we get $C_{\overline{A}}(v) = C_{\overline{A}}(V_i) = 1$ and so $C_A(v) = A \cap Y$. It follows that $[v, a] \neq 1$ for all $v \in Y \backslash A$ and $a \in A \backslash Y$. Hence $va$ is not an involution and so $Y$ and $A$ are the only maximal elementary abelian subgroups of $AY$. Since $M_2$ normalizes $AY$ and $Y$, $M_2$ normalizes $A$.

$8^\circ$. *$n = 3$ and $O^2(M)/Y \cong C_3 \times SL_3(2)$ or $3^{\cdot}Alt(6)$.*

By ($7^\circ$) $Y = C_M(Y)$. Thus if $Y$ is a natural $3^{\cdot}Alt(6)$-module, then ($8^\circ$) holds. So suppose that $Y$ is the direct sum of two $SL_n(2)$-modules, $n \geqslant 3$. In particular, $M/Y = \overline{M} = \overline{K} \times \overline{C}$ where $\overline{C}$ is isomorphic to a subgroup of $SL_2(2)$ with $O_2(\overline{C}) = 1$. Thus $\overline{C} \cong 1, C_3$ or $SL_2(2)$. Note that $M_2 \cap K$ centralizes $Y/Y \cap A$ and that $N_L(Y) \leqslant N_M(A \cap Y) = M_2$. Since by ($6^\circ$)(b) $L/A \cong SL_2(4)$, we infer that $N_L(Y)/C_{N_L(Y)}(Y/Y \cap A) \cong C_3$. Thus 3 divides $|M/K|$. Hence $\overline{C} \cong C_3$ or $SL_2(2)$ and $O^2(M)/Y \cong C_3 \times SL_n(2)$. It remains to show that $n = 3$.

If $n = 4$, then by ($5^\circ$) $|A| = 2^{3(n-1)} = 2^9$ and $|R| = 2^{n-1} = 2^3$, and so $|A/R| = 2^6$. Since $L/A \cong SL_2(4)$ all non-central simple $L$-modules have order $2^4$, and we conclude that $L$ has a central composition factor on $A/R$, a contradiction to 1.43(p).

Suppose that $n \geqslant 5$. Let $X \leqslant M$ such that $X \cong C_3$ and $XY \trianglelefteq M$. Since $[K_2, X] \leqslant Y$ and $X$ acts fix-point freely on $Y$, $K_2 = C_{K_2}(X)Y$. For $i = 1, 2$ put $A_i := A \cap V_i$. Then $A \cap Y = A_1 \times A_2$. Put $A_3 := C_A(X)$. Since $X \leqslant M_2$, $X$ normalizes $A$ and so $A = (A \cap Y) \times A_3 = A_1 \times A_2 \times A_3$. Let $v \in V_1 \backslash A_1$ and put $K_1 := C_{K_2}(v) \cap C_{K_2}(X)$. Note that $K_1$ is a complement to $A_3$ in $C_{K_2}(X)$, $K_1 \cong SL_{n-1}(2)$ and the $A_i$, $1 \leqslant i \leqslant 3$, are isomorphic natural $SL_{n-1}(2)$-modules for $K_1$.

According to 7.9 there exists $t \in A \backslash C_A(Y)$ such that $[C_D(t), L] \leqslant A$ for all 2-subgroups $D$ of $M$ with $[Y, D] \leqslant A$. Since $t \in A$, $t = t_1 t_2 t_3$ with $t_i \in A_i$. Since $n - 1 > 3$, there exists a transvection $d \in K_1$ with $[t_i, d] = 1$ for all $1 \leqslant i \leqslant 3$. Then

$$|[A, d]| = |[A_1, d]|^3 = 8.$$

Since $d \in K_1 \leqslant K_2$, $[Y, d] \leqslant Y \cap A \leqslant A$. Also $[d, t] = 1$, and the choice of $t$ implies $[d, L] \leqslant A \leqslant C_G(A)$. Thus $L$ normalizes $[A, d]$. Since $L/A \cong SL_2(4)$ and $|[A, d]| = 8$ we conclude that $[A, d, L] = 1$ and $[A, d] \leqslant C_A(L) = R \leqslant Y$, a contradiction since $1 \neq [A_3, t] \leqslant A_3$ and $A_3 \cap Y = 1$. Thus ($8^\circ$) is proved.

We are now able to derive a final contradiction. Since $n = 3$, (5°)(a) shows that $|A| = 2^{3(3-1)} = 2^6 = |Y|$. By (6°)(b) $U$ is the natural $\Omega_3(4)$-module for $L$ and $U = W$. Hence $|U| = 2^6$ and $A = W$. In particular, $A/R = \widehat{U}$ is a natural $SL_2(4)$-module of $L$,

Note that either $K/Y \cong SL_3(2)$ and $Y = V_1 \oplus V_2$, or $K/Y \cong 3{\cdot}Alt(6)$. Since $|\overline{W}| = 4$ and $|V_i| = 8$, it is straight forward to verify that $K_2/Y \cong Sym(4)$ and $V_i \cap A$ is a natural $SL_2(2)$-module for $K_2$. In particular, $Y \cap A$ is a direct sum of two natural $SL_2(2)$-modules for $K_2$, and $V_i \cap A, 1 \leqslant i \leqslant 3$, are simple $K_2$-submodules in $Y \cap A$.

Put $F := N_G(A)$. Then $L \leqslant F$ and by (7°) $A \trianglelefteq M_2$ and so $M_2 \leqslant F$. Also $F \cap M \leqslant N_M(C_Y(A)) = M_2$ and so $F \cap M = M_2$.

In particular, $L_2 := L \cap M = L \cap M_2$. Since $L/A \cong SL_2(4)$, $L_2/AY \cong C_3$ and $L_2$ acts transitively on $AY/Y \cong Y/Y \cap A = Y/C_Y(W)$. Hence $L_2$ also acts transitively on $\{V_1, V_2, V_3\}$. Since $V_i \cap A, 1 \leqslant i \leqslant 3$, are the simple $K_2$-submodules of $Y \cap A$ we conclude that $Y \cap A$ is a simple module for $L_2K_2/AY \cong C_3 \times SL_2(2)$. Also $L_2K_2$ acts transitively on the nine elements in $V_1^\sharp \cup V_2^\sharp \cup V_3^\sharp$. Let $1 \neq r \in R$. Note that $O^2(K_2)$ normalizes $L_2$ and so also $C_{Y\cap A}(L_2)$. Moreover, $O^2(K_2)$ acts fixed-point freely on $Y \cap A$, $R \leqslant C_{Y\cap A}(L_2)$ and $|R| = 4$. We conclude that $R = C_{Y\cap A}(L_2)$ and $O^2(K_2)$ acts transitively on $R$. Since $K_2L_2$ acts simply on $Y \cap A$ and $|L_2K_2/L_2O^2(K_2)| = 2$ we get $|R^{L_2K_2}| = 2$ and $|r^{L_2K_2}| = 6$.

Let $1 \neq z \in \Omega_1 Z(S)$. By 7.1(c), $\Omega_1 Z(S) \leqslant Y_H \cap Y_M = A \cap Y$ and by 7.5(g), $C_R(Q^g) = 1$ for all $g \in G$. Since $[z, Q] = 1$ we conclude that $z$ and $r$ are not conjugate in $G$. It follows that $z^{M_2} = z^{K_2L_2}$ has size nine and $r^{M_2} = r^{K_2L_2}$ has size six.

Put $F_1 := N_F(R)$ and note that $L \leqslant F_1$ and $L_2O^2(K_2) \leqslant F_1$. In particular, $z^{M_2\cap F_1} = z^{M_2}$. We now calculate the size of $z^F$, $z^{F_1}$ and $r^F$. Note that each of these sets is an $L$-invariant subset of $A$.

Since $A/R$ is the natural $SL_2(4)$-module for $L$, $A/R$ is partitioned by the five $L$-conjugates of $A \cap Y/R$. Also $z^{M_2} \cap R = \varnothing$ and $|r^{M_2} \cap R| = 3$. Hence $|z^F| \geqslant |z^{F_1}| \geqslant 5 \cdot 9$ and $|r^F| \geqslant 3 + 5 \cdot 3$. Now $|A^\sharp| = 2^6 - 1 = 45 + 18$ gives $|z^F| = |z^{F_1}| = 45$ and $|r^F| = 18$.

By (6°) $N_G(Q) \leqslant M$. Since $[z, Q] = 1$, $Q!$ implies $C_G(z) \leqslant M$. In particular, $C_F(z) \leqslant M \cap F = M_2$. Note that $K_2 \trianglelefteq M_2$ and $R^\sharp$ is one of the two orbits of $K_2$ on $r^{M_2}$. Thus $|M_2/M_2 \cap F_1| = |M_2/N_{M_2}(R^\sharp)| = 2$. Since $C_F(z) \leqslant M_2$ this gives $|C_F(z)/C_{F_1}(z)| \leqslant 2$. Together with $|F| = 45|C_F(z)|$ and $|F_1| = 45|C_{F_1}(z)|$ we conclude that $|F/F_1| \leqslant 2$. Thus $|R^F| \leqslant 2$ and $|r^F| \leqslant |R^\sharp||R^F| = 3 \cdot 2 = 6$, a contradiction to $|r^F| = 18$. □

Note that the three cases in 7.13 have been treated in 7.14, 7.15 and 7.16. Thus, the proof of Theorem G is complete.

CHAPTER 8

# The $Q$-Tall Asymmetric Case I

In this chapter we begin the investigation of the $Q$-tall asymmetric case. That is, $M \in \mathfrak{M}_G(S)$, $Y_M$ is asymmetric in $G$, and $Y_M \nleqslant O_p(N_G(Q))$. The main result of this chapter reduces the problem to what might be called the generic case, namely, where $[Y_M, M^\circ] \nleqslant Q$, $M/C_M(Y_M)$ possesses a unique component $K$, and $[Y_M, K]$ is a simple $K$-module, see Case (1) of Theorem H for more details. This is achieved by studying the action of $M$ on the Fitting submodule $I$ of $Y_M$, introduced in Appendix D, rather than on $Y_M$ itself. The Fitting submodule is close to being semisimple and so much easier to work with. And, since I is faithful for $M/C_M(Y_M)$, it still allows to identify $M/C_M(Y_M)$.

As in the previous chapter a member $H$ of $\mathfrak{H}_G(O_p(M))$ is used to obtain a subgroup $L$ of $H$ with $L \in \mathfrak{L}_G(Y_M)$. But in this chapter internal properties of $L$, like

$$A := O_p(L) = \langle (Y_M \cap O_p(L))^L \rangle \quad \text{and} \quad C_{Y_M}(L) = Y_M \cap Y_M^g \quad \text{for} \ \ g \in L \backslash L \cap M^\dagger,$$

are in the center of our attention. Due to $Q$-tallness, $H$ and thus also $L$ can be chosen in $N_G(Q)$. It is then easy to see that $Q$, $L$ and $A$ normalize each other. We subdivide the proof into three cases, treated in separate sections:

(1) $I \leqslant A$ (2) $I \nleqslant A$ and $[\Omega_1 Z(A), L] \neq 1$, (3) $I \nleqslant A$ and $[\Omega_1 Z(A), L] = 1$.

In the first case it is easy to see that $I$ is symmetric in $G$ (see 8.13(b)). So the main result of Chapter 4 can be applied to $I$, and the different outcomes of this result are then discussed.

In the second case the non-trivial action of $L$ on $\Omega_1 Z(A)$ shows that also $H$ acts non-trivially on $\Omega_1 Z(O_p(H))$, and similar to the previous chapter we get a strong offender that allows to apply the FF-module theorems from Appendix C.

In the third case we prove that $A$ acts nearly quadratically on $I$. We then apply the Nearly Quadratic $Q$!-Theorem proved in Appendix D, and treat each of its cases.

Here is the main result of this chapter.

Theorem H. *Let $G$ be a finite $\mathcal{K}_p$-group, $S \in Syl_p(G)$, and let $Q \leqslant S$ be a large subgroup of $G$. Suppose that $M \in \mathfrak{M}_G(S)$ such that $Y_M$ is asymmetric in $G$ and $Q$-tall.*

*Then $\mathfrak{H}_{N_G(Q)}(O_p(M)) \neq \varnothing$ and for every $H \in \mathfrak{H}_{N_G(Q)}(O_p(M))$ also $\mathfrak{L}_H(Y_M) \neq \varnothing$. Moreover, one of the following holds, where $Y := Y_M$, $\overline{M^\dagger} := M^\dagger/C_{M^\dagger}(Y)$, $I := F_Y(\overline{M})$ is the Fitting submodule of $Y$, and $q$ is some power of $p$:*

(1) *For every $H \in \mathfrak{H}_{N_G(Q)}(O_p(M))$ and every $L \in \mathfrak{L}_H(Y_M)$ and $A := O_p(L)$:*
   (a) *$Q$ normalizes $L$ and $A$,*
   (b) *$\overline{A}$ is a non-trivial elementary abelian subgroup of $\overline{M}$,*
   (c) *$Y = IC_Y(A)$, $I \nleqslant Q^\bullet$ and $C_Y(A) = Z(A) = C_Y(L)$,*
   (d) *$K := [F^*(\overline{M}), A]$ is the unique component of $\overline{M}$, $K \leqslant \overline{M^\circ}$, and $I$ is a simple $K$-module,*
   (e) *$A$ acts nearly quadratically on $Y$ and not quadratically on $I$, and $[Y, K\overline{A}] = I$,*
   (f) *$|Y/C_Y(A)| \leqslant |\overline{A}|^2$,*
   (g) *$AQ$ acts $\mathbb{K}$-linearly on $I$, where $\mathbb{K} := End_K(I)$,*
   (h) *If $g \in M$ and $C_Y(Q^g) \cap C_Y(A) \neq 1$, then $[\overline{Q^g}, \overline{A}] \leqslant \overline{Q^g} \cap \overline{A}$ and $[Y, Q^g] \leqslant [Y, A]C_Y(A)$.*

(2) *$p = 2$, $\overline{M^\circ} \cong L_3(2)$, $I$ is a corresponding natural module, $|Y/I| = 2$, $I$ is symmetric in $G$, and $I \leqslant Q$.*

(3) *$p = 2$, $\overline{M^\circ} \cong \Omega_6^+(2)$, $I$ is a corresponding natural module, $|Y/I| = 2$, $I$ is symmetric in $G$, $I \nleq Q^\bullet$, $Y = O_2(M)$, $M = M^\dagger$, and $C_G(t)$ is not of characteristic 2 for any non-singular $t \in I$.*
(4) *$p = 2$, $\overline{M^\circ} \cong Sp_{2n}(2)$, $n \geqslant 2$, $I$ is a corresponding natural module, $I \nleq Q^\bullet$ and $|Y/I| = 2$.*
(5) *$\overline{M^\circ} \cong SL_n(q)$, $n \geqslant 2$, and $Y$ is a corresponding natural module.*
(6) *$p = 2$, $\overline{M^\circ} \cong Sp_{2n}(q)$, $n \geqslant 2$, and $Y$ is a corresponding natural module.*
(7) *$p = 3$, $\overline{M^\circ} \cong \Omega_3(3)$, and $Y$ is a corresponding natural module for $\overline{M^\circ}$.*
(8) *$p = 2$, $\overline{M} \cong \Gamma SL_2(4)$, $\overline{M^\circ} \cong SL_2(4)$ or $\Gamma SL_2(4)$, $I$ is a corresponding natural module, $I \nleq Q^\bullet$ and $|Y/I| \leqslant 2$.*
(9) *$p = 2$, $\overline{M} \cong 3{\cdot}Sym(6)$, $\overline{M^\circ} \cong 3{\cdot}Alt(6)$ or $3{\cdot}Sym(6)$, and $Y$ is a simple $\overline{M}$-module of order $2^6$,*
(10) *There exists an $\overline{M}$-invariant set $\{\overline{K_1}, \overline{K_2}\}$ of subgroups of $\overline{M}$ such that $\overline{K_i} \cong SL_{m_i}(q)$, $[\overline{K_1}, \overline{K_2}] = 1$, $\overline{K_1 K_2} \trianglelefteq \overline{M}$, and $Y = I$ is the tensor product over $\mathbb{F}_q$ of corresponding natural modules for $K_1$ and $K_2$. Moreover, either $\overline{M} = \overline{M^\circ} \cong SL_2(2) \wr C_2$, or $\overline{M^\circ}$ is one of $\overline{K_1}, \overline{K_2}$ or $\overline{K_1 K_2}$.*

*In particular, $I = [Y_M, M^\circ]$, and (2) is the only case where $I \leqslant Q^\bullet$.*

Table 1 lists examples for $Y_M$, $M$ and $G$ fulfilling the hypothesis of Theorem H and one of the cases (2) – (10).

TABLE 1. Examples for Cases 2–10 of Theorem H

| | Case | $[Y_M, M^\circ]$ for $M^\circ$ | c | Remarks | examples for $G$ |
|---|---|---|---|---|---|
| * | 2 | nat $SL_3(2)$ | 2 | $G \neq G^\circ$ | $Aut(G_2(3))$ |
| * | 3 | nat $\Omega_6^+(2)$ | 2 | - | $\Omega_8^+(3).Sym(3)$ |
| * | 4 | nat $Sp_4(2)'$ or $Sp_4(2)$ | 2 | - | $P\Omega_6^-(3)\langle\omega\rangle$ or $PO_6^-(3)$ |
| | 5 | nat $SL_n(q)$ | 1 | - | $L_{n+1}(q)$ |
| | 5 | nat $SL_2(2)$ | 1 | - | $Sp_4(2)'$ |
| | 5 | nat $SL_2(3)$ | 1 | - | $Mat_{12}$ |
| | 5 | nat $SL_2(4)$ | 1 | - | $Mat_{22}$, $Mat_{23}$ |
| | 5 | nat $SL_3(2)$ | 1 | - | $Alt(9)$ |
| | 6 | nat $Sp_4(2)$ | 1 | - | $PSO_6^-(3)$, $P\Omega_6^-(3)\langle\omega\rangle$ |
| | 6 | nat $Sp_4(2)'$ | 1 | - | $\Omega_6^-(3)$, Suz |
| | 7 | nat $\Omega_3(3)$ | 1 | - | $\Omega_5(3)$ |
| * | 7 | nat $\Omega_3(3)$ | 1 | - | $Sp_6(2), \Omega_8^-(2)$ |
| | 8 | nat $\Gamma SL_2(4)$ | 1 | - | $\Gamma L_3(4)$, $Mat_{22}$ |
| * | 8 | nat $SL_2(4)[.2]$ | 2 | $\overline{M} \cong \Gamma SL_2(4)$ | $Aut(Mat_{22})$ |
| | 9 | $2^6$ for $3{\cdot}Alt(6)[.2]$ | 1 | $\overline{M} \sim 3{\cdot}Sym(6)$ | $Mat_{24}$ |
| * | 9 | $2^6$ for $3{\cdot}Sym(6)$ | 1 | $\overline{M} \sim 3{\cdot}Sym(6)$ | He |
| | 10 | nat $SL_{t_1}(q)[\otimes SL_{t_2}(q)]$ | 1 | - | $L_{t_1+t_2}(q)$, $L_{2t_1+1}(q)\Phi_2$ $t_1 = t_2$ |
| | 10 | nat $SL_2(2))[\otimes SL_3(2)]$ | 1 | - | $Mat_{24}$ |
| * | 10 | nat $SL_2(2))[\otimes SL_2(2)]$ | 1 | - | $Alt(9)$ |
| * | 10 | nat $SL_2(2) \otimes SL_2(2)$ | 1 | - | $Sym(9)$, $Alt(10)$ |
| * | 7 | nat $SL_2(3) \otimes SL_2(3)$ | 1 | - | HN |

In the table $c := |Y_M/[Y_M, M^\circ]|$ and $\Phi_2$ is a group of graph automorphisms of order 2. In the examples with $G = P\Omega_6^-(3)\langle\omega\rangle$, $\omega$ is a reflection in $PO_6^-(3)$. An entry of the form $A[B]$ in the $[Y_M, M^\circ]$ column indicates that there exists more than one choice for $Q$ in the example $G$. Depending on this choice the structure of $[Y_M, M^\circ]$ as an $M^\circ$-module is either described by $A$ or $AB$.

$*$ indicates that ($char\ Y_M$) fails in $G$.

## 8.1. Notation and Preliminary Results

In this section we assume the hypothesis and notation of Theorem H; in particular $Y = Y_M$ and $I = F_Y(M)$.

LEMMA 8.1. $Y_M \nleqslant O_p(N_G(Q))$.

PROOF. By Hypothesis, $Y_M$ is $Q$-tall and so by 2.6(e) $Y_M \nleqslant O_p(N_G(Q))$. □

LEMMA 8.2. *$\mathfrak{H}_{N_G(Q)}(O_p(M)) \neq \varnothing$, and for $H \in \mathfrak{H}_{N_G(Q)}(O_p(M))$, $\mathfrak{L}_H(Y_M) \neq \varnothing$.*

PROOF. By 1.55(a) $N_G(Q)$ has characteristic $p$, and by 8.1 $Y_M \nleqslant O_p(N_G(Q))$. Hence 2.9 implies that $\mathfrak{H}_{N_G(Q)}(O_p(M)) \neq \varnothing$.

Pick $H \in \mathfrak{H}_{N_G(Q)}(O_p(M))$, and let $L$ be minimal among all subgroups of $H$ satisfying $Y \leqslant L$ and $Y \nleqslant O_p(L)$. Then the Asymmetric $L$-Lemma 2.16(e) shows that $L \in \mathfrak{L}_H(Y_M)$. □

NOTATION 8.3. According to 8.2 we are allowed to fix $H \in \mathfrak{H}_{N_G(Q)}(O_p(M))$ and $L \in \mathfrak{L}_H(Y_M)$. Recall from the definition of $\mathfrak{L}_G(Y_M)$:

(i) $L$ is $Y$-minimal of characteristic $p$, and $N_L(Y)$ is the unique maximal subgroup of $L$ containing $Y$.

(ii) $L/A \cong SL_2(\widetilde{q}), Sz(\widetilde{q})$ or $Dih_{2r}$ and $|Y/Y \cap A| = \widetilde{q}$, where $p = 2$ in the last two cases, $r$ is an odd prime, and $\widetilde{q} = 2$ in the last case.

(iii) $A = \langle (Y \cap A)^L \rangle$.

Also observe that $L$ satisfies the hypothesis of 1.43, since by 1.42(b) $O_p(L) \leqslant N_L(Y)$.

LEMMA 8.4. *(a) $C_M(I) = C_M(Y) = C_M(I/rad_I(M))$.*

*(b) $N_G(I) = M^\dagger = N_G(Y) = MC_G(Y) = MC_G(I)$.*

*(c) $C_G(I) = C_G(Y) = C_{M^\dagger}(Y) = C_{M^\dagger}(I/rad_I(M))$.*

*(d) $M \nleqslant N_G(Q)$.*

*(e) $Y$, $I$ and $I/rad_I(M)$ are $Q!$-modules for $\overline{M}$ with respect to $\overline{Q}$.*

*(f) $I$ is a semisimple $M^\circ$-module, $C_Y(M^\circ) = C_Y(M_\circ) = 1$ and $I = [I, M^\circ] = [I, M_\circ]$.*

PROOF. (a): By D.6, $I$ and $I/rad_I(M)$ are faithful $\overline{M}$-modules, so $C_M(I) = C_M(I/rad_I(M)) = C_M(Y)$. This is (a).

(b): By the basic property of $M$, $M^\dagger = MC_G(Y)$. Since $I \leqslant Y$, this gives $M^\dagger = MC_{M^\dagger}(I)$. In particular, $M^\dagger \leqslant N_G(I)$ and $M^\dagger \leqslant N_G(Y)$. Again by the basic property of $M$, $M^\dagger$ is a maximal $p$-local subgroup of $G$, and so $M^\dagger = N_G(I) = N_G(Y)$. Hence $C_{M^\dagger}(I) = C_G(I)$, and (b) is proved.

(c): By (b) $N_G(I) = M^\dagger = MC_G(Y)$, and $C_G(Y)$ centralizes $I$ and $I/rad_I(M)$. Hence

$$C_G(I) = C_M(I)C_G(Y) \quad \text{and} \quad C_{M^\dagger}(I/rad_I(M)) = C_M(I/rad_I(M))C_G(Y).$$

Thus (c) follows from (a).

(d): Otherwise 1.24(f) implies $Y_M \leqslant Y_{N_G(Q)} \leqslant O_p(N_G(Q))$, contrary to 8.1.

(e): By (d) $M \nleqslant N_G(Q)$ and by (c) $C_G(Y) = C_G(I)$. Since $Q$ is a large subgroup of $G$, 1.57(b) shows that $Y$ and $I$ are faithful $Q!$-modules for $\overline{M}$ with respect to $\overline{Q}$. So we can apply D.10 with $V = Y$ and $H = \overline{M}$ and conclude that also $I/rad_I(M)$ is a $Q!$-module for $\overline{M}$ with respect to $\overline{Q}$.

(f): By (e), $Y$ is a faithful, $p$-reduced $Q!$-module for $\overline{M}$ with respect to $\overline{Q}$. Thus by D.8, $I$ is a semisimple $\overline{M^\circ}$-module and so also a semisimple $M^\circ$-module. Since by (d) $M \nleqslant N_G(Q)$, we get $Q \neq M^\circ$, and so by 1.55(d) $C_I(M^\circ) \leqslant C_G(M^\circ) = 1$. As $I$ is a semisimple $M^\circ$-module, this gives $I = [I, M^\circ] = [I, M_\circ]$. □

LEMMA 8.5. *(a) Let $g \in G$ with $Q^g \leqslant M^\dagger$ and $L \leqslant N_G(Q^g)$. Then $Q^g$ normalizes $L$ and $A$.*

*(b) $L$ and $A$ normalize $Q$, and $Q$ normalizes $L$ and $A$.*

PROOF. (a): Since $Q^g \leqslant M^\dagger$, $Q^g$ normalizes $Y$. Since $L$ normalizes $Q^g$, $Q^g$ also normalizes $Y^l$ for all $l \in L$, and we conclude that $Q^g$ normalizes $\langle Y^L\rangle$. As $L$ is $Y$-minimal, $L = \langle Y^L\rangle$ and so $Q^g$ normalizes $L$ and $O_p(L)$. Since $A = O_p(L)$ this gives (a).

(b): Since $L \in \mathfrak{L}_H(Y_M)$ and $H \in \mathfrak{H}_{N_G(Q)}(O_p(M))$, $L \leqslant H \leqslant N_G(Q)$. So (b) follows from (a). □

LEMMA 8.6. *Suppose that* $[\Omega_1 Z(A), L] \neq 1$. *Then* $I \nleqslant A$ *and* $[Y_{HQ}, HQ] \neq 1$.

PROOF. By 8.4(c) $C_G(I) = C_G(Y)$ and thus also $C_{\Omega_1 Z(A)}(Y) = C_{\Omega_1 Z(A)}(I)$. Since $L = \langle Y^L\rangle$, $[\Omega_1 Z(A), L] \neq 1$ implies $[\Omega_1 Z(A), Y] \neq 1$. Hence also $[\Omega_1 Z(A), I] \neq 1$ and $I \nleqslant A$. It remains to prove $[Y_{HQ}, HQ] \neq 1$.

Since $L \in \mathfrak{L}_G(Y_M)$, 1.43 applies to $L$. So 1.43(h) gives $C_A(L) = C_A(O^p(L))$. As $[\Omega_1 Z(A), L] \neq 1$ this implies $[\Omega_1 Z(A), O^p(L)] \neq 1$. By 2.17(b) $[L, O_p(H)] \leqslant O_p(L) = A \leqslant O_p(H)$. So $O_p(H)$ normalizes $L$ and $A$, and $[L, O_p(H)]$ centralizes $\Omega_1 Z(A)$.

Now the $P \times Q$-Lemma gives $[C_{\Omega_1 Z(A)}(O_p(H)), O^p(L)] \neq 1$. Since $A \leqslant O_p(H)$, we have $C_{\Omega_1 Z(A)}(O_p(H)) \leqslant \Omega_1 Z(O_p(H))$. Thus $[\Omega_1 Z(O_p(H)), O^p(L)] \neq 1$ and so $[\Omega_1 Z(O_p(H)), O^p(H)] \neq 1$. Since by 2.11(e) $H$ is $p$-irreducible, 1.35 gives $[Y_H, H] \neq 1$. As $O^p(HQ) \leqslant H$, 1.26(c) shows that $[Y_{HQ}, HQ] \neq 1$. □

LEMMA 8.7. *Let* $U \leqslant Y$ *be* $A$*-invariant and* $U \nleqslant A$. *Suppose that* $U$ *is* $N_L(Y)$*-invariant or* $Y \leqslant UA$.

(a) $YA = UA$ *and* $Y \cap A = [U, A]C_Y(L) = (U \cap A)C_Y(L)$.
(b) $[A, Y] = [A, u]C_{[A,U]}(L) = [A, U]$ *for every* $u \in U \backslash A$.

PROOF. By assumption, $U$ is $N_L(Y)$-invariant or $Y \leqslant UA$. We will first show that in either case $YA = UA$.

Suppose that $U$ is $N_L(Y)$-invariant. Since $L \in \mathfrak{L}_G(Y_M)$, 2.14 shows that $N_L(Y)/A$ has a unique non-trivial elementary abelian normal $p$-subgroup. Thus $YA/A = UA/A$ and so $YA = UA$. Suppose that $Y \leqslant UA$. Since $U \leqslant Y$, this gives $YA = UA$.

Since $YA = UA$ we get $Y = U(Y \cap A)$. Let $u \in U\backslash A$. Then 1.43(g) shows that

$$(*) \qquad Y \cap A = [A, u]C_Y(L).$$

In particular,

$$Y \cap A = [A, U]C_Y(L) = (U \cap A)C_Y(L) \quad \text{and} \quad Y = U(Y \cap A) = UC_Y(L).$$

This gives $[A, Y] = [A, U] \leqslant U$. Intersecting both sides of the equation in $(*)$ with $[A, U]$ gives

$$[A, U] = [A, u]([A, U] \cap C_Y(L) = [A, u]C_{[A,U]}(L).$$

So all parts of the lemma are proved. □

LEMMA 8.8. *Put* $U := C_I(L)$ *and* $E := \langle Q^g \mid g \in G, C_U(Q^g) \neq 1\rangle$. *Suppose that* $U \neq 1$. *Then*

(a) $Q \leqslant E \leqslant M^\circ$ *and* $[E, L] \leqslant A$. *In particular,* $E$ *normalizes* $L$.
(b) $[E, Y] \leqslant Y \cap A$.
(c) $E = N_M(U)^\circ = N_G(U)^\circ$.
(d) *Let* $x \in L\backslash N_L(Y)$. *If* $I \leqslant A$, *then* $I/U \cong \overline{I^x}$ *as an* $\mathbb{F}_pE$*-module.*

PROOF. (a): By 2.7(b) $E \leqslant M^\circ$, and by 8.5(a) $Q$ normalizes $L$ and so also $U$. Hence $C_U(Q) \neq 1$ and $Q \leqslant E$.

Let $g \in G$ with $C_U(Q^g) \neq 1$. Then $L \leqslant C_G(U) \leqslant C_G(C_U(Q^g))$, and $Q!$ implies $L \leqslant N_G(Q^g)$. Also $Q^g \leqslant E \leqslant M^\circ \leqslant M^\dagger$, and 8.5(a) shows that $Q^g$ normalizes $L$ and $A$. In particular,

$$[L, Q^g] \leqslant L \cap Q^g \leqslant O_p(L) = A,$$

and (a) follows.

(b): Since $Y \leqslant L$ (a) gives $[E, Y] \leqslant A$. By (a) $E \leqslant M^\circ \leqslant M$ and so also $[E, Y] \leqslant Y$.

(c): By (a) $E$ normalizes $L$. Since $L$ centralizes $U$, we conclude that $L$ centralizes $\langle U^E\rangle$. By (a), $E \leqslant M^\circ \leqslant M$. So $E$ normalizes $I$ and since $U \leqslant I$, $\langle U^E\rangle \leqslant I$. Thus $U \leqslant \langle U^E\rangle \leqslant C_I(L) = U$, and $E$ normalizes $U$. Hence $E \leqslant N_M(U)$. Since $E$ is generated by conjugates of $Q$ this gives

$$E \leqslant N_M(U)^\circ.$$

Clearly,

$$N_M(U)^\circ \leqslant N_G(U)^\circ,$$

and by 2.7(b)

$$N_G(U)^\circ \leqslant E,$$

so (c) holds.

(d): Let $x \in L\backslash N_L(Y)$. By 1.43(a), $A' \leqslant C_Y(L)$ and so, since $I \leqslant A$ and $I$ is $A$-invariant, $[I, A] \leqslant I \cap A' \leqslant C_I(L) = U$. Since by (a) $[E, x] \leqslant A$, we conclude that $[E, x]$ centralizes $IU/U$ and so $IU/U \to I^xU/U$, $yU \mapsto y^xU$, is an $E$-isomorphism. Note the $IU/U \cong I/I \cap U$. Also by 1.42(f), $L = \langle Y, Y^x\rangle$ and so since $Y$ is abelian, $C_{I^x}(Y) = C_{I^x}(\langle Y, Y^x\rangle) = C_{I^x}(L) = C_I(L) = U$. Hence $C_{I^x}(Y) = I^x \cap U$ and

$$I^xU/U \cong I^x/I^x \cap U = I^x/C_{I^x}(Y) \cong \overline{I^x}.$$

Thus (d) holds. □

Lemma 8.9. *Let $K \leqslant M$ with $1 \neq \overline{K} \trianglelefteq F^*(\overline{M})$ and $\overline{K} = [\overline{K}, \overline{Q}]$. Suppose that $I \leqslant A$ and $[F^*(\overline{M^\circ}), \overline{Q}] \leqslant \overline{N_M\big([K,Q]O_p(M^\circ)\big)}$. Then $C_{\overline{A}}(\overline{K}) = 1$.*

Proof. Let $F$ be the inverse image of $F^*(\overline{M})$ in $M^\dagger$ and $R := KC_{M^\dagger}(Y) \cap M^\circ$. Since $\overline{F}$ normalizes $\overline{K}$, $F$ normalizes $KC_{M^\dagger}(Y)$ and $R$. Note that $\overline{K} = [\overline{K}, Q]$ implies $\overline{K} \leqslant \overline{M^\circ}$, so $KC_{M^\dagger}(Y) \leqslant M^\circ C_{M^\dagger}(Y)$ and

$$KC_{M^\dagger}(Y) = KC_{M^\dagger}(Y) \cap M^\circ C_{M^\dagger}(Y) = \big(KC_{M^\dagger}(Y) \cap M^\circ\big)C_{M^\dagger}(Y) = RC_{M^\dagger}(Y).$$

Hence $\overline{K} = \overline{R}$. By 1.52(c) (applied with $L := M$),

$$[C_{M^\dagger}(Y), QR] \leqslant [C_G(Y), M^\circ] \leqslant O_p(M^\circ).$$

In particular $[C_{M^\dagger}(Y), Q] \leqslant O_p(M^\circ)$. Using $KC_{M^\dagger}(Y) = RC_{M^\dagger}(Y)$ we get

$$\text{(I)} \qquad [K, Q]O_p(M^\circ) = [R, Q]O_p(M^\circ) \quad \text{and} \quad [C_R(Y), RQ] \leqslant O_p(M^\circ).$$

Put $E := O^p([R, Q])$ and $N := N_G(E)$. Since $1 \neq K \leqslant F^*(\overline{M})$ and $O_p(\overline{M}) = 1$ we have $1 \neq \overline{K} = O^p(\overline{K})$. As $\overline{K} = [\overline{K}, Q] = [\overline{R}, Q]$ this gives $\overline{E} = \overline{K} \neq 1$. Since $F$ normalizes $R$, $N_F(Q) \leqslant N$. In particular, by $Q!$,

$$\text{(II)} \qquad O_p(M^\circ) \leqslant C_{M^\dagger}(Y) \leqslant N_F(Q) \leqslant N.$$

Thus $O_p(M^\circ)$ normalizes $[R, Q]$ and so

$$E = O^p([R, Q]) = O^p([R, Q]O_p(M^\circ)) \overset{\text{(I)}}{=} O^p([K, Q]O_p(M^\circ)).$$

It follows that $N_M\big([K, Q]O_p(M^\circ)\big) \leqslant N \cap M$. By assumption, $[F^*(\overline{M^\circ}), \overline{Q}] \leqslant \overline{N_M\big([K,Q]O_p(M^\circ)\big)}$ and so

$$\text{(III)} \qquad [F^*(\overline{M^\circ}), \overline{Q}] \leqslant \overline{N \cap M}.$$

By 1.8 $\overline{F} = [\overline{F}, \overline{Q}]C_{\overline{F}}(\overline{Q})$ and $[\overline{F}, \overline{Q}] = [\overline{F}, \overline{Q}, \overline{Q}]$. As $[\overline{F}, Q] \leqslant F^*(\overline{M^\circ}) \leqslant \overline{F}$ this gives

$$\text{(IV)} \qquad \overline{F} = [F^*(\overline{M^\circ}), \overline{Q}]C_{\overline{F}}(\overline{Q}).$$

Since by 1.52(b) $Q$ is a weakly closed subgroup of $G$, a Frattini argument gives

$$\text{(V)} \qquad C_{\overline{F}}(\overline{Q}) \leqslant N_{\overline{F}}(\overline{Q}) = \overline{N_F(Q)} \leqslant \overline{N \cap M^\dagger}.$$

Combining (III), (IV) and (V) we get $\overline{F} \leqslant \overline{N \cap M^\dagger}$, and since by (II) $C_{M^\dagger}(Y) \leqslant N$, $F \leqslant N$.

Note that $E$ is subnormal in $M$ and so, since $M$ is of characteristic $p$, by 1.2(a) also $E$ is of characteristic $p$. As $E \neq 1$ we get $1 \neq O_p(E) \trianglelefteq N$ and $O_p(N) \neq 1$. Clearly $Q \leqslant N$, and 1.55 shows that $N$ has characteristic $p$. Since $F \leqslant N$ 2.8 implies $Y = Y_M \leqslant Y_N$, so

$$\text{(VI)} \qquad Y \leqslant O_p(N).$$

By 1.43(a) $A' \leqslant C_Y(L)$. By the assumption of this lemma $I \leqslant A$. Put $B := C_A(\overline{K})$. Then

$$\text{(VII)} \qquad [I, B] \leqslant [A, A] \leqslant C_Y(L).$$

Suppose for a contradiction that $C_{\overline{A}}(\overline{K}) \neq 1$, so $\overline{B} \neq 1$ and $[Y, B] \neq 1$. By 8.4(c) $C_G(Y) = C_G(I)$ and so $[I, B] \neq 1$. By 8.5(b) $Q$ normalizes $A$. Since $Q$ also normalizes $\overline{K}$, $Q$ normalizes $B$. As seen above $\overline{R} = \overline{K}$ and so $B = C_A(\overline{R})$. Hence $R$ normalizes $\overline{B}$. We conclude that $RQ$ normalizes $\overline{B}$. As $[I, \overline{B}] = [I, B]$ this shows that $RQ$ also normalizes $[I, B]$. Hence, by 1.52(c) $C_G([I, B])$ normalizes $(RQ)^\circ$. By (VII) $L$ centralizes $[I, B]$, and so $L$ normalizes $(RQ)^\circ$. Since $Q$ is weakly closed 1.46(c) gives $(RQ)^\circ = \langle Q^R \rangle = [Q, R]Q$ and so $O^p((RQ)^\circ) = O^p([Q, R]) = E$. Thus $L$ normalizes $E$ and $L \leqslant N$. Since $Y \nleqslant O_p(L)$ we get $Y \nleqslant O_p(N)$, a contradiction to (VI). □

Lemma 8.10. *Suppose that $\overline{Q}$ is homocyclic abelian. Then $\overline{Q}$ is elementary abelian.*

Proof. Put $N := N_G(Q)$ and $F := \langle Y^N \rangle$. Note that $Q' \leqslant C_M(Y)$ and so $Y \leqslant C_M(Q')$. Also $[Q, Y] \leqslant Y$ is elementary abelian and 1.19 now shows that $O^p(F)$ centralizes $\Phi(Q)$.

Suppose for a contradiction that $\overline{Q}$ is not elementary abelian. Since $\overline{Q}$ is homocyclic this gives $\Omega_1(Q) \leqslant \Phi(Q)O_p(M)$. Then $[Q, Y] \leqslant Q \cap Y \leqslant \Omega_1(Q)$ and $[\Omega_1(Q), Y] \leqslant [\Phi(Q)O_p(M), Y] \leqslant \Phi(Q)$. Since $\Omega_1(Q)$ and $\Phi(Q)$ are $N$-invariant, we get that $[Q, F] \leqslant \Omega_1(Q)$ and $[\Omega_1(Q), F] \leqslant \Phi(Q)$. Hence $O^p(F)$ centralizes each factor of the series $1 \leqslant \Phi(Q) \leqslant \Omega_1(Q)\Phi(Q) \leqslant Q$. Coprime action shows that $O^p(F)$ centralizes $Q$. Since $C_G(Q) \leqslant Q$, we conclude that $O^p(F) = 1$. Hence $F$ is a $p$-group and $Y \leqslant F \leqslant O_p(N) = O_p(N_G(Q))$. This contradicts 8.1. □

## 8.2. The Case $I \leqslant A$

In this section we continue to assume the hypothesis and notation of Theorem H. Furthermore, we assume $I \leqslant A$. We start with a summary of the notation used in this section:

Notation 8.11.
- $x \in L$ with $1 \neq [I, I^x] \leqslant I \cap I^x$, see 8.12.
- $D := \langle I^L \rangle$, $U := C_I(L)$ and $W := C_I(Q)$.
- $\tilde{Y} := Y/I$.
- $E := \langle Q^g \mid g \in G \mid C_U(Q^g) \neq 1 \rangle$, as in 8.8.
- $\mathbb{K} := End_{M_\circ}(I)$, as in 8.18.

If $I$ is a natural $\Omega_6^+(2)$-module for $\overline{M^\circ}$:
- $I_0$ is natural $SL_4(2)$-module for $\overline{M^\circ}$.
- $W_0 := C_{I_0}(Q)$ and $U_0 := C_{I_0}(A)$, with $I_0$ chosen such that $U_0$ is a hyperplane of $I_0$, see the discussion before 8.23.
- $N := N_G(U)$, $C := C_G(U)$, $B := \langle I^N \rangle$, $\widehat{B} := B/U$, and $N_0 := C_N(\widehat{B})$.
- $X := \langle (B \cap O_2(M))^{M^\circ} \rangle$.
- $K := Hom_E(U_0, \widehat{B})$, and $s$ is a $C$-invariant symplectic form on $K$, see 8.28.
- $C_0 := C_C(K^\perp)$. For $F \leqslant C$, $\check{F}$ is the image of $F$ in $Sp(K/K^\perp)$.

Lemma 8.12. *Suppose that $I \leqslant A$. Then there exists $x \in L$ such that $1 \neq [I, I^x] \leqslant I \cap I^x$. Moreover, $I^x$ and $A$ are non-trivial quadratic offenders on $I$, and $Q$ normalizes $I^x$.*

Proof. Since $Y \nleqslant O_p(L)$, $\langle Y^L \rangle$ is not abelian. Thus there exists $x \in L$ with $[Y, Y^x] \neq 1$. By 8.4(b),(c),

$$(*) \qquad N_G(Y) = N_G(I) \qquad \text{and} \qquad C_G(Y) = C_G(I).$$

As $[Y, X^x] \neq 1$ this implies $[I, Y^x] \neq 1$, and since also $C_G(Y^x) = C_G(I^x)$, $[I, I^x] \neq 1$.

Since $A$ normalizes $Y$ and $I^x \leqslant A$ we conclude that $I^x \leqslant N_G(Y)$, so by $(*)$ $I^x \leqslant N_G(I)$. By symmetry also $I \leqslant N_G(I^x)$ and thus $[I, I^x] \leqslant I \cap I^x$. Since $I$ is abelian, this shows that $I$ acts quadratically on $I^x$. Possibly after replacing $x$ by $x^{-1}$, we also have $|I/C_I(I^x)| \geqslant |I^x/C_{I^x}(I)|$, so $I^x$ is a quadratic offender on $I$.

Again by $(*)$ $C_A(I) = C_A(Y)$, and 1.43(g), $C_A(Y) = Y \cap A$. Hence $C_A(I) = Y \cap A$. Put $\widehat{A} := A/C_Y(L)$. By 1.43(e) $\widehat{A} = \widehat{A \cap Y} \times \widehat{A \cap Y^l}$ for $l \in L \backslash N_L(Y)$. Thus

$$|A/C_A(I)| = |A/A \cap Y| = |\widehat{A}/\widehat{A \cap Y}| = |\widehat{A \cap Y^l}| = |\widehat{A \cap Y}| \geqslant |\widehat{I}| = |I/C_I(L)| \geqslant |I/C_I(A)|.$$

Also by 1.43(a) $[I, A] \leqslant A' \leqslant C_Y(L)$ and so $[I, A, A] = 1$. Thus also $A$ is a quadratic offender on $I$.

Finally, by 8.5(b) $L$ normalizes $Q$, and $Q \leqslant N_G(I)$. Hence $Q \leqslant N_G(I^x)$. □

Put $D := \langle I^L \rangle$, $U := C_I(L)$ and $W := C_I(Q)$, and (if $I \leqslant A$) let $x \in L$ be as in 8.12.

LEMMA 8.13. *Suppose that $I \leqslant A$.*

(a) *$D \leqslant A$ and $D$ is not abelian.*
(b) *$I$ is symmetric in $G$.*
(c) *$L = \langle Y, Y^x \rangle$. In particular, $L = \langle Y^L \rangle$.*
(d) *$C_L(D) = Z(L)$ and $C_Y(I^x) = C_Y(L)$.*
(e) *$C_I(I^x) = C_I(A) = C_I(D) = C_I(L) = U$.*
(f) *$D$ is a non-trivial quadratic offender on $I$.*
(g) *$[Y, A] \leqslant Y \cap A$.*
(h) *$[D, A] \leqslant C_I(L) = U$.*

PROOF. (a) and (b): By hypothesis $I \leqslant A$ and so $D = \langle I^L \rangle \leqslant A$. Let $x$ be as in 8.12 Then $1 \neq [I, I^x] \leqslant I \cap I^x$, so $D$ is not abelian and $I$ is symmetric in $G$.

(c): Since $I$ is abelian, $I \neq I^x$. By 8.4(b), $N_G(I) = N_G(Y)$ and thus $x \notin N_L(Y)$. Since $L \in \mathfrak{L}_G(Y_M)$, $N_L(Y)$ is the unique maximal subgroup of $L$ containing $Y$, and so $L = \langle Y, Y^x \rangle$ by 1.42(f).

(d): By 8.4(c) $C_G(I) = C_G(Y)$. Thus by (c)

$$C_L(D) = C_L(\langle I^L \rangle) = C_L(\langle Y^L \rangle) = C_L(L) = Z(L).$$

Since $Y$ is abelian,

$$C_Y(I^x) = C_Y(\langle I, I^x \rangle) = C_Y(\langle Y, Y^x \rangle) = C_Y(L).$$

(e): Note that $I^x \leqslant D \leqslant A \leqslant L$ and by (d) $C_I(I^x) = C_I(L)$. Hence (e) follows.

(f): By 8.12 $A$ is quadratic on $I$. Since $D \leqslant A$, also $D$ acts quadratically on $I$. By 8.12 $I^x$ is a non-trivial offender on $I$, and by (e) $C_I(D) = C_I(I^x)$. Since $I^x \leqslant D$ we get

$$|I/C_I(D)| = |I/C_I(I^x)| \leqslant |I^x/C_{I^x}(I)| \leqslant |D/C_D(I)|.$$

So $D$ is a non-trivial offender on $I$.

(g) and (h): By definition of $\mathfrak{L}_G(Y_M)$, $N_L(Y)$ is a maximal subgroup of $L$ and $A \leqslant N_L(Y)$. This gives (g) and $[I, A] \leqslant I$. By 1.43(a), $A' \leqslant C_Y(L)$, and since $I \leqslant A$, $[I, A] \leqslant I \cap C_Y(L) = C_I(L)$. Conjugation with $L$ gives $[D, A] \leqslant C_I(L) = U$. □

LEMMA 8.14. *Suppose that $I \leqslant A$ and $|D/C_D(Y)| < |Y/Y \cap A|^2$. Then $[Y, D] \leqslant I$.*

PROOF. By 1.43(h), (e), (g) applied with $B = D$,

(I) $\quad C_D(L) = D \cap C_Y(L) = C_{D \cap Y}(L),\ |D/D \cap Y| = |D \cap Y/C_{D \cap Y}(L)|$ and $C_D(Y) = D \cap Y,$

and so

(II) $\quad |D/C_D(L)| = |D/C_{D \cap Y}(L)| = |D/D \cap Y||D \cap Y/C_{D \cap Y}(L)| = |D/D \cap Y|^2 = |D/C_D(Y)|^2.$

Put $\widetilde{q} := |Y/Y \cap A|$. By assumption $|D/C_D(Y)| < |Y/Y \cap A|^2 = \widetilde{q}^2$. Thus (II) gives

(III) $$|D/C_D(L)| < \widetilde{q}^4.$$

Recall from 8.3 that

$$L/A \cong SL_2(\widetilde{q}), Sz(\widetilde{q}) \text{ or } Dih_{2r} \quad \text{ and } \quad |Y/Y \cap A| = \widetilde{q},$$

where $p = 2$ in the last two cases, $r$ is an odd prime, and $\widetilde{q} = 2$ in the last case.

Suppose that $p = 2$ and $L/O_2(L) \cong Dih_{2r}$. Then $\widetilde{q} = 2$ and by (III) $|D/C_D(L)| < 16$. Since $GL_3(2)$ has order $2^3 \cdot 3 \cdot 7$ and contains no dihedral group of order 14. We conclude that $L/O_2(L) \cong Dih_6 \cong SL_2(2)$.

So we may assume that $L/A \cong SL_2(\widetilde{q})$ or $Sz(\widetilde{q})$. Since $[D, Y, Y] \leqslant [Y, Y] = 1$, $Y$ acts quadratically on $D/C_D(L)$. Thus C.15 shows that all non-central chief factors of $L$ on $D/C_D(L)$ are natural $SL_2(\widetilde{q})$- and $Sz(\widetilde{q})$-modules, respectively. The natural $Sz(\widetilde{q})$- module has order $\widetilde{q}^4$, a contradiction to (III). Hence $L/A \cong SL_2(\widetilde{q})$.

The natural $SL_2(\widetilde{q})$-module has order $\widetilde{q}^2$, and so (III) shows that $L$ has a unique non-central chief factor on $D/C_D(L)$. By 1.43(p) $L$ has no central chief factors on $D/C_D(L)$. Thus $D/C_D(L)$ is a natural $SL_2(q)$-module. In particular, $L$ acts transitively on $D/C_D(L)$.

By (I) $C_D(L) \leqslant Y$, so $IC_D(L) \leqslant Y$, and $IC_D(L)$ is elementary abelian. The transitivity of $L$ on $D/C_D(L)$ now implies that $D$ has exponent $p$. As $D$ is not abelian by 8.13(a), this shows that $p$ is odd. Since $L/A \cong SL_2(\widetilde{q})$ and $D/C_D(L)$ is a natural $SL_2(\widetilde{q})$-module we conclude that there exists an involution $t \in L$ with $[t, L] \leqslant A$, and $t$ inverts $D/C_D(L)$. Thus $C_D(t) = C_D(L)$ and $C_I(t) = C_I(L)$. Since $t \in N_L(Y) = N_L(I)$, coprime action shows

$$I = [I, t]C_I(t) = [I, t]C_I(L) \leqslant [D, t]C_I(L).$$

By 8.13(h) $[D, A] \leqslant C_I(L)$. Thus $[D, \langle t\rangle A]C_I(L) = [D, t]C_I(L)$ is $L$-invariant and contains $I$. Since $D = \langle I^L\rangle$, this gives $D = [D, t]C_I(L)$. As $D' \leqslant [D, A] \leqslant C_I(L)$, $D/C_I(L)$ is abelian. Coprime action now shows

$$D/C_I(L) = [D, t]C_I(L)/C_I(L) \times C_D(t)/C_I(L).$$

Since $D = [D, t]C_I(L)$, this gives $C_D(t) = C_I(t)$ and so $C_D(L) = C_I(L)$. Thus $D/C_I(L)$ is a natural $SL_2(\widetilde{q})$-module. It follows that $N_L(Y)$ acts simple on $C_{D/C_I(L)}(Y)$. Note that

$$1 \neq I/C_I(L) \leqslant (Y \cap D)/C_I(L) \leqslant C_{D/C_I(L)}(Y)$$

and that $N_L(Y)$ normalizes this series. Thus $I/C_I(L) = Y \cap D/C_I(L)$ and $I = Y \cap D$. Hence $[Y, D] \leqslant Y \cap D \leqslant I$, and 8.14 is proved. □

Put $\widetilde{Y} := Y/I$, and recall from 8.3 and 8.4(b) that $D \leqslant N_G(Y) = N_G(I)$, so $D$ acts on $Y$ and $\widetilde{Y}$.

LEMMA 8.15. *Suppose that* $I \leqslant A$.

(a) $[Y, D, D] \leqslant C_I(L)$ *and* $[\widetilde{Y}, D, D] = 1$.
(b) *Either* $[\widetilde{Y}, D] = 1$ *or* $|\widetilde{Y}/C_{\widetilde{Y}}(D)|^2 \leqslant |Y/Y \cap A|^2 \leqslant |D/C_D(Y)|$.
(c) *If* $\overline{I^x} = \overline{A}$, *then* $Y \cap A = IC_Y(I^x) = IC_Y(L)$ *and* $A = II^xC_Y(L)$.

PROOF. (a): By 8.13(h) $[A, D] \leqslant C_I(L)$. Since $[Y, D] \leqslant D \leqslant A$, this gives $[Y, D, D] \leqslant [A, D] \leqslant C_I(L) \leqslant I$. Hence $[\widetilde{Y}, D, D] = 1$ and (a) holds.

(b): Suppose that $|Y/Y \cap A|^2 > |D/C_D(Y)|$. Then 8.14 shows that $[Y, D] \leqslant I$ and so $[\widetilde{Y}, D] = 1$. Suppose that $|Y/Y \cap A|^2 \leqslant |D/C_D(Y)|$. Since $[Y \cap A, D] \leqslant [A, D] \leqslant C_I(L) \leqslant I$ we have $\widetilde{Y \cap A} \leqslant C_{\widetilde{Y}}(D)$ and so $|\widetilde{Y}/C_{\widetilde{Y}}(D)|^2 \leqslant |Y/Y \cap A|^2 \leqslant |D/C_D(Y)|$.

(c): Assume that $\overline{I^x} = \overline{A}$, so $A = I^xC_A(Y)$. By 1.43(g) $C_A(Y) = A \cap Y$, so $A = I^x(A \cap Y)$ and $A \cap Y^x = I^x(A \cap Y^x \cap Y)$. By 1.43(h) $A \cap Y \cap Y^x = C_Y(L)$ and so $A \cap Y^x = I^xC_Y(L)$. Hence also $A \cap Y = IC_I(L)$, and 1.43(e) gives $A = (A \cap Y)(A \cap Y^x) = II^xC_Y(L)$. Finally, by 8.13(d) $C_Y(I^x) = C_Y(L)$, and (c) is proved. □

According to 8.13(b) $I$ is symmetric in $G$. Thus, we can apply Theorem D with $I$ in place of $Y$. We will do this considering the various outcomes of Theorem D separately, and we will use the notation of Theorem D.

LEMMA 8.16. *Suppose that $I \leqslant A$. Then Case (3) of Theorem D does not hold for $I$ in place of $Y$.*

PROOF. Assume that Case (3) of Theorem D holds. Then $I$ is a natural $SL_2(q)$-wreath product module for $\overline{M}$ with respect to some $\mathcal{K}$, $\overline{M_\circ} = O^p(\langle\mathcal{K}\rangle)\overline{Q}$, and $Q$ acts transitively on $\mathcal{K}$.

Put $P := M^\circ S$ and let $P^*$ be the inverse image of $\langle\mathcal{K}\rangle$ in $M$. Then $I$ is also a natural $SL_2(q)$-wreath product module for $\overline{P}$ and $O^p(\overline{P}) = \overline{M_\circ} = O^p(\langle\mathcal{K}\rangle)$. Hence A.28(b) shows

1°. *$\overline{P}$ is $p$-minimal.*

Moreover, by A.28(c), $O_p(P/C_P(I)) = 1$ and by 8.4(c) $C_P(I) = C_P(Y)$. Thus

2°. *$O_p(\overline{P}) = 1$.*

We now investigate the action of $P$ on $\widetilde{Y}$. Note that $C_P(Y) \leqslant C_P(\widetilde{Y})$, so $P/C_P(\widetilde{Y}) \cong \overline{P}/C_{\overline{P}}(\widetilde{Y})$. Since $\overline{P}$ is $p$-minimal and so $p$-irreducible, we either have

3°. *$C_{\overline{S}}(\widetilde{Y}) \leqslant O_p(\overline{P})$ or $O^p(\overline{P}) \leqslant C_{\overline{P}}(\widetilde{Y})$.*

We now discuss these two cases separately and show that both of them lead to a contradiction.

4°. *$C_{\overline{S}}(\widetilde{Y}) \leqslant O_p(\overline{P})$ does not hold.*

Suppose that $C_{\overline{S}}(\widetilde{Y}) \leqslant O_p(\overline{P})$. By (2°) $O_p(\overline{P}) = 1$ and so $C_{\overline{S}}(\widetilde{Y}) = 1$. In particular $C_D(Y) = C_D(\widetilde{Y})$, and $[\widetilde{Y}, D] \neq 1$ since $D \nleqslant C_M(Y)$. This gives

$$|\widetilde{Y}/C_{\widetilde{Y}}(D)| < |\widetilde{Y}/C_{\widetilde{Y}}(D)|^2 \overset{8.15(b)}{\leqslant} |D/C_D(Y)| = |D/C_D(\widetilde{Y})|.$$

So $D$ is an over-offender on $\widetilde{Y}$. On the other hand, since $C_{\overline{S}}(\widetilde{Y})$, $\widetilde{Y}$ is $p$-reduced for $P$. Moreover, since $\overline{P}$ is $p$-minimal, 1.38 shows that also $\overline{P}/C_{\overline{P}}(\widetilde{Y})$ is $p$-minimal. Hence C.13(e) yields a contradiction.

5°. *$O^p(\overline{P}) \leqslant C_{\overline{P}}(\widetilde{Y})$ does not hold.*

Suppose that $O^p(\overline{P}) \leqslant C_{\overline{P}}(\widetilde{Y})$. Then $[Y, M_\circ] = [Y, O^p(P)] \leqslant I$, and by 8.4(f) $C_Y(M_\circ) = 1$.

Since $I$ is a natural $SL_2(q)$-wreath product module for $\overline{M}$ with respect to $\mathcal{K}$,

$$\overline{P^*} = \langle\mathcal{K}\rangle = \bigtimes_{K\in\mathcal{K}} K, \qquad \text{and} \quad I = \bigtimes_{K\in\mathcal{K}} [I,K],$$

and for $K \in \mathcal{K}$, $K \cong SL_2(q)$ and $[I, K]$ is a natural $SL_2(q)$-module for $K$.

Assume first first that $p$ is odd or $q = 2$. Put $\overline{Z} := O_{p'}(\overline{P^*})$. Then $\overline{Z}$ is a normal $p'$-subgroup of $\overline{P}$ and $I = [I, \overline{Z}]$. Coprime action shows $Y = C_Y(\overline{Z}) \times I$. Since $M_\circ$ normalizes $C_Y(\overline{Z})$ and $[Y, M_\circ] \leqslant I$, $C_Y(\overline{Z}) \leqslant C_Y(M_\circ) = 1$. But then $Y = I$, which is impossible since $I \leqslant A$ and $Y \nleqslant A$.

Assume now that $p = 2$ and $q \neq 2$. Then $q \geqslant 4$ and $K \cong SL_2(q)$ is simple. Since $[Y, O^p(P)] \leqslant I$, $C_{\overline{P}}(I)$ is a $p$-group, and since $O_p(\overline{P}) = 1$, we conclude that $I$ is faithful $\overline{P}$-module.

Let $K \in \mathcal{K}$. Observe that $K \cap \overline{S}$ is an offender on $I$. Since $K$ is simple, $K$ is $J_K(V)$-component of $K$, and since $K \trianglelefteq\trianglelefteq \overline{P}$, we conclude from A.42 that $K$ is a $J_{\overline{P}}(I)$ component of $\overline{P}$. By C.13 there exists subgroups $E_1, \ldots, E_r$ of $\overline{P}$ such that

$$J_{\overline{P}}(I) = E_1 \times \cdots \times E_r, \qquad \mathcal{J}_{\overline{P}}(I) = \{E_1', \ldots, E_r',\}$$

$Q$ acts transitively on $\{E_1, \ldots, E_r\}$, and either $E_i \cong SL_2(q^*)$ and $[[I, E_i]/C[I, E_i](E_i)$ is a natural $SL_2(q^*)$-module for $E_i$ or $E_i \cong Sym(2^n + 1)$ and $[I, E_i]$ is natural $Sym(2^n + 1)$-module for $E_i$.

As we have seen, $K \in \mathcal{J}_{\overline{P}}(I)$ and so $K = E_i'$ for some $1 \leqslant i \leqslant r$. Since $[I, K]$ is natural $SL_2(q)$-module for $q \geqslant 4$, $[I, E_i]$ cannot be a natural $Sym(2^n + 1)$-module. It follows that $K = E_i$. Now the transitive action of $Q$ on $\mathcal{K}$ and $\{E_1, \ldots, E_r\}$ gives $\mathcal{K} = \{E_1, \ldots, E_r\}$ and $\mathcal{J}_{\overline{P}}(V) = \langle\mathcal{K}\rangle = O^p(\langle\mathcal{K}\rangle) = \overline{M_\circ}$.

By 8.12 $A$ is an offender on $I$ and so by C.13(g)

$$\overline{A} = (\overline{A} \cap E_1) \times \cdot \times (\overline{A} \cap E_n) \leqslant \langle\mathcal{K}\rangle = \overline{M_\circ}.$$

Since $[Y, M_\circ] \leqslant I$ this implies $[Y, A] \leqslant I$.

By 8.5(b) $Q$ normalizes $A$. Thus there exists $d \in A$ with $1 \neq \overline{d} \in C_{\overline{A}}(Q)$. Since $[I, \langle K \rangle] = I$ we have $[I, K, d] \neq 1$ for some $K \in \mathcal{K}$ and since $Q$ centralizes $\overline{d}$ and acts transitively $\mathcal{K}$, $[I, K, d] \neq 1$ for all $K \in \mathcal{K}$. Since $[I, K]$ is a natural $SL_2(q)$-module for $\overline{M_\circ}$ and $d$ is a 2-element, $[I, K, d] = C_{[I,K]}(d)$. As $I = \times_{K \in \mathcal{K}} [I, K]$ we get $[I, d] = C_I(d)$. On the other hand, $\overline{A}$ is elementary abelian and so $|\overline{d}| = p = 2$. Hence $d$ acts quadratically on $Y$ and

$$[Y, d] \leqslant [Y, A] \cap C_Y(d) \leqslant I \cap C_Y(d) = C_I(d) = [I, d].$$

Hence $[Y, d] = [I, d]$ and $Y = C_Y(d)I$. Note that $d \in A \backslash Y$, and so by 1.43(f) $C_Y(d) \leqslant A$. Now $Y = C_Y(d)I \leqslant A$, a contradiction. □

Recall that $M_\circ = O^p(M^\circ)$. For the definition of $J_{\overline{M}}(I)$ and a $J_{\overline{M}}(I)$-component of $\overline{M}$ see A.7.

Lemma 8.17. *Suppose that $I \leqslant A$. Then Case (4:4) of Theorem D does not hold for $Y$ in place of $I$.*

Proof. Assume case (4:4) of Theorem D. Then $p$ is odd, $\overline{M^\circ} = \overline{L_1 L_2}$ with $[\overline{L_1}, \overline{L_2}] = 1$, $\overline{L_i} \cong SL_{n_i}(q)$, $n_i \geqslant 2$ and $n_1 + n_2 \geqslant 5$, and $I \cong V_1 \otimes_{\mathbb{F}_q} V_2$, where $V_i$ is a natural $SL_{n_i}(q)$-module for $\overline{L_i}$. Note that for $n \geqslant 2$ and odd $q$:

1°. *$O^p(SL_n(q)) = SL_n(q)'$, and $SL_n(q)'$ is either quasisimple or isomorphic to $Q_8$ (and $n = 2$ and $q = 3$).*

Let $\{i, j\} = \{1, 2\}$, and let $L_i$ be the inverse image of $\overline{L_i}$ in $M^\circ$, and put $K_i := (L_i Q)_\circ$. Note that $M^\circ = L_1 L_2$ and $[L_1, L_2] \leqslant C_M(Y) \leqslant N_M(Q)$. Also $Q$ is a weakly closed subgroup of $M$, and so we can apply 1.47. It follows that

2°. *$K_i \trianglelefteq M^\circ$, $M_\circ = K_1 K_2$, $K_i = [K_i, Q]$ and $F^*(\overline{M}) \leqslant N_{\overline{M}}(\overline{K_i})$.*

In particular, $K_i = O^p(K_i) \leqslant O^p(L_i) \leqslant M_\circ = K_i K_j$ and so $O^p(L_i) = K_i(O^p(L_i) \cap L_j)$. Since $\overline{L_i} \cong SL_{n_i}(q)$ and $\overline{L_i} \cap \overline{L_j} \leqslant Z(\overline{L_i})$ we conclude from conclude from (1°) that

3°. *$\overline{K_i} = \overline{L_i}' = O^p(\overline{L_i}) \cong SL_{n_i}(q)'$.*

We will now verify the hypothesis of 8.9 with $K_i$ in place of $K$. By (2°), $K_i \trianglelefteq M^\circ$ and $K_i = [K_i, Q]$. Hence $[M, Q] \leqslant M^\circ \leqslant N_M([K_i, Q])$ and thus

$$[F^*(\overline{M^\circ}), Q] \leqslant [\overline{M}, Q] \leqslant \overline{N_M([K_i, Q])} \leqslant \overline{N_M\big([K_i, Q] O_p(M^\circ)\big)}.$$

Moreover, by (3°) $\overline{K_i} \neq 1$ and $\overline{K_i} = F^*(\overline{K_i})$. Since $\overline{K_i} \trianglelefteq \overline{M^\circ}$, $\overline{K_i}$ is subnormal in $\overline{M}$. Hence $\overline{K_i} = F^*(\overline{K_i}) \leqslant F^*(\overline{M})$. Now (2°) shows that $\overline{K_i} \trianglelefteq F^*(\overline{M})$. Thus, indeed $M$ and $K_i$ satisfies the hypothesis of 8.9. Hence

4°. *$C_{\overline{A}}(\overline{K_i}) = 1$.*

By 8.12 $A$ is a non-trivial quadratic offender on $I$. Thus, there exists a best offender $B \leqslant A$ on $I$ with $[I, B] \neq 1$. Then $\overline{B} \neq 1$ and so by (4°) $[\overline{K_i}, \overline{B}] \neq 1$. On the other hand, since $\overline{L_i} \cong SL_{n_i}(q)$, $[\overline{L_1}, \overline{L_2}] = 1$ and $\overline{M^\circ} = \overline{L_1 L_2}$ we conclude that $\overline{L_1}'$ and $\overline{L_2}'$ are the only minimal non-central normal subgroups of $\overline{M_\circ}$. Thus $\{\overline{L_1}', \overline{L_2}'\}$ is $\overline{M}$-invariant. In particular $O^2(\overline{M}) \leqslant N_{\overline{M}}(\overline{L_i}')$. Since $p$ is odd, we get that $\overline{B} \leqslant J_{\overline{M}}(I) \leqslant N_{\overline{M}}(\overline{L_i}')$. But then by (3°) $[\overline{K_i}, \overline{B}] = \overline{K_i} \trianglelefteq J_{\overline{M}}(I)$, and $\overline{K_i}$ is minimal with that property. Hence $\overline{K_1}$ and $\overline{K_2}$ are $J_{\overline{M}}(I)$-components of $\overline{M}$. Now The Other $P(G, V)$-Theorem [**MS1**] (or A.41(f)) implies $[I, \overline{K_1}, \overline{K_2}] = 1$, a contradiction to the fact that $I \cong V_1 \otimes_{\mathbb{F}_q} V_2$ as an $M^\circ$-module. □

Lemma 8.18. *Suppose that $I \leqslant A$. Then $\overline{M_\circ}$ is quasisimple, $I$ is a simple $M_\circ$-module, and $A$ acts $\mathbb{K}$-linearly on $I$, where $\mathbb{K} := End_{M_\circ}(I)$.*

PROOF. Note that we have excluded cases (3) and (4:4) of Theorem D, see 8.16 and 8.17. In all the remaining cases of Theorem D $\overline{M_\circ}$ is quasisimple and $[I, M^\circ]$ is a simple $M_\circ$-module. By 8.4(f) $I = [I, M^\circ] = [I, M_\circ]$, and so $I$ is a simple $M_\circ$-module. In particular, $\mathbb{K}$ is a field, and since $A$ normalizes $M_\circ$, $A$ acts $\mathbb{K}$-semilinearly on $I$. By 8.12 $A$ is an offender on $I$ and so by [**MS5**, 2.5] either $A$ acts $\mathbb{K}$-linearly on $I$ or $|I| = 4$. The latter case is impossible as $\overline{M_\circ}$ is quasisimple. $\square$

For the next step recall that $U = C_I(L)$, $W = C_I(Q)$ and $D = \langle I^L \rangle$. As in 8.8 define

$$E := \langle Q^g \mid g \in G \mid C_U(Q^g) \neq 1 \rangle.$$

Moreover $\mathbb{K} = End_{M_\circ}(I)$ as in 8.18.

LEMMA 8.19. *Suppose that $I \leqslant A$. Then*

(a) *$L$ normalizes $E$.*
(b) *$U$ is a non-trivial $\mathbb{K}$-subspace of $I$.*
(c) *$E = N_G(U)^\circ = N_M(U)^\circ$. In particular, $E \leqslant M$.*
(d) *$I/U \cong \overline{I^h}$ as an $\mathbb{F}_p E$-module for all $h \in L \backslash N_L(Y)$.*

PROOF. Since $L$ centralizes and so normalizes $U$, $L$ normalizes $E$. By 8.18 $A$ acts $\mathbb{K}$-linearly on $I$ and by 8.13(e), $U = C_I(L) = C_I(A)$. So $U$ is a non-trivial $\mathbb{K}$-subspace of $I$. By 8.8(c) $E = N_G(U)^\circ = N_M(U)^\circ$; in particular, $E \leqslant M$. Since $I \leqslant A$, 8.8(d) shows that $I/U \cong \overline{I^h}$ as an $\mathbb{F}_p E$-module. $\square$

LEMMA 8.20. *Suppose that $I \leqslant A$. Then $[Y, M_\circ] \leqslant I$.*

PROOF. We first show :

$1^\circ$. *$E$ normalizes $D$ and $[\widetilde{Y}, E, D] = 1$.*

By 8.19(c) $E \leqslant M$ and so $E$ normalizes $I$. By 8.19(a), $L$ normalizes $E$, whence $E$ normalizes $D = \langle I^L \rangle$. By 8.8(b), $[Y, E] \leqslant Y \cap A$ and by 8.13(h) $[A, D] \leqslant C_I(L) \leqslant I$. Thus $[Y, E, D] \leqslant [A, D] \leqslant I$ and $[\widetilde{Y}, E, D] = 1$.

For the next steps recall that $M_\circ = O^p(M^\circ)$ and $W = C_I(Q)$.

$2^\circ$. *Suppose that $[I, E, D] = 1$. Then $[Y, M_\circ] \leqslant I$.*

By 8.13(e) $C_I(D) = U$ and so, since $[I, E, D] = 1$, $[I, E] \leqslant C_I(D) = U$, and $E$ centralizes $I/U$. By 8.19(d) the $\mathbb{F}_p E$-modules $I/U$ and $\overline{I^h}$ are isomorphic for all $h \in L \backslash N_L(Y)$. This gives $[\overline{I^h}, E] = 1$ for all such $h$, and so also $[\overline{D}, \overline{E}] = 1$ and $[D, E, I] = 1$. The Three Subgroups Lemma now implies that $[I, D, E] = 1$. In particular, $[I, D] \leqslant C_I(Q^g)$ for all $g \in G$ with $Q^g \leqslant E$. By 8.13(a) $[I, D] \neq 1$, and so for all such $Q^g$, $1 \neq [I, D] \leqslant C_G(Q) \cap C_G(Q^g)$, and 1.52(e) gives $Q = Q^g$. Hence

$$\text{(I)} \qquad E = Q \quad \text{and} \quad [I, D, Q] = 1.$$

Put

$$T := \{s \in M \mid [I, s] \leqslant W \text{ and } [I, s, s] = 1\}.$$

Let $t \in T$ with $[I, t] \neq 1$. Since $W = C_I(Q)$, A.55(d) (with $V = I$) shows that $W = [I, t]$. From $[I, t, t] = 1$ we get $[W, t] = 1$. In particular

$$\text{(II)} \qquad [I, t] = W \text{ for all } t \in T \backslash C_T(I) \text{ and } T = C_M(W) \cap C_M(I/W).$$

By (I) $[I, D, Q] = 1$ and so $[I, D] \leqslant C_I(Q) = W$, and by 8.13(f) $\overline{D}$ is a non-trivial quadratic offender on $I$. This shows that $D \leqslant T$, so $[I, T] \neq 1$ and $C_I(T) \leqslant C_I(D)$. By 8.13(e) $C_I(D) = U$ and so $C_I(T) \leqslant U$. Since $[I, T] \neq 1$, (II) gives $[I, T] = W$. Moreover, since $N_M(Q)$ normalizes $C_I(Q) = W$, (II) shows that $N_M(Q)$ normalizes $T$, and $Q!$ shows that $T \leqslant N_M(Q)$. We record:

$$\text{(III)} \qquad D \leqslant T \trianglelefteq N_M(Q), \qquad C_I(T) \leqslant U \qquad \text{and} \qquad [I, T] = W.$$

Next we prove:

(IV) $\overline{T}$ is a weakly closed subgroup of $\overline{M}$.

Otherwise, 1.45 shows that there exists $g \in M$ such that $\overline{T^g} \neq \overline{T}$ and $[\overline{T^g}, \overline{T}] \leqslant \overline{T^g} \cap \overline{T}$. In particular $\overline{T^g} \leqslant N_{\overline{M}}(\overline{T})$. Then $T^g$ normalizes $[I, T]$ and so $[I, T, T^g] \leqslant [I, T]$. Thus $[I, T, T^g] \leqslant [I, T] \cap [I, T^g]$. By (III) $[I, T] = W$ and so

$$[I, T, T^g] \leqslant W \cap W^g.$$

By (III) $N_M(Q)$ normalizes $T$. Thus $\overline{T^g} \neq \overline{T}$ implies that $g \notin N_M(Q)$, so $Q \neq Q^g$, and 1.52(e) gives $C_G(Q) \cap C_G(Q^g) = 1$. Then also $W \cap W^g = 1$ and $[I, T, T^g] = 1$. By (III) $[I, T] = W$ and $C_I(T) \leqslant U$. Hence $W \leqslant C_I(T^g) \leqslant U^g$. Thus $C_{U^g}(Q) \neq 1$ and $Q \leqslant E^g$. By (I), $E = Q$ and so $Q \leqslant E^g = Q^g$ and $Q = Q^g$, a contradiction. Hence (IV) is proved.

Note that by (III), $\overline{D} \leqslant \overline{T} \trianglelefteq \overline{N_M(Q)}$, and by $Q!$, $N_{\overline{M}}(C_I(S)) \leqslant \overline{N_M(Q)}$, and so

$$\overline{D} \leqslant \overline{T} \leqslant O_p(N_{\overline{M}}(C_I(S))).$$

By 8.13(f) $\overline{D}$ is a non-trivial quadratic offender on $I$, and by [**MS6**, Corollary 3.7] every offender contained in $O_p(N_{\overline{M}}(C_I(S)))$ is a best offender. Thus $D$ is a best offender on $I$. Since by 8.18 $\overline{M_\circ}$ is quasisimple and $I$ is a simple $\overline{M_\circ}$-module, we are allowed to apply the Point-Stabilizer Theorem C.8 to $\overline{M_\circ D_0}$.

Now C.8 shows that $\overline{M_\circ D} \cong SL_n(q)$, $n \geqslant 2$, $Sp_{2n}(q)$,$n \geqslant 2$, $G_2(q)$ or $Sym(n)$, $n > 6$, and $I$ is a corresponding natural module for $\overline{M_\circ D}$. The last two cases are impossible since they do not appear in Theorem D.

Suppose that $I$ is a natural $Sp_{2n}(q)$ module with $n \geqslant 2$. By B.37, $W$ is 1-dimensional. Hence by (III) $[I, T] = W$ is 1-dimensional, and $\overline{T}$ acts as a transvection group on $I$. But then $\overline{T}$ is not a weakly closed subgroup of $\overline{M}$ since $n \geqslant 2$. Therefore $\overline{M_\circ D} \cong SL_n(q)$. Note that the natural $SL_2(q)$-module also is a natural $SL_2(q)$-wreath product module and so has been ruled out by 8.16. Thus $n \geqslant 3$ and $\overline{M_\circ D}$ is perfect. Hence $\overline{D} \leqslant \overline{M_\circ}$ and

(V) $\overline{M_\circ} \cong SL_n(q)$, $n \geqslant 3$, and $I$ is a corresponding natural module for $\overline{M_\circ}$.

Again by B.37, $W$ is 1-dimensional. Let $1 \neq u \in U$. Since $M$ acts transitively on $I$, $[u, Q^g] = 1$ for some $g \in M$. Thus $C_U(Q^g) \neq 1$ and $Q^g \leqslant E$. Since $E = Q$ by (I), this gives $Q^g = Q$ and $u \in W$. So $U = W$, and by 8.13(e)

$$W = U = C_I(I^x) = C_I(D) = C_I(A),$$

and since by 8.12 $A$ acts quadratically on $I$,

$$[I, I^x] \leqslant [I, D] \leqslant [I, A] \leqslant C_I(A) = U = W.$$

By B.37(1) $\overline{Q} = C_{\overline{M^\circ}}(W) \cap C_{\overline{M^\circ}}(I/W)$, and so $|\overline{Q}| = |q^{n-1}| = |I/U|$ and $\overline{I^x} \leqslant \overline{D} \leqslant \overline{A} \leqslant \overline{Q}$. Since by 8.19(d) $I/U \cong \overline{I^x}$, $|\overline{I^x}| = |I/U| = |\overline{Q}|$ and

(VI) $\overline{A} = \overline{D} = \overline{I^x} = \overline{Q}$.

As $\overline{I^x} = \overline{A}$, 8.15(c) shows that $Y \cap A = IC_Y(I^x)$, and since $\overline{Q} = \overline{I^x}$, $Y \cap A = IC_Y(Q)$. Put $a := |Y/Y \cap A|$ and $b := |W|$. (Actually $b = q$, but this will not be important.) Let $s \in Q$ with $\overline{s} \neq 1$. Then $[Y \cap A, s] = [IC_Y(Q), s] = [I, s] \leqslant W$ and so $|[Y \cap A, s]| \leqslant b$. Hence

$$|Y/C_Y(s)| \leqslant |Y/Y \cap A||Y \cap A/C_{Y \cap A}(s)| \leqslant a|[Y \cap A, s]| \leqslant ab.$$

Since $s \in Q$, $[C_Y(Q), s] = 1$. Now A.55(c) gives

$$|C_Y(Q)| \leqslant |[Y, s]| = |Y/C_Y(s)| \leqslant ab.$$

As $C_Y(Q) \cap I = W$ has order $b$, $|C_Y(Q)I/I| \leqslant \frac{ab}{b} = a$. Using $Y \cap A = IC_Y(Q)$ we get

$$|Y/I| = |Y/Y \cap A||Y \cap A/I| = |Y/Y \cap A||C_Y(Q)I/I| \leqslant aa = a^2.$$

We are now in the position to prove (2°).

Assume that that $|D/C_D(Y)| < |Y/Y \cap A|^2$. Then 8.14 implies $[Y, D] \leqslant I$. Since $\overline{D} = \overline{Q}$ and $M_\circ \leqslant M^\circ = \langle Q^M \rangle$, this gives $[Y, M_\circ] \leqslant I$, and (2°) holds.

Assume that $|D/C_D(Y)| \geqslant |Y/Y \cap A|^2$. Then

$$|Y/I| \leqslant a^2 = |Y/Y \cap A|^2 \leqslant |D/C_D(Y)| = |\overline{D}| = q^{n-1}.$$

Since $SL_n(q)$ has no non-central simple (FF-)modules of order at most $q^{n-1}$, we get $[Y/I, M_\circ] = 1$. So again $[Y, M_\circ] \leqslant I$, and (2°) is proved.

Suppose now for a contradiction that $[Y, M_\circ] \nleqslant I$ and choose an $M_\circ D$-submodule $X$ of $Y$ minimal with respect to $[X, M_\circ] \nleqslant I$. Put

$$X_1/I := C_{X/I}(M_\circ), \qquad V := X/X_1, \qquad \widehat{M_\circ D} := M_\circ D/C_{M_\circ D}(V),.$$

Next we show:

3°. *$V$ is a simple $M_\circ D$-module, $F^*(\widehat{M_\circ D}) = \widehat{M_\circ}$, $\widehat{M_\circ D} = \langle \widehat{D}^{\widehat{M_\circ D}} \rangle$, and*

(VII) $$|V/C_V(D)| \leqslant \sqrt{|D/C_D(V)|} < |D/C_D(V)|.$$

Note that $[\overline{M_\circ}, \overline{C_{M_\circ D}(V)}] = 1$ since $\overline{M_0}$ is quasisimple and $\overline{M_\circ} \nleqslant \overline{C_{M_\circ D}(V)}$,. Since also $I$ is a simple $\overline{M_0}$-module and $O_p(\overline{M}) = 1$, 1.14(c) shows that $\overline{C_{M_\circ D}(V)}$, is a $p'$-group and so $C_D(V) = C_D(Y)$. In particular, $\widehat{D} \neq 1$.

By the choice of $X$, $V$ is a simple $M_\circ D$-module with $[V, M_\circ] \neq 1$. Since $C_{M_\circ}(Y) \leqslant C_{M_\circ}(V)$, $\widehat{M_\circ}$ is a non-trivial quotient of the quasisimple group $\overline{M_\circ}$, and so also $\widehat{M_\circ}$ is quasisimple. As $V$ is a simple $M_\circ D$-module, $O_p(\widehat{M_\circ D}) = 1$. Thus 1.14(a) implies that $F^*(\widehat{M_\circ D}) = \widehat{M_\circ}$ is quasisimple, and $[\widehat{M_\circ}, \widehat{D}] = \widehat{M_\circ}$. Hence $\widehat{M_\circ D} = \langle \widehat{D}^{\widehat{M_\circ D}} \rangle$.

Moreover,

$$|V/C_V(D)|^2 \leqslant |\widetilde{Y}/C_{\widetilde{Y}}(D)|^2 \overset{8.15(b)}{\leqslant} |D/C_D(Y)| = |D/C_D(V)|,$$

and so

$$|V/C_V(D)| \leqslant \sqrt{|D/C_D(V)|} < |D/C_D(V)|.$$

Hence (3°) is proved.

4°. *$[I, E, D] \neq 1$, and $V$ is not selfdual as an $\mathbb{F}_p M_\circ D$-module.*

Since $[Y, M_\circ] \nleqslant I$, (2°) shows

(VIII) $$[I, E, D] \neq 1.$$

By 8.13(e) $C_I(D) = U$, and so $[I, E] \nleqslant U$. Since by 8.19(d) $\overline{I^x} \cong I/U$ as an $E$-module, also $[\overline{E}, \overline{I^x}] \neq 1$ and thus $[\overline{E}, \overline{D}] \neq 1$. Hence $[E, D] \nleqslant C_D(Y) = C_D(V)$ and $[E, D, V] \neq 1$. By (1°) $[\widetilde{Y}, E, D] = 1$ and hence also $[V, E, D] = 1$. Since $[E, D, V] \neq 1$, the Three Subgroups Lemma implies that $[V, D, E] \neq 1$.

Let $V^*$ be the $\mathbb{F}_p$-dual of the $\mathbb{F}_p M_\circ D$-module $V$. Since $[V, E, D] = 1$, B.8 gives $[V^*, D, E] = 1$. Hence $[V, D, E] \neq 1$ implies that $V$ is not isomorphic to $V^*$ as an $\mathbb{F}_p M_\circ D$-module. Thus (4°) has been established.

By 8.15(a), $D$ acts quadratically on $\widetilde{Y}$ and so also on $V$. Hence, according to (VII), $\widehat{D}$ is a quadratic (over-) offender on $V$. Now (3°) shows that we can apply the FF-Module Theorem C.3 to $\widehat{M_\circ D}$. We will discuss the various outcomes of this theorem.

In cases C.3(2)-(4) $V$ is a natural $Sp_{2n}(q)$-, $SU_n(q)$-, $\Omega_n(q)$-module,respectively. But then $V$ is selfdual over $\mathbb{F}_p$, which contradicts (4°).

In cases C.3(5)-(12), the Best Offender Theorem C.4 shows that either

$$|V/C_V(D)| = |D/C_D(V)|,$$

or

$$|V/C_V(D)| = q^4 \leqslant |D/C_D(V)| \leqslant q^5 \text{ (in the } Spin_7(q)\text{-case)},$$

or

$$2|V/C_V(D)| = |D/C_D(V)|,\ |D/C_D(V)| = 2^k \text{ and } n = 2k \geqslant 6 \text{ (in the } Sym(n)\text{-cases)}.$$

In either of these cases $|V/C_V(D)| > \sqrt{|D/C_D(V)|}$, which contradicts (VII).

Thus C.3(1) holds. So $V$ is a natural $SL_m(p^l)$-module, $m \geqslant 2$. If $m = 2$ we get (for example by C.13(g)) $|V/C_V(D)| = |D/C_D(V)|$, which again contradicts (VII). Thus $\widehat{M_\circ D} \cong SL_m(p^l)$, $m \geqslant 3$. In particular $\widehat{M_\circ D} = \widehat{M_\circ}$, so $\widehat{D} \leqslant \widehat{M_\circ}$. Since $\overline{C_{M_\circ D}(V)}$ is a $p'$-group, this gives $\overline{D} \leqslant \overline{M_\circ}$. Moreover, comparing $\widehat{M_\circ}$ with $\overline{M_\circ}$ in Theorem D, we get:

5°. $\quad \overline{D} \leqslant \overline{M^\circ} = \overline{M_\circ}$, *and one of the following holds:*

(A) *$I$ is a natural $SL_n(q)$-module for $\overline{M_\circ}$, $n = m \geqslant 3$, $q = p^l$.*
(B) *$I$ is a natural $\Omega_6^+(q)$ module for $\overline{M_\circ}$, $m = 4$ and $q = p^l$.*
(C) *$I$ is the exterior square of an natural $SL_n(q)$-module for $\overline{M_\circ}$, $n = m \geqslant 5$, $q = p^l$.*

We now derive a contradiction to our assumption $[Y, M^\circ] \nleqslant I$ by showing that none of the above three cases holds. And we do this by comparing the action of $\widehat{M_\circ}$ on $V$ with that of $\overline{M_\circ}$ on $I$.

Suppose that Case (A) holds, so $I$ is a natural $SL_n(q)$-module for $\overline{M_\circ}$. Then by B.38(b) $[I, E] \leqslant U$, a contradiction to (VIII).

Thus (A) does not hold and so $m \geqslant 4$. Hence C.18 shows that $H^1(\overline{M_\circ}, V^*) = 0$. Thus $X/I = [X/I, M_\circ] \times X_1/I$ and the minimality of $X$ shows $X_1 = I$ and $V = X/I$. So

6°. $\quad$ *$X/I$ is an natural $SL_m(q)$-module for $\overline{M_\circ}$, where $m \geqslant 4$.*

Suppose next that Case (B) holds, so $I$ is a natural $\Omega_6^+(q)$ module for $\overline{M_\circ}$ and $m = 4$. In particular $V$ has $\mathbb{F}_q$-dimension 4, where $\mathbb{F}_q := End_{\widehat{M_\circ}}(V)$ is a field of order $q$. By B.37, $W$ is 1-dimensional and $\overline{Q} = C_{\overline{M_\circ}}(W^\perp/W) \cap C_{\overline{M_\circ}}(W)$. It follows that $|\overline{Q}| = q^4$, and $C_V(Q) = [V, Q]$ is a 2-dimensional subspace of $V$. Since by (1°) $[V, E, D] = 1$, $[V, Q] \leqslant [V, E] \leqslant C_V(D)$.

If $C_V(D) = [V, Q]$, the quadratic action of $D$ shows $[V, D] \leqslant [V, Q]$ and so

$$\widehat{D} \leqslant C_{\widehat{M_\circ}}([V, Q]) \cap C_{\widehat{M_\circ}}(V/[V, Q]) = \widehat{Q}.$$

Thus $\overline{D} \leqslant \overline{Q}$, a contradiction, since, for example by the Point-Stabilizer Theorem C.8, no subgroup of $\overline{Q}$ is a non-trivial offender on $I$, while by 8.13(f) $\overline{D}$ is a non-trivial offender on $I$.

We have shown that $[V, Q] < C_V(D) < V$. Since $\dim_{\mathbb{F}_q}[V, Q] = 2$ and $\dim_{\mathbb{F}_q} V = m = 4$, we get that $C_V(D)$ is an $\mathbb{F}_q$-hyperplane of $V$.

Put $T := C_{M_\circ}(C_V(D)) \cap C_{M_\circ}(V/C_V(D))$, so $D \leqslant T$ and $\widehat{T}$ is the unipotent radical of the normalizer of a hyperplane in $\widehat{M_\circ}$. Note that $T$ centralizes a 3-dimensional singular subspace $W_0$ of $I$. Since $D \leqslant T$, $W_0 \leqslant C_I(D) = U$, and so by 2.7(b) $N_M(W_0)^\circ \leqslant E$. By 8.19(c) $E$ normalizes $U$, so also $N_M(W_0)^\circ$ normalizes $U$. Now B.38(a) shows that $W_0 \leqslant U \leqslant W_0^\perp$, so $U = W_0$ since $W_0 = W_0^\perp$. Thus $|\overline{I^x}| = |I/U| = q^3 = |\overline{T}|$, and $I^x \leqslant D \leqslant T$ gives $\overline{I^x} = \overline{T}$.

By 8.13(h) $[A, D] \leqslant C_I(L) \leqslant C_M(Y)$ and since $I^x \leqslant D$, $A$ centralizes $\overline{I^x} = \overline{T}$. Since $\overline{T}$ is a Sylow $p$-subgroup of $C_{\overline{M}}(\overline{T})$, we conclude that $A \leqslant T$, and

$$|\overline{T}| = |\overline{I^x}| \leqslant |\overline{A}| = |\overline{T}|,$$

so $\overline{A} = \overline{I^x}$ and by 8.15(c) $Y \cap A = IC_Y(L)$. Hence also $X \cap A = IC_X(L) = IC_X(A)$. In particular, $C_X(A) \nleqslant I$. Since a natural $\Omega_6^+(q)$-module is isomorphic to the exterior square of the natural $SL_4(q)$-module and since $\overline{A} = \overline{T}$, we can apply [**MS5**, 6.3]. We conclude that $X$ is not a $Q!$-module for $M^\circ$ with respect to any $p$-group, a contradiction to $Q!$. This shows that also Case (B) does not hold.

Suppose that Case (C) holds. Then $I$ is the exterior square of a natural $SL_n(q)$-module $V_0$ with $n \geqslant 5$. By 8.12 and 8.13(f) $I^x$, $D$ and $A$ are non-trivial offenders on $I$. Hence C.4 shows that there exist a $\mathbb{F}_q$-hyperplane $V_1$ of $V_0$ such that

$$\overline{D} = \overline{I^x} = \overline{A} = C_{\overline{M_\circ}}(V_1) \text{ and } |\overline{D}| = q^{n-1}.$$

If $V_0$ is dual to $V$ as an $\mathbb{F}_p M_\circ$-module we get $|C_V(D)| = q$ and so $|V/C_V(D)| = q^{n-1} = |\overline{D}|$. But this contradicts (VII). Thus, $V_0$ is isomorphic to $V$ as an $\mathbb{F}_p M_\circ$-module. As above, using $\overline{I^x} = \overline{A}$

and 8.15(c), we conclude that $X \cap A = IC_X(A)$. Since $[X, A] \leqslant X \cap A$, we have $X \cap A \nleqslant I$ and so also $C_X(A) \nleqslant I$. Applying [**MS5**, 6.3] shows that $X$ is not a $Q$!-module, a contradiction.

We have seen that each of the three cases in (5°) lead to a contradiction, and so 8.20 is proved. □

LEMMA 8.21. *Suppose that $I \leqslant A$. Then one of the following holds:*

(a) *$p = 2$, $\overline{M^\circ} \cong SL_3(2)$, $I$ is a corresponding natural module, $|Y/I| = 2$, and Case (2) of Theorem H holds.*
(b) *$p = 2$, $\overline{M^\circ} \cong \Omega_6^+(2) \cong Alt(8)$, $I$ is the corresponding natural module, $|Y/I| = 2$, and $Y$ is the central quotient of the permutation module on a set $\Lambda$ of eight objects.*

PROOF. According to 8.20 $[Y, M_\circ] \leqslant I$. By 8.4(f), $C_Y(M) = 1$ and so $Y$ does not split over $I$. Moreover, by 8.18 $\overline{M_\circ}$ is quasisimple. Comparing Theorem D (for quasisimple $\overline{M_\circ}$) with C.18 yields $p = 2$ and one of the following three cases:

(A) $I$ is a natural $SL_3(2)$-module for $M^\circ$, and $|Y/I| = 2$.
(B) $I$ is natural $Sp_{2n}(q)$- or $Sp_4(2)'$-module for $M^\circ$.
(C) $\overline{M^\circ} \cong \Omega_6^+(2) \cong Alt(8)$, $I$ is the corresponding natural module, $|Y/I| = 2$, and $Y$ is the central quotient of the permutation module on a set $\Lambda$ of eight objects.

Suppose that (A) holds. By B.37 $|W| = 2$ and $\overline{Q} = C_{\overline{M}}(I/W)$ has order 4. Suppose that $|[Y, Q]| = 2$. Then $|Y/C_Y(a)| = 2$ for any $1 \neq a \in \overline{Q}$. Since $\overline{Q}$ is generated by two such elements, $|Y/C_Y(Q)| \leqslant 4$ and $Q$ is an offender on $Y$. But this contradicts C.22. Hence $W < [Y, Q] \leqslant I$, and since $N_M(W)$ acts simply on $I/W$, $I = [Y, Q] \leqslant Q$. Thus, case (2) of Theorem H holds, and (a) is verified.

Suppose that (B) holds. Note that $I \leqslant A \trianglelefteq L$ and $A \leqslant M$. So 2.25 shows that $Y \leqslant O_p(L)$, a contradiction to $L \in \mathfrak{L}_H(Y_M)$.

Finally in Case (C), (b) holds, and so the lemma is proved. □

By the preceding lemma, $I$ is either a natural $SL_3(2)$-module or a natural $\Omega_6^+(2)$-module for $M^\circ$. Moreover, if $I$ is natural $SL_3(2)$-module then Theorem H holds. So we assume for the remainder of this subsection that $I$ is a natural $\Omega_6^+(2)$-module for $M^\circ$. In particular, Case (b) of 8.21 holds and so $Y$ is the central quotient of the permutation module on a set $\Lambda$ of eight objects.

We will make use of the fact that $\Omega_6^+(2) \cong SL_4(2) \cong Alt(\Lambda) \cong Alt(8)$. Let $I_0$ be a natural $SL_4(2)$-module for $\overline{M^\circ}$ and $W_0 := C_{I_0}(\overline{Q})$.

LEMMA 8.22. *Suppose that $I \leqslant A$ and $I$ is the natural $\Omega_6^+(2)$-module for $M^\circ$.*

(a) *$\overline{M} \cong \Omega_6^+(2) \cong Alt(8)$ or $\overline{M} \cong O_6^+(2) \cong Sym(8)$. In particular, $\overline{M} = \overline{M_\circ}\,\overline{S}$.*
(b) *$Y \cap A = I$ and $A = D$.*
(c) *$W$ is a singular 1-space in $I$, $\overline{Q} = \overline{Q^\bullet} = C_{\overline{M^\circ}}(W^\perp/W) \cap C_{\overline{M^\circ}}(W) = O_2(N_{\overline{M}}(W))$, and $\overline{Q}$ is a natural $\Omega_4^+(2)$-module for $M_{\overline{M^\circ}}(W)$.*
(d) *$|\overline{Q}| = 16$, $\overline{Q}$ has two orbits of length 4 on $\Lambda$, $W_0$ is a 2-subspace of $I_0$, $\overline{Q} = C_{\overline{M^\circ}}(W_0) \cap C_{\overline{M^\circ}}(I_0/W_0) = O_2\big(N_{\overline{M^\circ}}(W_0)\big)$.*
(e) *$\overline{I^x} = \overline{A}$ is elementary abelian of order 8, $\overline{A}$ acts regularly on $\Lambda$, $\overline{A} \leqslant \overline{M^\circ}$, and $I = [Y, A]$.*
(f) *$I \nleqslant Q^\bullet$.*

PROOF. (a): Since $I$ is natural $\Omega_6^+(2)$-module for $M_\circ$ and since $M$ normalizes $M_\circ$, $M$ fixes the unique $M_\circ$-invariant non-degenerate quadratic form on $I$. Now $|O_6^+(2)/\Omega_6^+(2)| = 2$ implies $\overline{M} \cong \Omega_6^+(2)$ or $\overline{M} \cong O_6^+(2)$.

(b): We have $I \leqslant Y \cap A < Y$ and $|Y/I| = 2$, thus $Y \cap A = I$. Since $L \in \mathfrak{L}_G(Y_M)$, $A = \langle (Y \cap A)^L \rangle$ and so $A = \langle I^L \rangle = D$.

(c): Since both $Q$ and $Q^\bullet$ are large subgroups of $G$, B.37 shows that $W$ is a singular 1-space in $I$ and $\overline{Q} = \overline{Q^\bullet} = C_{\overline{M^\circ}}(W^\perp/W) \cap C_{\overline{M^\circ}}(W)$. Now B.28 implies that $\overline{Q} = O_2(N_{\overline{M}}(W))$ and $\overline{Q}$ is a natural $\Omega_4^+(2)$-module for $M_{\overline{M^\circ}}(W)$.

(d): Since $\overline{Q}$ is a natural $\Omega_4^+(2)$-module, $|\overline{Q}| = 16$. Up to conjugacy

$$\langle (12)(34), (13)(24)\rangle \times \langle (56)(78), (57)(68)\rangle$$

is the only (elementary) abelian subgroup of order 16 in $Alt(8)$, and so $\overline{Q}$ has two orbits of length 4 on $\Lambda$. If $W_1$ is 2-subspace of $I_0$, then $O_2(N_{\overline{M^\circ}}(W_1))$ is elementary abelian of order 16, and so (d) holds.

(e): For $\lambda \in \Lambda$ let $y_\lambda$ be the unique non-trivial element in $Y$ fixed by $C_M(\lambda)$. Then $y_\lambda \notin I$ and since by (b) $Y \cap A = I$, $y_\lambda \notin A$. Hence 1.43(g) gives $Y \cap A = [y_\lambda, A]C_Y(A)$. Since $[Y \cap A, A] = [I, A] \neq 1$, we get $[y_\lambda, A, A] \neq 1$. Thus $|\lambda^A| \geqslant 4$. So either $\overline{A}$ acts regularly on $\Lambda$ or has two orbits of length 4. On the other hand by 8.12, $A$ is an offender on $I$. The Offender Theorem C.4(h) now shows that $A$ acts regularly on $\Lambda$. In particular, all orbits of $I^x$ on $\Lambda$ have the same length. Again by 8.12, $I^x$ is an offender on $I$, and C.4(h) shows that also $I^x$ acts regularly in $\Lambda$. Hence $\overline{A} = \overline{I^x}$. The regularity of $\overline{A}$ also gives

$$Y = \langle y_\lambda^A\rangle = \langle y_\lambda\rangle[Y, A] \leqslant \langle y_\lambda\rangle I = Y,$$

so $I = [Y, A]$. Moreover, every element of $\overline{A}$ is an even permutation, so $\overline{A} \leqslant \overline{M^\circ}$. Thus (e) holds.

(f): Suppose that $I \leqslant Q^\bullet$. Since $L \leqslant N_G(Q) \leqslant N_G(Q^\bullet)$ this gives $I^x \leqslant Q^\bullet$. By (c) $\overline{Q} = \overline{Q^\bullet}$, so $\overline{I^x} \leqslant \overline{Q}$, and by (d) $\overline{Q}$ has an orbit of length 4 on $\Lambda$. Hence $\overline{I^x}$ is not regular on $\Lambda$, which contradicts (e). □

Put $U_0 := C_{I_0}(\overline{A})$. Note that $\overline{M^\circ}$ has two classes of regular elementary abelian subgroups, interchanged by the outer automorphism. By (a) $\overline{M} \cong Alt(8)$ or $\overline{M} \cong Sym(8)$, and we conclude that $N_{\overline{M}}(\overline{A}) \leqslant \overline{M^\circ}$. Moreover, each member of one of these classes centralizes a hyperplane in $I_0$, each member of the other a 1-subspace. So replacing $I_0$ but its dual, if necessary, we may assume that $\overline{A}$ centralizes a hyperplane in $I_0$, so $U_0$ is a hyperplane of $I_0$.

LEMMA 8.23. *Suppose that $I \leqslant A$ and $I$ is the natural $\Omega_6^+(2)$-module for $M^\circ$. Then*

(a) *$U_0$ is hyperplane in $I_0$, $\overline{I^x} = \overline{A} = C_{\overline{M^\circ}}(U_0)$, $N_{\overline{M}}(\overline{A}) = N_{\overline{M^\circ}}(U_0)$, $N_{\overline{M}}(\overline{A})/\overline{A} \cong SL_3(2)$, and $\overline{A}$ is a natural $SL_3(2)$-module for $N_{\overline{M}}(\overline{A})$ isomorphic to $U_0$.*

(b) *$U$ is a singular 3-space in $I$, $N_{\overline{M}}(U) = N_{\overline{M}}(\overline{A}) = M_{\overline{M^\circ}}(U_0)$, $U$ is natural $SL_3(2)$-module for $N_{\overline{M}}(U)$ dual to $U_0$, $I/U$ and $\overline{A}$ are natural $SL_3(2)$-module for $M_{\overline{M}}(U)$ isomorphic to $U_0$, and $\overline{I^x} = \overline{A} = C_{\overline{M}}(U) = C_{\overline{M}}(I/U) = C_{\overline{M^\circ}}(U_0)$.*

PROOF. (a): By the choice of $I_0$, $U_0$ is a hyperplane of $I_0$, and by 8.22(e) $\overline{A} = \overline{I^x}$ has order eight. This gives $\overline{A} = C_{\overline{M}}(U_0)$, and (a) follows.

(b): Observe that $I \cong I_0 \wedge I_0$ as an $M_\circ$-module and recall from 8.13(e) that $U = C_I(A)$. Thus (b) follows from (a). □

LEMMA 8.24. *Suppose that $I \leqslant A$ and $I$ is the natural $\Omega_6^+(2)$-module for $M^\circ$.*

(a) *$\overline{E} = N_{\overline{M}}(U) = N_{\overline{M}}(\overline{A}) = N_{\overline{M^\circ}}(U_0)$.*

(b) *$\overline{E}/\overline{A} \cong SL_3(2)$, $U$ is a natural $SL_3(2)$-module for $E$ dual to $U_0$, and $I/U$ and $\overline{A}$ are natural $SL_3(2)$-modules for $E$ isomorphic to $U_0$.*

(c) *$A \trianglelefteq E$ and $\overline{A} = O_2(\overline{E}) = \overline{O_2(E)}$.*

(d) *$I = [I, E] = [Y, A] = [Y, O_2(E)]$.*

(e) *$U = [I, A] = [I, O_2(E)]$.*

PROOF. (a) and (b): Recall from 8.19(c) that $E = N_M(U)^\circ$. By B.38(c) $U$ is a natural $SL_3(2)$-module for $E$ and so $N_{\overline{M}}(U) = \overline{E}C_{\overline{M}}(U)$. By 8.23(b) $C_{\overline{M}}(U) = \overline{A}$ is a natural $SL_3(2)$-module and thus non-central simple module for $N_{\overline{M}}(U)$. Since $E \trianglelefteq N_M(U)$ we conclude that $\overline{A} \leqslant \overline{E}$ and $\overline{E} = N_{\overline{M}}(U)$. Now (a) and (b) follow from 8.23(b).

(c): From (b) we get $I = [I, E]$. Since $I$ normalizes $U$ and so $E$, we have $[I, E] \leqslant E$ and thus $I \leqslant E$. As $L$ normalizes $U$ and thus $E$, we conclude that $E$ normalizes $\langle I^L\rangle \leqslant E$. By 8.22(b)

$A = D = \langle I^L \rangle$ and so $A \trianglelefteq E$, in particular $A \leqslant O_2(E)$. By (b) $\overline{E}/\overline{A} \cong SL_3(2)$ and thus $O_2(\overline{E}) = \overline{A}$. Hence $\overline{A} \leqslant \overline{O_2(E)} \leqslant O_2(\overline{E}) = \overline{A}$, and (c) follows.

(d) and (e): As we have already seen above, (b) gives $I = [I, E]$, and by 8.22(e) $I = [Y, A]$. Moreover, since by (b) both, $U$ and $I/U$, are simple $\overline{E}$-modules, $[I, A] = U$. Since by (c) $\overline{A} = \overline{O_2(E)}$, (d) and (e) follow. □

Put

$$N := N_G(U), \quad C := C_G(U), \quad B := \langle I^N \rangle, \quad \widehat{B} := B/U, \quad N_0 := C_N(\widehat{B}).$$

LEMMA 8.25. *Suppose that $I \leqslant A$ and $I$ is the natural $\Omega_6^+(2)$-module for $M^\circ$.*

(a) *$E \trianglelefteq N$, and $L \leqslant C$.*
(b) *$A \leqslant B \leqslant O_2(E) \leqslant O_2(N)$ and $\overline{A} = \overline{B} = O_2(\overline{E}) = C_{\overline{M}}(U) = C_{\overline{M^\circ}}(U_0)$.*
(c) *$[B, O_2(E)] = B' = \Phi(B) = U \leqslant \Omega_1 Z(B)$.*
(d) *$O_2(E) \leqslant N_0 \leqslant C \cap M^\dagger$.*
(e) *$[B, Y] = I$.*
(f) *$N = EC$, $N_N(Y) = EC_G(Y)$ and $[E, C] \leqslant O_2(E) \leqslant N_0$.*

PROOF. (a): By 8.19(c) $E = N_G(U)^\circ$, so $E \trianglelefteq N$, and by definition, $U = C_I(L)$ and so $L \leqslant C$.

(b): By 8.22(b) $\langle I^L \rangle = A = D \leqslant B$, and by 8.24(c), $A \trianglelefteq E$, so $I \leqslant A \leqslant O_2(E)$. Since by (a) $E \trianglelefteq N$ also $O_2(E) \trianglelefteq N$, whence $B = \langle I^N \rangle \leqslant O_2(E) \leqslant O_2(N)$. By 8.24(c) $\overline{A} = O_2(\overline{E})$ and by 8.23(b) $\overline{A} = C_{\overline{M}}(U) = C_{\overline{M^\circ}}(U_0)$. So also the second part of (b) holds.

(c): Recall from (a) that $O_2(E) \trianglelefteq N$ and from (b) that $A \leqslant B \leqslant O_2(E)$. By 8.24(e), $U = [I, A] = [I, O_2(E)]$. Since $U$ and $O_2(E)$ are $N$-invariant and $B = \langle I^N \rangle$, this gives $[B, O_2(E)] = U$ and

$$U = [I, A] \leqslant [I, B] \leqslant [I, O_2(E)] = [B, O_2(E)] = U.$$

Since $[B, B] \leqslant [B, O_2(E)]$ we conclude that $U = B' = [B, O_2(E)]$. Moreover, as $I/U$ is elementary abelian and $[U, I] = 1$, also $U = \Phi(B)$ and $U \leqslant \Omega_1 Z(B)$.

(d): By (c) $[B, O_2(E)] = U$ and so $O_2(E) \leqslant C_N(B/U) = N_0$. Since $I \leqslant B$, we get $[I, N_0] \leqslant [B, N_0] \leqslant U \leqslant I$ and so $N_0 \leqslant N_G(I) = N_G(Y) = M^\dagger$. Since $[B, N_0] \leqslant U \leqslant \Omega_1 Z(B)$, $N_0$ centralizes $\Phi(B) = U$, see 1.18. Thus $N_0 \leqslant C$.

(e): By (b) $A \leqslant B \leqslant O_2(E)$ and by 8.24(d) $[Y, A] = [Y, O_2(E)] = I$. Hence $[Y, B] = I$, and (e) holds.

(f): By 8.24(b) $U$ is a natural $SL_3(2)$-module for $E$ and thus $E$ induces $Aut(U)$ on $U$, so $N = EC$. By 8.24(a) $\overline{E} = N_{\overline{M}}(U)$, and we conclude that $N_N(Y) = N_{N_G(Y)}(U) = EC_G(Y)$. From 1.52(c) we get $[N_G(U)^\circ, C_G(U)] \leqslant O_2(N_G(U)^\circ)$. As $E = N_G(U)^\circ$, this gives $[E, C] \leqslant O_2(E)$. Also by (d) $O_2(E) \leqslant N_0$. □

LEMMA 8.26. *Suppose that $I \leqslant A$ and $I$ is the natural $\Omega_6^+(2)$-module for $M^\circ$.*

(a) *$N_N(Y)$ is a parabolic subgroup of $N$, and $N_N(YN_0) = N_N(Y)$.*
(b) *$\widehat{B}$ is the direct sum of $m$ natural $SL_3(2)$-modules isomorphic to $I/U$ (and $U_0$) for $E$, for some $m \geqslant 2$.*
(c) *$[B, C_E(Y)] \leqslant U \leqslant I$ and $C_E(U) = E \cap C \leqslant N_0$.*
(d) *$F = [F, E] = [F, E_\circ]$ for any $E$-invariant subgroup of $F$ of $B$. In particular, $B = [B, E_\circ] \leqslant E_\circ \leqslant M_\circ$.*
(e) *$\widehat{B}$ is a 2-reduced $N$-module.*

PROOF. (a): Since $O_2(M) \leqslant N$ and $Y$ is asymmetric, $N_N(Y)$ is a parabolic subgroup of $N$ (see 2.6(c)). By definition of $N_0$, $[B, N_0] \leqslant U \leqslant I$, and by 8.25(e), $[B, Y] = I$, so

$$N_N(Y) \leqslant N_N(YN_0) \leqslant N_N([\widehat{B}, YN_0]) = N_N([\widehat{B}, Y]) = N_N(\widehat{I}) = N_N(I) = N_N(Y).$$

(b): By 8.25(f) $[E, C] \leqslant N_0$ and $N = EC$. Hence $\widehat{I}^c \cong \widehat{I}$ as an $E$-module for every $c \in C$, and $\widehat{B} = \langle \widehat{I}^N \rangle = \langle \widehat{I}^C \rangle$. Since by 8.24(b) $\widehat{I} = I/U \cong U_0$ as an $E$-module, (b) follows.

(c): Since $C_E(Y)$ centralizes $I/U$, (b) gives $C_E(Y) \leqslant C_E(\widehat{B}) \leqslant N_0$. Hence $[B, C_E(Y)] \leqslant U \leqslant I$ and $C_E(Y) \leqslant C \cap N_0$.

(d): This is a direct consequence of (b).

(e): By 8.25(f) $N_N(Y) = EC_G(Y)$. As $\widehat{I} = I/U$ is a natural $SL_3(2)$-module for $E$, we conclude that $\widehat{I}$ is 2-reduced for $N_N(Y)$. Since $\widehat{B} = \langle \widehat{I}^N \rangle$ and by (a) $N_N(Y)$ is a parabolic subgroup of $N$, A.12 shows that $\widehat{B}$ is a 2-reduced $N$-module. □

Put $X := \langle (B \cap O_2(M))^{M^\circ} \rangle$. Moreover the integer $m$ is chosen as in 8.26(b).

LEMMA 8.27. *Suppose that $I \leqslant A$ and $I$ is the natural $\Omega_6^+(2)$-module for $M^\circ$.*

(a) $X = O_2(M_\circ) = [X, M_\circ] = [O_2(M), M_\circ]$ *and* $M_\circ/X \cong SL_4(2)$. *In particular,* $C_{M_\circ}(Y) = X$.
(b) $X' \leqslant \Phi(X) \leqslant I$.
(c) $[X, E_\circ] = X \cap B$.
(d) *$X/I$ is the direct sum of $m-2$ natural $SL_4(2)$ modules for $M_\circ$ isomorphic to $I_0$. In particular,* $|X/I| = 2^{4(m-2)}$ *and* $|X \cap B/I| = 2^{3(m-2)}$.
(e) $Y \cap X = I$, $|X/X \cap B| = 2^{m-2}$ *and* $|YX/X \cap B| = 2^{m-1}$.

PROOF. (a): Note that $[B, O_2(M)] \leqslant B \cap O_2(M) = B \cap X$ and $X = \langle (B \cap X)^{M^\circ} \rangle$. Since $\overline{M^\circ}$ is simple, 1.54(c) shows that $M_\circ \leqslant \langle B^{M^\circ} \rangle$. Thus

$$[O_2(M), M_\circ] \leqslant [O_2(M), \langle B^{M^\circ} \rangle] = \langle [O_2(M), B]^{M^\circ} \rangle \leqslant X. \tag{I}$$

Since $B \cap O_2(M)$ is $E$-invariant, 8.26(d) gives $B \cap O_2(M) = [B \cap O_2(M), E_\circ]$, and since $E_\circ \leqslant M_\circ$, we get $B \cap O_2(M) \leqslant [X, M_\circ]$, so

$$X = \langle (B \cap O_2(M))^{M^\circ} \rangle \leqslant [X, M_\circ] \leqslant X.$$

It follows that $X = [X, M^\circ] = [X, M_\circ] \leqslant O_2(M_\circ)$; in particular $X \leqslant O_2(M_\circ) \leqslant O_2(M^\circ) \leqslant O_2(M)$. By (I) $[O_2(M), M_\circ] \leqslant X$ and thus

$$X = [X, M_\circ] = [O_2(M_\circ), M_\circ] = [O_2(M), M_\circ].$$

As $M^\circ$ is normal in $M^\dagger$, also $[O_2(M_\circ), M_\circ]$ is normal in $M^\dagger$, so $X \trianglelefteq M^\dagger$.

Since $\overline{M^\circ}$ is simple, 1.54(b) shows that $M_\circ/[O_2(M_\circ), M_\circ]$ is quasisimple, that is, $M_\circ/X$ is quasisimple. Note that

$$C_{BX}(Y) = BX \cap O_2(M) = (B \cap O_2(M))X = X.$$

By 8.25(b) $\overline{B} = C_{\overline{M^\circ}}(U_0)$ and by 8.26(d) $B \leqslant M_\circ$. Together with $\overline{M_\circ} = \overline{M^\circ}$ we get $BC_{M_\circ}(Y) = C_{M_\circ}(U_0)$. Since by 8.24(a) $N_{M^\circ}(U_0) = N_{M^\circ}(U) = M_\circ \cap N$, $N_{M_\circ}(U_0)$ normalizes $B$. Hence $BX/X$ is a $N_{M_\circ}(U_0)$-invariant complement to $C_{M_\circ}(Y)/X$ in $C_{M^\circ}(U_0)/X$. Now C.21 shows that $C_{M_\circ}(Y)/X = 1$ and so $M_\circ/X \cong SL_4(2)$. So (a) holds.

Before proving (b) – (e) we have a closer look at the structure of $E$.

$1^\circ$. $E = E_\circ C_E(Y)$, $M_\circ \cap N = E_\circ X$ *and* $O_2(M_\circ \cap N) = BX = C_{M_\circ}(U) = C_{M_\circ}(U_0)$.

By 8.24(a) $\overline{M \cap N} = \overline{N_M(U)} = \overline{E}$ and so $M \cap N = EC_M(Y)$, and by 8.24(b) $\overline{E}/\overline{A} \cong SL_3(2)$ and $\overline{A}$ is a natural $SL_3(2)$-module for $E$. Thus $O^2(\overline{E}) = \overline{E}$ and so $E = E_\circ C_E(Y)$ and $M \cap N = E_\circ C_M(Y)$. Since $E_\circ \leqslant M_\circ$, this gives $M_\circ \cap N = E_\circ C_{M_\circ}(Y)$. Moreover, (a) shows that $X = C_{M_\circ}(Y)$ and so $M_\circ \cap N = E_\circ X$.

By 8.26(d), $B \leqslant E_\circ$ and so $BX \leqslant O_2(M^\circ \cap N)$. Since $\overline{B} = O_2(\overline{E}) = O_2(\overline{M_\circ \cap N})$, we get $BX = O_2(M_\circ \cap N)$. By 8.25(b), $\overline{B} = C_{\overline{M}}(U) = C_{\overline{M^\circ}}(U_0)$ and hence $C_{M_\circ}(U) = BC_{M_\circ}(Y) = BX = C_{M_\circ}(U_0)$.

$2^\circ$. $[X, E, B] \leqslant I$ *and* $[X \cap B, B] \leqslant [X \cap B, O_2(E)] \leqslant I$.

Note that $[X, E] \leqslant X \cap E \leqslant O_2(E)$, $B \leqslant O_2(E)$ and by 8.25(c) $[O_2(E), B] = U \leqslant I$. Thus

$$[X, E, B] \leqslant [O_2(E), B] \leqslant I \quad \text{and} \quad [X \cap B, B] \leqslant [X \cap B, O_2(E)] \leqslant [B, O_2(E)] \leqslant I.$$

3°. $[X, E_\circ] \leqslant X \cap B$ *and* $X' \leqslant \Phi(X) \leqslant I$.

Let $g \in M_\circ \backslash N$. Since $M_\circ$ is doubly transitive on the hyperplanes of $I_0$ and $N_{M_\circ}(U_0) = N_{M_\circ}(U) = M_\circ \cap N \leqslant N_{M_\circ}(B \cap X)$,

$$(B \cap X)^{M_\circ} = (B \cap X)^{M_\circ \cap N^g} \cup \{B^g \cap X\}.$$

Also by (1°) $M_\circ \cap N = E_\circ X$, and $X$ normalizes $B \cap X$. Thus $(B \cap X)^{M_\circ \cap N^g} = (B \cap X)^{E_\circ^g}$ and

$$\text{(II)} \qquad \begin{aligned} X = \langle (B \cap X)^{M_\circ} \rangle &= (B^g \cap X)\langle (B \cap X)^{M_\circ \cap N^g} \rangle \\ = (B^g \cap X)\langle (B \cap X)^{E_\circ^g} \rangle &= (B^g \cap X)(B \cap X)[X, E_\circ^g]. \end{aligned}$$

By (2°) $[B^g \cap X, B^g] \leqslant I$ and $[X, E^g, B^g] \leqslant I$. Also $B^g \cap N$ normalizes $B \cap X$. Hence (II) yields

$$[X, B^g \cap N] = [(B^g \cap X)(B \cap X)[X, E^g], B^g \cap N] \leqslant [B \cap X, B^g \cap N]I \leqslant (B \cap X)I = B \cap X.$$

By 8.25(b) $\overline{B} = C_{\overline{M^\circ}}(U_0)$. It follows that $\overline{B^g \cap N} = C_{\overline{M^\circ}}(U_0^g) \cap C_{\overline{M^\circ}}(I_0/U_0 \cap U_0^g)$ has index 2 in $\overline{B^g}$ and acts faithfully on $U_0$. Thus $[U_0, B^g \cap N] \neq 1$ and so also $[U, B^g \cap N] \neq 1$. Note that $E/C_E(U) \cong SL_3(2)$ is simple and $E = N_G(U)^\circ = E^\circ$. Hence $U$ and $N_G(U)$ satisfy the hypothesis of 1.54, and 1.54(c) shows that $E_\circ \leqslant \langle (B^g \cap N)^{E_\circ} \rangle$. As $[X, B^g \cap N] \leqslant B \cap X$ and $B \cap X$ is $E$-invariant, this implies $[X, E_\circ] \leqslant B \cap X$, and the first statement in (3°) is proved.

Then also $[X, E_\circ^g] \leqslant B^g \cap X$, and (II) gives

$$X = (B \cap X)(B^g \cap X).$$

Again using that $[B^g \cap X, B^g] \leqslant I$ we have $[B^g \cap X, B^g \cap N] \leqslant I \leqslant B \cap B^g \cap X$ and since $B \cap X$ and $B^g \cap N$ normalize each other, $[B \cap X, B^g \cap N] \leqslant B \cap B^g \cap X$. We conclude that $[X, B^g \cap N] \leqslant B \cap B^g \cap X$. Since by 8.25(c) $\Phi(B) = B' \leqslant U$,

$$[X, B^g \cap N, X] = [B \cap B^g \cap X, X] = [B \cap B^g \cap X, (B \cap X)(B^g \cap X)] \leqslant B'B'^g = UU^g \leqslant I.$$

As before, 1.54 gives $M_\circ = \langle (B^g \cap N)^{M_\circ} \rangle$, and as $X$ is $M_\circ$-invariant, $[X, M_\circ, X] \leqslant I$ follows. By (a) $X = [X, M_\circ]$ and so $[X, X] \leqslant I$, and by 8.25(c) $\Phi(B) = B' \leqslant U \leqslant I$. Since $\Phi(X \cap B) \leqslant \Phi(B)$ and $X = \langle (X \cap B)^{M^\circ} \rangle$, $X/I$ is elementary abelian. Thus, (3°) is proved.

4°. $[X, E_\circ] = [X, M_\circ \cap N] = X \cap B$ *and* $[X \cap B, O_2(M_\circ \cap N)] \leqslant I$.

By 8.26(d), $X \cap B = [X \cap B, E_\circ]$ and by (3°) $[X, E_\circ] \leqslant X \cap B$. Hence $[X, E_\circ] = X \cap B$. Since by (1°) $M_\circ \cap N = E_\circ X$ and again by (3°) $X' \leqslant I \leqslant X \cap B$, we also get $[X, M_\circ \cap N] = [X, E_\circ X] = X \cap B$.

By 8.25(c) $[X \cap B, B] \leqslant B' \leqslant I$ and by (1°) $O_2(M_\circ \cap N) = BX$. Hence

$$[X \cap B, O_2(M_\circ \cap N)] = [X \cap B, BX] = [X \cap B, B]X' \leqslant I.$$

After this preparation we are now able to prove (b) – (e).

(b) and (c): This follows from (3°) and (4°), respectively.

(d): By (3°) $\Phi(X/I) = 1$, so $X$ centralizes $X/I$, and $X/I$ is an $M_\circ/X$-module. Moreover, by (a) $M_\circ/X \cong SL_4(2)$. By (1°) $E_\circ X = M_\circ \cap N = N_{M_\circ}(U) = N_{M_\circ}(U_0)$, and so $E_\circ X$ is the normalizer of the hyperplane $U_0$ of the natural $SL_4(2)$-module $I_0$ for $M_\circ$. Also by (1°) $BX = O_2(E_\circ X) = C_{M_\circ}(U_0)$.

Let $R_0$ be a 1-dimensional subspace of $U_0$. Put $P := C_{M_\circ}(R_0)$ and note that

$$R_0 = [I_0, O_2(P)] = [U_0, O_2(P)] \leqslant U_0.$$

Hence $O_2(P) \leqslant N_{M_\circ}(U_0) = M_\circ \cap N$. Thus using both statements in (4°):

$$[X, O_2(P)] \leqslant [X, M_\circ \cap N] = X \cap B \text{ and } [X, O_2(P), O_2(M_\circ \cap N)] \leqslant [X \cap B, O_2(M_\circ \cap N)] \leqslant I.$$

Note that $P/X \sim 2^3 SL_3(2)$ and $X \leqslant O_2(M_\circ \cap N) \nleqslant O_2(P)$. Thus $P = \langle O_2(M_\circ \cap N)^P \rangle$, and since $X$ and $O_2(P)$ are $P$-invariant, $[X, O_2(P), P] \leqslant I$.

Let $U_1$ be an $E_\circ$-submodule of $B \cap X/I$ isomorphic to $U_0$. Since $X$ centralizes $X/I$ we conclude that $E_\circ X$ normalizes $U_1$. Thus $U_1 \cong U_0$ as an $EX$-module and so $R_1 := [U_1, O_2(P)]$ is an 1-dimensional subspace of $U_1$. As $[X, O_2(P), P] \leqslant I$ we get $[R_1, P] = 1$. Let $1 \neq r \in R_1$ and $h \in E_\circ X \backslash P$. Since $E_\circ$ acts transitively on $U_1$, $rr^h \in r^{E_\circ} \subseteq r^{M_\circ}$. Since $M_\circ$ acts doubly transitive on the

1-spaces in $I_0$ and since $C_{M_\circ}(r) = P = C_{M_\circ}(R_0)$, $M_\circ$ also acts doubly transitive on $r^{M_\circ}$. It follows that $r^{M_\circ} \cup \{1\}$ is closed under multiplication and so $I_1 := \langle r^{M_\circ}\rangle$ has order $|M_\circ/P|+1 = 15+1 = 2^4$.

Note that $U_1 \leqslant I_1$. So $I_1 = \langle U_1^{M_\circ}\rangle$, and $I_1$ is a natural $SL_4(2)$-module for $M_\circ$ isomorphic to $I_0$. As $B/B \cap X$ and $I/U$ are natural $SL_3(2)$-modules for $E_\circ$ and as by 8.26(a) $X/I$ is the direct sum of $m$ natural $SL_3(2)$-modules for $E_\circ$ isomorphic to $U_0$, $B \cap X/I$ is the direct sum of $m-2$ $E_\circ$-submodules isomorphic to $U_0$. Since $X/I = \langle (B \cap X/I)^{M_\circ}\rangle$, we conclude that $X/I$ is the direct sum of $m-2$ natural $SL_4(2)$-modules for $M_\circ$ isomorphic to $I_0$. So (d) is proved.

(e): Recall from 8.21 that $|Y/I| = 2$ and so $[Y, M_\circ] \leqslant I$, and by (d) $M_\circ$ has no central chief factors on $X/I$. Thus $Y \cap X \leqslant I \leqslant Y \cap X$, so $Y \cap X = I$ and $|YX/X| = 2$. By (d) $|X/I| = 2^{4(m-2)}$ and $|X \cap B/I| = 2^{3(m-2)}$, so $|X/X \cap B| = 2^{m-2}$ and $|YX/X \cap B| = |YX/X||X/X \cap B| = 2\cdot 2^{m-2} = 2^{m-1}$. Hence (e) holds. □

Recall that $\widehat{B} = B/U$ and $N_0 = C_N(\widehat{B})$. We now investigate $\widehat{B}$ as an $N/N_0$-module.

LEMMA 8.28. *Suppose that $I \leqslant A$ and $I$ is the natural $\Omega_6^+(2)$-module for $M^\circ$.*

(a) $N/N_0 \;=\; C/N_0 \times EN_0/N_0$.

(b) *Put $K := Hom_E(U_0, \widehat{B})$. View $K$ as an $\mathbb{F}_2N$-module with $E \leqslant C_N(K)$ and $U_0$ as a natural $SL_3(2)$-module for $N$ with $C$ acting trivially. Then $|K| = 2^m$ and there exists an $\mathbb{F}_2N$ isomorphism*

$$K \otimes_{\mathbb{F}_2} U_0 \;\to\; \widehat{B} \quad \textit{with} \quad \alpha \otimes v \;\mapsto\; \alpha v.$$

(c) *For $a, b \in B$ define $[\widehat{a}, \widehat{b}] = [a, b]$ and $\widehat{a}^2 = a^2$.* [1] *Put $\mathbb{F} := Hom_E(U_0 \wedge U_0, U)$. Then $|\mathbb{F}| = 2$ and there exists a $C$-invariant symplectic form*

$$s:\; K \times K \mapsto \mathbb{F}, \quad (\alpha, \beta) \;\mapsto\; s(\alpha, \beta)$$

*on $K$ such that*

$$s(\alpha, \beta)(v \wedge w) = [\alpha v, \beta w]$$

*for all $v, w \in U_0$ and $\alpha, \beta \in K$.*

PROOF. (a): By 8.25(d) $N_0 \leqslant C$, by 8.25(f) $N = EC$ and $[E, C] \leqslant N_0$, and by 8.26(c) $E \cap C \leqslant N_0$. Thus, $C \cap EN_0 = N_0$ and

$$N/N_0 \;=\; C/N_0 \;\times\; EN_0/N_0$$

(b): By 8.26(b) $\widehat{B}$ is the sum of $m$ natural $SL_3(2)$-modules isomorphic to $U_0$ for $E$. Since $End_E(U_0) = \mathbb{F}_2$ this gives (b).

(c): Let $1 \neq v \in U_0$. By 8.24(b) $U$ is dual to $U_0$ as an $E$-module. So $C_E(v)$ is the normalizer in $E$ of a hyperplane of $U$ and so $C_U(C_E(v)) = 1$. Let $\alpha, \beta \in K$. Since $\alpha$ and $\beta$ are $E$- homomorphisms from $U_0$ to $\widehat{B}$, $C_E(v)$ centralizes $\alpha v$, $\beta v$ and so also $(\alpha v)^2$ and $[\alpha v, \beta v]$. As $C_U(C_E(v)) = 1$ this gives $(\alpha v)^2 = 1$ and $[\alpha v, \beta v] = 1$. Thus the inverse image of $\alpha(U_0)$ in $B$ is elementary abelian, and for given $\alpha, \beta \in K$ we obtain a well defined $E$-linear function

$$s(\alpha, \beta):\; U_0 \wedge U_0 \to U, \quad v \wedge w \;\mapsto\; [\alpha v, \beta w].$$

Thus $s(\alpha, \beta) \in \mathbb{F} = Hom_E(U_0 \wedge U_0, U)$. Note that $U_0 \wedge U_0$ is a natural $SL_3(2)$-module for $E$ dual to $U_0$ and so isomorphic to $U$. Thus $|\mathbb{F}| = 2$ and so

$$s:\; K \times K \mapsto \mathbb{F}, \quad (\alpha, \beta) \;\mapsto\; s(\alpha, \beta),$$

is a well-defined $C$-invariant bilinear form on $K$.

Since the inverse image of $\alpha(U_0)$ is abelian, it follows that $s(\alpha, \alpha) = 0$ and $s$ is a (possible degenerate) symplectic form on $K$. □

Note that $s$ induces a non-degenerate symplectic form on $K/K^\perp$. Put $C_0 := C_C(K^\perp)$. For $F \leqslant C$ let $\check{F}$ be the image of $F$ in $Sp(K/K^\perp)$.

[1] Note that this is well-defined since $U \leqslant \Omega_1 Z(B)$.

LEMMA 8.29. *Suppose that $I \leqslant A$ and $I$ is the natural $\Omega_6^+(2)$-module for $M^\circ$.*

(a) *$K$ is a faithful 2-reduced $C/N_0$-module, and $K/K^\perp$ is faithful 2-reduced $C_0/N_0$-module.*
(b) $C_{YX}(K/K^\perp) = YX \cap N_0 = C_{YX}(K) = X \cap B$.
(c) *$K^\perp = 1$ and $C = C_0$.*
(d) $C/N_0 \cong \check{C} = Sp(K)$.
(e) *$N_C(Y)$ is the normalizer in $C$ of a 1-subspace of $K$.*
(f) *$m = 2$ or 4.*
(g) *$O_2(N_C(Y)/N_0) = YXN_0/N_0$ and $N_C(Y)/YXN_0 \cong Sp_{m-2}(2)$.*

PROOF. (a): By 8.26(e) $\widehat{B}$ is a 2-reduced $N$-module, and since $C \trianglelefteq N$, also a 2-reduced $C$-module. Since by 8.28(b) $\widehat{B} \cong K \otimes U_0$ as an $N$-module, $\widehat{B}$ is as an $C$-module the direct sum of (three) copies of $K$. Hence $C_C(K) = C_C(\widehat{B}) = N_0$, and $K$ is a faithful 2-reduced $C/N_0$-module.

Since $C_0 \trianglelefteq C$ we conclude that $K$ is a 2-reduced $C_0$-module. Note that $C_{C_0}(K/K^\perp)$ acts nilpotently on $K$ and so centralizes $K$. It follows that $K/K^\perp$ is a faithful 2-reduced $C_0/N_0$-module and (a) holds.

(b): Put

$$K_1 := Hom_E(U_0, \widehat{I}) \quad \text{and} \quad K_2 := Hom_E(U_0, \widehat{B \cap X}).$$

By 8.25(e), $[B, Y] = I$ and so $[\widehat{B}, Y] = \widehat{I}$ is isomorphic to $U_0$. Hence $K_1$ is 1-subspace of $K$. Since $Y \leqslant C$ and $\widehat{B} \cong K \otimes U_0$ we have $[\widehat{B}, Y] \cong [K, Y] \otimes U_0$ and so

$$K_1 = [K, Y].$$

As $B/B \cap X \cong \overline{B} \cong U_0$, $K_2$ is hyperplane of $K$. From $[B \cap X, Y] = 1$ we get $[\widehat{B \cap X}, Y] = 1$ and $[K_2, Y] = 1$. Thus

$$K_2 \leqslant C_{K_2}(Y) \leqslant [K, Y]^\perp = K_1^\perp.$$

Suppose that $K_1^\perp = K$. Then $s(\alpha, \beta) = 0$ for all $\alpha \in K_1$, $\beta \in K$ and

$$1 = s(\alpha, \beta)(v \wedge w) = [\alpha(v), \beta(w)] \text{ for all } v, w \in U_0.$$

But this implies $[I, B] = 1$, a contradiction.

Hence $K_1^\perp \neq K$, and since $K_1^\perp$ contains the hyperplane $K_2$, we get $K_2 = K_1^\perp$. Moreover, since $K_2 \leqslant C_K(Y)$,

$$K_1 = [K, Y] \quad \text{and} \quad K_2 = C_K(Y) = K_1^\perp.$$

Since $[B \cap X, YX] \leqslant I$ and $[I, YX] = 1$, $YX$ centralizes $K_2/K_1 = K_1^\perp/K_1$ and $K_1$. Note that $K^\perp \leqslant K_1^\perp = K_2$, $[K_2, YX] \leqslant K_1$ and $K_1 \cap K^\perp = 1$. Thus $[K^\perp, YX] = 1$ and $YX \leqslant C_0$. Put $Z_1 := K_1K^\perp/K^\perp$. Then $Z_1$ is a 1-space in $K/K^\perp$, $Z_1^\perp = K_1^\perp/K^\perp$ and

$$\check{Y}\check{X} \leqslant C_{\check{C_0}}(Z_1^\perp/Z_1). \tag{I}$$

Observe that $C_{I_0}(B) = U_0 = [I_0, E_\circ]$, and by 8.27(d) $X/I$ is a direct sum of copies of $I_0$, so $C_{X/I}(B) = [X/I, E_\circ]$. By 8.27(c), $[X, E_\circ] = X \cap B$ and thus

$$C_{X/I}(B) = (X \cap B)/I.$$

Regarding the action of $X$ on $B/I$, this means $C_X(B/I) = X \cap B$ and so

$$C_{YX}(K/K_1) = C_{YX}(B/I) = YC_X(B/I) = Y(X \cap B).$$

Note that $|Y/I| = 2$ and $I$ but not $Y$ centralizes $B/U$. So $C_Y(B/U) = I$ and

$$C_{YX}(K) = (X \cap B)C_Y(B/U) = (X \cap B)I = X \cap B.$$

By 8.29(a) both $K$ and $K/K^\perp$ are faithful $C_0/N_0$-modules. So

$$C_{YX}(K/K^\perp) = YX \cap N_0 = C_{YX}(K) = X \cap B,$$

and (b) holds.

(c): By 8.27(e) $|YX/X \cap B| = 2^{m-1}$, and we get

$$|\check{Y}\check{X}| = 2^{m-1}. \tag{II}$$

Put $c := \dim_{\mathbb{F}_2}(K/K^\perp)$. Then $c \leqslant \dim_{\mathbb{F}_2} K = m$ and

$$|C_{Sp(K/K^\perp)}(Z_1^\perp/Z_1)| = 2^{c-1}.$$

Since $\check{Y}\check{X}$ has order $2^{m-1}$ and by (I) is contained in $C_{Sp(K/K^\perp)}(Z_1^\perp/Z_1)$, we conclude that $2^{m-1} \leqslant 2^{c-1}$. Now $c \leqslant m$ gives $c = m$,

$$K^\perp = 1,\ C_0 = C \ \text{ and } \ \check{Y}\check{X} = C_{Sp(K/K^\perp)}(Z_1^\perp/Z_1) = C_{Sp(K)}(K_1^\perp/K_1);$$

in particular (c) holds.

(d): Since $[I, I^x] \neq 1$ we have $K_1 \not\leqslant K_1^x$. By B.26(a)

$$\langle\ C_{Sp(K)}(K_1^\perp/K_1),\ C_{Sp(K)}(K_1^{x\perp}/K_1^x)\ \rangle \ = \ Sp(K),$$

and so $\langle YX, (YX)^x\rangle$ induces $Sp(K)$ on $K$. Thus $\check{C} = Sp(K)$, and (d) holds.

(e): Since $[K, Y] = K_1$, $\check{Y}$ is a transvection group on $K$. It follows that

$$\check{Y} = C_{\check{C}}(K_1^\perp) \quad \text{and} \quad N_{\check{C}}(\check{Y}) = N_{\check{C}}(K_1),$$

and (e) holds.

(f): Since $C_C(K) = N_0$, $N_C(\check{Y}) = N_C(YN_0)$. By 8.26(a) $N_N(YN_0) = N_N(Y)$ and so $N_C(K_1) = N_C(Y) \leqslant M^\dagger$. Since $M^\dagger$ normalizes $Y$ and $X$ we conclude that $\check{X}$ is an $N_{\check{C}}(K_1)$-invariant complement to $\check{Y}$ in $\check{Y}\check{X}$. In particular, $\check{Y}$ is not the only $N_{Sp(K)}(K_1)$-invariant subgroup of $C_{Sp(K)}(K_1^\perp/K_1)$. Hence B.30 shows that $m \leqslant 4$, and (f) holds.

(g): Note that $O_2(N_{Sp(K)}(K_1)) = C_{Sp(K)}(K_1^\perp/K_1) = \check{Y}\check{X}$ and that $K_1^\perp/K_1$ is a natural $Sp_{m-2}(2)$-module for $N_{Sp(K)}(K_1)) = N_C(Y)$, so also (g) holds. □

LEMMA 8.30. *Suppose that $I \leqslant A$ and $I$ is the natural $\Omega_6^+(2)$-module for $M^\circ$. Then $M^\circ = M_\circ$, $N_0 = B$, and one of the following holds:*

(1) *$m = 2$, $O_2(M) = Y$, $M^\dagger = M$, $\overline{M^\dagger} \cong \Omega_6^+(2)$ or $O_6^+(2)$ and*

$$M^\dagger \sim 2^{6+1}\Omega_6^+(2) \text{ or } 2^{6+1}O_6^+(2) \quad \text{and} \quad N \sim 2^{3+\overline{3}\cdot 2}SL_3(2) \times SL_2(2),$$

(2) *$m = 4$, $[M^\circ, C_G(Y)] = X$, $M^\dagger = M_\circ C_G(Y)$,*

$$M^\dagger/YX = M_\circ/YX \times C_G(Y)/YX \cong SL_2(2) \times SL_4(2),$$

*$X/I \cong YX/Y$ is a tensor product over $\mathbb{F}_2$ of corresponding natural modules and*

$$M^\dagger \sim 2^{6+1+4\cdot 2}SL_4(2) \times SL_2(2) \quad \text{and} \quad N \sim 2^{3+\overline{3}\cdot 4}SL_3(2) \times Sp_4(2).$$

PROOF. We first show:

1°. *$N/N_0 \cong Sp_m(2) \times SL_3(2)$, $m = 2$ or $4$, and $\widehat{B}$ is a tensor product over $\mathbb{F}_2$ of corresponding natural modules.*

By 8.28(a) $N/N_0 = C/N_0 \times EN_0/N_0$, and by 8.28(b) $\widehat{B} \cong K \otimes_{\mathbb{F}_2} U_0$, where $K = End_E(U_0, \widehat{B})$. By 8.29(d) $C/N_0 \cong Sp(K)$ and so $K$ is a natural $Sp_m(2)$-module for $C$, and by 8.29(f) $m = 2$ or 4. Also $U_0$ is a natural $SL_3(2)$-module for $E$, and so (1°) holds.

2°. *$C_{M^\dagger}(X/I) \cap C_{M^\dagger}(I) = YX$ and $N_0 = B$.*

Put

$$X_1 := C_{O_2(M)}(X/I) \quad \text{and} \quad X_2/I := C_{X_1/I}(M_\circ).$$

Then $[X_1, X] \leqslant I$ and so $M_\circ/X$ acts on $X_1/I$. By 8.27(b) $X' \leqslant I$ and thus $X \leqslant X_1$, and by 8.27(a), $[O_2(M), M^\circ] = X$ and so $[X_1, M_\circ] = X$. Since $I_0$ is a natural $SL_4(2)$-module for $M_\circ/X \cong SL_4(2)$, C.18 shows that $H^1(M_\circ/X, I_0) = 1$. By 8.27(d) $X/I$ is a direct sum of copies of $I_0$. Hence also $H^1(M_\circ/X, X/I) = 1$ and so $X_1 = X_2X$.

Pick $t \in X_2$. Then $[t, M_\circ] \leqslant I \leqslant Z(X)$, so $[t, X]$ and $C_X(t)$ are $M_\circ$-invariant. Hence $[t, M_\circ]$ and $X/C_X(t)$ are isomorphic $M_\circ$-modules. But $I$ is the natural $\Omega_6^+(2)$-module for $M_\circ$ and by 8.27(d) each $M_\circ$-chief factor of $X/I$ is a natural $SL_4(2)$-module. It follows that $[X, t] = 1$ and so $[X_2, X] = 1$.

Since $[O_2(M), M_\circ] = X$ and $[M_\circ, X_2] \leqslant I$ we get $[O_2(M), M_\circ, X_2] = 1$ and $[M^\circ, X_2, O_2(M)] = 1$. The Three Subgroup Lemma now implies $[X_2, O_2(M), M^\circ] = 1$. By 1.55(d) $C_G(M^\circ) = 1$, so $[X_2, O_2(M)] = 1$ and $X_2 \leqslant Z(O_2(M))$. Since $[X_2, M_\circ] \leqslant I \leqslant \Omega_1 Z(X_2)$, 1.18 gives $[\Phi(X_2), M_\circ] = 1$. As $C_G(M^\circ) = 1$, $X_2$ is elementary abelian. Therefore $X_2 \leqslant \Omega_1 Z(O_2(M))$, and by 2.2(e), $\Omega_1 Z(O_2(M)) = Y$. Hence $X_2 = Y$ and so

$$X_1 = X_2 X = YX.$$

Let $X_3 := C_{M^\dagger}(X/I) \cap C_{M^\dagger}(I)$. Then

$$[O_2(M^\dagger), X_3] \leqslant [O_2(M), X_3] \leqslant O_2(M) \cap X_3 = X_1 = YX,$$

$[YX, X_3] = [Y, X_3][X, X_3] \leqslant I$ and $[I, X_3] = 1$. Hence $X_3$ acts nilpotently on $O_2(M^\dagger)$, and since $M^\dagger$ is of characteristic 2, we conclude that $X_3$ is a 2-group. So $X_3 \leqslant O_2(M^\dagger) \leqslant O_2(M)$ and $X_3 \leqslant X_1 = YX \leqslant X_3$. Thus

$$X_3 = YX.$$

This is the first part of (2°).

By 8.25(d) $N_0 \leqslant M^\dagger$. Put $N_2 := C_{N_0}(Y)$. By 1.52(c) $[M^\circ, C_{M^\dagger}(Y)] \leqslant O_2(M^\circ)$ and so $[M_\circ, N_2] \leqslant O_2(M^\circ) \cap M_\circ = O_2(M_\circ)$. By 8.27(a) $O_2(M_\circ) = X$ and so $M_\circ$ normalizes $N_2X$. Since $[B, N_0] \leqslant U \leqslant I$ and by 8.27(b) $[X, X] \leqslant I$, we get $[X \cap B, N_2X] \leqslant I$. As $M_\circ$ normalizes $N_2X$, we conclude that

$$[X, N_2X] = [\langle (X \cap B)^{M_\circ} \rangle, N_2X] \leqslant I.$$

Thus $N_2$ centralizes $X/I$ and $I$ and so $N_2 \leqslant X_3 = YX$. Hence

$$N_2 = YX \cap N_2 = YX \cap C_{N_0}(Y) = YX \cap N_0.$$

By 8.29(b) $YX \cap N_0 = X \cap B$ and so $N_2 = X \cap B$. Since $\overline{N_0} \leqslant C_{\overline{M}}(U) = \overline{B}$, this gives $N_0 = BN_2 = B(X \cap B) = B$, and (2°) is proved.

3°. $\quad E = E_\circ$ *and* $M^\circ = M_\circ$.

By 8.26(c) $C_E(U) \leqslant N_0$, by (2°) $N_0 = B$, and by 8.26(d) $B \leqslant E_\circ$. Thus $E = E_\circ C_E(U) = E_\circ B = E_\circ$. Since $E_\circ \leqslant M_\circ$, this gives $Q \leqslant E \leqslant M_\circ$ and so $M^\circ = \langle Q^M \rangle \leqslant M_\circ$.

4°. $\quad O_2(M^\dagger) = YX$ *and* $C_G(Y)/YX \cong Sp_{m-2}(2)$.

By 8.29(g), $O_2(N_C(Y)/N_0) = YXN_0/N_0$ and by (2°) $B = N_0$. It follows that

$$O_2(N_C(Y)) \leqslant YXN_0 = YXB \leqslant O_2(N_C(Y))$$

and so

$$O_2(N_C(Y)) = YXB. \tag{I}$$

Thus

$$YX \leqslant O_2(M^\dagger) \leqslant O_2(N_C(Y)) = YXB = YXN_0$$

and

$$O_2(M^\dagger) = YX(O_2(M^\dagger \cap B)).$$

Also $O_2(M^\dagger) \cap B \leqslant O_2(M) \cap B \leqslant X$ and so $O_2(M^\dagger) = YX$.

Note that $\overline{M^\dagger} = \overline{M}$ and by 8.25(b) $C_{\overline{M}}(U) = \overline{B}$. As $N_C(Y) = C_{M^\dagger}(U)$ this gives $N_C(Y) = C_G(Y)B = C_G(Y)YXB$. Hence

$$\begin{aligned} N_C(Y)/YXN_0 &\overset{(\mathrm{I})}{=} N_C(Y)/YXB = C_G(Y)YXB/YXB \\ &\cong C_G(Y)/YXB \cap C_G(Y) = C_G(Y)/YXC_B(Y). \end{aligned}$$

Since $C_B(Y) = C_M(Y) \cap B = B \cap O_2(M) \leqslant X$, we get that $N_C(Y)/YXN_0 \cong C_G(Y)/YX$. By 8.29(g), $N_C(Y)/YXN_0 \cong Sp_{m-2}(2)$ and so $C_G(Y)/YX \cong Sp_{m-2}(2)$, and (4°) is proved.

We are now able to prove the lemma. By 8.22(a) $\overline{M} \cong \Omega_6^+(2)$ or $\overline{M} \cong O_6^+(2)$, and $\overline{M} = \overline{M_\circ}\ \overline{S}$, and by 8.29(f) $m = 2$ or 4. Moreover, $(1^\circ)$ shows that $N/N_0 \cong Sp_m(2) \times SL_3(2)$ and $\widehat{B} = B/U$ is a tensor product over $\mathbb{F}_2$ of corresponding natural modules. By $(2^\circ)$ $N_0 = B$. Also $U$ is the natural $SL_3(2)$-modules for $E$ dual to $U_0$, and so the structure of $N$ is as described in (1) (for $m = 2$) and in(2) (for $m = 4$).

By 8.27(d) $X/I$ is a direct sum of $m - 2$ natural $SL_4(2)$-modules for $M_\circ$ isomorphic to $I_0$, and by $(4^\circ)$ $C_G(Y)/YX \cong Sp_{m-2}(2)$.

Suppose first that $m = 2$. Then $m - 2 = 0$ and so $X = I$, $YX = I$ and $C_G(Y) = Y$. Thus $M^\dagger = MC_G(Y) = M$ and since $Y \leqslant O_2(M) \leqslant C_G(Y)$, $Y = O_2(M)$. Thus (1) holds if $m = 2$.

Suppose next that $m = 4$. Then $X/I$ is a direct sum of two natural $SL_4(2)$-modules for $M_\circ$ and $C_G(Y)/XY \cong SL_2(2)$. By $(2^\circ)$ $C_{M^\dagger}(X/I) \cap C_{M^\dagger}(I) = YX$ and so $C_G(Y)/XY$ acts faithfully on $X/I$. By 1.52(c) $[M^\circ, C_G(Y)] \leqslant O_2(M^\circ)$ and so $[M_\circ, C_G(Y)] \leqslant X$. Thus $C_G(Y)M_\circ/YX \cong SL_2(2) \times SL_4(2)$, and $X/I$ is a tensor product over $F_2$ of corresponding natural modules.

Note that $S$ normalizes at least one of the three simple $M_\circ$-submodules of $X/I$. Let $R$ be such a simple $M_\circ$-module. Since $M_\circ$ induces $SL_4(2) \cong Aut(R)$ on $R$ we conclude that $S$ induces inner automorphism on $M_\circ/C_{M_\circ}(R) = M_\circ/X$. Since $\overline{M} = \overline{M_\circ}\,\overline{S}$ this gives $\overline{M^\dagger} = \overline{M} = \overline{M_\circ}$ and $M^\dagger = M_\circ C_G(Y)$. Thus (2) holds. □

LEMMA 8.31. *Suppose that $I \leqslant A$ and $I$ is the natural $\Omega_6^+(2)$-module for $M^\circ$. Then $m = 2$. In particular, 8.30(1) holds.*

PROOF. Suppose not. Then 8.30(2) holds. In particular, $m = 4$, $M^\dagger = M_\circ C_G(Y)$ and $N_0 = B$. Since $\overline{M} \cong SL_4(2) \cong \Omega_6^+(2)$ and $I$ is a natural $\Omega_6^+(2)$-module for $M$, $N_M(U)$ is a parabolic subgroup of $M$. So we may choose notation such that $S$ normalizes $U$. Then $S \leqslant N$. Recall from 8.29(d) that $K$ is a natural $Sp_4(2)$-module for $C/N_0 = C/B$.

Let $K_1$ be as in the proof of 8.29, that is, $K_1 = [K, Y]$ and $K_1$ is 1-space in $K$. In particular, $S$ normalizes $K_1$. Let $K_3$ be the 2-subspace of $K$ such that $K_1 < K_3 < K_1^\perp$ and $S$ normalize $K_3$. Then $K_2$ is a singular 2-subspace of $K$. Put

$$C_3 := N_C(K_3), \quad Y_3 := \langle Y^{C_3} \rangle, \quad I_3 := \langle I^{C_3} \rangle.$$

Note that $K_3$ is the natural $SL_2(2)$-module for $C_3$. Thus $K_3 = \langle K_1^{C_3} \rangle$ and so

$$I_3/U \cong K_3 \otimes U_0 \text{ and } I_3/U = \langle \alpha(v) \mid \alpha \in K_3, v \in U_0 \rangle.$$

Since $K_3$ is a 2-space, we get $|I_3/U| = 2^6$ and $|I_3| = 2^9$, and since $K_3$ is singular, $I_3$ is abelian. As $C_G(I) = C_G(Y)$, we conclude that $Y_3$ is abelian.[2]

Since $C_3$ acts transitively on the 1-spaces in $K_3$, $C_3$ also acts transitively on the corresponding transvections. It follows that $Y_3B = C_C(K_3)$ and $|Y_3B/B| = 2^3$. Hence $C_K(Y_3) = K_3$, and since $\widehat{B} \cong K \otimes U_0$, we infer $C_{\widehat{B}}(Y_3) \cong K_3 \otimes U_0$ and $C_{\widehat{B}}(Y_3) = I_3/U$.

Since $Y_3$ is abelian we get $Y_3 \cap B/U \leqslant C_{B/U}(Y_3) = I_3/U$ and so $Y_3 \cap B = I_3$. Hence $Y_3$ has order $2^{12}$. As $Y_3$ is abelian and generated by involutions, $Y_3$ is elementary abelian.

Since $Y_3$ is abelian, $Y_3 \leqslant C_G(Y)$. Note that $YB/B$ is the only transvection group contained in $O_2(N_C(K_1)/B)$. As $N_C(K_1) = N_C(Y)$ we get $Y_3 \nleqslant O_2(C_G(Y)) = XY$. Since $C_G(Y)/XY \cong SL_2(2)$ and $YX/Y$ is the tensor product of natural modules for

$$M^\dagger/XY = M_\circ Y/X \times C_G(Y)/XY \cong SL_2(2) \times SL_4(2),$$

we get

$$|Y_3X/YX| = 2 \quad \text{and} \quad |C_{YX/Y}(Y_3)| = 2^4.$$

Since $Y_3$ has order $2^{12}$ and $Y$ has order $2^7$, we conclude that $Y_3 \cap YX$ has order $2^{11}$, $|Y_3 \cap YX/Y| = 2^4$ and

$$C_{YX/Y}(Y_3) = Y_3 \cap XY/Y.$$

It follows that $Y_3X/Y$ has exactly two maximal elementary abelian subgroups, namely $YX/Y$ and $Y_3/Y$.

---

[2] This also follows from the fact that $Y$ is asymmetric in $G$

Since $[M_\circ, C_G(Y)] = X$, $M_\circ$ normalizes $Y_3X$. As $M_\circ$ normalizes $YX/Y$, it also normalizes the unique other maximal elementary abelian subgroup of $Y_3/Y$. Hence $M_\circ$ normalizes $Y_3$. Since $S$ normalizes $K_3$, $S$ normalizes $Y_3$ and so $M = M_\circ S \leqslant N_G(Y_3)$. The basic property of $M$ now implies $N_G(Y_3) \leqslant M^\dagger = N_G(Y)$, a contradiction since $C_3 \leqslant N_G(Y_3)$ and $Y \ntrianglelefteq C_3$.

This completes our proof-by-contradiction, and the lemma holds. □

It remains to analyze Case 8.30(1).

LEMMA 8.32. *Suppose that $I \leqslant A$ and $I$ is the natural $\Omega_6^+(2)$-module for $M^\circ$. Let $t$ be a non-singular vector in $I$. Then $C_G(t) \nleqslant M$.*

PROOF. Recall that $U = C_Y(L)$ and so $L \leqslant C$. Since $C/N_0 = C/B \cong SL_2(2)$ we infer $N_C(Y) = YB$ and $|C/N_C(Y)| = 3$. So $|I^C| = 3$. Let $I^C =: \{I_1, I_2, I_3\}$ with $I = I_1$. Let $V_0$ be a 2-subspace of $U$. Note that for $i \in \{1,2,3\}$, $U$ is singular 3-subspace of $I_i$ and so $V_0$ is a singular 2-space in $I_i$. Hence $V_0$ is contained in a unique singular 3-space $V_i$ of $I_i$ different from $U$. Note also that $V_0^\perp = UV_i$ in $I_i$. Define

$$M_i := N_G(I_i), \quad Y_i := C_G(I_i), \quad E_i := N_G(V_i)^\circ, \quad B_i := O_2(E_i).$$

So $M_1 = M$ and $Y_1 = Y$. Note that by 8.30, $M^\circ = M_\circ$. Since $O_2(M_\circ) = X = I$ we have $M_i^\circ/I_i \cong \Omega_6^+(2)$. By 2.7(b) $E_i \leqslant M_i^\circ$, and B.38(c) shows that $V_i$ is a natural $SL_3(2)$-module for $E_i$. Note that $E_i \trianglelefteq N_{M_i^\circ}(V_i)$ and both $I_i/V_i$ and $C_{M_i^\circ}(V_i)/I_i$ are natural $SL_3(2)$-modules (dual to $V_i$). Hence $C_{M_i^\circ}(V_i) = [C_{M_i^\circ}(V_i), E_i] \leqslant E_i$. $E_i = N_{M_i^\circ}(V_i)$, $B_i = C_{M_i^\circ}(V_i)$, and $B_i$ has order $2^9$.

Put

$$E_0 := N_G(V_0)^\circ \quad \text{and} \quad B_0 := O_2(E_0).$$

Since $V_0 \leqslant V_i \leqslant I_i$, 2.7(b) shows that $E_0 \leqslant E_i \leqslant M_i^\circ$. By B.38(c) $V_0$ is natural $SL_2(2)$-module for $E_0$. Also $I_i = [I_i, E_0] \leqslant E_0$, $C_{M_i^\circ}(V_0)/I_i$ is extra special of order $2^5$ with center $C_{M_i^\circ}(UV_i)/I_i$, and $C_{M_i^\circ}(V_0)/C_{M_i^\circ}(UV_i)$ is the direct sum of two natural $SL_2(2)$-modules for $E_0$. Hence $C_{M_i^\circ}(V_0) = [C_{M_i^\circ}(V_0), E_0] \leqslant E_0$. $E_0 = N_{M_i^\circ}(V_0)$, $B_0 = C_{M_i^\circ}(V_0)$ and $B_0$ has order $2^{6+5} = 2^{11}$. Note also that $E = N_{M_i^\circ}(U)$ and so $E_i \cap E = E_0$. Since $C$ centralizes $U$, $C$ centralizes $V_0$ and so $C$ normalizes $E_0$. Moreover, since $C$ acts as $Sym(3)$ on $\{I_1, I_2, I_3\}$ it also act as $Sym(3)$ on $\{V_1, V_3, V_3\}$ with $B$ the kernel of the action.

Let $\{i,j,k\} = \{1,2,3\}$. Note that $Y_i$ fixes $I_i$, $Y_i \leqslant C$ and $Y_i \nleqslant B$, so $Y_i$ acts non-trivially on $\{I_j, I_k\}$ and $\{V_j, V_k\}$.

Put

$$V_{ij} := \langle V_j^{E_i} \rangle.$$

1°. *$V_{ij}$ is the unique elementary abelian subgroup of $2^6$ in $B_0$ containing $V_iV_j$. In particular, $V_{ij} = V_{ji}$.*

Put $Z/V_0 = Z(E_0/V_0)$. Note that $[U, E_0] \leqslant V_0$ and $[V_i, E_0] \leqslant V_0$. So $UV_i \leqslant Z \cap I_i$. Since $I_i/UV_i = I_i/V_0^\perp$ is a natural $SL_2(2)$ module for $E_0$ we conclude that $Z \cap I_i = UV_i$. Also $ZI_i/I_i \leqslant Z(E_0/I_i) = C_{E_0}(UV_i)/I_i$. The latter group has order 2. As $E$ normalizes $I_i$ and $I_j$, $I_i \cap I_j = U$ and so $V_j \nleqslant I_i$. Since $V_j \leqslant Z$ we conclude that $|ZI_i/I_i| = 2$ and $Z = UV_iV_j$ is elementary abelian of order $2^5$. In particular,

$$ZI_i/I_i = C_{E_0}(UV_i)/I_i \quad \text{and} \quad C_{I_i}(Z) = UV_i.$$

Since $B_i = C_{M_i^\circ}(V_i) \leqslant E_0$, we have $B_i = C_{E_0}(V_i)$ and so

$$\text{(I)} \qquad B_i \cap I_j = C_{I_j}(V_i) = C_{I_j}(Z) = UV_j.$$

Thus $|I_jB_i/B_i| = |I_j/UV_j| = 4 = |B_0/B_i|$ and so $B_0 = I_jB_i$. Since $[Z, B_i] \leqslant [Z, B_0] \leqslant V_0 \leqslant V_i$, $Z/V_i \leqslant \Omega_1 Z(B_i/V_i)$. Also $Z \cap I_i/V_i \neq 1$ and $ZI_i/I_i \neq 1$. Since $I_i/V_i$ and $B_i/I_i$ are simple $E_i$-module we conclude that $B_i = \langle Z^{E_i} \rangle$ and thus $B_i/V_i$ is elementary abelian. Since $[B_i, I_j] = [B_iI_j, I_j] = [B_0, I_j] = V_jU$, we get

$$[B_i/V_i, B_0] = [B_i/V_i, I_jB_i] = [B_i, I_j]V_i/V_i = V_iV_jU/I_i = Z/I_i.$$

Note that $C_{E_i}(B_i/I_i) = B_i$ and so $|B_0/C_{B_0}(B_i/I_i)| = |B_0/B_i| = 4$ and

$$|[B_i/V_i, B_0]| = |Z/I_i| = 4 = |B_0/C_{B_0}(B_i/I_i)|.$$

Hence $B_0$ is an offender on the dual of $B_i/V_i$. The General FF-Module Theorem C.2(d) now implies that $B_i/V_i$ is the direct sum of natural $SL_3(2)$-modules for $E_i$. Since $B_i/I_i$ and $I_i/V_i$ are both dual to $V_i$, the summands are isomorphic. It follows that there exists three simple $E_i$-submodules in $B_i/V_i$. As $[B_i/V_i, B_0] = Z/V_i$ has order four, each of the simple submodules intersects $Z/V_i$ in a subgroups of order 2. Hence each subgroup of order 2 of $Z/V_i$ lies is a simple $E_i$ submodule. Recall that $V_{ij} = \langle V_j^{E_i}\rangle$ and note that $V_iV_j \leqslant V_{ij}$. Since $V_iV_j \leqslant Z$ and $V_jV_i/V_i$ has order 2 we conclude that $V_{ij}/V_i$ is a simple $E_i$-submodule of $B_i/V_i$. So $V_{ij}/V_i$ is a natural $SL_3(2)$-module for $E_i$. Note that $I_iI_j$ is elementary abelian and $E_i$ acts transitively on $V_{ij}/V_i$. Thus all non-trivial elements in $V_{ij}$ have order 2 and so $V_{ij}$ is elementary abelian of order $2^6$. Since both $I_i/V_i$ and $V_{ij}/V_i$ are simple $E_i$-submodules of $B_i/V_i$, $B_i = I_iV_{ij}$ and so

$$B_i \cap B_j = C_{B_i}(V_j) = C_{I_i}(V_j)V_{ij} \overset{\text{(I)}}{=} UV_iV_{ij} = UV_iV_jV_{ij} = ZV_{ij}.$$

Note that $C_{B_0}(Z) \leqslant C_{B_0}(UV_i) = I_iZ$ and $C_{I_i}(Z) = UV_i \leqslant Z$. So $C_{B_0}(Z) = Z$. Also $Z \cap V_{ij} = V_iV_j$ has index 2 in $Z$. It follows that $Z$ and $V_{ij}$ are the only maximal elementary abelian subgroups of $ZV_{ij} = B_i \cap B_j$. Since $Z$ has order $2^5$, $V_{ij}$ is the only elementary abelian subgroup of order $2^6$ in $B_i \cap B_j$. As $B_i \cap B_j = C_{B_0}(V_iV_j)$ this shows that $V_{ij}$ is the only elementary abelian subgroups in $B_0$ of order $2^6$ containing $V_iV_j$. Thus (1°) holds.

Recall that that $\overline{M^\circ} \cong Alt(8)$ acts on a set $\Lambda$ of 8 objects and $Y$ is the central quotient of the permutation module on $\Lambda$. Let $1 \neq y_\lambda \in Y$ with $C_{\overline{M^\circ}}(y_\lambda) \cong Alt(7)$. Note $\overline{Z}$ is 2-central in $\overline{M^\circ}$ and so we may assume that $\overline{Z}$ corresponds to $\langle(12)(34)(56)(78)\rangle$ in $Alt(8)$. Then $[Y, Z] = \langle y_{12}, y_{34}, y_{56}, y_{78}\rangle$. It follows that $[Y, Z]$ is a non-singular isotropic 3-space of $I$. Hence the elements of $[Y, Z]\backslash V_0$ are non-singular. Since $C/B \cong SL_2(2)$ and $YB \in Syl_2(C)$ we conclude that $[C, Z]/V_0$ has order four and the elements in $[C, Z]\backslash V_0$ are not 2-central in $G$. Recall here that since $O_2(M)$ is weakly closed, elements of $Y$ are conjugate in $G$ if and only if they are conjugate in $M$, see 2.6(d). Also $Z/V_0 = U/V_0 \times [C, Z]/V_0$ as an $C$-module. Since $V_i \neq U$ and $V_i$ is a singular 3-space we conclude that $\langle V_i, V_j, V_k\rangle = Z$ and $V_iV_j \cap I_k = [Y_k, Z]$ is a non-singular isotropic 3-space in $I_k$. It follows that $E_0$ has two orbits on $I_iI_j\backslash I_i$, namely the four 2-central involutions in $I_j\backslash V_0$ and the four non-2-central involutions in $[Y_k, Z]\backslash V_0$.

Put $M_{ij} := N_G(V_{ij})$. Then by (1°) $V_{ij} = V_{ji}$ and so $\langle E_i, E_j\rangle \leqslant M_{ij}$. Let $v \in I_iI_j\backslash I_i$. Then $C_{E_i}(vI_i/I_i) = E_0$ and $|E_i/E_0| = 7$. We conclude that $E_i$ has two orbits on $V_{ij}\backslash I_i$, namely the twenty-eight 2-central involutions and the twenty-eight non-2-central involutions. Also $E_i$ acts transitively on the seven 2-central involutions in $I_i$. Note that the same holds with $i$ and $j$ interchanged. Since $I_i \neq I_j$ we conclude that $M_{ij}$ acts transitively on thirty-five 2-central involutions and transitively on the twenty-eight non-2-central involutions in $V_{ij}$. It follows that 35 and so also 5 divides $|M_{ij}|$.

Let $t_k \in [Y_K, Z]\backslash U_0$. Then $|t_k^{M_{ij}}| = 28$ and we conclude that 5 divides $|C_{M_{ij}}(t)|$. Since $M_{ij} \cap M_k \leqslant N_{M_k}(V_{ij} \cap I_k) = N_{M_k}([Y_k, Z])$, 5 does not divide $M_{ij} \cap M_k$ and so $C_{M_{ij}}(t_k) \nleqslant M_k$. Since $t_k$ is non-singular in $I_k$ this gives $C_G(t) \nleqslant M$, and the lemma is proved. □

Lemma 8.33. *Suppose that $I \leqslant A$ and $I$ is the natural $\Omega_6^+(2)$-module for $M^\circ$. Then Case (3) of Theorem H holds.*

Proof. Recall that case 8.30(1) holds, in particular $p = 2$, $|Y/I| = 2$, $\overline{M^\circ} \cong \Omega_6^+(2)$, and $I$ is a natural $\Omega_6^+(2)$-module for $M^\circ$. Let $t$ be a non-singular vector of $I$. If $C_G(t)$ is not of characteristic 2, then Case (3) of Theorem H holds.

So suppose for a contradiction that $C_G(t)$ is of characteristic 2. Since $Y = O_2(M) \leqslant C_G(t)$, 2.6(c) shows $M^\dagger \cap C_G(t)$ is a parabolic subgroup of $C_G(t)$. Since $M^\dagger = M$, this gives $P := O_2(C_G(t)) \leqslant O_2(C_M(t))$. Since $t$ is non-singular in $I$ and $\overline{M} \cong \Omega_6^+(2)$ or $O_6^+(2)$, we have $C_{\overline{M}}(t) \cong Sp_4(2)$ or $C_2 \times Sp_4(2)$. Hence either $\overline{P} = 1$, or $|\overline{P}| = 2$ and $[Y, \overline{P}] = \langle t\rangle$. In either case $[Y, P] \leqslant \langle t\rangle$ and so also $[\langle Y^{C_G(t)}\rangle, P] \leqslant \langle t\rangle$. As $C_G(t)$ is of characteristic 2, this implies $Y \leqslant P \trianglelefteq C_G(t)$. Since $Y = O_2(M)$ is weakly closed, we conclude that $C_G(t) \leqslant N_G(Y) = M^\dagger = M$. But this contradicts 8.32. □

PROPOSITION 8.34. *Suppose that $I \leqslant A$. Then Case (2) or Case (3) of Theorem H holds.*

PROOF. By 8.21 either $I$ is a natural $SL_3(2)$-module for $M^\circ$ and Theorem H(2) holds, or $I$ is a natural $\Omega_6^+(2)$-module for $M^\circ$. In the latter case 8.33 shows that Theorem H(3) holds. $\square$

## 8.3. The Case $I \nleqslant A$ and $\Omega_1 Z(A) \nleqslant Z(L)$

In this short section we continue to assume the hypothesis and notation of Theorem H. Furthermore, we assume that $I \nleqslant A$ and $\Omega_1 Z(A) \nleqslant Z(L)$.

PROPOSITION 8.35. *Suppose that $I \nleqslant A$ and $\Omega_1 Z(A) \nleqslant Z(L)$. Then Theorem H(4), (5) or (6) holds.*

PROOF. Then $[\Omega_1 Z(A), L] \neq 1$ and so by 8.6 $[Y_{HQ}, HQ] \neq 1$. Note that $Q \leqslant O_p(HQ)$ and so $[Y_{HQ}, Q] = 1$, $HQ = HC_N(Y_{HQ})$ and $[Y_{HQ}, H] \neq 1$. Let $V$ be an $H$-submodule of $Y_{HQ}$ minimal with $[V, O^p(H)] \neq 1$. Since $H \in \mathfrak{H}_G(O_p(M))$, 2.11(e) shows that $H$ is $p$-irreducible and so by 1.34(c), $V$ is $H$-quasisimple. Note that $V = [V, H] \leqslant H$ and since $V$ is $p$-reduced for $H$, $V \leqslant Y_H$. Hence, according to 2.17 there exists a non-trivial strong offender $W$ on $Y$ such that $W \leqslant V \leqslant Y_{HQ}$ and

$$[X, W] = [Y, W] \text{ for all } X \leqslant Y \text{ with } |X/C_X(W)| > 2.$$

Since $[Y_{HQ}, Q] = 1$, $Y_{HQ} \leqslant C_G(Q) = Z(Q)$ and $[Y, W] \leqslant W \leqslant Z(Q)$; in particular, $W \trianglelefteq Q$ and $[Y, W, Q] = 1$. So we can apply C.26. Since $[V, W, Q] = 1$ we get that $\overline{M^\circ} \cong SL_n(q)$ or $Sp_{2n}(q)$, $n \geqslant 2$, and $[Y, M^\circ]$ is a corresponding natural module. Moreover, either $Y = [Y, M^\circ]$ or $\overline{M^\circ} \cong Sp_{2n}(2)$, $n \geqslant 2$, and $|Y/[Y, M^\circ]| = 2$. By the definition of the Fitting submodule, $I = [Y_M, M^\circ]$ since $[Y, M^\circ]$ is the unique $M$-component of $Y$.

If $\overline{M^\circ} \cong SL_n(q)$, then Theorem H(5) holds.

If $\overline{M^\circ} \cong Sp_{2n}(q)$ and $I = Y$, then $I \nleqslant Q^\bullet$ and so by 2.26 $p$ is even. Thus Theorem H(6) holds.

If $I \neq Y$, then Theorem H(4) holds. $\square$

## 8.4. The Case $I \nleqslant A$ and $\Omega_1 Z(A) \leqslant Z(L)$

In this section we continue to assume the hypothesis and notation of Theorem H. Furthermore, we assume that $I \nleqslant A$ and $\Omega_1 Z(A) \leqslant Z(L)$.

LEMMA 8.36. *Suppose that $I \nleqslant A$ and $\Omega_1 Z(A) \leqslant Z(L)$. Then*

(a) $Z(A) = C_Y(L) = C_Y(A) \leqslant Y \cap A$.
(b) $YA = IA$ *and* $[Y, A] = [I, A] \leqslant I \cap A$.
(c) $C_A(I \cap A) = Y \cap A = [I, A]C_Y(L) = (I \cap A)C_Y(L)$; *in particular* $C_A(Y) = C_A(I) = C_A(I \cap A)$.
(d) $|I/C_I(A)| \leqslant |A/C_A(I)|^2$,
(e) *$A$ acts nearly quadratically but not quadratically on $I$.*

PROOF. (a): By 1.43(q) $\Omega_1 Z(A) = Z(A)$. As $\Omega_1 Z(A) \leqslant Z(L)$ this gives $Z(A) = A \cap Z(L)$. By 1.43(b) $A \cap Z(L) = C_Y(L)$, and by 1.43(j)

$$C_Y(A) = Y \cap Z(A) \leqslant C_Y(L) \leqslant C_Y(A),$$

so $C_Y(A) = C_Y(L)$.

(b) and (c): Since $I$ is $N_L(Y)$-invariant, 8.7 implies that $YA = IA$, $[Y, A] = [I, A] \leqslant I \cap A$ and $Y \cap A = [I, A]C_Y(L)$. By 8.4(c), $C_G(Y) = C_G(I)$ and so also $C_A(Y) = C_A(I)$. Moreover, by 1.43(g) $C_A(Y) = Y \cap A$ and by 1.43(j) $C_A(Y \cap A) = Z(A)(Y \cap A)$. As $Z(A) \leqslant Y \cap A$ by (a), this gives $C_A(Y \cap A) = Y \cap A$, and since $Y \cap A = [I, A]C_Y(L) = (I \cap A)C_Y(L)$, $C_A(Y \cap A) = C_A(I \cap A)$ follows.

(d) and (e): By (b) $A = \langle (Y \cap A)^L \rangle = \langle ([I, A]C_Y(L))^L \rangle$. Since by (a) $[A, A] \neq 1$, this gives $[I, A, A] \neq 1$, i.e. $A$ does not act quadratically on $I$. Moreover, by 1.43(n) $|Y/C_Y(A)| \leqslant |A/C_A(Y)|^2$. Since $|I/C_I(A)| \leqslant |Y/C_Y(A)|$ and by (c) $C_A(Y) = C_A(I)$, this gives $|I/C_I(A)| \leqslant |A/C_A(I)|^2$.

By 1.43(m) $A$ acts nearly quadratically on $Y$ and so also on $I$. $\square$

LEMMA 8.37. *Suppose that $I \nleqslant A$ and $\Omega_1 Z(A) \leqslant Z(L)$. Then*
(a) *$Y = IC_Y(A)$.*
(b) *$A$ is a non-trivial offender on $I \cap A$.*
(c) *Suppose that no subgroup of $A$ is a non-trivial offender on $I$. Then $A$ is a non-trivial best offender on $I \cap A$.*

PROOF. Recall that $\widetilde{q} = |Y/Y \cap A|$. By 8.36(b) $IA = YA$, and so

$$|I/I \cap A| = |Y/Y \cap A| = \widetilde{q}, \tag{I}$$

and by 8.36(c)

$$Y \cap A = C_Y(I \cap A). \tag{II}$$

Moreover, by 1.43(e) $|A \cap Y/C_{A\cap Y}(L)| = |A/A \cap Y|$. Since by 8.36(a) $C_Y(A) = Z(A) = C_{A\cap Y}(L)$, we get

$$|A \cap Y/C_Y(A)| = |A/A \cap Y|. \tag{III}$$

(a): By 8.36(b) $YA = IA$ and so $Y = I(Y \cap A)$. Now (a) follows from 8.36(c).

(b): By 8.36(e) $A$ does not acts quadratically on $I$. So $[I, A, A] \neq 1$ and $[I \cap A, A] \neq 1$. Also

$$\begin{aligned} |A/C_A(I \cap A)| &\overset{\text{(II)}}{=} |A/Y \cap A| \overset{\text{(III)}}{=} |Y \cap A/C_Y(A)| \\ &\overset{\text{(II)}}{=} |(I \cap A)C_Y(A)/C_Y(A)| = |I \cap A/C_{I\cap A}(A)| \end{aligned}$$

and thus $A$ is a non-trivial offender on $I \cap A$.

(c): Observe that by 1.43(a) $\Phi(A) \leqslant C_Y(L)$ and so $A/C_A(I \cap A)$ is elementary abelian. Since $I \cap A$ and $A$ are $N_L(Y)$-invariant and $A$ is a non-trivial offender on $I \cap A$, A.29(b) shows that there exists a non-trivial $N_L(Y)$-invariant best offender $D$ on $I \cap A$ with $C_A(I \cap A) \leqslant D \leqslant A$ such that $|B||C_{I\cap A}(B)| \leqslant |D||C_{I\cap A}(D)|$ for all $B \leqslant A$. Since $[I \cap A, D] \neq 1$ and $Y$ is abelian, we have $D \nleqslant Y \cap A$. Thus by 1.43(f), $C_Y(D) \leqslant A$ and we conclude that

$$C_I(D) = C_{I\cap A}(D). \tag{IV}$$

Note that by the choice of $D$, $C_A(I \cap A) \leqslant D$ and so $C_A(I \cap A) = C_D(I \cap A)$. By 8.36(c), $C_A(I) = C_A(I \cap A) = Y \cap A$, and we conclude that

$$C_A(I) = C_D(I) = C_D(I \cap A) = C_A(I \cap A) = Y \cap A. \tag{V}$$

By 8.36(a) $C_Y(L) \leqslant Y \cap A$. Thus $C_Y(L) \leqslant C_D(I) \leqslant D \leqslant A$ and so $D/C_D(I)$ is an $N_L(Y)$-invariant section of $A/C_Y(L)$. Since

$$I \cap A/C_{I\cap A}(L) = I \cap A/(I \cap A) \cap C_Y(L) \cong (I \cap A)C_Y(L)/C_Y(L)$$

as $N_L(Y)$-modules and $C_{I\cap A}(L) \leqslant C_{I\cap A}(D)$, also $I \cap A/C_{I\cap A}(D)$ is (as an $N_L(Y)$-module) isomorphic to a section of $A/C_Y(L)$.

By 2.18 any chief factor for $N_L(Y)$ on $A/C_Y(L)$ has order $\widetilde{q}$ and so $|D/C_D(I)|$ and $|I \cap A/C_{I\cap A}(D)|$ both are powers of $\widetilde{q}$. As

$$\widetilde{q}|I \cap A/C_{I\cap A}(D)| \overset{\text{(I)}}{=} |I/I \cap A||I \cap A/C_{I\cap A}(D)| \overset{\text{(IV)}}{=} |I/I \cap A||I \cap A/C_I(D)| = |I/C_I(D)|, \tag{VI}$$

we get that $|I/C_I(D)|$ is a power of $\widetilde{q}$.

On the other hand, by the assumption of (c), $D$ is not an offender on $I$. Thus $|D/C_D(I)| < |I/C_I(D)|$ and so, since both sides are powers of $\widetilde{q}$,

$$\widetilde{q}|D/C_D(I)| \leqslant |I/C_I(D)| \overset{\text{(VI)}}{=} \widetilde{q}|I \cap A/C_{I\cap A}(D)| \overset{\text{(b)}}{\leqslant} \widetilde{q}|D/C_D(I \cap A)|. \tag{VII}$$

By (V) $C_D(I) = C_D(I \cap A)$ and so $|D/C_D(I)| = |D/C_D(I \cap A)|$. Thus equality must hold in (VII). Hence

$$|I \cap A/C_{I\cap A}(D)| = |D/C_D(I \cap A)|$$

and so

$$|D||C_{I\cap A}(D)| = |I \cap A||C_D(I \cap A)| \leqslant |I \cap A||C_A(I \cap A)|.$$

Since $A$ is an offender on $I \cap A$, $|I \cap A||C_A(I \cap A)|| \leqslant |A||C_{I\cap A}(A)|$. So

$$|D||C_{I\cap A}(D)| \leqslant |A||C_{I\cap A}(A)|,$$

and the maximality of $|D||C_{I\cap A}(D)|$ shows that $A$ is a best offender on $I \cap A$. □

Proposition 8.38. *Suppose that $I \not\leqslant A$ and $\Omega_1 Z(A) \leqslant Z(L)$. Then Case (1), (5) (for $n = 2$ and $q = 4$), (7),(8), (9) or (10) of Theorem H holds.*

Proof. We will first show:

1°. *Let $g \in M$ such that $C_Y(Q^g) \cap C_Y(A) \neq 1$. Then*

$$[\overline{Q^g}, \overline{A}] \leqslant \overline{Q^g} \cap \overline{A} \text{ and } [Y, Q^g] \leqslant [A, I]C_Y(A).$$

By 8.36(a) $C_Y(L) = C_Y(A)$ and so $[C_Y(Q^g) \cap C_Y(A), L] = 1$. Now $Q!$ implies $L \leqslant N_G(Q^g)$ and thus by 8.5(a) $Q^g$ normalizes $A$. So

$$[A, Q^g] \leqslant A \cap Q^g \text{ and } [Y, Q^g] \leqslant Y \cap Q^g \leqslant Y \cap O_p(L) = Y \cap A.$$

By 8.36(c) $Y \cap A = [I, A]C_Y(A)$ and so (1°) holds.

2°. *Suppose that $I$ is a vector space over the field $\mathbb{K}$, $Q$ acts $\mathbb{K}$-semilinearly on $I$ and $A$ acts $\mathbb{K}$-linearly on $I$. Then $Q$ acts $\mathbb{K}$-linearly on $I$.*

As $A$ acts non-trivially and $\mathbb{K}$-linearly on $I$, $[I, A]C_I(A)$ is a proper $\mathbb{K}$-subspace of $I$. Since $Q$ normalizes $A$, $C_Y(Q) \cap C_Y(A) \neq 1$ and

$$[I, Q] \leqslant I \cap [Y, Q] \overset{(1°)}{\leqslant} I \cap [A, I]C_Y(A) = [A, I]C_I(A).$$

Thus $Q$ centralizes the non-trivial $\mathbb{K}$-space $I/[I, A]C_I(A)$. Hence $Q$ acts $\mathbb{K}$-linearly on $I$.

Note that $Y$ is a $p$-reduced faithful $Q!$-module for $\overline{M}$ with respect to $\overline{Q}$. By 8.36 we have that $[Y, A] \leqslant I$, and $A$ acts nearly quadratically but not quadratically on $I$. By 8.5(b) $Q$ normalizes $A$, and $A$ normalizes $Q$. Thus the assumptions of the Nearly Quadratic $Q!$-Theorem D.11 are fulfilled for $\overline{M}$, $\overline{Q}$ and $\overline{A}$. We will now discuss the seven cases of that Theorem.

Case 1. $K := [F^*(\overline{M}), A]$ *is the unique component of $\overline{M}$, $K \leqslant \overline{M^\circ}$, $I$ is a simple $K$-module, $I = [Y, K\overline{A}]$ and $A$ acts $\mathbb{K}$-linearly on $I$, where $\mathbb{K} := End_K(I)$.*

By 8.5(b) $Q$ normalizes $L$ and $A$, by 1.43(m), $\overline{A} \cong A/C_A(Y)$ is elementary abelian and $[Y, A] \neq 1$, and by 8.37(a), $Y = IC_Y(A)$. Moreover, by 8.36(a), $Z(A) = C_Y(L) = C_Y(A)$, and by 8.36(d) $|I/C_I(A)| \leqslant |A/C_A(I)|^2$. Since $A$ acts $\mathbb{K}$-linearly on $I$, by (2°) also $Q$ acts $\mathbb{K}$-linearly on $I$. As seen above, $A$ acts nearly quadratically but not quadratically on $I$. Together with (1°) this shows that Case (1) of Theorem H holds.

Case 2. $M^\circ \cong \Omega_3(3)$, *and $Y$ is the corresponding natural module for $M^\circ$.*

Then Case (7) of Theorem H holds.

Case 3. $Y = I$, *and there exists an $\overline{M}$-invariant set $\{K_1, K_2\}$ of subnormal subgroups of $\overline{M}$ such that $K_i \cong SL_{m_i}(q)$, $m_i \leqslant 2$, $q$ a power of $p$, $[K_1, K_2] = 1$ and as a $K_1K_2$-module $Y \cong Y_1 \otimes_{\mathbb{F}_q} Y_2$ where $Y_i$ is a natural $SL_{m_i}(q)$-module for $K_i$. Moreover, $\mathbb{K} := End_{K_1K_2}(I) \cong \mathbb{F}_q$, and one of the following holds:*

(1) *$\overline{M^\circ}$ is one of $K_1, K_2$ or $K_1K_2$,*
(2) *$m_1 = m_2 = q = 2$, $\overline{M} \cong SL_2(2) \wr C_2$, $\overline{M^\circ} = O_3(\overline{M})\overline{Q}$ and $\overline{Q} \cong C_4$ or $D_8$.*
(3) *$m_1 = m_2 = p = 2$, $q = 4$, $\overline{M^\circ} = K_1K_2\overline{Q} \cong SL_2(4) \wr C_2$, $A$ acts $\mathbb{K}$-linearly on $I$ and $M^\circ$ does not.*

If $\overline{Q}$ is homocyclic, then 8.10 shows that $\overline{Q}$ is elementary abelian. This rules out the case $\overline{Q} \cong C_4$ in (2). So $\overline{Q} \cong D_8$ and $\overline{M^\circ} = K_1K_2\overline{Q} = \overline{M} \cong SL_2(2) \wr C_2$ in (2). In (3), since $K_1K_2 \trianglelefteq \overline{M}$, $M^\circ$ acts $\mathbb{K}$-semilinearly on $I$, but not $\mathbb{K}$-linear. Hence also $Q$ acts $\mathbb{K}$-semilinearly but not $\mathbb{K}$-linearly on $I$. Since $A$ acts $\mathbb{K}$-linearly on $I$ this contradicts $(2^\circ)$.

Now (1) and (2) show that Case (10) of Theorem H holds.

Case 4. *$\overline{M} \cong \Gamma SL_2(4)$, $\overline{M^\circ} \cong SL_2(4)$ or $\Gamma SL_2(4)$, $I$ is the corresponding natural module, and $|Y/I| \leqslant 2$.*

Then Case (8) of Theorem H holds.

Case 5. *$\overline{M} \cong \Gamma GL_2(4)$, $\overline{M^\circ} \cong SL_2(4)$, $I$ is the corresponding natural module, and $Y = I$.*

Then Case (5) of Theorem H holds with $n = 2$ and $q = 4$.

Case 6. *$\overline{M} \cong 3^{\cdot}Sym(6)$, $\overline{M^\circ} \cong 3^{\cdot}Alt(6)$ or $3^{\cdot}Sym(6)$, and $Y = I$ is simple of order $2^6$.*

Then Case (9) of Theorem H holds.

Case 7. *$M \cong Frob(39)$ or $C_2 \times Frob(39)$, $\overline{M^\circ} \cong Frob(39)$ and $Y = I$ is simple of order $3^3$.*

Note that $|\overline{A}| = 3$, $|[Y, A]| = 3^2$, $|C_Y(A)| = 3$ and $C_Y(A) \leqslant [Y, A]$. By 8.36(a), $C_Y(A) = C_Y(L)$, and by 8.36(c), $Y \cap A = [Y, A]C_Y(L) = [Y, A]$. Hence $|Y \cap A/C_Y(L)| = 3$ and $|A/C_Y(L)| = |A/Y \cap A||Y \cap A/C_Y(L)| = 9$. It follows that $L/O_3(L) \cong SL_2(3)$ and $A/C_Y(L)$ is the natural $SL_2(3)$-module for $L$. In particular, there exists an involution $t \in L \cap M^\dagger$ that inverts $A/C_Y(L)$ and so also $\overline{A}$. Thus $\bar{t} \notin Z(\overline{M})$, a contradiction since $\overline{M}/Z(\overline{M})$ has odd order. $\square$

## 8.5. The Proof of Theorem H

Clearly, one of the cases $I \leqslant A$, $I \nleqslant A$ and $\Omega_1Z(A) \nleqslant Z(L)$, and $I \nleqslant A$ and $\Omega_1Z(A) \leqslant Z(L)$ holds. Hence Theorem H follows from 8.34, 8.35, and 8.38, respectively.

CHAPTER 9

# The $Q$-tall Asymmetric Case II

In this chapter we continue the discussion of the $Q$-tall asymmetric case. More precisely, we discuss Case (1) of Theorem H proved in Chapter 8. As there we use a subgroup $L \in \mathfrak{L}_G(Y_M)$ with $L \leqslant N_G(Q)$ and investigate the action of $A$ $(= O_p(L))$ on $Y_M$.

At this point in the proof of the Local Structure Theorem we have already left behind all cases where one might have detected a non-trivial offender on $Y_M$ or its Fitting submodule $I$ by using properties of conjugates of $Y_M$ or the subgroups of $\mathfrak{H}_G(O_p(M))$ and $\mathfrak{L}_G(Y_M)$. Also the theorems on nearly quadratic action have already been exploited by showing that $\overline{M} = MC_G(Y_M)/C_G(Y_M)$ has a unique component $K$, that $I$ is a simple $K$-module and that $AQ$ acts $\mathbb{K}$-linearly on $I$, where $\mathbb{K} = End_K(I)$.

So in this chapter we need to apply the Theorems of Guralnick and Malle [**GM1**] and [**GM2**] on simple modules $V$ for almost quasisimple groups that allow a non-trivial $2F$-offender. In our case, $A$ is such a $2F$-offender on $I$. That is,

$$[I, A] \neq 0 \quad \text{and} \quad |I/C_I(A)| \leqslant |A/C_A(I)|^2.$$

But not all the pairs $(K, I)$ which we obtain by applying the Guralnick-Malle Theorems appear in the conclusion of the main theorem of this chapter. In section 9.1 we therefore provide some generic arguments which help to trim down the Guralnick-Malle list: If $K$ is a genuine group of Lie type in characteristic $p$ we show that $\overline{A} \leqslant K$ by using information about the outer automorphism group of $K$; and if $I$ is a selfdual $K$-module we obtain a wealth of additional information and are able to give a fairly precise description of the action of $A$ on $I$.

Here is the main result of this chapter.

Theorem I. *Let $G$ be a finite $\mathcal{K}_p$-group, $S \in Syl_p(G)$, and let $Q \leqslant S$ be a large subgroup of $G$. Suppose that $M \in \mathfrak{M}_G(S)$ such that $Y_M$ is asymmetric in $G$ and $Q$-tall and that Case 1 of Theorem H holds. Then one of the following holds, where $Y := Y_M$, $\overline{M^\dagger} := M^\dagger/C_{M^\dagger}(Y)$, $I := F_Y(\overline{M})$, and $q$ is some power of $p$:*

(1) *$\overline{M^\circ} \cong SL_n(q)$, $n \geqslant 3$, and $I$ is a corresponding natural module.*
(2) *$p = 2$, $\overline{M^\circ} \cong Sp_{2n}(q)$, $n \geqslant 2$, or $Sp_4(2)'$, and $I$ is a corresponding natural module.*
(3) *$\overline{M^\circ} \cong \Omega_n^\epsilon(q)$, $n \geqslant 3$, $(n, q) \neq (3, 3)$, $p$ is odd if $n$ is odd, and $I$ is a corresponding natural module.*
(4) *$\overline{M^\circ} \cong SL_n(q)/\langle(-id)^{n-1}\rangle$, $n \geqslant 5$, and $I$ is the exterior square of a corresponding natural module.*
(5) *$p$ is odd, $\overline{M^\circ} \cong SL_n(q)/\langle(-id)^{n-1}\rangle$, $n \geqslant 3$, and $I$ is the symmetric square of a corresponding natural module.*
(6) *$\overline{M^\circ} \cong SL_n(q)/\langle \lambda id \mid \lambda \in \mathbb{F}_q, \lambda^n = \lambda^{q_0+1} = 1\rangle$, $n \geqslant 3$, $q = q_0^2$, and $I$ is the unitary square of a corresponding natural module.*
(7) *$\overline{M^\circ} \cong Spin_{10}^+(q)$, and $I$ is a corresponding half-spin module.*
(8) *$\overline{M^\circ} \cong E_6(q)$, and $I$ is one of the (up to isomorphism) two simple $\mathbb{F}_pM^\circ$-modules of $\mathbb{F}_q$-dimension 27.*
(9) *$p = 2$, $\overline{M} = \overline{M^\circ} = Mat_{24}$, and $I$ is the simple Todd or Golay-code module of $\mathbb{F}_2$-dimension 11.*
(10) *$p = 2$, $\overline{M^\circ} \cong Mat_{22}$, and $I$ is the simple Golay-code module of $\mathbb{F}_2$-dimension 10.*
(11) *$p = 2$, $\overline{M} = \overline{M^\circ} \cong Aut(Mat_{22})$, and $I$ is the simple Todd module of $\mathbb{F}_2$-dimension 10.*

(12) $p = 3$, $\overline{M^\circ} \cong Mat_{11}$, *and* $I$ *is the simple Golay-code module of* $\mathbb{F}_3$*-dimension* 5.
(13) $p = 3$, $\overline{M^\circ} \cong 2{\cdot}Mat_{12}$, *and* $I$ *is the simple Golay-code module of* $\mathbb{F}_3$*-dimension* 6.

COROLLARY 9.1. *Assume the hypothesis and notation of Theorem I. Suppose in addition that* $Y \neq I$. *Then one of the following holds:*

(1) $\overline{M^\circ} \cong Sp_{2n}(q)$ *or* $Sp_4(2)'$, $p = 2$, $I$ *is the corresponding natural module and* $|Y/I| \leqslant q$.
(2) $\overline{M^\circ} \cong \Omega_4^-(3)$, $I$ *is the corresponding natural module,* $|Y/I| = 3$, *and* $Y$ *is isomorphic to the* 5*-dimensional quotient of a six dimensional permutation module for* $\overline{M^\circ} \cong Alt(6)$.
(3) $\overline{M^\circ} \cong \Omega_5(3)$, $I$ *is the corresponding natural module, and* $|Y/I| = 3$.
(4) $\overline{M^\circ} \cong \Omega_6^+(2)$, $I$ *is the corresponding natural module, and* $|Y/I| = 2$.
(5) $p = 2$, $\overline{M} = \overline{M^\circ} \cong Mat_{24}$, $I$ *is the simple Todd-module of* $\mathbb{F}_2$*-dimension* 11, *and* $|Y/I| = 2$.

COROLLARY 9.2. *Assume the hypothesis and notation of Theorem I. Suppose in addition that* $C_G(y)$ *is of characteristic* $p$ *for all* $1 \neq y \in Y_M$. *Then* $Y = I$. *Moreover, the cases (11) (Todd-module for* $Aut(Mat_{22})$*) and and (13) (Golay-module for* $2{\cdot}Mat_{12}$*) of Theorem I do not occur.*

Table 1 lists examples for $Y_M$, $M$ and $G$ fulfilling the hypothesis of Theorem I.

TABLE 1. Examples for Theorem I

| | Case | $[Y_M, M^\circ]$ for $M^\circ$ | c | Remarks | examples for $G$ |
|---|---|---|---|---|---|
| | 3 | nat $\Omega_n^\epsilon(q)$ | 1 | - | $P\Omega_{n+2}^\epsilon(q)$ |
| * | 3 | nat $\Omega_3(5)$) | 1 | - | $Co_1$ |
| * | 3 | nat $\Omega_4^-(2)$ | 1 | - | $L_4(3)$ |
| * | 3 | nat $\Omega_4^-(3)$ | $\leqslant 3$ | - | $U_6(2).c(.2)$ |
| | 3 | nat $\Omega_4^-(3)$ | 1 | - | McL |
| * | 3 | nat $\Omega_5(3)$ | 1 | - | $Fi_{22}(.2)$ |
| * | 3 | nat $\Omega_5(3)$ | $\leqslant 3$ | - | ${}^2E_6(2).c(.2)$ |
| * | 3 | nat $\Omega_6^+(2)$ | $\leqslant 2$ | - | $P\Omega_8^+(3).c(.2)$ |
| * | 3 | nat $\Omega_7^+(3)$ | 1 | - | $Fi_{24}'(.2)$ |
| * | 3 | nat $\Omega_{10}^+(2)$ | 1 | - | M |
| | 3, 4 | $\Lambda^2(\text{nat})SL_n(q)$ | 1 | $n \geqslant 4$ | $P\Omega_{2n}^+(q)$, $\Omega_{2n+1}(q)$ $p$ odd $P\Omega_{2n+2}^-(q)$, $O_{2n}^+(q)$ $p = 2$ |
| | 5 | $S^2(\text{nat})SL_n(q)$ | 1 | - | $PSp_{2n}(q)$ |
| | 6 | $U^2(\text{nat})SL_n(q_0^2)$ | 1 | - | $U_{2n}(q_0)$, $U_{2n+1}(q_0)$ |
| | 7 | half-spin $Spin_{10}^+(q)$ | 1 | - | $E_6(q)$ |
| | 8 | $q^{27}$ for $E_6(q)$ | 1 | - | $E_7(q)$ |
| | 9 | Golay $2^{11}$ for $Mat_{24}$ | 1 | - | $Co_1$ |
| | 9 | Todd $2^{11}$ for $Mat_{24}$ | 1 | - | $J_4$ |
| * | 9 | Todd $2^{11}$ for $Mat_{24}$ | $\leqslant 2$ | - | $Fi_{24}'.c$ |
| | 10 | Golay $2^{10}$ for $Mat_{22}$ | 1 | - | $Co_2$ |
| * | 11 | Todd $2^{10}$ for $Aut(Mat_{22})$ | 1 | - | $Aut(Fi_{22})$ |
| | 12 | Golay $3^5$ for $Mat_{11}$ | 1 | - | $Co_3$ |
| * | 12 | Golay $3^6$ for $2{\cdot}Mat_{12}$ | 1 | - | $Co_1$ |

Here $c := |Y_M/[Y_M, M^\circ]|$, and $*$ indicates that ($char\ Y_M$) fails in $G$.

## 9.1. Notation and Preliminary Results

NOTATION 9.3. We will use the notation introduced in Theorem I and in 8.3. In particular, since $L \in \mathfrak{L}_G(Y_M)$,

$$L/A \cong SL_2(\widetilde{q}),\ Sz(\widetilde{q}),\ \text{or } D_{2r} \text{ and } \widetilde{q} = |Y/Y \cap A|.$$

Moreover, by our hypothesis we are in case (1) of Theorem H. Summing up we have:

(a) $A$ is $Q$-invariant, $\overline{A}$ is elementary abelian, and $A$ acts nearly quadratically on $Y$, but not quadratically on $I$.
(b) $K := [F^*(\overline{M}), \overline{A}]$ is the unique component of $\overline{M}$, $K \leqslant \overline{M^\circ}$, and $I$ is a simple $K$-module.
(c) $|Y/C_Y(A)| \leqslant |\overline{A}|^2$ and $[Y, K\overline{A}] = I$.
(d) $AQ$ acts $\mathbb{K}$-linearly on $I$, where $\mathbb{K} := End_K(I)$.
(e) If $g \in M$ with $C_Y(Q^g) \cap C_Y(A) \neq 1$, then $[\overline{Q^g}, \overline{A}] \leqslant \overline{Q^g} \cap \overline{A}$ and $[Y, Q^g] \leqslant [Y, A]C_Y(A)$.
(f) $Y = IZ(A) = IC_Y(A)$ and $C_Y(A) = Z(A) = C_Y(L)$. In particular, $I \nleqslant A$ and $[Z(A), I] = 1$.

For any group $H$ and finite dimensional $\mathbb{F}_pH$-module $V$ we denote by $Y_V(H)$ the largest $p$-reduced submodule of $V$, i.e., the largest submodule $U$ satisfying $O_p(H/C_H(U)) = 1$.

Lemma 9.4. *Let $X$ be a non-trivial $p$-subgroup of $\overline{M}$. Then $C_X(K) = C_X(K/Z(K)) = C_X(\overline{M^\circ}) = 1$, $X \cap C_{\overline{M}}(K/Z(K))K \leqslant K$ and $K = [K, X]$.*

Proof. From 9.3(b) we get that $K$ is the unique component of $\overline{M}$, $K \leqslant \overline{M^\circ}$ and $I$ is a simple $K$-module. The last fact implies that $C_{\overline{M}}(K)$ is a $p'$-group. Thus $C_X(K) = 1$ and so $C_X(\overline{M^\circ}) = 1$ since $K \leqslant \overline{M^\circ}$.

As $K$ is quasisimple, $C_{\overline{M}}(K/Z(K)) = C_{\overline{M}}(K)$ and $X \cap C_{\overline{M}}(K/Z(K))K = X \cap C_{\overline{M}}(K)K$. Since $C_{\overline{M}}(K)$ is a $p'$-group, we conclude that $O^{p'}(C_{\overline{M}}(K)K) \leqslant K$ and so $X \cap C_{\overline{M}}(K)K \leqslant K$.

Note that $K$ is quasisimple, $K \trianglelefteq \overline{M}$ and $[K, X] \neq 1$. Thus $K = [K, X]$, and 9.4 is proved. □

Lemma 9.5. (a) *$A' = \Phi(A) \leqslant C_Y(L)$. In particular, $A$ acts quadratically on $I \cap A$.*
(b) *$I \cap A = [I, A]C_I(A)$. In particular, $I \cap A$ is a $\mathbb{K}$-subspace of $I$.*
(c) *$C_A(I \cap A) = Y \cap A = (I \cap A)C_Y(A)$.*
(d) *$A$ is a non-trivial offender on $I \cap A$.*
(e) *Suppose that no subgroup of $A$ is a non-trivial offender on $I$. Then $A$ is a non-trivial best offender on $I \cap A$.*

Proof. By 9.3(f) $I \nleqslant A$ and $Z(A) = C_Y(L)$. Thus also $\Omega_1 Z(A) \leqslant Z(L)$ and we can apply 8.36 and 8.37.

(a): By 1.43(a) $A' = \Phi(A) \leqslant C_Y(L)$ and so (a) holds.

(b): By 8.36(b) $[I, A] \leqslant I \cap A$ and by 8.36(d) $Y \cap A = [I, A]C_Y(L)$. Hence $I \cap A = [I, A]C_I(L)$.

(c), (d) and (e): These claims follow from 8.36(c), 8.37(b) and 8.37(c), respectively. □

Lemma 9.6. *Let $P \leqslant M$ with $AQ \leqslant P$ and $\overline{A} \leqslant O_p(\overline{P})$. Then $P^\circ$ normalizes $\overline{A}$ and $[I, P^\circ] \leqslant I \cap A$.*

Proof. Since $\overline{A} \leqslant O_p(\overline{P})$, $C_Y(O_p(\overline{P})) \leqslant C_Y(A)$. Thus, for $g \in P$,
$$1 \neq C_Y(O_p(\overline{P})) \cap C_Y(Q^g) \leqslant C_Y(A) \cap C_Y(Q^g),$$
and so 9.3(e) gives
$$[\overline{Q^g}, \overline{A}] \leqslant \overline{A} \cap \overline{Q}^g \text{ and } [Y, Q^g] \leqslant [Y, A]C_Y(A).$$
Hence $Q^g$ normalizes $\overline{A}$ and since $[Y, A]C_Y(A) \leqslant Y \cap A$, $[I, Q^g] \leqslant I \cap A$. Thus $P^\circ$ normalizes $\overline{A}$ and $[I, P^\circ] \leqslant I \cap A$. □

Lemma 9.7. *Suppose that $I$ is selfdual as an $\mathbb{F}_pK$-module. Put $D := [I, A] \cap C_I(A)$. Then there there exists a non-degenerate $K$-invariant symplectic, symmetric or unitary $\mathbb{K}$-form $s$ on $I$. Moreover, for any such form $s$ the following hold:*
(a) *$M^\circ\langle A^M\rangle$ acts $\mathbb{K}$-linearly on $I$, and $s$ is $M^\circ\langle A^M\rangle$-invariant.*
(b) *$|\mathbb{K}| = \widetilde{q}$.*
(c) *$D$ is 1-dimensional over $\mathbb{K}$ and $I \cap A = [I, A]C_I(A) = D^\perp$.*
(d) *$\dim_{\mathbb{K}}[I, a] \leqslant 2$ for all $a \in A$.*

(e) $D \leqslant C_I(Q)$.
(f) $A$ *centralizes* $D^{\perp}/D$ *and* $\overline{A} \leqslant O_p(N_{\overline{M}}(D))$.
(g) *Let* $X$ *be a* $\mathbb{K}$*-subspace of* $C_I(A)$ *with* $C_I(A) = D \times X$. *Put*

$$T := \{g \in GL_{\mathbb{K}}(X^{\perp}) \mid s(u,v) = s(u^g, v^g) \text{ for all } u, v \in X^{\perp}\},$$

*and let* $\breve{A}$ *be the image of* $A$ *in* $T$. *Then* $X \cap X^{\perp} = 1$ *and*

$$\breve{A} C_T(X^{\perp} \cap D^{\perp}) = C_T(D) \cap C_T(X^{\perp}/D).$$

(h) $s$ *is symmetric.*
(i) $|Z(K)| \leqslant 2$.
(j) *Let* $R_1 \leqslant M$ *with* $QA \leqslant R_1$, $Q \not\trianglelefteq R_1$ *and* $O_p(\overline{R_1}) \neq 1$, *and let* $I_1 := Y_I(R_1)$ *be the largest* $p$*-reduced* $R_1$*-submodule of* $I$. *Then* $I_1$ *is a natural* $SL_n(\widetilde{q})$*-module for* $R_1^{\circ}$ *and for* $\langle A^{R_1}\rangle$. *Moreover,* $D = [I_1, A] = C_{I_1}(Q)$.

Proof. By 9.3(b), $I$ is a simple $K$-module and by assumption $I$ is a selfdual $\mathbb{F}_pK$-module. So we can apply B.7 with $(M, K, I, \mathbb{F}_p)$ in place of $(H, N, V, \mathbb{F})$. In particular, the existence of $s$ follows from B.7(a).

For $U \subseteq I$ put

$$U^{\perp} := \{v \in I \mid s(u,v) = 0 \text{ for all } u \in U\}.$$

Recall from basic linear algebra :

1°. *Let* $U$ *and* $V$ *be* $\mathbb{K}$*-subspaces of* $I$. *Then*
(a) $U^{\perp\perp} = U$.
(b) $U^{\perp} \cap V^{\perp} = (UV)^{\perp}$.
(c) $(U \cap V)^{\perp} = U^{\perp}V^{\perp}$
(d) $\dim I = \dim U + \dim U^{\perp}$.

and

2°. *Let* $N$ *be a group acting* $K$*-linearly on* $I$ *and suppose that* $s$ *is* $N$*-invariant. Then*
(a) $C_I(N)^{\perp} = [I, N]$.
(b) $[I, N]^{\perp} = C_I(N)$.
(c) *Let* $U$ *be an* $N$*-submodule of* $I$. *Then* $C_N(U) = C_N(I/U^{\perp})$.

Next we prove:

3°. *(a) holds.*

By B.7(c) $M$ acts $\mathbb{K}$-semilinearly on $I$. Let $M_1$ consists of those elements in $M$ that act $\mathbb{K}$-linearly on $I$. Then by B.7(f), $s$ is $O^{p'}(M_1)$-invariant. By 9.3(d) $QA$ is $\mathbb{K}$-linear on $I$ and so is contained in $O^{p'}(M_1)$. It follows that $M^{\circ}\langle A^M\rangle = \langle (QA)^M\rangle \leqslant O^{p'}(M_1)$ and (a) holds.

4°. *(b) and (c) hold.*

By 9.5(b) $I \cap A = [I, A]C_I(A)$, and so (using (1°) and (2°))

$$(I \cap A)^{\perp} = [I, A]^{\perp} \cap C_I(A)^{\perp} = C_I(A) \cap [I, A] = D$$

and

$$I \cap A = (I \cap A)^{\perp\perp} = D^{\perp}.$$

Thus the the second part of (c) holds.

Since

$$\dim_{\mathbb{K}} I = \dim_{\mathbb{K}} D + \dim_{\mathbb{K}} D^{\perp} = \dim_{\mathbb{K}} D + \dim_{\mathbb{K}} I \cap A,$$

we have $|D| = |I/I \cap A|$. By 2.14 $YA/A$ is the unique non-trivial elementary abelian normal $p$-subgroup of $N_L(Y)/A$. It follows that $YA = IA$, and $N_L(Y)$ acts simply on

$$Y/Y \cap A \cong YA/A = IA/A \cong I/I \cap A = I/D^{\perp}.$$

In particular, $|D| = |I/I \cap A| = |Y/Y \cap A| = \widetilde{q}$. In addition, by 9.3(f), $C_Y(A) = C_Y(L)$. Since $D \leqslant C_I(A) \leqslant C_Y(A)$ we conclude that $N_L(Y)$ centralizes $D$ and so $C_M(D)$ acts simply on $I/D^{\perp}$. Now B.7(e) shows that $D$ is 1-dimensional over $\mathbb{K}$. Thus $|\mathbb{K}| = |D| = \widetilde{q}$, and (b) and (c) are proved.

5°. $[D^\perp, A] = [I \cap A, A] = [I, A, A] = D$.

From (c),

$$[D^\perp, A] = [I \cap A, A] = [[I, A]C_I(A), A] = [I, A, A].$$

By 9.3(a), $A$ is nearly quadratic but not quadratic on $I$. So $A$ is cubic on $I$ and

$$1 \neq [I, A, A] \leqslant C_I(A) \cap [I, A] = D.$$

By (c), $D$ is 1-dimensional over $\mathbb{K}$, and we get $[I, A, A] = D$. Thus (5°) is proved.

6°. *(d) and (e) hold.*

Let $a \in A$. Since $a$ acts $\mathbb{K}$-linearly on $I$ and $\dim_{\mathbb{K}} D = 1$, (5°) gives $\dim_{\mathbb{K}}[I \cap A, a] \leqslant 1$. As by (c) also $\dim_{\mathbb{K}} I/I \cap A = \dim_{\mathbb{K}} I/D^\perp = 1$, we get that $\dim_{\mathbb{K}}[I, a] \leqslant 2$. So (d) holds.

By 9.3(a) $A$ and so also $D$ is $Q$-invariant, and by (c) $Q$ acts $\mathbb{K}$-linearly on $I$. As $D$ is 1-dimensional over $\mathbb{K}$, this gives $D \leqslant C_I(Q)$. Hence (e) is proved.

7°. *(f) holds.*

By (5°) $[D^\perp, A] = D$, by definition $D \leqslant C_I(A)$, and by (c), $[I, A]C_I(A) = D^\perp$. Hence $A$ centralizes $I/D^\perp$, $D^\perp/D$ and $D$. Moreover, by B.7(d), $N_M(D)$ normalizes the chain $D \leqslant D^\perp \leqslant I$. Thus $\langle A^{N_M(D)}\rangle$ centralizes all factors of this series and so acts as a $p$-group on $I$. By 8.4(a) $C_M(Y) = C_M(I)$ and so $\overline{A} \leqslant O_p(N_{\overline{M}}(D))$.

8°. *(g) holds.*

Note that $X \leqslant C_I(A) = [I, A]^\perp$ and

$$[I, A] = C_I(A)^\perp = (D \times X)^\perp \overset{(1°)(a)}{=} D^\perp \cap X^\perp \text{ and } D^\perp X^\perp \overset{(1°)(c)}{=} (D \cap X)^\perp = I.$$

In particular, $X^\perp \nleqslant D^\perp$ and by (c) $X \leqslant C_I(A) \leqslant D^\perp$, so $X = X \cap D^\perp$ and $X = X \cap C_I(A)$. It follows that

$$X \cap X^\perp = X \cap D^\perp \cap X^\perp = X \cap [I, A] = X \cap C_I(A) \cap [I, A] = X \cap D = 1.$$

Hence $X$ is a non-degenerate subspace of $I$, and $I = X^\perp \times X$. Let $i \in X^\perp \backslash D^\perp$. As $D^\perp = [I, A]C_I(A)$ and $A$ acts nearly quadratically on $I$, we have $[i, A]C_I(A) = [I, A]C_I(A) = D^\perp$. Intersecting with $[I, A]$ gives $[I, A] = [i, A](C_I(A) \cap [I, A]) = [i, A]D$. As $[I, A] = X^\perp \cap D^\perp$ we conclude that

$$(*) \qquad [i, A]D = X^\perp \cap D^\perp.$$

Put $T_1 := C_T(D) \cap C_T(X^\perp \cap D^\perp/D)$ and $T_2 = C_T(X^\perp \cap D^\perp)$. Recall from (5°) that $[D^\perp, A] = D$, so $\breve{A} \leqslant T_1$. Since by (c) $D$ is 1-dimensional, $D^\perp$ is a $\mathbb{K}$-hyperplane of $I$, and since $X^\perp \nleqslant D^\perp$, $X^\perp/X^\perp \cap D^\perp$ is 1-dimensional. Hence by the choice of $i \in X^\perp$, $X^\perp = (\mathbb{K}i)(X^\perp \cap D^\perp)$. Since $T_1$ centralizes $X^\perp \cap D^\perp/D$, this gives $C_{T_1}(iD/D) = C_{T_1}(X^\perp/D)$.

Observe that $T = Cl(X^\perp)$ (in the notation of Appendix B). Hence by B.6(a) $X^\perp/X^\perp \cap D^\perp \cong D^*$ as $\mathbb{K}T_1$-modules and $C_{T_1}(X^\perp/D) = C_{T_1}(X^\perp \cap D^\perp)$, and so $T_2 = C_{T_1}(iD/D)$. By $(*)$ $A$ acts transitively on $i(X^\perp \cap D^\perp)/D$ and so a Frattini argument implies that $T_1 = \breve{A}T_2$ and (g) holds.

9°. *(h) holds.*

Let $X$, $T$ and $\breve{A}$ be as in (g), and let $T_1$ and $T_2$ be as in the proof of (g). Suppose that $s$ is not symmetric. Then $s$ is a unitary or symplectic form, where in the latter case $p$ is odd since $s$ is not symmetric. Hence B.28(c:a) and B.28(b:a) show that $\Phi(T_1) = T_2$. On the other hand, by (g) $T_1 = \breve{A}T_2$. This gives $T_1 = \breve{A}$, and $T_1$ is abelian, since $\breve{A}$ is abelian by 9.3(a). This contradiction shows that $s$ is symmetric and so (h) holds.

10°. *(i) holds.*

Let $k \in Z(K)$. By 9.3(b) $I$ is a simple $K$-module, and by 9.3(d) $\mathbb{K} = End_K(I)$, so $k$ acts as scalar $\lambda \in \mathbb{K}$ on $I$. By (h), $s$ is $\mathbb{K}$-bilinear and so for any $v, w \in I$:

$$s(v, w) = s(v^k, w^k) = s(\lambda v, \lambda w) = \lambda^2 s(v, w).$$

Since $s$ is non-zero we conclude that $\lambda^2 = 1$, and (i) holds.

We now begin with the proof of (j). Put $R := \langle A^{R_1}\rangle$.

11°. *$RR_1^\circ$ acts $\mathbb{K}$-linearly on $I$, and $s$ is $RR_1^\circ$-invariant.*

Note that $RR_1^\circ \leqslant M^\circ\langle A^M\rangle$. Hence (a) implies (11°).

12°. *$C_I(R_1^\circ) = 1$ and $I = [I, R_1^\circ]$.*

Since $Q \ntrianglelefteq R_1$, $Q!$ implies $C_I(R_1^\circ) = 1$. By (11°) $R_1^\circ$ acts $\mathbb{K}$-linearly on $I$ and $s$ is $R_1^\circ$-invariant. Hence

$$[I, R_1^\circ] = C_I(R_1^\circ)^\perp = I,$$

and (12°) follows.

13°. *$C_I(R) = 1$, and $[W, A] \neq 1$ and $W \neq I$ for every non-trivial $p$-reduced $R_1$-submodule of $I$. In particular $[I_1, A] \neq 1$ and $I_1 \neq I$.*

Set $I_0 := C_I(R)$ and suppose that $I_0 \neq 1$. Let $l \in R_1$. Since $A \leqslant R \trianglelefteq R_1$ and $Q \leqslant R_1$, $I_0$ is $R_1$ invariant and $1 \neq C_{I_0}(Q^l) \leqslant C_Y(Q^l) \cap C_Y(A)$. Now 9.3(e) shows that $[Y, Q^l] \leqslant [Y, A]C_Y(A)$. By 9.3(f), $Y = IZ(A)$. So $[Y, A] = [I, A]$ and thus

$$[I, Q^l] \leqslant I \cap [Y, A]C_Y(A) = [I, A]C_I(A).$$

By (c) $[I, A]C_I(A) = I \cap A$. Hence $[I, Q^l] \leqslant I \cap A$ and so $[I, R_1^\circ] \leqslant I \cap A$. But by (12°), $I = [I, R_1^\circ]$ and by 9.3(f) $I \nleqslant A$, a contradiction. Hence $C_I(R) = 1$.

Let $W$ be a non-trivial $p$-reduced $R_1$-submodule of $I$. Then $C_W(R) = 1$, and since $W \neq 1$ and $R = \langle A^{R_1}\rangle$, $[W, A] \neq 1$. Moreover, since $O_p(\overline{R_1}) \neq 1$, $I$ is not $p$-reduced for $R_1$, and $W \neq I$.

14°. *$I = [I, R]$.*

By (11°) $R$ acts $\mathbb{K}$-linearly on $I$ and $s$ is $R$-invariant, and by (13°), $C_I(R) = 1$. Hence $[I, R] = C_I(R)^\perp = I$.

15°. *$A$ is a best offender and a strong dual offender on every $A$-submodule of $I \cap A$.*

By 9.5(d), $A$ is a non-trivial offender on $I \cap A$. By (5°) $[I \cap A, A] = D$ and so $[I \cap A, A]$ is 1-dimensional over $\mathbb{K}$ by (c). Hence A.33(c) shows that $A$ is a best offender and a strong dual offender on every $A$-submodule of $I \cap A$. So (15°) holds.

16°. *$D = [I_1 \cap A, A] \leqslant I_1$.*

Since $I_1$ is $A$-invariant, $[I_1 \cap A, A] \leqslant [I_1, A] \leqslant I_1$. Recall from (5°) that $D = [I \cap A, A]$. By (15°) $A$ is a strong dual offender on $I \cap A$, so either $D = [I_1 \cap A, A] \leqslant I_1$ or $[I_1 \cap A, A] = 1$. In the former case we are done. So suppose the latter. Then $I_1 \nleqslant A$ since by (13°) $[I_1, A] \neq 1$. Then

$$[I_1, A]C_I(A) = [I, A]C_I(A),$$

since $A$ is nearly quadratic on $I$. By (c) $I \cap A = [I, A]C_I(A)$, and so $I \cap A[I_1, A]C_I(A)$. Thus

$$D = [I \cap A, A] = [[I_1, A]C_I(A), A] = [I_1, A, A] \leqslant [I_1 \cap A, A] \leqslant [I \cap A, A].$$

So again $D = [I_1 \cap A, A]$, and (16°) is proved.

17°. *$I_1$ is a $\mathbb{K}$-subspace of $I$. In particular, $I_1$ is a $\mathbb{K}RQ$-submodule of $I$.*

Put $R_0 := C_{R_1}(I_1)$. By (16°) $D \leqslant I_1$, and since $D$ is a non-trivial $\mathbb{K}$-subspace of $I$ and $M$ acts $\mathbb{K}$-semilinearly on $I$, we conclude that $R_0$ acts $\mathbb{K}$-linearly on $I$. Thus $R_0$ centralizes $\mathbb{K}I_1$ and so $C_{R_1}(I_1) = R_0 = C_{R_1}(\mathbb{K}I_1)$. Since $I_1$ is $p$-reduced for $R_1$ we get

$$O_p\big(R_1/C_{R_1}(\mathbb{K}I_1)\big) = O_p\big(R_1/C_{R_1}(I_1)\big) = 1,$$

and $\mathbb{K}I_1$ is $p$-reduced. Thus $I_1 = \mathbb{K}I_1$, and $I_1$ is $\mathbb{K}$-subspace of $I$.

Clearly $I_1$ is $R_1$-invariant and so also $RQ$-invariant, and by (11°) $RQ$ acts $\mathbb{K}$-linearly on $I$. Thus $I_1$ is a $\mathbb{K}RQ$-submodule of $I$.

18°. *$D = [I_1, A] \leqslant I_1 \leqslant A$.*

If $I_1 \leqslant A$, then $D = [I_1 \cap A, A] = [I_1, A]$ by (16°). So we may assume for a contradiction that $I_1 \nleqslant A$. Then

$$[I_1, A]C_I(A) = [I, A]C_I(A) = I \cap A$$

since $A$ is nearly quadratic on $I$.

By (c) $I \cap A = D^\perp$ is a $\mathbb{K}$-hyperplane in $I$, and by (17°) $I_1$ is a $\mathbb{K}$-space. Hence

$$I = I_1(I \cap A) = I_1[I_1, A]C_I(A) = I_1C_I(A),$$

and so $[I, A] \leqslant [I_1, A] \leqslant I_1$. Thus $[I, R] \leqslant I_1$. But $[I_1, R] = I$ by (14°) and $I \neq I_1$ by (13°), a contradiction.

19°. *$U := [I_1, R]$ is a simple $\mathbb{F}_pR$-submodule of $I_1$, $[U, A] = D$, and $A$ is a non-trivial strong dual offender on $U$. In particular, $R$ is generated by strong dual offenders on $U$.*

Let $U_1$ be a simple $\mathbb{F}_pR$-submodule of $I_1$. Since $C_I(R) = 1$ by (13°), we have $[U_1, A] \neq 1$, and since $A$ is a strong dual offender on $I_1$ by (15°), $[I_1, A] = [U_1, A] \leqslant U_1$ and so $U = [I_1, R] \leqslant U_1$. The simplicity of $U_1$ implies $U_1 = U$, so $U$ is a simple $\mathbb{F}_pR$-submodule of $I_1$.

By (17°) $I_1$ and thus also $U = [I_1, R]$ are $\mathbb{K}R$-modules. Since by (c) $D$ is a 1-dimensional $\mathbb{K}$-space, $1 \neq [U, A]$ shows that $[U, A] = D$.

By (18°) $I_1 \leqslant A$ and so $U \leqslant I \cap A$. Thus (15°) shows that $A$ is a strong dual offender on $U$. Since $R = \langle A^{R_1} \rangle$ we conclude that $R$ is generated strong dual offenders on $U$ and so (19°) holds.

Observe that $Q$ normalizes $I_1$ and $R$, so $U$ is an $RQ$-module. Put $H := RQ$, $\widetilde{H} := H/C_H(U)$ and $\mathbb{F} := End_R(U)$.

20°. *$U$ is a simple $Q!$-module for $\widetilde{H}$ with respect to $\widetilde{Q}$.*

By (19°) $U$ is a simple $R$-module. Since $U = [I_1, R]$ and $R \trianglelefteq R_1$, $U$ is also an $H$-module. Suppose that $Q \trianglelefteq H$. Since $U$ is a simple $R$-module, we conclude that $[U, Q] = 1$. But then also $[U, R_1^\circ] = 1$, a contradiction since $C_I(R_1^\circ) = 1$ by (12°). Hence $Q \ntrianglelefteq H$, and 1.57(b) shows that $U$ is a $Q!$-module for $\widetilde{H}$ with respect to $\widetilde{Q}$.

21°. *$|\mathbb{F}| = |\mathbb{K}| = \widetilde{q}$, $Q$ acts $\mathbb{F}$-linearly on $U$, and* $\dim_{\mathbb{F}} D = 1$.

Let $\mathbb{K}_U$ be the image of $\mathbb{K}$ in $End(U)$. By (17°) $H$ acts $\mathbb{K}$-linearly on $I_1$, so $\mathbb{K}_U \leqslant \mathbb{F}$. As $A \leqslant R$, $[U, A]$ is $\mathbb{F}$-invariant. By (19°) $[U, A] = D$ and so $[U, A]$ is 1-dimensional over $\mathbb{K}$. Hence $d\mathbb{F} \subseteq D = d\mathbb{K}_U$ for $1 \neq d \in D$. By Schur's Lemma $\mathbb{F}$ is a division ring and we conclude that $\mathbb{F} = \mathbb{K}_U$. Since $Q$ acts $\mathbb{K}$-linearly on $I$ we conclude that $Q$ acts $\mathbb{F}$-linearly on $U$. Moreover, (d) gives $|\mathbb{F}| = |\mathbb{K}_U| = |\mathbb{K}| = \widetilde{q}$.

22°. *(j) holds.*

By (19°) $U$ is a simple $\widetilde{R}$-module and $\widetilde{R}$ is generated by strong dual offender on $U$. So we can apply C.5. We conclude that one of the following holds:

(1) $\widetilde{R} \cong SL_n(q)$, $n \geqslant 2$, or $Sp_{2n}(q)$, $n \geqslant 2$, and $U$ is a corresponding natural module.
(2) $p = 2$, $\widetilde{R} \cong Alt(6)$ or $Alt(7)$, $U$ is a spin-module of order $2^4$ and $\widetilde{A} \cong \langle (12)(34), (13)(24) \rangle$
(3) $p = 2$, $\widetilde{R} \cong O^\epsilon_{2n}(2)$, $n \geqslant 3$, or $Sym(n)$, $n = 5$ or $n \geqslant 7$, $U$ is a corresponding natural module, and $|\widetilde{A}| = 2$.

Recall from 9.3(a) that $A$ is $Q$-invariant. Hence $\widetilde{Q}$ normalizes $\widetilde{A}$, $[U, A]$ and $C_U(A)$. Moreover, by (e) $D \leqslant C_I(Q)$ and by (19°) $D = [U, A]$, so $Q!$ shows that $Q \trianglelefteq N_H([U, A])$. In particular

$$(*) \qquad Q \trianglelefteq N_H([U, A]) \quad \text{and} \quad \widetilde{Q} \leqslant O_p(N_{\widetilde{H}}([U, A])).$$

Suppose that (2) holds. Then $|\mathbb{F}| = 2$ and $|[U, A]| = 4$. But this is a contradiction since $|\mathbb{K}| = |\mathbb{F}|$ by (21°) and $|[U, A]| = |D| = |\mathbb{K}|$ by (19°).

Suppose that (3) holds. Then $p = 2$, and $N_{\widetilde{R}}([U, A]) \cong C_2 \times \widetilde{E}$, $\widetilde{E} \cong Sp_{2n}(2)$ or $Sym(n-2)$, and $C_U(A)/[U, A]$ is a simple $N_{\widetilde{R}}([U, A])$-module. By $(*)$ $[\widetilde{Q}, N_{\widetilde{R}}([U, A])] \leqslant \widetilde{A} \leqslant C_{\widetilde{R}}(C_U(A))$, and by the simplicity of $C_U(A)/[U, A]$, $[C_U(A), Q, N_{\widetilde{R}}([U, A])] = 1$. Hence the Three Subgroups Lemma gives $[N_{\widetilde{H}}([U, A]), C_U(A), \widetilde{Q}] = 1$, and so, since $C_U(A) = [N_{\widetilde{H}}([U, A]), C_U(A)][U, A]$, $C_U(A) \leqslant C_H(Q)$. But this contradicts $Q!$ since $C_U(A)$ is a hyperplane in $U$.

Suppose that (1) holds. Then $|\mathbb{F}| = q$, and by (21°) $D$ is a 1-dimensional $\mathbb{F}$-space and $Q$ acts $\mathbb{F}$-linearly on $U$. If $\widetilde{R} \cong SL_n(q)$ then, since $GL_n(q)/SL_n(q)$ is a $p'$-group, $\widetilde{Q} \leqslant \widetilde{R}$ and so $\widetilde{H} = \widetilde{R}$. If $\widetilde{R} \cong Sp_{2n}(q)$, then the $R$-invariant symplectic forms on $U$ form a 1-dimensional $\mathbb{F}$-space. Since $Q$ is a $p$-group acting $\mathbb{F}$-linearly, $Q$ centralizes this 1-space. Thus again $\widetilde{Q} \leqslant \widetilde{H}$ and $\widetilde{H} = \widetilde{R}$.

We have shown that $\widetilde{H} = \widetilde{R}$. Suppose that $\widetilde{R} \cong Sp_{2n}(q)$, $n \geqslant 2$. Since $U$ is a $Q$!-module, B.37 yields $D = C_U(Q)$. Note that $\dim_{\mathbb{F}} U = 2n \geqslant 4$, $\dim_{\mathbb{F}}[U, A] = \dim_{\mathbb{F}} D = 1$ and $\dim_{\mathbb{F}}[U, A] + \dim_{\mathbb{F}} C_V(A) = \dim_{\mathbb{F}} U$. Thus $\dim_{\mathbb{F}} C_U(A) \geqslant 3$, and we can choose a $Q$-invariant 2-dimensional $\mathbb{F}$-subspace $I_2$ of $C_U(A)$. Put $R_2 := N_{R_1}(I_2)$. Then $R_2$ induces a group on $I_2$ that contains $SL_2(q)$. Hence, $I_2$ is a simple and thus $p$-reduced $R_2$-module. Moreover, since $[I_2, Q] \neq 1$, $Q$ is not normal in $R_2$, and since $O_p(\overline{R_1}) \leqslant O_p(\overline{R_2})$ we have $O_p(\overline{R_2}) \neq 1$. So (13°) applied to $(R_2, I_2)$ in place of $(R_1, W)$ gives $[I_2, A] \neq 1$, a contradiction to $I_2 \leqslant C_U(A)$.

Hence $\widetilde{R} \cong SL_n(q)$, $n \geqslant 2$. Suppose for a contradiction that $U \neq I_1$. Since $C_{I_1}(R) = 1$ and $[I_1, R] = U$, C.22 shows that $R/C_R(I_1) \cong SL_3(2)$ and the commutator $[U, A]$ is 2-dimensional, a contradiction since by (19°) $[U, A] = D$, and by (21°) $D$ has dimension 1. Thus $U = I_1$.

Finally if $n \geqslant 3$ or $n = 2$ and $|\mathbb{F}| > 3$ then $\widetilde{H}$ is quasisimple and so $\widetilde{H} = \widetilde{R_1^\circ}$. In the exceptional case $\widetilde{H} \cong SL_2(q)$, $q \leqslant 3$, the equality $\widetilde{H} = \widetilde{R_1^\circ}$ is easy to check. By (21°), $|\mathbb{F}| = q = \widetilde{q}$, so (j) holds. □

## 9.2. The Proof of Theorem I

We will use the notation given in 9.3 and Theorem I.

LEMMA 9.8. *Suppose that one of the following holds, where $q$ is a power of $p$.*

(a) *$K \cong SU_n(q)$ or $\Omega_n^\epsilon(q)$, $n \geqslant 3$, and $I$ is the corresponding natural module.*
(b) *$K \cong G_2(q)'$, and $I$ has $\mathbb{F}_q$-dimension 6 or 7, depending on $q$ being even or odd.*
(c) *$I$ is an $FF$-module for $\overline{M}$.*

*Then Theorem I holds.*

PROOF. Suppose first that (a) holds. If $I$ is a natural $SU_n(q)$-module for $K$, then $\mathbb{K} = \mathbb{F}_{q^2}$ and there exists a $K$-invariant non-degenerate unitary $\mathbb{K}$-form $s$ on $I$. So we can apply 9.7(h) and conclude that $s$ is symmetric, a contradiction.

Thus $I$ is a natural $\Omega_n^\epsilon(q)$-module for $K$, $n \geqslant 3$. As $K$ is quasisimple, $(n, q) \neq (3, 3)$. Since $I$ is a simple $K$-module, $p$ is odd if $n$ is odd. By 9.3(d), $Q$ acts $\mathbb{K}$-linearly on $I$ and thus B.35(d) shows that either $\overline{Q} \leqslant K$ or $p = 2$ and $K \cong \Omega_n^\epsilon(q)$. Suppose the latter, then $n \geqslant 4$, and since $K$ is quasisimple, $K \not\cong \Omega_4^+(q)$. Thus B.37 shows that $\overline{Q} \leqslant K$ also in this case. As $K$ is quasisimple we conclude that $K = \overline{M}^\circ$. Thus Theorem I(3) holds.

Suppose next that (b) holds, that is, $K \cong G_2(q)'$ and $I$ has dimension 6 or 7. Then $\mathbb{K} = \mathbb{F}_q$ and $I$ is selfdual. In particular, we again can apply 9.7. Then for $D := [I, A] \cap C_I(A)$

$$q = \widetilde{q} \qquad \text{and} \qquad \dim_{\mathbb{K}} D = 1.$$

Since $A$ acts $\mathbb{K}$-linearly on $I$, $A$ does not induce any non-trivial field automorphisms on $K$. It follows that either $\overline{A} \leqslant K$ or $q = 2$ and $K\overline{A} \cong G_2(2)$. Thus either $K\overline{A} \cong G_2(q)$ or $q = 2$ and $K\overline{A} = K \cong G_2(2)'$. Put $R := C_{K\overline{A}}(D)$. Since $D$ is a singular 1-subspace of $I$ and $K$ acts transitively on the singular 1-spaces, $D$ is centralized by a Sylow $p$-subgroup of $K\overline{A}$. Thus $R \sim q^{2+1+2}SL_2(q)$ (if $K\overline{A} \cong G_2(q)$) or $2^{2+2}SL_2(2)$ (if $q = 2$ and $K\overline{A} \cong G_2(2)'$). In either case, $C_R(D^\perp/D)$ is the unique elementary abelian normal subgroup of order $q^2$ in $R$ and acts quadratically on $I$. This is a contradiction, since $A$ does not act quadratically on $I$ by 9.3(a) and $A \leqslant C_R(D^\perp/D)$ by 9.7(f)

Suppose now that (c) holds and let $X$ be a non-trivial best offender in $\overline{M}$ on $I$. By 9.4 $K = [K, X]$ and $C_X(\overline{M}^\circ) = 1$. In particular, C.24 applies to $\overline{M}$ and $I$. Put $J = J_{\overline{M}}(I)$. Then $K = [K, X] \leqslant J$.

Assume that C.24(1) holds. Then $J \cong SL_2(q)^n$ and $I$ is a direct sum of natural $SL_2(q)$-modules for $J$. Since $I$ is a simple $K$-module and $K \leqslant J$ we conclude that $J \cong SL_2(q)$ and $I$ is a natural $SL_2(q)$-module. It follows that $\mathbb{K} = \mathbb{F}_q$ and $\dim_{\mathbb{K}} I = 2$. Since $A$ acts $\mathbb{K}$-linearly on $I$ this implies that $A$ acts quadratically on $I$, a contradiction.

Thus C.24(2) holds. Then $F^*(J)$ is quasisimple and so $K = F^*(J)$. In the cases C.24(2:c:1) and (2:c:3) $I$ is a direct sum of at least two non-trivial $F^*(J)$-submodules, a contradiction since $I$ is a simple $K$-module.

Hence C.24(2:c:2) holds. So by C.24(2:c:2:b)

$$(*) \qquad \text{either} \quad \overline{M^\circ} = K \quad \text{or} \quad \overline{M^\circ} \cong Sp_4(2),\ 3^{\cdot}Sym(6),\ SU_4(q).2,\ \text{or}\ G_2(2),$$

where $I$ is the natural $SU_4(q)$-module for $K$ in the $SU_4(q).2$-case. Moreover, by C.24(2:c:2:c) one of the cases C.3 (1) - (9), (12) applies to $(J, I)$, with $n \geqslant 3$ in case (1), $n \geqslant 2$ in case (2), and $n = 6$ in case (12). We will now treat these cases of C.3 one by one.

Suppose that C.3 (1) holds with $n \geqslant 3$. Then $I$ is a natural $SL_n(q)$-module for $J$. Thus $K = F^*(J) = J$ and by $(*)$, $K = \overline{M^\circ}$. So $I$ is a natural $SL_n(q)$ for $\overline{M^\circ}$, and Theorem I(1) holds.

Suppose that C.3 (2) holds with $n \geqslant 2$. Then $I$ is a natural $Sp_{2n}(q)$-module for $J$. Moreover, $K = F^*(J) \cong Sp_{2n}(q)'$, $\mathbb{K} = \mathbb{F}_q$, and there exists a $K$-invariant non-degenerate symplectic $\mathbb{K}$-form on $I$. By 9.7(h) this form is symmetric, and we conclude that $p = 2$. By $(*)$ either $\overline{M^\circ} = K \cong Sp_{2n}(q)'$ or $\overline{M^\circ} \cong Sp_4(2)$. Thus Theorem I(2) holds.

Suppose that C.3 (3) holds. Then $I$ is natural $SU_4(q)$-module for $J$. Hence $I$ is also a natural $SU_4(q)$-module for $K = F^*(J) = J$, a case we have already treaded assuming (a).

Suppose that C.3 (4) holds. Then $I$ is a natural $\Omega_n^\epsilon(q)$- or $O_n^\epsilon(q)$-module for $J$ for various $(n, q, \epsilon)$ with $n \geqslant 4$. Since $K = F^*(J)$ is quasisimple we conclude that $K \cong \Omega_n^\epsilon(q)$, a case we have already treaded assuming (a).

Suppose that C.3 (5) holds. Then $J \cong G_2(q)$, $p = 2$ and $I$ is the natural $G_2(q)$-module of order $q^6$. Then $K = F^*(J) = J' \cong G_2(q)'$, a case we have already treaded assuming (b).

Suppose that C.3 (6) holds. Then $J \cong SL_n(q)/\langle -id^{n-1}\rangle$, $n \geqslant 5$, and $V$ is the exterior square of a natural $SL_n(q)$-module. Thus $K = F^*(J) = J$ and by $(*)$ $\overline{M^\circ} = K$. Hence Theorem I(4) holds.

Suppose that C.3 (7) holds. Then $J \cong Spin_7(q)$ and $I$ is the spin module of order $q^8$. Thus $K = F^*(J) = J \cong Spin_7(q)$. Let $R$ be the centralizer in $K$ of a $QA$-invariant 1-dimensional singular subspace of the natural $\Omega_7(q)$-module. Put $R_1 = R\overline{QA}$ and $I_1 = C_I(O_p(R))$. Then $I_1$ is a natural $Sp_4(q)$-module for $R$. In particular, $I_1$ is a simple $R_1$ module and so $I_1 = Y_I(R_1)$. Then $Q!$ implies that $[I_1, Q] \neq 1$ and so $\overline{Q} \ntrianglelefteq R_1$. Thus 9.7(j) shows that $I_1$ is a natural $SL_n(\tilde{q})$-module for $R_1^\circ$. Since both $R$ and $R_1^\circ$ are normal in $R_1$, this is a contradiction.

Suppose that C.3(8) holds. Then $J \cong Spin_{10}^+(q)$ and $I$ is the half-spin module. Thus $K = F^*(J) = J$ and by $(*)$, $\overline{M^\circ} = K$. Hence Theorem I(7) holds.

Suppose that C.3(9) holds. Then $J \cong 3^{\cdot}Alt(6)$ and $|V| = 2^6$; in particular, $K = F^*(J) = J$ and $\mathbb{K} = \mathbb{F}_4$. Since $A$ acts $\mathbb{K}$-linearly on $I$ and any elementary abelian 2-subgroup of $GL_3(4)$ acts quadratically, we conclude that $A$ acts quadratically on $I$, a contradiction.

Suppose finally that C.3(12) holds with $n = 6$. Then $J \cong Alt(6)$ or $Sym(6)$ and $I$ is a corresponding natural module; in particular, $K = F^*(J) = Alt(6) = Sp_4(2)'$. By $(*)$, $\overline{M^\circ} = K \cong Sp_4(2)'$ or $\overline{M^\circ} \cong Sp_4(2)$ and Theorem I(2) holds. □

Lemma 9.9. *Suppose that $K$ is a quasisimple genuine group of Lie-type*[1] *defined over a field of characteristic $p$ and $I$ is not an $FF$-module for $\overline{M}$. Then $\overline{A} \leqslant K$.*

Proof. Let $K = {}^d\Sigma(q)$ (see A.58(b) for the definition). So $q$ is a power of $p$ and $d \in \{1, 2, 3\}$. By way of contradiction we assume $\overline{A} \nleqslant K$. Since $K \trianglelefteq \overline{M}$ by 9.3(b), the action of $\overline{M}$ on $K$ induces a chain of homomorphisms

$$\overline{M^\dagger} = \overline{M} \ \to Aut(K/Z(K)) \ \to Out(K/Z(K)) := Aut(K/Z(K))/Inn(K/Z(K)).$$

Let $\phi$ be the resulting homomorphism from $\overline{M}$ to $Out(K/Z(K))$, and for $X \leqslant \overline{M^\dagger}$ let $\widehat{X} := \overline{X}\phi$. Note that $C_{\overline{M}}(K/Z(K))K$ is the kernel of $\phi$ in $\overline{M}$.

1°. *Let $X \leqslant S$. Then $\widehat{X} \cong \overline{X}/\overline{X} \cap K$. In particular, $\widehat{A}$ is a non-trivial elementary abelian $p$-subgroup of $Out(K/Z(K))$ of order $|\overline{A}/\overline{A} \cap K|$.*

[1]For the definition see A.58.

This holds since by 9.4, $\overline{X} \cap \ker \phi = \overline{X} \cap C_{\overline{M}}(K/Z(K))K = \overline{X} \cap K$.

We fix the following notation:

Let $\Delta$ be the Dynkin diagram of $K$. We often identify $\Delta$ with its set of vertices. For a subdiagram $\Lambda \subseteq \Delta$, let $P_\Lambda$ be the corresponding Lie-Parabolic subgroup of $K$ with $\overline{S} \cap K \leqslant P_\Lambda$. In case of a minimal Lie-parabolic subgroup; i.e., if $\Lambda = \{\lambda\}$, we also write $P_\lambda$ rather than $P_\Lambda$.

Put $K_\Lambda := O^{p'}(P_\Lambda)$ and $Z_\Lambda := C_K(K/O_p(K))$. If $\Lambda$ is connected then $K_\Lambda/O_p(K_\Lambda)$ and $K_\Lambda/Z_\Lambda$ are genuine groups of Lie-type with Dynkin diagram $\Lambda$ and defined over $\mathbb{F}_{q^d}$ or $\mathbb{F}_q$. If $\Lambda_1, \dots \Lambda_l$ are the connected components of $\Lambda$ then $K_\Lambda/O_p(K_\Lambda)$ is isomorphic to a central product of the groups $K_{\Lambda_i}/O_p(K_{\Lambda_i})$, $1 \leqslant i \leqslant l$. Note that $\Lambda = \varnothing$ iff $P_\Lambda$ is $p$-closed and iff $K_\Lambda = \overline{S} \cap K$. Also $\Lambda = \Delta$ iff $K_\Lambda = K$ and iff $O_p(K_\Lambda) = 1$.

If $\Lambda$ is $QA$-invariant put $R_\Lambda := K_\Lambda \overline{QA}$; in particular $R_\Delta = K\overline{QA}$. Observe that $K \cap \overline{QA} \leqslant K \cap \overline{S} \leqslant P_\Lambda$ and so

$$R_\Lambda \cap K = K_\Lambda(K \cap \overline{QA}) \leqslant O^{p'}(P_\Lambda).$$

It follows that $R_\Lambda \cap K = K_\Lambda$ and that $R_\Lambda$ is a parabolic subgroup of $R_\Delta$ with $\overline{S} \cap R_\Delta = (\overline{S} \cap K)\overline{QA} \leqslant R_\Lambda$.

Conversely, let $R$ be a parabolic subgroup of $R_\Delta$ with $\overline{S} \cap R_\Delta \leqslant R$ and $O^{p'}(R \cap K) = R \cap K$. Then by A.63 $N_K(R \cap K)$ is a Lie-parabolic subgroups of $K$ and so $R = R_\Lambda$ for a unique $QA$-invariant $\Lambda \subseteq \Delta$. We denote this $\Lambda$ by $\Delta(R \cap K)$.

Finally, let $I^* := Hom_{\mathbb{F}_p}(I, \mathbb{F}_p)$ be the dual module of the $\mathbb{F}_p$-module $I$.

In the following we fix a proper (possible empty) $QA$-invariant subdiagram $\Lambda \subseteq \Delta$. Put $R := R_\Lambda$, and let $I_R := Y_I(R)$ be the largest $p$-reduced $R$-submodule of $I$. If $QA$ acts transitively on $\Delta$, observe that $\Lambda = \varnothing$ and $R = \overline{S} \cap K\overline{QA}$.

From A.60 applied to the adjoint version $K/Z(K)$:

2°. *There exist subgroups $Diag$ and $\Phi$ and a subset $\Gamma$ of $Out(K/Z(K))$ such that*

(a) *$\Phi\Gamma$ is a subgroup of $Out(K/Z(K))$, $\Phi \trianglelefteq \Phi\Gamma$, $Out(K/Z(K)) = Diag\Phi\Gamma$, $Diag \cap \Phi\Gamma = 1$ and $Diag \trianglelefteq Out(K/Z(K))$, and*
(b) *$Diag$ is a $p'$-group.*
(c) *$\Phi \cong Aut(\mathbb{F}_{q^d})$. In particular, $\Phi$ is cyclic.*
(d) *$C_{Diag\Phi\Gamma}(\Delta) = Diag\Phi$.*

Observe that $\Phi\Gamma$ contains a Sylow $p$-subgroup of $Out(K/Z(K))$, since $Out(K/Z(K)) = Diag\Phi\Gamma$ and $Diag$ is $p'$-group. Thus, replacing $\Phi\Gamma$ by a suitable conjugate under $Diag$, we may assume that

3°. *$\widehat{S} \leqslant \Phi\Gamma$. In particular, $\widehat{S} \cap Diag = 1$.*

By A.65

4°. *There exists $\tau \in \Gamma$ such that $\tau^2 = 1$ and $I^* \cong I^\tau$ as an $\mathbb{F}_pK$-module. Moreover,*

(1) *If $K = A_n(q)$, $n \neq 2$, $D_{2n+1}(q)$, $n \geqslant 2$, or $E_6(q)$, then $\Gamma = \langle\tau\rangle$ and $\tau$ induces the unique non-trivial graph automorphism on $\Delta$,*
(2) *otherwise $\tau = 1$.*

Next we show:

5°. *Let $s \in S$. Suppose that $s$ acts trivially on $\Delta$ and induces an inner automorphism on $K_\delta/Z_\delta$ for each $\delta \in \Delta$. Then $\overline{s} \in K$.*

By (1°) it suffices to show that $\widehat{s} = 1$. Since $s$ acts trivially on $\Delta$, (2°)(d) shows that $\widehat{s} \in Diag\Phi$. By (3°), $\widehat{s} \in \Phi\Gamma$ and so $\widehat{s} \in \Phi$. Choose $\delta \in \Delta$ such that $K_\delta/Z_\delta$ is defined over $\mathbb{F}_{q^d}$. Then $s$ induces the same field automorphism on $K_\delta/Z_\delta$ as on $K$ (see the description of field automorphism in [**GLS3**, 2.5].) As $s$ induces inner automorphism on $K_\delta/Z_\delta$ we conclude that $\widehat{s} = 1$.

6°. *There exists $\epsilon \in \Delta$ such that either $A$ does not fix $\epsilon$ or $A$ fixes $\epsilon$ and induces some non-trivial outer automorphism group on $K_\epsilon/Z_\epsilon$. In particular, $[K_\epsilon, A]$ is not a $p$-group.*

We first show the existence of an $\epsilon \in \Delta$ with the required property. For this we may assume that $A$ acts trivially on $\Delta$. Since $\overline{A} \nleqslant \overline{K}$, (5°) shows that $A$ induces a non-trivial outer automorphism group on $K_\epsilon/Z_\epsilon$ for every $\epsilon \in \Delta$. This establishes the existence of $\epsilon$.

Assume that $[K_\epsilon, A]$ is $p$-group. Then $[K_\epsilon, A](\overline{S} \cap K)$ is $p$-subgroup of $K$. Hence

$$[K_\epsilon, A] \leqslant \overline{S} \cap K \leqslant K_\epsilon \quad \text{and} \quad [K_\epsilon, A] \leqslant O_p(K_\epsilon) \leqslant Z_\epsilon.$$

It follows that $A$ normalizes $K_\epsilon$, so $A$ fixes $\epsilon$ and centralizes $K_\epsilon/Z_\epsilon$. In particular, $A$ does not induce a non-trivial outer automorphism group on $K_\epsilon/Z_\epsilon$, a contradiction.

In the following let $\epsilon$ be any element of $\Delta$ such that either $A$ does not fix $\epsilon$ or $A$ fixes $\epsilon$ and induces some non-trivial outer automorphism group on $K_\epsilon/Z_\epsilon$.

7°. *$\widetilde{q} = p \leqslant 3$; and if $\widetilde{q} = 3$ then $K = D_4(q)$, $\widehat{A} = \Gamma' \cong C_3$, and $\widehat{A}$ acts non-trivially on $\Delta$.*

Suppose that $\widetilde{q} > 2$. Then by 2.18 $\overline{A} = [\overline{A}, N_L(Y)]$ and any composition factor of $N_L(Y)$ on $\overline{A}$ has order $\widetilde{q}$. Thus $\widehat{A} = [\widehat{A}, \widehat{N_L(Y)}] \leqslant Out(K/Z(K))'$, and $\widetilde{q}$ divides $|\widehat{A}|$ since by (1°) $|\overline{A}/\overline{A} \cap K| = |\widehat{A}|$. By (2°) $Out(K/Z(K))$ is a semidirect product of $Diag$ by $\Phi\Gamma$, and by (3°) $\widehat{A} \leqslant \widehat{S} \leqslant \Phi\Gamma$. It follows that $\widehat{A} \leqslant (\Phi\Gamma)'$. Thus $\Phi\Gamma$ is not abelian, and we can apply A.61. So $\Delta$ is of type $D_4$, $\Gamma \cong Sym(3)$ and $(\Phi\Gamma)' \cong C_3$. Thus $\widehat{A} \cong C_3$. Since $\widetilde{q}$ divides $|\widehat{A}|$, we conclude that $\widetilde{q} = 3$, and so (7°) holds.

8°. *$\mathbb{K} = \mathbb{F}_p$, $I_R = C_I(O_p(R \cap K))$ is a simple $R \cap K$-module, and $C_I(S) = C_{I_R}(S) = C_{I_R}(\overline{S} \cap K) = C_I(\overline{S} \cap K)$ has order $p$.*

By (7°) $\widetilde{q} = p$ and so $|I/I \cap A| = \widetilde{q} = p$. Since by 9.5(b) $I \cap A$ is $\mathbb{K}$-subspace of $I$ we conclude that $|\mathbb{K}| = p$ and the first statement in (8°) holds.

Clearly $1 \neq I_R \leqslant C_I(O_p(R \cap K))$. Recall from 9.3(b) that $I$ is a simple $K$-module. By Smith's Lemma A.63 $C_I(O_p(R \cap K))$ is a simple $\mathbb{K}(R \cap K)$-module. Since $\mathbb{K} = \mathbb{F}_p$ we conclude that $C_I(O_p(R \cap K))$ is a simple $R \cap K$-module. So $I_R = C_I(O_p(R \cap K))$, and the second statement holds.

Steinberg's Lemma A.62 shows that $C_I(\overline{S} \cap K)$ is 1-dimensional over $\mathbb{K}$ and so has order $p$. Since $C_{I_R}(\overline{S} \cap K) \neq 1$ and $C_I(S) \leqslant C_I(\overline{S} \cap K)$ also the last statement holds.

9°. *$QA$ does not act transitively on $\Delta$.*

Suppose that $QA$ acts transitively on $\Delta$. Then every vertex of $\Delta$ has the same valency, and since $\Delta$ has vertices of valency 1, we get $|\Delta| = 1$ or $|\Delta| = 2$. This rules out the case $p = 3$ in (7°) and so $p = \widetilde{q} = 2$. By (8°) $\mathbb{K} = \mathbb{F}_p = \mathbb{F}_2$ and $C_I(S) = C_I(\overline{S} \cap K)$ has order 2. Hence $[C_I(\overline{S} \cap K), N_K(\overline{S} \cap K)] = 1$. Let $P_1$ be a minimal Lie-parabolic subgroup of $K$ containing $\overline{S} \cap K$ and put $R_1 := O^{2'}(P_1)$. The transitive action of $QA$ on $\Delta$ implies $K = \langle R_1^{QA} \rangle$. Since $C_I(R_1) \leqslant C_I(\overline{S} \cap K)$ and $QA$ centralizes $C_I(\overline{S} \cap K)$, this gives $C_I(R_1) = C_I(K) = 1$ and so

$$[C_I(S), R_1] = [C_I(\overline{S} \cap K), R_1] \neq 1.$$

Hence A.66 shows that

$$(*) \qquad I \text{ is the Steinberg module of } \mathbb{F}_2\text{-dimension } |\overline{S} \cap K|,$$

and $I$ is, as an $\overline{S} \cap K$ module, isomorphic to the regular permutation module $\mathbb{F}_2[\overline{S} \cap K]$. The latter fact shows that

$$|I| = |[I, t]|^2 \quad \text{for every involution } t \in K.$$

Note that $I$ is selfdual (for example $I^* \cong I^\tau$ by (4°) and $I^\tau$ is the Steinberg module by A.66). Let $1 \neq a \in A$. Then 9.7(d) gives $\dim_{\mathbb{K}}[I, a] \leqslant 2$ and so

$$(**) \qquad |[I, a]| \leqslant 4 \quad \text{for all } 1 \neq a \in \overline{A}.$$

Suppose that there exists $1 \neq a \in \overline{A} \cap K$. Then $|I| = |[I, a]|^2 \leqslant 4^2 = 2^4$. By $(*)$ $I$ has $\mathbb{F}_2$-dimension $|\overline{S} \cap K|$ and we conclude $|\overline{S} \cap K| \leqslant 4$. Hence $K \cong SL_2(4)$ and $I$ is the natural $Sym(5)$-module for $K\overline{A}$. But $Sym(5)$ has two classes of maximal elementary abelian 2-subgroups, one acts quadratically on the natural $Sym(5)$-module, and the other is contained in $Alt(5)$. Since $\overline{A} \nleqslant K$ we conclude that $A$ acts quadratically on $I$, which contradicts 9.3(a).

Hence $\overline{A} \cap K = 1$ and (1°) gives $|\widehat{A}| = |\overline{A}/\overline{A} \cap K| = |\overline{A}|$. Since $A$ does not act quadratically on $I$, we have $|\overline{A}| \geqslant 4$ and so $|\widehat{A}| \geqslant 4$. Note that $\widehat{A}$ is elementary abelian, $\widehat{A} \leqslant \Gamma\Phi$ (by (3°)), $\Phi$ is cyclic (by (2°)) and $|\Gamma| \leqslant |\Delta| \leqslant 2$. We conclude that $|\Delta| = |\Gamma| = 2$, $\Gamma \leqslant \widehat{A}$, $|\widehat{A}| = 4$ and $\Phi \neq 1$. In particular, $A$ acts transitively on $\Delta$. As seen above $[C_I(S), R_1] \neq 1$ and so $[C_I(O_p(R_1)), O^p(R_1)] \neq 1$. Let $a \in A \backslash C_A(\Delta)$. Since $\Phi \neq 1$ and $A$ acts transitively on $\Delta$, $R_1/O_p(R_1)$ is a group of Lie-type defined over a field of order larger than 2. Hence $|C_I(O_p(R_1))| \geqslant 2^4$. Observe that

$$|C_I(O_p(R_1)) \cap C_I(O_p(R_1))^a| \leqslant |C_I(O_p(R_1)O_p(R_1)^a)| = |C_I(\overline{S} \cap K)| = 2,$$

and so $|[C_I(O_p(R_1)), a]| \geqslant \frac{2^4}{2} = 8 > 4$, a contradiction to (∗∗). Thus (9°) is proved.

10°. *Suppose that $O^p(R) \neq 1$. Then $[I, O^p(R)] \nleqslant I_R$.*

Suppose for a contradiction that $[I, O^p(R)] \leqslant I_R$. Since $K$ is a group of Lie-type defined over a field of characteristic $p$, $K$ has parabolic characteristic $p$. So $O^p(R) \neq 1$ gives $E := O_p(O^p(R)) \neq 1$. In particular, $[I, E] \neq 1$. Moreover, by (8°) $I_R$ is a simple $R$-module and thus $[I_R, E] = 1$. As

$$[I, E] \leqslant [I, O^p(R)] \leqslant I_R \leqslant C_I(E),$$

we have $C_{I/C_I(E)}(R) \neq 1$. Hence there exists $i \in I$ such that $[i, R] \leqslant C_I(E)$ and $[i, E] \neq 1$. Since $E$ and $iC_I(E)$ are $R$-invariant, also $[i, E]$ is $R$-invariant. As $I_R$ is a simple $R$-module and $[i, E] \leqslant [I, E] \leqslant I_R$ this gives $[i, E] = I_R$. Thus

$$|E| \geqslant |E/C_E(i)| = |[i, E]| = |I_R| = |[I, E]|,$$

so $E$ is an offender on $I^*$. Since $[I, E] \leqslant C_I(E)$, $E$ acts quadratically on $I$ and so $E$ is elementary abelian. By (4°) $I^* \cong I^\tau$ as an $\mathbb{F}_pK$-module and therefore $E^{\tau^{-1}}$ is an elementary abelian offender on $I$, contrary to the assumption that $I$ is not an $FF$-module.

11°. *Suppose that $O^p(R) \leqslant \langle \overline{A}^R \rangle$. Then $I_R \leqslant I \cap A$, and $A$ is a quadratic best offender on $I_R$.*

We first apply 9.5. By 9.5(a) $A$ acts quadratically on $I \cap A$ and so also on $I_R$. Since $I$ is not an FF-module for $\overline{M}$, 9.5(e) shows that $A$ is a best offender on $I \cap A$. Thus, by A.31 $A$ is also a best offender on every $A$-submodule of $I \cap A$. Hence, it suffices to prove $I_R \leqslant I \cap A$.

So suppose for a contradiction that $I_R \nleqslant A$. Since $\tilde{q} = p$ by (7°) we have $Y \leqslant YA = I_RA$. Hence 8.7 applied with $U = I_R$ gives $[Y, A] \leqslant [I_R, A]$ and thus $[I, A] \leqslant I_R$. By assumption $O^p(R) \leqslant \langle \overline{A}^R \rangle$, and we conclude that $[I, O^p(R)] \leqslant I_R$. If $O^p(R) = 1$, then $R$ is a $p$-group and $I_R \leqslant C_I(R) \leqslant C_I(A)$, a contradiction as $C_I(A) \leqslant I \cap A$ by 9.5(b). Thus $O^p(R) \neq 1$. But then (10°) shows that $[I, O^p(R)] \nleqslant I_R$, again a contradiction.

12°. *Suppose that $\Delta(R \cap K) \neq \varnothing$ and $\overline{QA}$ acts transitively on $\Delta(R \cap K)$. Then $R$ is $p$-minimal.*

Since $\Delta$ has no closed circuits, $\Delta(R \cap K)$ contains a vertex of valency 1. Now the transitivity of $\overline{QA}$ shows that the connected components of $\Delta(R \cap K)$ have size 1.

Note that $O^p(R) \leqslant R \cap K$ and since $R \cap K = O^{p'}(R \cap K)$,

$$R \cap K/O_p(R \cap K) = E_1 \circ E_2 \circ \ldots \circ E_n,$$

where $n$ is the number of connected components of $\Delta(R \cap K)$, and $E_i$ is a rank 1 group of Lie-type. The latter fact shows that $E_i$ is also $p$-minimal. Hence the hypothesis of 1.39 is satisfied, and we conclude that $R$ is $p$-minimal.

Case 1. *$I$ is selfdual as an $\mathbb{F}_pK$-module.*

We now refine the choice of $R$ from the beginning of the proof. By (9°) $QA$ is not transitive on $\Delta$. Thus, every $QA$-orbit of $\Delta$ is a proper subset of $\Delta$. Choose $R$ in addition such that

(i) $\Delta(R \cap K)$ is a $QA$-orbit on $\Delta$;
(ii) if $C_K(C_I(S))$ is not $p$-closed then $R \cap K \leqslant C_K(C_I(S))$; and
(iii) if $C_K(C_I(S))$ is $p$-closed and $N_K(\overline{Q})$ is not $p$-closed then $R \cap K \leqslant N_K(\overline{Q})$.

Put $R_1 := R_{\Delta\backslash\Delta(R\cap K)}$ and $I_1 := Y_I(R_1)$. Observe that also $\Delta\backslash\Delta(R\cap K)$ is a proper $QA$-invariant subset of $\Delta$.

If $R\cap K \leqslant C_K(C_I(S))$ then $Q!$ gives $R\cap K \leqslant N_{R\cap K}(Q)$. So the choice of $R$ implies $\overline{Q} \trianglelefteq R$ unless $N_K(\overline{Q})$ is $p$-closed, and if $N_K(\overline{Q})$ is $p$-closed then $R_1 \nleqslant N_{\overline{M}}(\overline{Q})$. Therefore, since $\overline{Q}$ is not normalized by $K$, $\overline{Q} \ntrianglelefteq R_1$.

Put $D := [I,A]\cap C_I(A)$. Since $I$ is selfdual we can apply 9.7 and get:

13°.

(a) *$D$ is 1-dimensional over $\mathbb{K}$, $|\mathbb{K}| = \tilde{q}$, and $D^\perp = [I,A]C_I(A) = I\cap A$. In particular, by (7°), $|D| = \tilde{q} = p$.*
(b) *$A$ centralizes $D^\perp/D$ and $\overline{A} \leqslant O_p(N_{\overline{M}}(D))$.*
(c) *$I_1$ is a natural $SL_m(\tilde{q})$-module for $R_1^\circ$ and $\langle \overline{A}^{R_1}\rangle$.*
(d) *$D = [I_1,A] = C_{I_1}(Q)$.*

Next we show:

14°. *$D = C_I(S) \leqslant I_1$ and $K_\epsilon \nleqslant N_K(D)$. In particular, $[C_I(S),K_\epsilon] \neq 1$.*

Note that $\overline{S}\cap K$ normalizes $C_{I_1}(Q)$. By (13°)(a),(d) $|D| = p$ and $C_{I_1}(Q) = D$, and by (8°) $C_I(\overline{S}\cap K) = C_I(S)$ and $|C_I(S)| = p$, so $D = C_I(S) \leqslant I_1$.

By (6°) $[K_\epsilon, A]$ is not a $p$-group and by (13°)(b), $\overline{A} \leqslant O_p(N_{\overline{M}}(D))$. Thus $K_\epsilon \nleqslant N_K(D)$.

15°. *For $X \subseteq R_1$ let $X^\triangle$ be the image of $X$ in $Aut(I_1)$. Then $(R_1\cap K)^\triangle = R_1^\triangle$ and $I_1$ is a natural $SL_m(p)$-module for $R_1\cap K$.*

Since $\tilde{q} = p$, (13°)(c) shows that $R_1^{\circ\triangle} \cong SL_m(p)$ and $|I_1| = p^m$. So $Aut(I_1) \cong GL_m(p)$. As $R_1 = (R_1\cap K)\overline{QA} = O^{p'}(R_1)$ and $GL_m(p)/SL_m(p)$ is a $p'$-group, we conclude that $R_1^\triangle = R_1^{\circ\triangle} \cong SL_m(p)$. Note that, for $m \geqslant 3$ or $p > 3$, $SL_m(p) = O^p(SL_m(p))$; and for $m = 2$ and $p \leqslant 3$, $O^p(SL_m(p))$ is a $p'$-group and $|SL_m(p)/O^p(SL_m(p))| = p$. Since $O^p(R_1) \leqslant R_1\cap K$ and $R_1\cap K = O^{p'}(R_1\cap K)$ we conclude that $(R_1\cap K)^\triangle = R_1^\triangle$.

16°. *$\epsilon \in \Delta(R\cap K)$ and $K_\epsilon \leqslant R\cap K$.*

Clearly, $\epsilon\in\Delta(R\cap K)$ implies $K_\epsilon \leqslant R\cap K$. Assume that $\epsilon \notin \Delta(R\cap K)$. Then $\epsilon\in\Delta(R_1\cap K)$ and $K_\epsilon \leqslant R_1\cap K$. By (14°) $K_\epsilon \notin N_K(D)$ and since $D \leqslant I_1$ we get $[I_1, O^p(K_\epsilon)] \neq 1$. Thus $C_{K_\epsilon}(I_1) \leqslant Z_\epsilon$. By (15°), $(R_1\cap K)^\triangle = R_1^\triangle$ and so $A^\triangle \leqslant (\overline{S}\cap K)^\triangle$. Hence $A$ normalizes $K_\epsilon^\triangle$ and induces inner automorphisms on $K_\epsilon^\triangle$. It follows that $A$ fixes $\epsilon$ and induces inner automorphism on $K_\epsilon/Z_\epsilon$, contrary to the choice of $\epsilon$.

17°. *$N_K(D)$ is $p$-closed.*

By (16°) $K_\epsilon \leqslant R\cap K$ and by (14°) $[C_I(S),K_\epsilon]\neq 1$. Thus $R\cap K \nleqslant C_K(C_I(S))$ and choice of $R$ implies that $C_K(C_I(S))$ is $p$-closed. Since $\overline{S}\cap K \in Syl_p(K)$, also $N_K(C_I(S))$ is $p$-closed. By (14°) $C_I(S) = D$ and so (17°) holds.

18°. *$R$ is $p$-minimal, $O^p(R) \leqslant \langle\overline{A}^R\rangle$, $[I_R, O^p(R)] \neq 1$ and $[I_R, A] = D$.*

By (16°) $K_\epsilon \leqslant R$. Moreover, by (14°) $[C_I(S),K_\epsilon]\neq 1$, and by A.12 $C_I(S) \leqslant C_I(\overline{S}\cap R) \leqslant I_R$. Hence $[I_R,K_\epsilon]\neq 1$ and thus also $[I_R, O^p(R)] \neq 1$.

By (15°) $I_1$ is a natural $SL_m(p)$-module for $R_1\cap K$, by (13°)(a) $|D| = p$ and by (14°) $D \leqslant I_1$. Hence $C_{R_1\cap K}(D)^\triangle$ is the stabilizer of a point of $I_1$. On the other hand, by (17°) $N_K(D)$ is $p$-closed and thus also $C_{R_1\cap K}(D)$ is $p$-closed. This shows that $m = 2$ and $\Delta(R_1\cap K) = \{\delta\}$ for some $\delta\in\Delta$. Note that $\Delta$ is connected, $QA$ normalizes $\delta$, and $QA$ acts transitively $\Delta(R\cap K) = \Delta\backslash\{\delta\}$. Hence (12°) shows that $R$ is $p$-minimal. In particular, $R$ is $p$-irreducible by 1.37.

By the choice of $K_\epsilon$, $[K_\epsilon, A]$ is not a $p$-group. Since $K_\epsilon \leqslant R$, we conclude that $\overline{A} \nleqslant O_p(R)$, and so, since $R$ is $p$-irreducible, $O^p(R) \leqslant \langle\overline{A}^R\rangle$. As $[I_R, O^p(R)] \neq 1$ this gives $[I_R, A] \neq 1$, and by (11°) $A$ acts quadratically on $I_R$. So $1 \neq [I_R,A] \leqslant [I,A]\cap C_I(A) = D$. Since $|D| = p$, we conclude that $[I_R, A] = D$, and (18°) is proved.

We are now able to derive a contradiction which shows that (Case 1) does not occur. By (18°) $R$ is $p$-minimal, $O^p(R) \leqslant \langle \overline{A}^R \rangle$, and by (13°)u10a $|D| = p$. Hence (11°) shows that $A$ is a non-trivial quadratic best offender on $I_R$, and we are allowed to apply C.13 with $\widetilde{R} := R/C_R(I_R))$ and $J := J_{\widetilde{R}}(I_R)$. Hence

$$J = E_1 \times \cdots \times E_r, \qquad I_R = C_{I_R}(J) \prod_{i=1}^{r} [I_R, E_i], \quad \text{and} \quad [I_R, E_i, E_j] = 1 \text{ for } i \neq j,$$

where for $i = 1, \ldots, r$, $E_i \cong SL_2(p^k)$ or $Sym(2^k + 1)$ (and $p = 2$), and $[I_R, E_i]/C_{[I_R,E_i]}(E_i)$ is a corresponding natural module for $E_i$, and $\overline{S} \cap R$ acts transitively on $\{E_1, \ldots, E_r\}$. In particular, $C_{[I_R,E_i]}(E_i) \leqslant C_{I_R}(J)$. By (8°), $I_R$ is a simple $R$-module, so $C_{I_R}(J) = 1$. Thus $[I_R, E_i]$ is natural $SL_2(p^k)$- or $Sym(2^k+)$-module for $E_i$ and

$$I_R = [I_R, E_1] \times \cdots \times [I_R, E_r].$$

As $\overline{A} \leqslant J$ and $[I_R, A] = D$ has order $p$, there exists a unique $E_j$ such that $D \leqslant [I_R, E_j]$. Since by (14°) $D = C_S(I)$ we conclude that $\overline{S} \cap R$ normalizes $[I_R, E_j]$ and so $E_j$. Hence the transitivity of $\overline{S} \cap R$ on $\{E_1, \ldots, E_r\}$ gives $J = E_1$.

If $J \cong Sym(2^k + 1)$ then $[I_R, A]$ is not centralized by a Sylow 2-subgroup of $J$, a contradiction since $[I_R, A] = D = C_I(S)$. Thus $J \cong SL_2(p^k)$. As $[I_R, A] = D$ has order $p$ we get $k = 1$. It follows that $J = \widetilde{R \cap K} \cong SL_2(p)$. Thus $\Delta(R \cap K) = \{\epsilon\}$ and $R \cap K = K_\epsilon$. In particular, $K_\epsilon/O_p(K_\epsilon) \cong SL_2(p)$ and $C_{K_\epsilon}(I_R) \leqslant Z_\epsilon$. Now $\overline{A} \leqslant K_\epsilon C_R(I_R)$ shows that $\overline{A}$ induces inner automorphisms on $K_\epsilon C_R(I_R)/C_R(I_R) \cong K_\epsilon/C_{K_\epsilon}(I_R)$ and thus on $K_\epsilon/Z_\epsilon$, a contradiction to the choice of $\epsilon$.

Case 2. *$I$ is not selfdual as an $\mathbb{F}_pK$-module.*

19°.

(a) *$K = A_n(t^2)$, $n \geqslant 2$, $D_{2n+1}(t^2)$, $n \geqslant 2$, or $E_6(t^2)$; in particular, $\Delta$ has only single bonds.*
(b) *$p = \widetilde{q} = 2$, $S$ acts trivially on $\Delta$, and $\widehat{S} \leqslant \Phi \cong Aut(\mathbb{F}_{t^2})$.*
(c) *$\widehat{A}$ is the unique subgroup of order 2 in $\Phi$.*

By (4°) $I^* \cong I^\tau$ with $\tau^2 = 1$, and since $I$ is not selfdual, we have $\tau \neq 1$. Thus (4°) implies

$$K \;=\; A_n(q),\ n \geqslant 2,\ D_{2n+1}(q),\ n \geqslant 2,\ \text{or } E_6(q),$$

and $\tau$ induces the unique non-trivial graph automorphism of $\Delta$, so $\Gamma = \langle \tau \rangle$ has order 2. In particular, (a) holds, except that we still need to show that $q$ is a square. Also $K \neq D_4(q)$, and (7°) shows that $p = \widetilde{q} = 2$.

Let $s \in S$. Recall from (3°) that $\widehat{S} \leqslant \Gamma\Phi$. If $\widehat{s} \notin \Phi$, we conclude that $\tau \in \widehat{s}\Phi$ since $\Gamma$ has order 2. But $I \cong I^x$ as an $\mathbb{F}_2K$-module for all $x \in \Phi$ and so $I^* \cong I^\tau \cong I^{\widehat{s}} = I$ as an $\mathbb{F}_2K$-module; a contradiction since $I$ is not selfdual. Thus $\widehat{S} \leqslant \Phi$, and by (4°) $S$ acts trivially on $\Delta$. So (b) is proved.

Recall that $\widehat{A}$ is non-trivial and elementary abelian, $\widehat{A} \leqslant \widehat{S} \leqslant \Phi$ and $\Phi$ is cyclic. Thus (c) follows.

Note that $d = 1$ for the groups in (19°)(a) (see A.60), and so by (2°)(c) $\Phi \cong Aut(\mathbb{F}_q)$. We conclude that $\mathbb{F}_q$ has an automorphism of order 2 and so $q = t^2$ for some power $t$ of $p$, which completes the proof of (a).

By (19°)(c) $QA$ acts trivially on $\Delta$, so all subdiagrams of $\Delta$ are $QA$-invariant. Hence we can choose $R$ such $\overline{Q} \ntrianglelefteq R$, $\Delta(R \cap K)$ is connected and either $|\Delta| \geqslant 3$ and $|\Delta(R \cap K)| = 2$, or $|\Delta| = 2$ and $|\Delta(R \cap K)| = 1$. Put

$$m := |\Delta(R \cap K)|, \qquad \widetilde{R} := R/C_R(I_R), \qquad P := \overline{A}(R \cap K),$$

and let $\widetilde{A}$ be the image of $\overline{A}$ in $\widetilde{R}$. Recall from (19°)(b) that $p = 2$.

20°. *$\widetilde{R \cap K} = A_m(t^2)$, $m \leqslant 2$, $|\widetilde{P}/\widetilde{R \cap K}| = 2$, each $a \in \widetilde{A} \backslash \widetilde{R \cap K}$ acts as a field automorphism of order 2 on $\widetilde{R \cap K}$, $F^*(\widetilde{P}) = \widetilde{R \cap K}$ is quasisimple, $\langle \widetilde{A}^{\widetilde{P}} \rangle = \widetilde{P}$ and $O^2(R) \leqslant \langle A^R \rangle$.*

Since $\Delta$ has only single bonds and $\Delta(R \cap K)$ is connected of size $m \leqslant 2$, $\Delta(R \cap K)$ is of type $A_m$. As by (19°) $K$ is defined over $\mathbb{F}_{t^2}$ we conclude that $R \cap K/O_2(R \cap K) = A_m(t^2)$. In particular, $R \cap K/O_2(R \cap K)$ is quasisimple. Since $\overline{Q} \ntrianglelefteq R$ and $R = (R \cap K)\overline{QA}$, $Q!$ shows that $[I_R, R \cap K] \neq 1$. It follows that $C_{R \cap K}(I_R) \leqslant Z_{\Delta(R \cap K)}$. Hence, also $\widetilde{R \cap K}$ is a version of $A_m(t^2)$ and so quasisimple. Let $a \in \overline{A} \backslash K$. Note that $|\overline{A}/\overline{A} \cap K| = |\widehat{A}| = 2$ and $a$ acts as a field automorphism of order 2 on $K$. Hence $a$ also acts as a field automorphism of order 2 on $\widetilde{R \cap K}$, and $\widetilde{P} = \widetilde{R \cap K}\langle \widetilde{a} \rangle$. In particular, $\widetilde{R \cap K} = [\widetilde{R \cap K}, a]$ and so

$$\widetilde{P} = \widetilde{R \cap K}\widetilde{A} = [\widetilde{R \cap K}, \widetilde{A}]\widetilde{A} = \langle \widetilde{A}^{\widetilde{P}} \rangle.$$

Since $R \cap K/O_2(R \cap K)$ is quasisimple, this implies $O^2(R) = O^2(R \cap K) \leqslant \langle \overline{A}^R \rangle$, and (20°) is proved.

21°. *$P/O_2(P) \cong Sym(5)$, and $I_R$ is the corresponding natural module. In particular, $m = 1$, $t^2 = 4$, and $K/Z(K) \cong L_3(4)$.*

By (20°) $O^2(R) \leqslant \langle \overline{A}^R \rangle$, and so (11°) shows that $I_R \leqslant I \cap A$ and $A$ is a quadratic best offender on $I_R$. Moreover, since by (20°) $[I_R, O^2(R)] \neq 1$, $A$ is a non-trivial best offender on $I_R$.

By (20°) $\widetilde{P} = \langle \widetilde{A}^{\widetilde{P}} \rangle$ and so $J_{\widetilde{P}}(I_R) = \widetilde{P}$. As $I_R$ is a simple $R \cap K$-module by (8°), $I_R$ is simple a $\widetilde{P}$-module. Thus we can apply the FF-Module Theorem C.3. Since by (20°) $|\widetilde{P}/\widetilde{R \cap K}| = 2$, $m \leqslant 2$, and $\widetilde{R \cap K} = A_m(t^2)$ is a central quotient of $SL_{m+1}(t^2)$ $(2 \leqslant m+1 \leqslant 3)$, we conclude that $m+1 = 2$, $t^2 = 4$, $\widetilde{P} \cong Sym(5)$, and $I_R$ is the corresponding natural module. (Note here that the natural $Sym(5)$-module also appears as the natural $O_4^-(2)$-module in the FF-Module Theorem.)

Since $|\Delta(R \cap K)| = m = 1$, the choice of $R$ shows that $|\Delta| = 2$. The only rank 2 group of Lie-type listed in (19°)(a) is $L_3(t^2) = A_2(t^2)$, and so (21°) is proved.

22°. *There exists an involution $t$ in $R \cap K$ with $t \notin O_2(R \cap K)$ and $|I/C_I(t)| \leqslant 2^3$.*

Recall that $|I/I \cap A| = \widetilde{q} = 2$ and that by (11°) $A$ is best offender on $I \cap A$. Thus

$$(***) \qquad |I/C_I(A)| = |I/I \cap A||I \cap A/C_{I \cap A}(A)| \leqslant 2|A/C_A(I)| = 2|\overline{A}|.$$

Put $B := O_2(R \cap K)$. Suppose first that $\overline{A} \cap B = 1$. Since $P/B \cong Sym(5)$ and $\overline{A}$ is elementary abelian we conclude that $|\overline{A}| \leqslant 4$. As $A$ does not act quadratically on $I$, $|\overline{A}| \geqslant 4$ and so $\overline{A} \cap K \neq 1$. Let $1 \neq t \in \overline{A} \cap K$. Then $(***)$ gives $|I/C_I(t)| \leqslant 2|\overline{A}| = 8$, and (22°) holds.

Suppose next that $\overline{A} \cap B \neq 1$. Since $K/Z(K) \cong L_3(4)$, $B$ is a natural $\Gamma SL_2(4)$-module for $P$ and so $|C_B(A)| \leqslant 4$. In particular, $|\overline{A} \cap B| \leqslant 4$. Note that $\overline{A} \cap B = C_{\overline{A}}(I_R)$ and $\overline{A}$ is not an over-offender on $I_R$. Thus $|I_R/C_{I_R}(\overline{A})| \geqslant |\overline{A}|/|\overline{A} \cap B|$, and so using $(***)$

$$|I/C_I(\overline{A} \cap B)| \leqslant |I/C_I(\overline{A})I_R| = \frac{|I/C_I(\overline{A})|}{|I_R/C_{I_R}(\overline{A})|} \leqslant \frac{2|\overline{A}|}{|\overline{A}|/|\overline{A} \cap B|} = 2|\overline{A} \cap B| \leqslant 2^3.$$

Since $\overline{A} \cap B \neq 1$ and all involutions in $L_3(4)$ are conjugate, and since there exist involutions in $R \cap K \backslash B$, we again conclude that (22°) holds.

We are now able to derive a final contradiction. Choose $t$ as in (22°). Note that, for example since $t$ inverts an elements of order five in $R \cap K/B \cong Alt(5)$, $|W/C_W(t)| \geqslant 4$ for any non-central simple $\mathbb{F}_2(R \cap K)$-module. On the other hand $|I/C_I(t)| \leqslant 2^3$, and so $I$ has at most one non-central $R \cap K$-composition factor. Thus $[I, O^2(R)] = [I, O^2(R \cap K)] \leqslant I_R$, a contradiction to (10°). □

In the following we will use a result of Guralnick-Malle on simple $2F$-modules for quasisimple groups $H$, [**GM1**] and [**GM2**]. Here an $\mathbb{F}_pH$-module $V$ is a $2F$-module for $H$ if there exists an elementary abelian $p$-subgroup $A \leqslant H$ such that

$$|V/C_V(A)| \leqslant |A/C_A(V)|^2 \text{ and } [V, A] \neq 0.$$

According to 9.3(b) $I$ is a simple module for $K$. By 9.3(c) $\overline{A}$ satisfies the above inequality with respect to $Y$. Clearly $|I/C_I(A)| \leqslant |Y/C_Y(A)|$ and by 8.4(c) $C_A(I) = C_A(Y)$. Hence $\overline{A}$ satisfies the above inequality also with respect to $I$. Moreover, the case where $I$ is an $FF$-module has been treated already in 9.8. In the remaining case, if $K$ is the genuine group of Lie-type, $\overline{A} \leqslant K$ by 9.9, and the pair $(K, I)$ satisfies the hypothesis of [**GM1**] or [**GM2**].

We will distinguish the cases, where $K$ is a genuine group of Lie type, a non-genuine group of Lie type, an alternating group, and a sporadic group, respectively. For this purpose we break up the result of Guralnick-Malle into four parts which we will quote separately.

THEOREM 9.10 (**Guralnick-Malle**). *Let $H$ be a genuine quasisimple group of Lie-type defined over a field of characteristic $p$ and $V$ a faithful simple $2F$-module for $\mathbb{F}_pH$. Put $\mathbb{F} := End_H(V)$ and $d := \dim_{\mathbb{F}} V$. Let $\delta_{x|y} = 1$, if $x$ divides $y$, and $\delta_{x|y} = 0$, otherwise. Then $H$, $V$, $d$ and $\mathbb{F}$ are given in the following table:*

| $H$ | $d$ | $V$ | $\lvert\mathbb{F}\rvert$ | conditions |
|---|---|---|---|---|
| $SL_n(p^a)$ | $n$ | $V_{nat}$ | $p^a$ | |
| $SL_n(p^a)$ | $\binom{n}{2}$ | $\Lambda^2 V_{nat}$ | $p^a$ | $n \geqslant 3$ |
| $SL_n(p^a)$ | $\binom{n+1}{2}$ | $Sym^2 V_{nat}$ | $p^a$ | $p$ *odd*, $n \geqslant 3$ |
| $SL_n(p^{2a})$ | $n^2$ | $\frac{\mathbb{F}_{p^a}}{\mathbb{F}_{p^{2a}}} V_{nat} \otimes V_{nat}^{p^a}$ | $p^a$ | |
| $SL_6(p^a)$ | $20$ | $\bigwedge^3 V_{nat}$ | $p^a$ | |
| $SU_n(p^a)$ | $n$ | $V_{nat}$ | $p^{2a}$ | |
| $Sp_{2n}(p^a)$ | $2n$ | $V_{nat}$ | $p^a$ | |
| $Sp_{2n}(p^a)$ | $\binom{n}{2} - 1 - \delta_{p\mid n}$ | $\widetilde{\bigwedge}^2 V_{nat}$ | $p^a$ | $n = 2, 3$ or $p = 2, n = 4$ |
| $Sp_4(p^{2a})$ | $16$ | $\frac{\mathbb{F}_{p^a}}{\mathbb{F}_{p^{2a}}} V_{nat} \otimes V_{nat}^{p^a}$ | $p^a$ | |
| $\Omega_n^{\pm}(p^a)$ | $n$ | $V_{nat}$ | $p^a$ | |
| $Spin_{2n+1}(p^a)$ | $2^n$ | $Spin$ | $p^a$ | $n = 3, 4, 5$ |
| $Spin_{2n}^{+}(p^a)$ | $2^{n-1}$ | $Half-Spin$ | $p^a$ | $n = 4, 5, 6$ |
| $Spin_{2n}^{-}(p^a)$ | $2^{n-1}$ | $Spin$ | $p^{2a}$ | $n = 4, 5$ |
| $Sz(2^{2a+1})$ | $4$ | $M(\lambda_1)$ | $2^{2a+1}$ | |
| $G_2(p^a)$ | $7 - \delta_{2\mid n}$ | $M(\lambda_2)$ | $p^a$ | |
| $F_4(2^a)$ | $26$ | $M(\lambda_1), M(\lambda_4)$ | $2^a$ | |
| $E_6(p^a)$ | $27$ | $M(\lambda_1), M(\lambda_6)$ | $p^a$ | |
| $F_4(p^a)$ | $26 - \delta_{3\mid p}$ | $M(\lambda_4)$ | $p^a$ | $p$ odd |
| ${}^2E_6(p^a)$ | $27$ | $M(\lambda_1)$ | $p^{2a}$ | |
| $E_7(p^a)$ | $56$ | $M(\lambda_7)$ | $p^a$ | |

We remark that it has been shown in [**GLM**] that the last three cases of the table do not occur. But since they only add two lines of arguments to our proof, we prefer to work with the original list.

LEMMA 9.11. *Suppose that $K$ is a quasisimple genuine group of Lie-type defined over a field of characteristic $p$. Then Theorem I holds.*

PROOF. By 9.8 we may assume that

1°. *$I$ is not an $FF$-module for $K\overline{QA}$.*

Thus by 9.9 $\overline{A} \leqslant K$. So we can apply 9.10 with $(K, I, \mathbb{K})$ in place of $(H, V, \mathbb{F})$. Removing all the $FF$-modules and all the modules which have been treated in 9.8 we are left with the following list:

2°.

| $K$ | $d$ | $I$ | $\|\mathbb{K}\|$ | conditions |
|---|---|---|---|---|
| $SL_n(p^a)$ | $\binom{n+1}{2}$ | $Sym^2 V_{nat}$ | $p^a$ | $odd, n \geqslant 3$ |
| $SL_n(p^{2a})$ | $n^2$ | $\frac{\mathbb{F}_{p^a}}{\mathbb{F}_{p^{2a}}} V_{nat} \otimes V_{nat}^{p^a}$ | $p^{2a}$ | $n \geqslant 3$ |
| $SL_6(p^a)$ | 20 | $\bigwedge^3 V_{nat}$ | $p^a$ | |
| $Sp_{2n}(p^a)$ | $\binom{n}{2} - 1 - \delta_{p\|n}$ | $\widetilde{\bigwedge}^2 V_{nat}$ | $p^a$ | $n = 2, 3$ or $p = 2, n = 4$ |
| $Sp_4(p^{2a})$ | 16 | $\frac{\mathbb{F}_{p^a}}{\mathbb{F}_{p^{2a}}} V_{nat} \otimes V_{nat}^{p^a}$ | $p^a$ | |
| $Spin_{2n+1}(p^a)$ | $2^n$ | $Spin$ | $p^a$ | $n = 3, 4, 5$ |
| $Spin_{2n}^+(p^a)$ | $2^{n-1}$ | $Half - Spin$ | $p^a$ | $n = 6$ |
| $Spin_{2n}^-(p^a)$ | $2^{n-1}$ | $Spin$ | $p^{2a}$ | $n = 4, 5$ |
| $Sz(2^{2a+1})$ | 4 | $M(\lambda_1)$ | $p^{2a+1}$ | |
| $F_4(2^k)$ | 26 | $M(\lambda_1), M(\lambda_4)$ | $p^a$ | |
| $F_4(p^a)$ | $26 - \delta_{3\|p}$ | $M(\lambda_4)$ | $p^a$ | $p$ odd |
| ${}^2E_6(p^a)$ | 27 | $M(\lambda_1)$ | $p^{2a}$ | |
| $E_7(p^a)$ | 56 | $M(\lambda_7)$ | $p^a$ | |

If $K$ has rank 1 we see that $K \cong Sz(2^{2a+1})$ and $\dim_{\mathbb{K}} I = 4$. But then every elementary abelian 2-subgroup of $K$ acts quadratically on $I$, which contradicts 9.3(a) since $\overline{A} \leqslant K$. Thus we may assume:

3°. *$K$ has Lie rank at least two.*

Put $U := C_I(K \cap \overline{S})$ and $R := N_{K\overline{Q}}(U)$. By Smith's Lemma A.63, $U$ is 1-dimensional over $\mathbb{K}$. Since $Q$ acts $\mathbb{K}$-linear and $U$ is 1-dimensional, $Q$ centralizes $U$ and $Q!$ implies $\overline{Q} \trianglelefteq R$.

Let $\Delta$ be the Dynkin diagram of $K$. Observe that in all cases there exists $i \in \Delta$ such that either $I \cong M(\lambda_i)$ or $I$ is a simple $\mathbb{K}K$-submodule of $M(\lambda_i) \otimes_{\mathbb{F}} M(\lambda_i)^\sigma$, where $\mathbb{F}$ is the field used to define $K$ and $\sigma$ is an automorphism of $\mathbb{F}$ with $C_{\mathbb{F}}(\sigma) = \mathbb{K}$. Thus $R \cap K$ is the maximal parabolic corresponding to $\Delta \backslash \{i\}$. In particular, $R$ is a maximal subgroup of $K\overline{Q}$ and so $R = N_{K\overline{Q}}(\overline{Q})$. Let $P$ be the $p$-minimal subgroup of $K\overline{Q}$ corresponding to the node $i$ and containing $(\overline{S} \cap K)\overline{Q}$. Then $P \nleqslant R$ and so $\overline{Q} \ntrianglelefteq P$. Since $Q$ is weakly closed in $S$, $\overline{Q} \nleqslant O_p(P)$.

Suppose that one of the first two cases of (2°) holds. As $[(R \cap K)/O_p(R \cap K), Q] = 1$ we conclude that $Q$ induces inner automorphisms on $K$. Thus $\overline{M^\circ} = K$ and Theorem I(5) or (6) holds.

So assume for a contradiction that one of the remaining cases of (2°) holds. We prove next:

4°. *$I$ is selfdual as $\mathbb{F}_pK$-module.*

In the third case of (2°) $I$ is that the exterior cube of a natural $SL_6(q)$-module, and so selfdual. In all other cases A.65 shows that $I$ is selfdual.

As $Q$ fixes $i$ we can choose a proper $Q$-invariant connected subdiagram $\Lambda$ of $\Delta$ with $i \in \Lambda$, which is maximal with respect to these properties. Let $R_1$ be the corresponding parabolic subgroup of $\overline{M^\circ}$ with $(\overline{S} \cap K)\overline{Q} \leqslant R_1$ and note that $P \leqslant R_1$. Put $R_1^\circ := \langle \overline{Q}^{R_1} \rangle$. Since $I$ is a selfdual $\mathbb{F}_pK$-module, we can apply 9.7(j) and conclude that $I_{R_1}$ is a natural $SL_m(q)$-module for $R_1^\circ$. As $\Lambda$ is connected, $\Lambda$ is an $A_{m-1}$-diagram.

We will now derive a contradiction by showing that in all (remaining) cases $R_1$ can be chosen such that either $\Lambda$ is not of type $A_{m-1}$ or $I_{R_1}$ is not a natural $SL_m(q)$-module for $R_1^\circ$ or $I$ is an $FF$-module for $K$.

If $K \cong SL_6(q)$ and $I$ is the exterior cube of a natural $SL_6(q)$-module, then $I_{R_1}$ is the exterior square of a natural $SL_5(q)$-module, a contradiction.

If $\mathbb{K} \cong Sp_{2n}(q)$, $n \geqslant 3$, and $I$ is a section of the exterior square of the natural module, we can choose $\Lambda$ to be a $B_{n-1}$-diagram, a contradiction since $n \geqslant 3$.

If $K \cong Sp_4(q^2)$ and $I$ appears in $V_{nat} \otimes V_{nat}^q$, then $P \cap K/O_p(P \cap K) \cong SL_2(q^2)$ and $R_1$ is a natural $\Omega_4^-(q)$-module, a contradiction.

If $K \cong Spin_n^\epsilon(q)$, $n \geqslant 7$, we can choose $R_1$ such that $I_{R_1}$ is a natural $Spin_{n-2}^\epsilon(q)$-module. Since $I_{R_1}$ is also a natural $SL_m(q)$-module, we get $n = 8$ and $\epsilon = +$. Thus $I$ is an FF-module, contrary to (1°).

Suppose that $\overline{K} \cong F_4(q)$, ${}^2E_6(q)$ or $E_7(q)$. Then we can choose $\Lambda$ to be a $B_3$- or $C_3-$diagram (in the first two cases) or a $D_6$-diagram (in the last case), a contradiction. This completes the proof of the lemma. □

Theorem 9.12 (**Guralnick-Malle**). *Let $H$ be a finite group and $V$ a faithful simple $2F$-module for $\mathbb{F}_pH$. Suppose that $F^*(H)$ is a perfect central extension of an alternating group, but $F^*(H)$ is not a genuine group of Lie-type over a field of characteristic $p$. Put $\mathbb{F} := End_{\mathbb{F}^*(H)}(V)$, $d := \dim_{\mathbb{F}} V$, $\delta_{p|n} = 1$, if $p \mid n$ and $\delta_{p|n} = 0$ otherwise. Then one of the following holds:*

| $H$ | $d$ | $V$ | $\lvert\mathbb{F}\rvert$ |
|---|---|---|---|
| $Alt(n), Sym(n)$ | $n-1-\delta_{2\mid n}$ | natural | 2 |
| $3{\cdot}Alt(6), 3{\cdot}Sym(6)$ | 3 | ovoid | 4 |
| $Alt(7)$ | 4 | half-spin | 2 |
| $Sym(7)$ | 8 | spin | 2 |
| $Alt(9)$ | 8 | spin | 2 |
| $Alt(n), Sym(n)$ | $n-1-\delta_{3\mid n}$ | natural | 3 |
| $2{\cdot}Alt(5), 2{\cdot}Sym(5)$ | 2 | spin | 9 |
| $2{\cdot}Alt(9), 2{\cdot}Sym(9)$ | 8 | spin | 3 |

Lemma 9.13. *Suppose that $K/Z(K)$ is an alternating group. Then Theorem I holds.*

Proof. Since $K$ is quasisimple, we have $K/Z(K) \cong Alt(n)$ with $n \geqslant 5$. By 9.11 we may assume that $K/Z(K)$ is not a genuine group of Lie-Type defined over a field of characteristic $p$, and we may also assume that we are not in one of the cases treated in 9.8. We use 9.12 with $(\overline{AQ}K, I, \mathbb{K})$ in place of $(H, V, \mathbb{F})$. In particular, we have $p = 2$ or 3.

Case 1. *The case $p = 2$.*

Assume that $I$ is a natural $Alt(n)$-module. Since $I$ is also a $Q!$-module, C.23 shows that $n = 5, 6$ or 8 and $K \cong SL_2(4)$, $Sp_4(2)'$, and $SL_4(2)$, respectively. In the first and third case $K/Z(K)$ is a genuine group of Lie type in characteristic 2 contradicting our assumption. In the second case $|I| = 2^4$, and $I$ is an $FF$-module for $K$, a case which has been treated in 9.8.

If $K \sim 3{\cdot}Alt(6)$ and $|I| = 2^6$, or $K \cong Alt(7)$ and $|I| = 2^4$, then $I$ is an $FF$-module for $K$. Hence, also these cases have been treated in in 9.8.

Observe that the fourth case of 9.12 is excluded by the fact that $I$ is a simple $K$-module.

Assume that $K \cong Alt(9)$ and $I$ is the spin-module of order $2^8$. Then $\widetilde{q} = 2$ and $I$ is selfdual. Note that all involutions in $\overline{M}$ invert a 3-cycle in $K$. As the 3-cycles in $K$ act fix-point freely on $I$ we conclude $|[I, a]| \geqslant 2^4$ for all $a \in A \backslash C_A(I)$. But by 9.7(d), $|[I, a]| \leqslant \widetilde{q}^2 = 2^2$, a contradiction.

Case 2. *The case $p = 3$.*

If $I$ is the a natural $Alt(n)$-module for $K$, then again C.23 shows that $n = 6$. But then $K \cong L_2(9)$ is a genuine group of Lie-type, contrary to the assumptions.

If $K \sim 2{\cdot}Alt(5)$ and $\dim_{\mathbb{K}} I = 2$ then $A$ acts quadratically on $I$, a contradiction.

Suppose that $K \sim 2{\cdot}Alt(9)$. Then $I$ is selfdual and $\mathbb{K} = \mathbb{F}_3$. Now 9.7 shows that $\widetilde{q} = 3$ and $|[I, a]| \leqslant 9$ for all $a \in A$, a contradiction since $|[I, k]| \geqslant 3^4$ for all $k \in K$ with $|k| = 3$. (Indeed, there exists $E \leqslant K$ with $E \cong Q_8$, $Z(E) = Z(K)$ and $E = [E, k]$. Hence $Z(K) \leqslant \langle k, k^e \rangle$ for some $e \in E$ and so $3^8 = |I| = |[I, Z(K)]| \leqslant |[I, k]|^2$. Thus $|[I, k]| \geqslant 3^4$.) □

Theorem 9.14 (**Guralnick-Malle**). *Let $H$ be a finite group and $V$ a faithful simple 2F-module for $\mathbb{F}_pH$. Suppose that $F^*(H)/Z(F^*(H))$ is neither an alternating group nor a genuine group of Lie-type over a field of characteristic $p$, but $F^*(H)$ is a perfect central extension of a group of Lie-type.*

*Put* $\mathbb{F} := End_{\mathbb{F}^*(H)}(V)$ *and* $d := \dim_{\mathbb{F}} V$. *Then one of the following holds:*

| $F^*(H)$ | $d$ | $\lvert\mathbb{F}\rvert$ |
|---|---|---|
| $U_3(3)$ | 6 | 2 |
| $3{\cdot}U_4(3)$ | 6 | 4 |
| $2{\cdot}L_3(4)$ | 6 | 3 |
| $Sp_6(2)$ | 7 | 3 |
| $2{\cdot}Sp_6(2)$ | 8 | 3 |
| $2{\cdot}\Omega_8^+(2)$ | 8 | 3 |

LEMMA 9.15. *Suppose that* $K/Z(K)$ *is a group of Lie-type. Then* $K/Z(K)$ *is a genuine group of Lie type, and Theorem I holds.*

PROOF. If $K/Z(K)$ is a genuine group of Lie typ, then 9.11 shows that Theorem I holds. So assume for a contradiction that $K/Z(K)$ is not a genuine group of Lie-type defined over a field of characteristic $p$. Thus 9.14 can be applied with $(\overline{QA}K, I, \mathbb{K})$ in place of $(H, V, \mathbb{F})$. In particular, $p = 2$ or 3.

Case 1. *The case* $p = 2$.

The case $K \cong U_3(3) \cong G_2(2)'$ has been ruled out in (the proof of) 9.8.

Suppose $K \cong 3{\cdot}U_4(3)$. Then $\dim_{\mathbb{K}} I| = 6$, and by [**JLPW**] $I$ is selfdual as an $\mathbb{F}_2K$-module. Since $|Z(K)| = 3$ this contradicts 9.7(i).

Case 2. *Suppose that* $p = 3$.

In all cases we have that $|\mathbb{K}| = 3$, and by [**JLPW**] $I$ is selfdual. So 9.7 applies. In particular, $\tilde{q} = |\mathbb{K}| = 3$ and $|[I, a]| \leqslant 9$ for all $a \in A$. Let $a \in A$ with $[I, a] \neq 1$.

Suppose that $K \cong 2{\cdot}L_3(4)$ and $|I| = 3^6$. Since the diagonal automorphism of order 3 of $K/Z(K)$ does not normalize $Z(K)$, $\overline{A} \leqslant K$. Hence there exists $T \leqslant K$ with $|T| = 7$ and $T = [T, a]$. Since $I$ is selfdual, $I = [I, T]$ and so $|[I, a]| \geqslant 3^3$, a contradiction.

Suppose that $K \cong Sp_6(2)$ and $|I| = 3^7$. By [**JLPW**], $I$ is the unique simple 7-dimensional $F_3K$-module, and so $I$ is the module arising from the isomorphism $C_2 \times Sp_6(2) \cong Weyl(E_7)$, the Weyl-group of type $E_7$. Choose $T \leqslant K$ with $T \cong O_6^-(2) \cong Weyl(E_6)$. Then $T$ normalizes a 1-space in $I$. Since $T$ contains a Sylow 3-subgroup of $K$ we may assume that $\overline{Q} \leqslant T$. But then $Q!$ implies $\overline{Q} \trianglelefteq T$, a contradiction to $O_3(T) = 1$.

Suppose that $K \cong 2{\cdot}Sp_6(2)$ and $|I| = 3^8$. Then we can choose $T \leqslant K$ with $\overline{a} \in T$, $T \sim 2.(Sp_2(2) \times Sp_4(2))$ and $\overline{a} \notin O_3(T)$. It follows that there exists $E \leqslant T^\infty \sim 2{\cdot}Alt(6)$ with $E = [E, a]$, $E \cong Q_8$ and $Z(E) = Z(K)$. Thus $[|I, a]| \geqslant 3^4$, a contradiction.

Suppose that $K \cong 2{\cdot}\Omega_8^+(2)$. Then $|I| = 3^8$. By [**JLPW**], $I$ is the unique simple 8-dimensional $F_3K$-module, and so $I$ is the module arising from the isomorphism $2{\cdot}\Omega_8^+(2) \cong Weyl(E_8)'$. Since the graph automorphism of order three does not centralize $Z(K)$, $\overline{Q} \leqslant K$ and there exists $\overline{Q} \leqslant D \leqslant K$ with $D \cong C_3 \times \Omega_6^-(2) \cong Weyl(A_2 \times E_6)'$. Since $p = 3$, we see that $D$ normalizes a 1-space in $I$. So by $Q!$, $\overline{Q} = O_p(D)$, a contradiction since $\overline{Q}$ is weakly closed in $K$ and $O_p(D)$ is not. □

THEOREM 9.16 (**Guralnick-Malle**). *Let* $H$ *be a finite group and* $V$ *a faithful simple 2F-module for* $\mathbb{F}_pH$. *Suppose that* $F^*(H)$ *is a perfect central extension of a sporadic simple group. Put* $\mathbb{F} := End_{\mathbb{F}^*(H)}(V)$ *and* $d := \dim_{\mathbb{K}} V$. *Then one of the following holds:*

| $F^*(H)$ | $d$ | $\lvert\mathbb{F}\rvert$ |
|---|---|---|
| $Mat_{12}, Mat_{22}$ | 10 | 2 |
| $Mat_{23}, Mat_{24}$ | 11 | 2 |
| $3{\cdot}Mat_{22}$ | 6 | 4 |
| $Co_2$ | 22 | 2 |
| $Co_1$ | 24 | 2 |
| $Mat_{11}$ | 5 | 3 |
| $2{\cdot}Mat_{12}$ | 6 | 3 |

The cases $Co_2$ and $Co_1$ have been ruled out in [**GLM**], but again we decided to only refer to the original list.

LEMMA 9.17. *Suppose that $K/Z(K)$ is a sporadic simple group. Then Theorem I holds.*

PROOF. We can apply 9.16 with $(\overline{AQ}K, I, \mathbb{K})$ in place of $(H, V, \mathbb{F})$. In particular, $p = 2$ or 3.

Case 1. *The case $p = 3$.*

By [**JLPW**], $Mat_{11}$ has two simple 5-dimensional modules over $\mathbb{F}_3$. Also $2^{\cdot}Mat_{12}$ has two simple 6-dimensional modules over $\mathbb{F}_3$ interchanged by the outer automorphism of $2^{\cdot}Mat_{12}$. Thus either $K \cong Mat_{11}$ and $I$ is the simple Todd or Golay code module, or $K \cong 2^{\cdot}Mat_{12}$ and $I$ is the simple Golay code module. Note that $K$ has no outer automorphism of order 3, and so $\overline{M^\circ} = K$. We need to rule out the case where $K \cong Mat_{11}$ and $I$ is Todd-module. Then $Mat_{11}$ has an orbit of length 11 on the 1-spaces in $I$. Hence $Mat_{10}$ normalizes a 1-space in $I$, but this contradicts $Q!$, since $O_3(Mat_{10}) = 1$ and $Mat_{10}$ contains a Sylow 3-subgroup of $Mat_{11}$.

Case 2. *The case $p = 2$.*

Let $Z := C_I(S)$ and $R := C_{\overline{M}}(Z)$. Then by $Q!$, $\overline{Q} \leqslant O_p(R)$.

Suppose first that $K \cong Mat_{24}$. By [**JLPW**], $Mat_{24}$ has two simple 11-dimensional modules over $\mathbb{F}_2$. Thus, $I$ is the simple Todd or Golay code module. Since $Out(Mat_{24}) = 1$, we get that $\overline{M} = \overline{M^\circ} = K$, and Theorem I(9) holds.

Suppose next that $K \cong Mat_{22}$. By [**JLPW**], $Mat_{22}$ has two simple 10-dimensional modules over $\mathbb{F}_2$. Thus, $I$ is the simple Todd or Golay code module. Also $\overline{M^\circ} = K$ or $\overline{M} = \overline{M^\circ} \cong Aut(Mat_{22})$. Assume that $I$ is the Golay-code module. Then $R \sim 2^4 Alt(6)$ or $2^4 Sym(6)$ with $O_2(R) \leqslant K$, so $\overline{Q} \leqslant K$. Hence $\overline{M^\circ} = K \cong Mat_{22}$ and Theorem I(10) holds.

Assume that $I$ is the Todd module. If $\overline{M^\circ} \cong Aut(Mat_{22})$, then Theorem I(11) holds. So suppose that $\overline{M^\circ} = K$. Then $R \sim 2^4 Sym(5)$ and there exists $F \leqslant \overline{M^\circ}$ with $F \cong L_3(4)$, $O_2(R) \leqslant F$ and $C_I(F) \neq 1$. Since $Q \leqslant O_2(R) \leqslant F$ and $O_2(F) = 1$, we get a contradiction to $Q!$.

It remains to rule out the cases $K \cong Mat_{12}$, $3^{\cdot}Mat_{22}$, $Mat_{23}$, $Co_2$ and $Co_1$ in 9.16.

Suppose that $K \cong Mat_{12}$. By [**JLPW**], $Mat_{12}$ has a unique simple 10-dimensional modules over $\mathbb{F}_2$. Hence, $I$ is the non-central simple section of a natural permutation module on 12 letters. In particular, $I$ is selfdual and $|\mathbb{K}| = 2$. Thus by 9.7(j), $|[V, a]| \leqslant 4$, a contradiction, since no involution fixes more than 4 of the 12 letters.

Suppose that $K \cong 3^{\cdot}Mat_{22}$. By [**JLPW**], any 6-dimensional simple $3^{\cdot}Mat_{22}$ module over $\mathbb{F}_4$ is selfdual as an $\mathbb{F}_2K$-module. As $|Z(K)| = 3$, this contradicts 9.7(i).

Suppose that $K \cong Mat_{23}$. By [**JLPW**], $Mat_{23}$ has two simple 11-dimensional modules over $\mathbb{F}_2$. Thus, $I$ is the simple Todd or Golay code module of $\mathbb{F}_2$-dimension 11. Since $Out(K) = 1$ we have $\overline{M} = K$. If $I$ is the Todd-module, then there exists $\overline{Q} \leqslant E \leqslant K$ with $E \cong Mat_{22}$ and $C_I(E) \neq 1$, a contradiction to $Q!$. Thus $I$ is the Golay code module and so $R \sim 2^4 Alt(7)$, $\overline{Q} = O_2(R)$ is elementary abelian of order $2^4$, and $|C_I(Q)| = 2$. Suppose that $\overline{A} \leqslant \overline{Q}$. Since $A$ is an $2F$-offender and $R$ acts simply on $O_2(R)$, A.29(a) implies that $\overline{Q}$ is a $2F$-offender. But $|I/C_I(Q)| = 2^{10} > (2^4)^2 = |\overline{Q}|^2$, a contradiction.

Hence $\overline{A} \nleqslant \overline{Q}$. Let $\Omega$ be a set of size 23 with $\overline{M} \cong Mat_{23}$ acting faithfully on $\Omega$. Then $R = N_{\overline{M}}(\Theta)$ for some $\Theta \subseteq \Omega$ with $|\Theta| = 7$. Let $\Lambda \subseteq \Theta$ with $|\Lambda| = 3$ and put $R_1 = N_{\overline{M}}(\Lambda)$. Then $R_1/C_{R_1}(\Lambda) \cong Sym(3)$ and $C_{R_1}(\Lambda) \cong Mat_{20} \sim 2^4 SL_2(4)$, where $O_2(R_1)$ is a natural $SL_2(4)$-module for $C_{R_1}(\Lambda)/O_2(R_1)$. Also $\overline{Q} = C_{\overline{M}}(\Theta) \leqslant C_{R_1}(\Lambda)$ and so $R_1^\circ = C_{R_1}(\Lambda)$.

Since $R$ induces $Alt(7)$ on $\Theta$, $R_1 \cap R/C_{R_1 \cap R}(\Lambda) \cong Sym(3)$ and $C_{R_1 \cap R}(\Lambda)/\overline{Q} \cong Alt(4)$. Thus $|R_1 \cap R| = 2^7 3^2$, $|R/R \cap R_1| = 5 = |R_1^\circ / R_1^\circ \cap R|$ and $O_2(R_1 \cap R) \in Syl_2(R_1^\circ)$. Note also that $O_2(R_1 \cap R_2)/\overline{Q}$ corresponds to $\langle (12)(34), (14)(23) \rangle$ in $Alt(7)$ and so by [**MS5**, 7.5] is, up to conjugacy, the unique maximal quadratically acting subgroup of $R/\overline{Q}$ on $\overline{Q}$.

Since $\overline{Q}$ normalizes $\overline{A}$ by 8.5(b) and $\overline{A}$ is elementary abelian, we conclude that $\overline{A}$ acts quadratically on $\overline{Q}$. Hence $\overline{A}$ is contained in an $R$-conjugate of $O_2(R \cap R_1)$. So we may choose $\Lambda$ such that $\overline{A} \leqslant O_2(R \cap R_1)$. As seen above, $O_2(R \cap R_1) \in Syl_2(R_1^\circ)$ and $O_2(R_1)$ is a natural $SL_2(4)$-module for $R_1^\circ$. Hence $O_2(R \cap R_1) = \overline{Q} O_2(R_1)$, and $\overline{Q}$ and $O_2(R_1)$ are the only maximal elementary abelian subgroups of $O_2(R \cap R_1)$, so $\overline{A} \leqslant O_2(R_1)$ since $\overline{A} \nleqslant \overline{Q}$. Thus 9.6 implies that $R_1^\circ \leqslant N_{R_1}(\overline{A})$. Since $R_1^\circ$ acts simply on $O_2(R_1)$ this gives $\overline{A} = O_2(R_1)$,

Put $U := \langle Z^{R_1} \rangle$. Since $R$ centralizes $Z$ and $|R/R \cap R_1| = 5$, $U$ is a quotient of the $Sym(5)$-permutation module for $R_1$. Note that $C_U(R_1) = 1$ by $Q$! and that the permutation module is the direct sum of simple submodules of order 2 and $2^4$. Thus $|U| = 2^4$.

Since $A$ is not an offender on $I$, $|I/C_I(A)| > |\overline{A}| = 2^4$ and so $|C_I(A)| \leqslant \frac{2^{11}}{2^5} = 2^6$. Note that $U \leqslant C_I(A)$ since $\overline{A} = O_2(R_1)$. Thus $|C_I(A)/U| \leqslant \frac{2^6}{2^4} = 2^2$. Since $R_1^\circ$ is perfect, this gives $[C_I(A), R_1^\circ] = U$. Observe that $H^1(U, R_1^\circ/A) = 1$ (for example by C.18) and since $C_I(R_1^\circ) = 0$ we get $U = C_I(A)$, so $|C_I(A)| = 2^4$.

Since $I$ is not an $FF$-module, 9.5(a) shows that $A$ is an offender on $I \cap A$ and therefore $|I \cap A/C_I(A)| \leqslant |\overline{A}| = 2^4$. Thus $|I \cap A| \leqslant 2^8$ and $\widetilde{q} = |I/I \cap A| \geqslant 2^3$. Note that $\overline{A} = A/C_A(Y)$ and $I/C_I(A)$ both are $N_L(Y)$-invariant sections of $AY/C_Y(L)$. Thus by 2.18(c), both, $|\overline{A}| = 2^4$ and $|I/C_I(A)| = 2^7$, are powers of $\widetilde{q}$. But then $\widetilde{q} = 2$, a contradiction to $\widetilde{q} \geqslant 8$.

Suppose that $\overline{K} \cong Co_2$ or $Co_1$. By [**SW**], $Co_2$ has a unique simple 22-dimensional module over $\mathbb{F}_2$, and by [**Gr2**], $Co_1$ has a unique simple 24-dimensional module over $\mathbb{F}_2$. Hence $I$ is selfdual and isomorphic to the non-central simple section of the Leech-lattice modulo 2. Also $\widetilde{q} = |\mathbb{K}| = 2$. Thus by 9.7(j), $|[I, a]| \leqslant 4$ for all $a \in A$. But the commutator space of any involution in $Co_1$ on the Leech-lattice modulo 2 is at least 8-dimensional, and since $2^{24}/|I| \leqslant 2^2$, we conclude that $|[I, a]| \geqslant 2^8/2^2 = 2^6$, a contradiction. □

## 9.3. The Proof of Corollary 9.1

In this section we will proof Corollary 9.1. So we continue to assume the hypothesis of Theorem I and use the notation introduced in 9.3.

LEMMA 9.18. *Suppose that* $Y = IC_Y(\overline{S} \cap K)$. *Then* $Y = I$. *In particular,* $Y = I$ *if* $\overline{A} \in Syl_p(K\overline{A})$.

PROOF. By 9.3(c) $[Y, K] \leqslant I$ and, by $Q$!, $C_Y(K) = 1$. As $\overline{S} \cap K \in Syl_p(K)$, Gaschütz' Theorem gives $Y = IC_Y(\overline{S} \cap K) = C_Y(K)I = I$, see C.17.

Note that by 9.3(f) $Y = IC_Y(\overline{A})$. Thus if $\overline{A} \in Syl_p(K\overline{A})$ then $\overline{A} \cap K = \overline{S} \cap K$ and $Y = IC_Y(\overline{S} \cap K)$, and so $Y = I$. □

9.19. **Proof of Corollary 9.1:**

Suppose that $Y \neq I$. By $Q$!, $C_Y(K) = 1$, and by 9.3(b),(c) $[Y, K] = I$, so $|Y/I| \leqslant |H^1(K, I)|$. Comparing Theorem I with C.18 we obtain one of the following cases:

(A) $\overline{M} \cong L_3(2)$, $|Y| = 2^4$ and $I$ is a natural $SL_3(2)$-module for $\overline{M}$.
(B) $\overline{M^\circ} \cong Sp_{2n}(q)$ or $Sp_4(2)'$, $p = 2$, $I$ is the corresponding natural module and $|Y/I| \leqslant q$.
(C) $\overline{M^\circ} \cong \Omega_3(5), \Omega_4^-(3), \Omega_5(3)$ or $\Omega_6^+(2)$, $I$ is the corresponding natural module, and $|Y/I| \leqslant 5, 9, 3$ and 2 respectively.
(D) $\overline{M^\circ} \cong L_3(4)$, $I$ is the unitary square of corresponding natural module and $|Y/I| \leqslant 4$.
(E) $p = 2$, $\overline{M^\circ} \cong Mat_{24}$, $I$ is the simple Todd-module of $\mathbb{F}_2$-dimension 11 and $|Y/I| = 2$.
(F) $p = 2$, $\overline{M^\circ} \cong Mat_{22}$, $I$ is the simple Golay code module of $\mathbb{F}_2$-dimension 10 and $|Y/I| = 2$.
(G) $p = 3$, $\overline{M^\circ} \cong Mat_{11}$, $I$ is the simple of Golay code module of $\mathbb{F}_3$-dimension 5 and $|Y/I| = 3$.

It remains to treat each of these seven cases. Recall first that by 9.3(f) $Y = IC_Y(A)$ and so we can pick $t \in Y \backslash C_Y(A)$.

In Case (A) $I$ is a natural $SL_3(2)$-module for $\overline{M}$ and so $C_{\overline{M}}(t) \cong Frob(21)$ has odd order, a contradiction since $\overline{A} \leqslant C_{\overline{M}}(t)$.

In Case (B), $I$ is a natural $Sp_{2n}(2)$- or $Sp_4(2)'$-module for $\overline{M^\circ}$ and so Corollary 9.1(1) holds.

Suppose that Case (C) holds with $\overline{M^\circ} \cong \Omega_3(5)$. By B.35(d) we conclude that $\overline{A} \leqslant \overline{M^\circ}$. Thus $\overline{A} \in Syl_3(\overline{M^\circ})$ and 9.18 gives $Y = I$, contradiction.

Suppose that Case (C) holds with $\overline{M^\circ} \cong \Omega_4^-(3) \cong Alt(6)$. Again B.35(d) gives $\overline{A} \leqslant \overline{M^\circ}$. Let $W$ be an $\mathbb{F}_3\overline{M^\circ}$-module with $Y \leqslant W$, $C_W(\overline{M^\circ}) = 1$ and $|W/I| = 3^2$. Let $X_1$ and $X_2$ be non-conjugate subgroups of $\overline{M^\circ}$ with $X_i \cong Alt(5)$. Choose notation such that $\overline{A_1} := \overline{A} \cap X_1 \neq 1$. For $i = 1, 2$, put $W_i := C_W(X_i)I$ and note that $W_i$ is a $\overline{M^\circ}$-module isomorphic to the 5-dimensional quotient of permutation module $\mathbb{F}_3^{\overline{M^\circ}/X_i}$. Then $W_1 = IC_{W_1}(\overline{A_1})$ and, since $\overline{A_1}$ acts fixed-point freely on $\overline{M^\circ}/X_2$, $C_{W_2}(\overline{A_1})) \leqslant I$. It follows that $IC_W(\overline{A}_1) = W_1$. As $Y = IC_Y(A) \leqslant IC_W(\overline{A_1})$ this gives $Y = W_1$. Thus 9.1(2) holds in this case.

Suppose that Case (C) holds with $\overline{M^\circ} \cong \Omega_5(3)$ or $\Omega_6^+(2)$. Then $I$ is the corresponding natural module, $|Y/I| \leqslant 3$ and 2, respectively, and 9.1(3) or 4 holds.

Suppose that Case (D) holds. Then $I$ is the unitary square of a natural $SL_3(4)$-module for $K \cong L_3(4)$. Since $I$ is not an FF-module, we can apply 9.9 and conclude that $\overline{A} \leqslant K$. Let $P_1$ and $P_2$ be the two maximal subgroups of $K$ containing $\overline{S} \cap K$ such that $C_I(K \cap \overline{S}) \trianglelefteq P_1$, and let $Q_i := O_p(P_i)$. Then $C_I(Q_1) = C_I(K \cap \overline{S})$ has order 2, and so $|I/C_I(Q_1))| = 2^8 = |Q_1|^2$. By $Q$!, $\overline{Q} \trianglelefteq P_1$ and the simple action of $P_1$ on $Q_1$ implies $\overline{Q} = Q_1$.

Suppose that $\overline{A} \leqslant Q_1$. Since $|Y/C_Y(A)| \leqslant |\overline{A}|^2$, A.29 shows that $|Y/C_Y(B)| \leqslant |B|^2$ for some non-trivial $P_1$-invariant subgroup $B$ of $Q_1$. As $P_1$ acts simply on $Q_1$ we get $B = Q_1$ and

$$|Y/C_Y(Q_1)| \leqslant |Q_1|^2 = |I/C_I(Q_1)|.$$

Hence $Y = C_Y(Q_1)I$. Since $P_1$ is perfect and $[C_Y(Q_1), P_1, P_1] \leqslant [C_I(Q_1), P_1] = 1$, we get $C_Y(Q_1) \leqslant C_Y(P_1)$ and $Y = IC_Y(P_1)$. Since $\overline{S} \cap K \leqslant P_1$, 9.18 gives $I = Y$, a contradiction.

Suppose now that $\overline{A} \nleqslant Q_1$. Since $Q_1$ and $Q_2$ are the only maximal elementary abelian subgroups of $\overline{K} \cap S$, $\overline{A} \leqslant Q_2$. Thus 9.6 shows that $P_2^\circ$ $(:= \langle \overline{Q}^{P_2} \rangle)$ normalizes $\overline{A}$. As $\overline{Q} \nleqslant Q_2$, $P_2 = P_2^\circ Q_2$, and as $Q_2$ is a simple $P_2$-module, $\overline{A} = Q_2$ and so $Y = C_Y(Q_2)I$. Since $C_I(Q_2)$ is a natural $Alt(5)$-module for $P_2$, $H^1(C_I(P_2), P_2/Q_2) = 1$ (seeC.18) and so $C_Y(Q_2) = C_I(Q_2)C_Y(P_2)$ and $Y = IC_Y(P_2)$, again a contradiction to 9.18 since $\overline{S} \cap K \leqslant P_2$.

Suppose that Case (E) holds. Then $\overline{M^\circ} \cong Mat_{24}$, $I$ is the simple Todd-module, and $|Y/I| \leqslant 2$. So 9.1(5) holds.

Suppose that Case (F) holds. Then $\overline{M} \cong Mat_{22}$ or $Aut(Mat_{22})$, and $I$ is the simple Golay code module. Hence $Y$ is isomorphic to the restriction of the 11 dimensional simple Golay-code module for $Mat_{24}$ to $\overline{M}$. Let $(\Omega, \mathcal{B})$ be a Steiner system of type $(24, 8, 5)$, $H := Aut(\Omega, \mathcal{B}) = Mat_{24}$, $T \subseteq \Omega$ with $|T| = 2$. Then $N_H(T) \cong Aut(Mat_{22})$. Let $V$ be the simple Golay code module for $H$. Then $H$ has two orbits on $V^\sharp$, one orbit corresponding to the octads in $\Omega$ and the other to the partitions of $\Omega$ into two dodecads. Also $[V, N_H(T)]^\sharp$ consists of all elements in $V$ 'perpendicular' to $T$, that is, the elements corresponding to octads and pairs of dodecads, each intersecting $T$ in a subset of even size. So $V \backslash [V, N_H(T)]$ consists of all octads and pairs of dodecads intersecting $T$ in exactly one element.

Let $B$ be an octad with $|B \cap T| = 1$. Then $N_H(B) \sim 2^4 Alt(8)$ induces $Alt(8)$ on $B$ while $C_H(B)$ acts regularly on $\Omega \backslash B$. Thus $N_H(B) \cap N_H(T) \cong Alt(7)$. Let $\{C, D\}$ be a partition of $\Omega$ into two dodecads with $|C \cap T| = |D \cap T| = 1$. Then $N_H(C) \cong Mat_{12}$ acts transitively on $C$, and $N_H(C) \cap N_H(C \cap T) \cong Mat_{11}$ acts transitively on $D$ with point-stabilizer $L_2(11)$. Hence $N_H(T) \cap N_H(\{C, D\}) \cong Aut(L_2(11))$.

It follows that $C_{\overline{M}}(t)$ is isomorphic to a subgroup of index at most two of $Alt(7)$ or $Aut(L_2(11))$. In particular, $C_{\overline{M}}(t)$ has dihedral Sylow 2 subgroups. Since $\overline{A}$ is elementary abelian, we conclude that $|\overline{A}| \leqslant 4$. Thus $|I/C_I(A)| \leqslant |\overline{A}|^2 \leqslant 2^4$. Since $A$ does not act quadratically, $C_I(A) \neq C_I(a)$ for some $1 \neq a \in \overline{A}$ and so $|I/C_I(a)| \leqslant 2^3$. But this is a contradiction, for example, since each involution in $C_{\overline{M}}(t)$ inverts an elements of order 5, and all elements of order five in $\overline{M^\circ}$ have an 8-dimensional commutator on $I$.

Suppose that Case (G) holds. Then $\overline{M^\circ} \cong Mat_{11}$ and $I$ is the simple Golay code module. Since $Out(Mat_{11}) = 1$ and $C_M(I) = C_M(Y)$, we get $O^{3'}(\overline{M}) = K$. By 9.18 $\overline{A}$ is not a Sylow 3-subgroup of $K$. Thus $|\overline{A}| = 3$ and so $|I/C_I(A)| \leqslant |\overline{A}|^2 = 9$. Let $R \leqslant K$ with $R \cong Mat_{10} \sim Alt(6).2$ and $\overline{A} \leqslant R$, and let $g \in R \backslash R'$. Then $\overline{A}$ and $\overline{A}^g$ are not conjugate in $R'$ and we choose $g$ such that $\overline{A}$ and

$\overline{A}^g$ correspond to $\langle(123)\rangle$ and $\langle(124)(356)\rangle$ in $Alt(6)$. Then $R' = \langle\overline{A}, \overline{A}^g\rangle$. Thus $|I/C_I(R')| \leqslant 3^4$ and so $|C_I(R')| = 3$, a contradiction since $I$ is the Golay-code module (or to $Q!$).

## 9.4. The Proof of Corollary 9.2

In this section we will proof Corollary 9.2. For this we continue to assume the hypothesis of Theorem I and use the notation introduced in 9.3. In addition, we assume

(*char* $Y_M$) $\qquad C_G(y)$ is of characteristic $p$ for all $y \in Y^\sharp$.

The following lemma is crucial for the proof of the corollary. Exactly here property (*char* $Y_M$) is used.

Lemma 9.20. *Suppose that property (char $Y_M$) holds. Then $O_p(N_{\overline{M}}(B)) \neq 1$ for all $1 \neq B \leqslant C_Y(A)$.*

Proof. Suppose that $O_p(N_{\overline{M}}(B)) = 1$ for some $1 \neq B \leqslant C_Y(A)$. Note that $O_p(M)$ normalizes $O_p(N_G(B))$ and that by 2.6(b) $O_p(M)$ is weakly closed in $G$. Hence $O_p(N_G(B)) \leqslant N_G(O_p(M)) \leqslant M^\dagger$ and so $\overline{O_p(N_G(B))} \leqslant O_p(N_{\overline{M}}(B)) = 1$. Thus $[Y_M, O_p(N_G(B))] = 1$.

By (*char* $Y_M$) $C_G(b)$ is of characteristic $p$ for $1 \neq b \in B$ [2] and so by 1.2(c) $C_G(b)$ is of local characteristic $p$. It follows that $C_G(B)$ has characteristic $p$. In particular,

$$Y_M \leqslant C_G(O_p(N_G(B))) \leqslant C_G(O_p(C_G(B))) \leqslant O_p(C_G(B)) \leqslant O_p(N_G(B)).$$

On the other hand, by 9.3(f) $B \leqslant Z(A) \leqslant Z(L)$ and so $L \leqslant N_G(B)$. This contradicts $Y_M \nleqslant O_p(L)$. □

Lemma 9.21. *Suppose that property (char $Y_M$) holds. Then Case (11) of Theorem I does not occur.*

Proof. Suppose that Case (11) of Theorem I holds. Then $p = 2$, $\overline{M^\dagger} = \overline{M} = \overline{M^\circ} \cong Aut(Mat_{22})$, and $I$ is the Todd module of order $2^{10}$. Choose a set $\Omega$ of size 22 with $\overline{M}$ acting faithfully and 4-transitively on $\Omega$. Let $\alpha, \beta \in \Omega$ with $\alpha \neq \beta$. Since $\overline{M}$ acts 4-transitively in $\Omega$, $F := C_{\overline{M}}(\alpha)$ and $P := N_{\overline{M}}(\{a, \beta\})$ are maximal subgroups of $\overline{M}$. Since $I$ is the Todd-module there exists $1 \neq x \in C_I(F)$. Define $\{x, y\} := x^P$ and $z := xy$. By the maximality of $F$ and $P$, $F = C_{\overline{M}}(x)$, $x \neq y$, and $P = C_M(z)$.

Note that

$$F \cap K \cong Mat_{21} \cong L_3(4),\ P \sim 2^{4+1}\Gamma SL_2(4),\ P' = P \cap K \cap F \cong Mat_{20} \sim 2^4 SL_2(4).$$

Moreover, $O_2(P')$ is a natural $SL_2(4)$-module for $P'$, and $Z(F) = Z(P) = 1$. Since $|F/F \cap K| = 2$ we conclude that $O_2(F) = 1$. Also $|\overline{M}/P| = \binom{22}{2} = 21 \cdot 11$, and so $P$ is a parabolic subgroup of $\overline{M}$. Thus we may assume that $\overline{S} \leqslant P$. Then $z \leqslant C_I(\overline{S}) \leqslant C_I(A)$ and so by 9.3(f), $z \in C_I(L)$.

If $x \in C_I(A)$ then by 9.20 $O_2(C_{\overline{M}}(x)) \neq 1$, which contradicts $O_2(F) = 1$. Thus

1°. $\quad$ *$x \notin C_I(A)$ and $\overline{A} \nleqslant F$.*

Since $z \in C_I(L)$, we have $\overline{N_L(Y)} \leqslant C_{\overline{M}}(z) = P$. It follows that $\langle x, y\rangle/\langle z\rangle$ is a composition factor for $N_L(Y)$ on $AY/C_Y(L)$ of order 2. On the other hand by 2.18(b) any such composition factor has order $\tilde{q}$. Hence $\tilde{q} = 2$ and so $|I/I \cap A| = 2$. Since by 9.5(b), $I \cap A = [I, A]C_I(A)$, this shows that

2°. $\quad |I/[I, A]C_I(A)| = 2.$

As $Q$ centralizes $z$, $Q!$ implies that $\overline{Q} \trianglelefteq P$. Since $\overline{M^\circ} \neq K$ we have $\overline{Q} \nleqslant K$. Also $P$ acts simply on $R := O_2(P')$, $|O_2(P)/R| = 2$ and $Z(P) = 1$. Thus $\overline{Q} = O_2(P)$. It follows that (see for example [**MSt**, Theorem 3])

3°.

(a) $C_I(Q) = [I, Q, Q] = \langle z\rangle$,
(b) $[I, Q]/\langle x, y\rangle$ *is a natural $SL_2(4)$-module for $P'$,*

[2]This is the only place in the proof of Corollary 9.2 where (*char* $Y_M$) is used.

(c) $\widetilde{I} := I/[I,Q]$ *is a natural* $Sym(5)$*-module for* $P$*. In particular,* $\widetilde{I}$ *is a selfdual* $P$*-module.*

We claim that $\overline{A} \cap R \neq 1$. If $\overline{A} \nleqslant \overline{Q}$, then $1 \neq [\overline{Q}, \overline{A}] \leqslant \overline{A} \cap R$. So suppose that $\overline{A} \leqslant \overline{Q}$. Since $A$ does not act quadratically on $I$, $|\overline{A}| \geqslant 4$. As $|\overline{Q}/R| = 2$ this gives $\overline{A} \cap R \neq 1$.

So we can choose $a \in A$ with $1 \neq \overline{a} \leqslant R \leqslant K$. Since all involutions in $K$ are conjugate, $\overline{a}^g \in P'\backslash O_2(P)$ for some $g \in K$. By (3°) $P'$ has two non-central composition factors on $I$ and so $|I/C_I(a)| \geqslant 2^4$ and $|C_I(a)| \leqslant 2^6$. Since $\overline{a} \in R \leqslant \overline{Q}$, (3°)(a) gives $[[I,Q],a] \leqslant \langle z \rangle$ and thus $|C_I(a) \cap [I,Q]| \geqslant 2^5$. Hence $|C_I(a)[I,Q]/[I,Q]| \leqslant 2$ and so $|\widetilde{C_I(a)}| \leqslant 2$.

By (2°) $[\widetilde{I}, A]\widetilde{C_I(A)}$ has index at most 2 in $\widetilde{I}$. Suppose that $A$ acts quadratically on $\widetilde{I}$. Then $\widetilde{C_I(A)}[\widetilde{I}, A] \leqslant C_{\widetilde{I}}(A)$ and so $|\widetilde{I}/C_{\widetilde{I}}(A)| \leqslant 2$. As by (3°)(c) $\widetilde{I}$ is selfdual this gives $|[\widetilde{I}, A]| \leqslant 2$ and so $\widetilde{C_I(A)}[\widetilde{I}, A]$ has order at most 4, a contradiction. Hence $A$ does not act quadratically on $\widetilde{I}$. Note that the only elementary abelian subgroups of $Sym(5)$, which do not act quadratically on the natural $Sym(5)$-module, are the Sylow 2-subgroups of $Alt(5)$. Thus $T := \overline{A}O_2(P) \in Syl_2(P'O_2(P))$.

As $Z(P) = 1$, Gaschütz' Theorem gives $C_{O_2(P))}(T) \leqslant R$, see C.17. Since $R$ is a natural $SL_2(4)$-module for $P'$, $T \cap P'$ has exactly two maximal elementary abelian subgroups, namely $R$ and say $R^*$. Moreover, $RR^* = T \cap P'$ and so $C_{O_2(P)}(R^*) \leqslant R$. Since $|\overline{A}O_2(P)/O_2(P)| = 4$ and $|P'O_2(P)/P'| = 2$ we can choose $b \in \overline{A} \cap P'\backslash O_2(P)$. Then $b \in R^*$. Also $[O_2(P), b] \leqslant C_R(b) = [R, b] = R \cap R^*$ has order 4 and so $B := C_{O_2(P)}(b) \nleqslant R$. In particular, since $C_{O_2(P)}(R^*) \leqslant R$, $[B, R^*] \neq 1$. It follows that $C_T(b) = BR^*$ and $C_T(b)$ has exactly two maximal elementary abelian subgroups, namely $B\langle b \rangle$ and $R^*$. As $\overline{A} \leqslant C_T(b)$ and $\overline{A}O_2(P) = T$ this gives $A \leqslant R^* \leqslant P'$. Since $P' = P \cap K \cap F$, we have $\overline{A} \leqslant F$, a contradiction to (1°). □

LEMMA 9.22. *Suppose that property* ($char\ Y_M$) *holds. Then (13) of Theorem I does not occur.*

PROOF. Suppose that $p = 3$, $\overline{M^\circ} \sim 2{\cdot}Mat_{12}$ and $I$ is the simple Golay code module of order $3^6$. Observe that there exists a subgroup $P$ of $\overline{M^\circ}$ with $\overline{A} \leqslant P$ such that $P \sim 3^2SL_2(3)$, $C_I(O_3(P))$ is a natural $\Omega_3(3)$-module, and $[I, O_3(P)]/C_I(O_3(P))$ is a natural $SL_2(3)$-module for $P$. If $\overline{A} \leqslant O_3(P)$, then we can choose $1 \neq x \in C_I(O_3(P)) \leqslant C_I(A)$ with $C_{\overline{M}}(x) \cong Mat_{11}$, a contradiction to 9.20. Thus $\overline{A} \nleqslant O_3(P)$. Let $1 \neq e \in \overline{A}$. Then $e^g \in P\backslash O_3(P)$ for some $g \in \overline{M}$. In particular, $e^g$ acts non-trivially on $C_I(O_3(P))$ and $[I, Q]/C_I(O_3(P))$. Hence $|[I, e]| \geqslant 3^3$ and $|C_I(e)| \leqslant 3^3$. Since $|I/C_I(A)| \leqslant |\overline{A}|^2$ by 9.3(c), this gives $\overline{A} \neq \langle e \rangle$. Hence, as $\overline{A}$ is abelian, $|\overline{A}| = 3^2$ and $|\overline{A} \cap O_3(P)| = 3$. Since

$$3^3 = |C_I(O_3(P))| \leqslant |C_I(\overline{A} \cap O_3(P))| \leqslant |C_I(e)| \leqslant 3^3,$$

$C_I(\overline{A} \cap O_3(P)) = C_I(O_3(P))$ and so $|C_I(A)| = |C_I(O_3(P)A)| = 3$. But this contradicts $|I/C_I(A)| \leqslant |\overline{A}|^2 = 3^4$. □

9.23. **Proof of Corollary 9.2:**

In view of 9.21 and 9.22 it remains to show that $Y = I$. Hence, we assume property ($char\ Y_M$) and $Y \neq I$ and discuss the five cases of Corollary 9.1. By 9.3(f) $Y = IC_Y(A)$, so we can pick $t \in C_Y(A)\backslash I$.

Suppose that case 9.1(1) holds. Then $p = 2$ and $I$ is a natural $Sp_{2n}(q)$- or $Sp_4(2)'$-module for $\overline{M^\circ}$. In the first case $C_{\overline{M^\circ}}(t) \cong O^\epsilon_{2n}(q)$ and the second case $C_{\overline{M^\circ}}(t) \sim 3^2.C_4$ or $\Omega^-_4(2)$. In either case we conclude that $C_{\overline{M^\circ}}(t)$ and so also $C_{\overline{M}}(t)$ acts simply on $I$. Thus $O_2(C_{\overline{M}}(t))$ centralizes $I$, and since $C_M(Y) = C_M(I)$, we get $O_2(C_{\overline{M}}(t)) = 1$, a contradiction to 9.20.

Suppose that case 9.1(2) or (3) holds. Then $I$ is a natural $\Omega^-_4(3)$- or $\Omega_5(3)$-module for $K$. Thus B.35(d) shows that $\overline{S} \leqslant K$. Let $x$ be a non-singular vector in $I$. Then $O_3(C_{\overline{M}}(x)) \leqslant K$ and $C_K(x) \cong \Omega^\epsilon_{n-1}(3)$, where $n = \dim_{\mathbb{F}_3}(I)$. Thus $O_3(C_{\overline{M}}(x)) = 1$, and 9.20 shows that $x \notin C_I(A)$. Hence $C_I(A)$ does not contain any non-singular vectors. Since 3 is odd we conclude that $C_I(A)$ is singular.

Put $D := [I, A] \cap C_I(A)$. Observe that $\mathbb{K} := End_K(I) \cong \mathbb{F}_3$, and let $X$ be a $\mathbb{K}$-subspace of $C_I(A)$ with $C_I(A) = D \times X$. Since $C_I(A)$ is singular, $X \leqslant X^\perp$. On the other hand, $I$ is a selfdual $K$-module, and so 9.7(g) gives $X \cap X^\perp = 1$. Thus $X = 1$ and $D = C_I(A)$. By 9.7(c), $|D| = 3$.

Hence $C_K(D^\perp) = 1$, and by 9.7(g), $\overline{A} = \overline{A}C_K(D^\perp) = C_K(D^\perp/D) \cap C_K(D)$. If $K \cong \Omega_4^-(3)$ this gives $\overline{A} \in Syl_3(K)$, a contradiction to 9.18.

Thus $K \cong \Omega_5(3)$. Let $E$ be 2-dimensional subspace of $I$ with $D \leqslant E$ and $E \nleqslant D^\perp$. Then $C_K(E) \cong \Omega_3(3)$ is a complement to $\overline{A}$ in $C_K(D)$. Let $g \in K$ with $D^g \leqslant E^\perp$ and put $B := C_K(E) \cap C_K(D^g) \cap C_K(E^\perp/D^g)$ . Then $B$ is a Sylow 3-subgroup of $C_K(E)$, $B \leqslant \overline{A}^g$ and $\overline{A}B \in Syl_3(K)$. Also $[I, B] \leqslant E^\perp$ and $[I, B] \cap D = 1$. Since $Y = C_Y(A)I = C_Y(\overline{A}^g)I$ we have $[Y, A] = [I, A]$ and $[Y, B] = [I, B]$. Thus

$$[C_Y(A), \overline{A}B] \leqslant [Y, B] \cap C_I(A) = [I, B] \cap D = 1,$$

and so $Y = IC_Y(\overline{A}B)$. Since $\overline{A}B \in Syl_3(K)$, 9.18 shows that $Y = I$, a contradiction.

Suppose that case 9.1(4) holds. Then $I$ is a natural $\Omega_6^+(2)$-module for $K \cong \Omega_6^+(2) \cong Alt(8)$. Hence $Y$ is the central quotient of the permutation module on eight objects, and so $C_{\overline{M}}(t)$ is isomorphic to a subgroup of index at most two of $Sym(7)$ or $Sym(3) \times Sym(5)$. It follows that $O_2(C_{\overline{M}}(t)) = 1$, a contradiction to 9.20.

Suppose that case 9.1(5) holds. Then $\overline{M^\circ} \cong Mat_{24}$ and $I$ is the simple Todd-module. It follows that $\overline{M} = K$ and $Y$ is the quotient of the 24-dimensional permutation module by the Golay-code module. Hence $C_K(t)$ is isomorphic to $Mat_{23}$ or $L_3(4).Sym(3)$. So $O_2(C_{\overline{M}}(t)) = 1$, a contradiction to 9.20.

CHAPTER 10

# Proof of the Local Structure Theorem

In this chapter we prove the Local Structure Theorem and its corollary stated in the introduction. But before doing this we prove the Structure Theorem for Maximal Local Parabolic Subgroups, which combines the theorems proved in Chapters 4 – 9 into one.

THEOREM J (Structure Theorem for Maximal Local Parabolic Subgroups). *Let $G$ be a finite $\mathcal{K}_p$-group and $S \in Syl_p(G)$. Suppose that $|\mathcal{M}_G(S)| > 1$ and there exists a large subgroup $Q$ of $G$ in $S$. Then there exists $M \in \mathfrak{M}_G(S)$ with $Q \ntrianglelefteq M$. Moreover, for every $M \in \mathfrak{M}_G(S)$ with $Q \ntrianglelefteq M$ one of the following cases holds, where $Y := Y_M$, $\overline{M} := M/C_M(Y_M)$, $Q^\bullet := O_p(N_G(Q))$, and $q$ is a power of $p$:*

(1) **The linear case.**
  (a) $\overline{M^\circ} \cong SL_n(q)$, $n \geqslant 3$, *and* $[Y, M^\circ]$ *is a corresponding natural module for* $\overline{M^\circ}$.
  (b) *If* $Y \neq [Y, M^\circ]$ *then* $\overline{M^\circ} \cong SL_3(2)$, $|Y/[Y, M^\circ]| = 2$ *and* $[Y_M, M^\circ] \leqslant Q \leqslant Q^\bullet$.
(2) **The symplectic case.**
  (a) $\overline{M^\circ} \cong Sp_{2n}(q)$, $n \geqslant 2$, *or* $Sp_4(q)'$ *(and* $q = 2$*), and* $[Y, M^\circ]$ *is the corresponding natural module for* $\overline{M^\circ}$
  (b) *If* $Y \neq [Y, M^\circ]$, *then* $p = 2$ *and* $|Y/[Y, M^\circ]| \leqslant q$.
  (c) *If* $Y \nleqslant Q^\bullet$, *then* $p = 2$ *and* $[Y, M^\circ] \nleqslant Q^\bullet$.
(3) **The wreath product case.**
  (a) *There exists a unique* $\overline{M}$*-invariant set* $\mathcal{K}$ *of subgroups of* $\overline{M}$ *such that* $[Y, M^\circ]$ *is a natural* $SL_2(q)$*-wreath product module for* $\overline{M}$ *with respect to* $\mathcal{K}$. *Moreover,* $\overline{M^\circ} = O^p(\langle\mathcal{K}\rangle)\overline{Q}$ *and* $Q$ *acts transitively on* $\mathcal{K}$.
  (b) *If* $Y \neq [Y, M^\circ]$, *then* $p = 2$, $\overline{M} \cong \Gamma SL_2(4)$, $\overline{M^\circ} \cong SL_2(4)$ *or* $\Gamma SL_2(4)$, $|Y/[Y, M^\circ]| = 2$ *and* $[Y, M^\circ] \nleqslant Q^\bullet$.
(4) **The orthogonal case.** $Y \nleqslant Q^\bullet$, $\overline{M^\circ} \cong \Omega_n^\epsilon(q)$, $n \geqslant 5$, *where* $q$ *is odd if* $n$ *is odd, and* $Y$ *is a corresponding natural module for* $\overline{M^\circ}$.
(5) **The tensor product case.** $Y \nleqslant Q^\bullet$, *and there exist subgroups* $\overline{K_1}, \overline{K_2}$ *of* $\overline{M}$ *such that*
  (a) $\overline{K_i} \cong SL_{m_i}(q)$, $m_i \geqslant 2$, $[\overline{K_1}, \overline{K_2}] = 1$, *and* $\overline{K_1 K_2} \trianglelefteq \overline{M}$,
  (b) $Y$ *is the tensor product over* $\mathbb{F}_q$ *of corresponding natural modules for* $\overline{K_1}$ *and* $\overline{K_2}$,
  (c) $\overline{M^\circ}$ *is one of* $\overline{K_1}, \overline{K_2}$, *or* $\overline{K_1 K_2}$.
(6) **The non-natural $SL_n(q)$-case.** $Y \nleqslant Q^\bullet$ *and one of the following holds:*
  (1) $\overline{M^\circ} \cong SL_n(q)/\langle(-id)^{n-1}\rangle$, $n \geqslant 5$, *and* $Y$ *is the exterior square of a natural* $SL_n(q)$*-module.*
  (2) $p$ *is odd,* $\overline{M^\circ} \cong SL_n(q)/\langle(-id)^{n-1}\rangle$, $n \geqslant 2$, *and* $Y$ *is the symmetric square of a natural module.*
  (3) $\overline{M^\circ} \cong SL_n(q)/\langle \lambda id \mid \lambda \in \mathbb{F}_q, \lambda^n = \lambda^{q_0+1} = 1\rangle$, $n \geqslant 2$, $q = q_0^2$, *and* $Y$ *is the unitary square of a natural module.*
(7) **The exceptional case.** $Y \nleqslant Q^\bullet$ *and one of the following holds:*
  (1) $\overline{M^\circ} \cong Spin_{10}^+(q)$, *and* $Y$ *is a half-spin module.*
  (2) $\overline{M^\circ} \cong E_6(q)$, *and* $Y$ *is one of the (up to isomorphism) two simple* $\mathbb{F}_p M^\circ$*-modules of order* $q^{27}$.
(8) **The sporadic case.** $Y \nleqslant Q^\bullet$ *and one of the following holds:*
  (1) $\overline{M} \sim 3^{\cdot}Sym(6)$, $\overline{M^\circ} \sim 3^{\cdot}Alt(6)$ *or* $3^{\cdot}Sym(6)$, *and* $Y$ *is simple of order* $2^6$.
  (2) $p = 2$, $\overline{M^\circ} \cong Mat_{22}$, *and* $Y$ *is the simple Golay-code module of* $\mathbb{F}_2$*-dimension* 10.

(3) *$p = 2$, $\overline{M^\circ} \cong Mat_{24}$, and $Y$ is the simple Todd or Golay-code module of $\mathbb{F}_2$-dimension 11.*
(4) *$p = 3$, $\overline{M^\circ} \cong Mat_{11}$, and $Y$ is the simple Golay-code module of $\mathbb{F}_3$-dimension 5.*

(9) **The non-characteristic $p$ case.** *There exists $1 \neq y \in Y$ such that $C_G(y)$ is not of characteristic $p$ and one of the following holds:*
(1) *$Y$ is tall and asymmetric in $G$, but $Y$ is not char $p$-tall in $G$.*
(2) *$p = 2$, $\overline{M^\circ} \cong Aut(Mat_{22})$, $Y$ is the simple Todd module of $\mathbb{F}_2$-dimension 10, and $Y \nleqslant Q^\bullet$.*
(3) *$p = 3$, $\overline{M^\circ} \cong 2{\cdot}Mat_{12}$, $Y$ is the simple Golay-code module of $\mathbb{F}_3$-dimension 6, and $Y \nleqslant Q^\bullet$.*
(4) *$p = 2$, $\overline{M} \cong O^\epsilon_{2n}(2)$, $\overline{M^\circ} \cong \Omega^\epsilon_{2n}(2)$, $2n \geqslant 4$, $(2n, \epsilon) \neq (4, +)$, $Y$ is a corresponding natural module and $Y \leqslant Q^\bullet$.*
(5) *$p = 3$, $\overline{M^\circ} \cong \Omega^-_4(3)$, $[Y, M^\circ]$ is the corresponding natural module, $|Y/[Y, M^\circ]| = 3$, $Y$ is isomorphic to the 5-dimensional quotient of a six dimensional permutation module for $\overline{M^\circ} \cong Alt(6)$, and $[Y, M^\circ] \nleqslant Q^\bullet$.*
(6) *$p = 3$, $\overline{M^\circ} \cong \Omega_5(3)$, $[Y, M^\circ]$ is the corresponding natural module, $|Y/[Y, M^\circ]| = 3$ and $[Y, M^\circ] \nleqslant Q^\bullet$.*
(7) *$p = 2$, $\overline{M^\circ} \cong \Omega^+_6(2)$, $[Y, M^\circ]$ is the corresponding natural module, and $|Y/[Y, M^\circ]| = 2$.*
(8) *$p = 2$, $\overline{M^\circ} \cong Mat_{24}$, $[Y, M^\circ]$ is the simple Todd-module of $\mathbb{F}_2$-dimension 11, $|Y/[Y, M^\circ]| = 2$ and $[Y, M^\circ] \nleqslant Q^\bullet$.*

We remark that there is some overlap between the different cases and that the last case is not the only case, where $C_G(y)$ may not be of characteristic $p$ for some $1 \neq y \in Y$. See the comment after the Local Structure Theorem (Theorem A) in the introduction for more details.

## 10.1. Proof of Theorem J

In this section we prove Theorem J, so we assume the hypothesis and notation given there.

The existence of $M$ follows from 1.56(c). Now let $M \in \mathfrak{M}_G(S)$ with $Q \ntrianglelefteq M$.

Suppose first that $Y$ is symmetric in $G$. Then we can apply Theorem D. Assume that Case 5 of Theorem D holds. Then $\overline{M} \cong O^\epsilon_{2n}(2)$, $\overline{M^\circ} \cong \Omega^\epsilon_{2n}(2)$, $(2n, \epsilon) \neq (4, +)$, $[Y, M]$ is a corresponding natural module, $C_G(y)$ is not of characteristic 2 for every non-singular $y \in [Y, M]$, and either $Y = [Y, M]$ or $\overline{M} = O^+_6(2)$ and $|Y/[Y, M]| = 2$. If $Y \neq [Y, M^\circ]$, we conclude that J(9:7) holds. If $Y = [Y, M]$ and $Y \nleqslant Q^\bullet$ either J(4) (for $2n \geqslant 6$) or J(6:3) (for $(2n, \epsilon) = (4, -)$) holds. If $Y = [Y, M]$ and $Y \leqslant Q^\bullet$, then J(9:4) holds. All other case of Theorem D also appear in Theorem J.

Suppose next that $Y$ is asymmetric in $G$ and short. Then Theorem E implies Theorem J, where the $O^\epsilon_{2n}(2)$-Case of Theorem E is treated as in the previous paragraph.

Suppose that $Y$ is asymmetric in $G$ and tall. Assume that $Y$ is not *char* $p$-tall in $G$. Then Theorem F shows that $C_G(y)$ is not of characteristic $p$ for some $1 \neq y \in Y$ and thus J(9:1) holds. So we may assume that $Y$ is *char* $p$-tall. If, in addition, $Y$ is $Q$-short we can apply Theorem G and conclude that Theorem J holds.

Suppose finally that $Y$ is asymmetric in $G$ and $Q$-tall. Then we can apply Theorems H and I. Put $I = F_Y(\overline{M})$. Then by Theorem H $I = [Y, M^\circ]$ and $I \nleqslant Q^\bullet$ except in Case H(2), where $I$ is a natural $SL_3(2)$-module for $M^\circ$, $I \leqslant \overline{Q}$ and $|Y/I| = 2$.

Assume that one of the cases of Theorem I holds and $Y \neq I$. Then Corollary 9.2 shows $C_G(x)$ is not of characteristic $p$ for some $1 \neq x \in Y$. Now Corollary 9.1 implies that Case 2 or one of the Cases 9:5 – 9:8 of Theorem J holds. Also by Corollary 9.2 the Cases 11 (Todd-module for $Aut(Mat_{22})$) and 13 (Golay code module for $2 \cdot Mat_{12}$) of Theorem 9.2 only occur if $C_G(x)$ is not of characteristic $p$ for some $1 \neq x \in Y$, and so Cases 9:2 and 9:3 of Theorem J hold, respectively.

In all remaining cases of Theorems H and I a careful comparison shows that Theorem J holds, see Tables 1 and 2.

TABLE 1. The Cases of Theorem H and Theorem J

| Th H | $I$ | $\lvert Y/I\rvert$ | Remark | Th J |
|---|---|---|---|---|
| (1) | — | — | leads to Theorem I | — |
| (2) | nat $SL_3(2)$ | 2 | $I \leqslant Q$, $Y \nleqslant Q^\bullet$ | ( 1:b) |
| (3) | nat $\Omega_6^+(2)$ | 2 | $C_G(x)$ not of characteristic $p$ | (9:7) |
| (4) | nat $Sp_{2n}(2)$ | 2 | $Y > I \nleqslant Q^\bullet$ | (2) |
| (5) | nat $SL_n(q)$, $n \geqslant 3$ | 1 | $p = 2, Y = I \nleqslant Q^\bullet$ | (1) |
| (5) | nat $SL_2(q)$ | 1 | $I = Y \nleqslant Q^\bullet$ | (3) |
| (6) | nat $Sp_{2n}(q)$ | 1 | $Y = I \nleqslant Q^\bullet$ | (2) |
| (7) | nat $\Omega_3(3) \;\cong\; S^2(\text{nat})\, SL_2(3)$ | 1 | $Y = I \nleqslant Q^\bullet$ | (6:2) |
| (8) | nat $(\Gamma)SL_2(4)$ | 2 | $Y > I \nleqslant Q^\bullet$ | (3:b) |
| (9) | $2^6$ for $3{\cdot}Alt(6), 3{\cdot}Sym(6)$ | 1 | $I = Y \nleqslant Q^\bullet$ | (8:1) |
| (10) | nat $SL_{m_1}(q) \otimes SL_{m_2}(q))$ | 1 | $I = Y \nleqslant Q^\bullet$ | (5) |
| (10) | $2^4$ for $SL_2(2) \wr C_2$ | 1 | $I = Y \nleqslant Q^\bullet$ | (3) |

TABLE 2. The Cases $Y = I$ of Theorem I and Theorem J

| Th I | I | Remark | Th J |
|---|---|---|---|
| (1) | nat $SL_m(q)$ | $Y = I \nleqslant Q^\bullet$ | (1) |
| (2) | nat $Sp_{2n}(q), Sp_4(2)'$, $p = 2$ | $Y = I \nleqslant Q^\bullet$ | (2) |
| (3) | nat $\Omega_n^\epsilon(q), n \geqslant 5$ | $Y = I \nleqslant Q^\bullet$ | (4) |
| (3) | nat $\Omega_3(q) \;\cong\; S^2(\text{nat})\, SL_2(q)$ | $Y = I \nleqslant Q^\bullet$ | (6:2) |
| (3) | nat $\Omega_4^+(q) \;\cong \text{nat}\, SL_2(q) \otimes \text{nat}\, SL_2(q)$ | $Y = I \nleqslant Q^\bullet$ | (5) |
| (3) | nat $\Omega_4^-(q) \;\cong\; U^2(\text{nat})\, SL_2(q)$ | $Y = I \nleqslant Q^\bullet$ | (6:3) |
| (4) | $\Lambda^2(\text{nat})\ SL_m(q)$ | $Y = I \nleqslant Q^\bullet$ | (6:1) |
| (5) | $S^2(\text{nat})\ SL_m(q)$ | $Y = I \nleqslant Q^\bullet$ | (6:2) |
| (6) | $U^2(\text{nat})\ SL_m(q)$ | $Y = I \nleqslant Q^\bullet$ | (6:3) |
| (7) | half spin $Spin_{10}^+(q)$ | $Y = I \nleqslant Q^\bullet$ | (7:1) |
| (8) | $q^{27}$ for $E_6(q)$ | $Y = I \nleqslant Q^\bullet$ | (7:2) |
| (9) | Todd or Golay for $Mat_{24}$ | $Y = I \nleqslant Q^\bullet$ | (8:3) |
| (10) | Golay for $Mat_{22}$ | $Y = I \nleqslant Q^\bullet$ | (8:2) |
| (11) | Todd for $Aut(Mat_{22})$ | not (*char* $Y$) by Cor. 9.2 | (9:2) |
| (12) | Golay for $Mat_{11}$ | $Y = I \nleqslant Q^\bullet$ | (8:4) |
| (13) | Golay for $2{\cdot}Mat_{12}$ | not (*char* $Y$) by Cor. 9.2 | (9:3) |

## 10.2. Proof of the Local Structure Theorem

This section is devoted to the proof of the Local Structure Theorem (Theorem A).

Let $p$ be a prime, $G$ a finite $\mathcal{K}_p$-group, $S \in Syl_p(G)$ and $Q \leqslant S$. Suppose that $Q$ is a large subgroup of $G$ and $|\mathcal{M}_G(S)| > 1$. Recall that $Q$ is a weakly closed subgroup of $G$ by 1.52(b).

Let $L \leqslant G$ with $S \leqslant L$, $O_p(L) \neq 1$ and $Q \ntrianglelefteq L$. Since $L$ is a parabolic subgroup of $H$ with $O_p(L) \neq 1$, 1.55(b) shows that $C_G(O_p(L)) \leqslant O_p(L)$. Hence $L \in \mathcal{L}_G(S)$.

By 1.56(a), (b) there exists $M \in \mathfrak{M}_G(S)$ and $L^* \in \mathcal{L}_G(S)$ such that $L^* \leqslant M$ and

$$Y_L = Y_{L*} \leqslant Y_M,\ LC_G(Y_L) = L^*C_G(Y_L),\ L^\circ = (L^*)^\circ,\ Q \ntrianglelefteq L^*,\ \text{and } Q \ntrianglelefteq M.$$

Since $L^\circ = (L^*)^\circ \leqslant M$ and $L/C_L(Y_L) \cong L^*/C_{L*}(Y_L)$ we are allowed to replace $L$ be $L^*$, so we may assume that $L \leqslant M$. Put $\overline{M} := M/C_M(Y_M)$ and $\widetilde{L} := L/C_L(Y_L)$. Then $\overline{S} \leqslant \overline{L} \leqslant \overline{M}$. Hence by 1.24(f) $Y_L \leqslant Y_M$ and so $Y_L = Y_{Y_M}(\overline{L})$ (the largest $p$-reduced $\overline{L}$-submodule of $Y_M$).

Put

$$Y := Y_M,\ V := [Y, M^\circ],\ U := C_V(O_p(L \cap M^\circ)),\ \mathbb{K} := End_{M^\circ}(V),\ Z := C_V(S \cap M^\circ),\ k := \dim_{\mathbb{K}} U.$$

Then $Y_L \leqslant C_Y(O_p(L)) \leqslant C_Y(O_p(L \cap M^\circ)) \leqslant C_Y(O_p(L^\circ))$ and $Y_L \cap V \leqslant U$. Moreover, if $U$ is a simple $\mathbb{F}_pL$-module, then $Y_L \cap V = U$.

$1^\circ$.

(a) $C_Y(L^\circ) = 1$.
(b) $\widetilde{Q} \neq 1$ *and* $\overline{Q} \nleqslant O_p(\overline{L^\circ})$.
(c) $\overline{L \cap M^\circ}$ *is not $p$-closed.*
(d) $V \cap Y_L$ *is a faithful* $\widetilde{L^\circ}$*-module. In particular,* $[V \cap Y_L, Q] \neq 1$.

(a): Since $Q \ntrianglelefteq L$, 1.55(d) gives $C_G(L^\circ) = 1$; in particular $C_Y(L^\circ) = 1$.

(b): If $\widetilde{Q} = 1$, then $1 \neq Y_L \leqslant C_G(Q)$ and so $Q!$ gives $L \leqslant N_G(Q)$, a contradiction. Thus $\widetilde{Q} \neq 1$. Since $Y_L$ is $p$-reduced, $O_p(\widetilde{L^\circ}) \leqslant O_p(\widetilde{L}) = 1$. Hence $\widetilde{Q} \nleqslant O_p(\widetilde{L^\circ})$ and so also $\overline{Q} \nleqslant O_p(\overline{L^\circ})$.

(c): If $\overline{L \cap M^\circ}$ is $p$-closed then $\overline{Q} \leqslant O_p(\overline{L \cap M^\circ})$. Since $Q \leqslant L^\circ \leqslant L \cap M^\circ$, this gives $\overline{Q} \leqslant O_p(\overline{L^\circ})$, which contradicts (b).

(d) : Since $Y_L$ is faithful $p$-reduced $\widetilde{L}$-module, A.9(d) shows that $[Y_L, L^\circ]$ is faithful $\widetilde{L^\circ}$-module. As $[Y_L, L^\circ] \leqslant [Y, M^\circ] \cap Y_L = V \cap Y_L$, also $V \cap Y_L$ is a faithful $\widetilde{L^\circ}$-module.

Note that we can apply Theorem J to $M$. Our strategy is to discuss each of the cases of Theorem J, where we first determine all the subgroups $\overline{L}$ of $\overline{M}$ with $\overline{S} \leqslant \overline{L}$ and $\overline{Q} \ntrianglelefteq \overline{L}$ and then the module structure of $Y_L = Y_Y(\overline{L})$.

Moreover, in some of the cases we will use the following observation to prove that $Y_L \nleqslant Q^\bullet$:

$2^\circ$. *Suppose that* $V \nleqslant Q^\bullet$ *and* $N_M(Q)$ *acts simply on* $V/[V, Q]$.

(a) $V \cap Q^\bullet = [V, Q]$.
(b) *If* $[V, Q, Q] = 1$*, then* $V \cap Y_L \nleqslant Q^\bullet$.
(c) *If* $[V, Q, Q, Q] = 1$ *and* $[V \cap Y_L, Q, Q] \neq 1$*, then* $V \cap Y_L \nleqslant Q^\bullet$.

Indeed, we have $[V, Q] \leqslant V \cap Q^\bullet < V$, and so the simple action of $N_M(Q)$ on $V/[V, Q]$ implies $(2^\circ)$(a).

Suppose that $[V, Q, Q] = 1$. By $(2^\circ)$(a) $V \cap Q^\bullet = [V, Q]$ and so $[V \cap Q^\bullet, Q] = 1$. Since by $(1^\circ)$(d) $[V \cap Y_L, Q] \neq 1$, this gives $V \cap Y_L \nleqslant Q^\bullet$, and $(2^\circ)$(b) holds.

Suppose next that $[V, Q, Q, Q] = 1$ and $[V \cap Y_L, Q, Q] \neq 1$. By $(2^\circ)$(a) $V \cap Q^\bullet = [V, Q]$ and so $[V \cap Q^\bullet, Q, Q] = [V, Q, Q, Q] = 1$. By hypothesis, $[V \cap Y_L, Q, Q] \neq 1$, and we conclude that $V \cap Y_L \nleqslant Q^\bullet$. Hence, $(2^\circ)$(c) holds.

Case 1. *Suppose that the wreath product case of Theorem J holds for $M$.*

Then $Y$ is a natural $SL_2(q)$-wreath product module for $\overline{M}$ with respect to some $\overline{M}$-invariant set of subgroups $\mathcal{K}$ of $\overline{M}$. Moreover, $Q$ acts transitively on $\mathcal{K}$. Put $r = |\mathcal{K}|$, $\{K_1, \ldots, K_r\} := \mathcal{K}$ and $V_i := [Y, K_i]$. Then

$$Y = V_1 \times \ldots \times V_r, \qquad K := \langle \mathcal{K} \rangle = K_1 \times \ldots \times K_r$$

with $K_i \cong SL_2(q)$, and $V_i$ is a natural $SL_2(q)$-module for $K_i$.

Assume that $\overline{M^\circ} \leqslant \overline{L}$. Then $M^\circ = L^\circ$. Now 1.58 shows that $Y_L = Y$ and that Theorem A(3) holds. So assume that $\overline{M^\circ} \nleqslant \overline{L}$. By A.28 $N_{K\overline{S}}(\overline{S} \cap K)$ is the unique maximal subgroup of $K\overline{S}$ containing $\overline{S}$. It follows that $O_p(\overline{L} \cap K) = \overline{S} \cap K$. In particular, $Y_L \leqslant C_Y(\overline{S} \cap K)$. Since $K_i \cong SL_2(q)$, $N_{K_i}(\overline{S} \cap K_i)/\overline{S} \cap K_i \cong C_{q-1}$ and so $N_K(\overline{S} \cap K)/\overline{S} \cap K \cong C_{q-1}^r$. As $\overline{L_\circ} \leqslant O^p(\overline{M^\circ}) \leqslant K$, we conclude that $\widetilde{L_\circ}$ is abelian and every cyclic quotient of $\widetilde{L_\circ}$ has order dividing $q-1$. In particular, $q > 2$.

Let $U_1, U_2, \ldots, U_s$ be the Wedderburn components of $L_\circ$ on $Y_L$. Since $\widetilde{L_\circ}$ is an abelian $p'$-group, $Y_L = U_1 \oplus U_2 \oplus \ldots \oplus U_s$ and $L_\circ/C_{L_\circ}(U_i)$ is cyclic. Let $W_i := C_{V_i}(\overline{S} \cap K)$. Then $Y_L \leqslant W_1 \oplus \ldots \oplus W_r$, and each $W_j$ is a homogeneous $N_K(\overline{S} \cap K)$-module. Hence $W_i$ is also a homogeneous $L^\circ$-module. For $1 \leqslant i \leqslant s$, let $R_i$ consists of all $1 \leqslant j \leqslant r$ such that the projection of $U_i$ onto $W_j$ is non-trivial. Put $W_{R_i} := \bigoplus_{j \in R_i} W_j$. Then $W_{R_i}$ is an homogeneous $L^\circ$-submodule of $C_V(\overline{S} \cap K)$ and $R_i \cap R_k = \varnothing$ for $1 \leqslant i < k \leqslant r$. Note that $Q$ normalizes $L_\circ$ and so also $\bigcup_{i=1}^s R_i$. Since $Q$ acts transitively on the subgroups $W_i$, we conclude that, $R_1, R_2, \ldots, R_s$ is a $Q$-invariant partition of $\{1, \ldots, r\}$ and that $Q$ acts transitively on $R_1, \ldots, R_s$. It follows that $Q$ acts transitively on $U_1, U_2 \ldots, U_s$. Since $\overline{S} \cap K \leqslant O_p(\overline{L})$, $\overline{Q} \nleqslant K$. In particular, $\overline{M^\circ} \ncong SL_2(q)$, and Theorem A(4) holds.

Case 2. *Suppose that the tensor product case of Theorem J holds for $M$.*

If $\overline{M^\circ} = \overline{M} \cong SL_2(2) \wr C_2$, then $\overline{S}$ is a maximal subgroup of $\overline{M}$ and $\overline{M} = \overline{L}$. Thus, Theorem A(6) holds. So assume that $\overline{M^\circ}$ is one of $\overline{K_1}$, $\overline{K_2}$ or $\overline{K_1K_2}$. Let $K_i$ be the inverse image of $\overline{K_i}$ in $M$, and let $V_i$ be a natural $SL_{m_i}(q)$-module for $K_i$ such that $Y \cong V_1 \otimes_{\mathbb{F}_q} V_2$ as a $K_1K_2$-module.

Note that either $\overline{K_i} \trianglelefteq \overline{M}$ or $p = 2$ and $\overline{K_1}^{\overline{x}} = \overline{K_2}$ for some $\overline{x} \in \overline{S}$. In particular $N_L(K_1) = N_L(K_2)$. Put

$$L_0 := N_L(K_1) = N_L(K_2),\ L_i := \langle (S \cap K_i)^{L_0} \rangle,\ U_i := C_{V_i}(O_p(L_i)).$$

Then $L_1L_2 \trianglelefteq L$, $[\overline{L_1}, \overline{L_2}] = 1$ and $Q \leqslant L_1L_2$. Since $\overline{L_i}$ is a parabolic subgroup of $\overline{K_i}$ generated by $p$-elements and $\overline{K_i} \cong SL_{m_i}(q)$, we get that $U_i$ is a natural $SL_{t_i}(q)$-module for $L_i$, where $1 \leqslant t_i \leqslant m_i$. Moreover, $Y_L \leqslant C_Y(O_p(\overline{L_1L_2})) \cong U_1 \otimes_{\mathbb{F}_q} U_2$ as an $L_1L_2$-module. Since $Q \ntrianglelefteq L$, $t_i \geqslant 2$ for some $i \in \{1,2\}$. It follows that $U_1 \otimes_{\mathbb{F}_q} U_2$ is a simple $\mathbb{F}_pL_1L_2$-module and so $Y_L \cong U_1 \otimes_{\mathbb{F}_q} U_2$ as an $L_1L_2$-module. Let $\{i,j\} := \{1,2\}$.

Assume that $t_j = 1$. Then $Y_L$ is a natural $SL_{t_i}(q)$-module for $L_i$ and $\overline{S} \cap \overline{K_j} \trianglelefteq \overline{L}$; in particular $Y_L = [Y_L, L^\circ]$. Since $\overline{Q} \ntrianglelefteq \overline{L}$ we conclude that $\widetilde{L^\circ} = \widetilde{L_i}$. Hence Theorem A(1) holds, if $t_i \geqslant 3$, and A(3) holds if $t_i = 2$.

Assume next that $t_j \geqslant 2$ and $\overline{M^\circ} = \overline{K_r}$ for some $r \in \{1,2\}$. Then $\widetilde{L^\circ} = \widetilde{L_r}$. Let $\{r,s\} := \{1,2\}$. Then $\overline{K_s}$ normalizes $\overline{Q^\bullet}$ and $N_{\overline{K_r}}(Z) \sim q^{m_r-1}SL_{m_{r-1}}(q)$, where $O_p(N_{\overline{K_r}}(Z))$ is a natural module for $SL_{m_{r-1}}(q)$. Thus, $N_{\overline{K_r}}(Z)$ acts simply on $O_p(N_{\overline{K_r}}(Z))$, and $\overline{Q} = O_p(N_{\overline{K_r}}(Z))$. It follows that $N_M(Q)$ acts simply on $C_V(Q)$ and $V/C_V(Q)$. In particular, $C_V(Q) = [V,Q]$ and $[V,Q,Q] = 1$, and so by (2°), $Y_L \nleqslant Q^\bullet$. Thus Theorem A(6) holds.

Assume now that $t_j \geqslant 2$ and $\overline{M^\circ} = \overline{K_1K_2}$. Then $\widetilde{L^\circ} = \widetilde{L_1}\widetilde{L_2}$ and, for $r \in \{1,2\}$, $N_{\overline{K_r}}(Z) \sim q^{m_r-1}SL_{m_{r-1}}(q)$. Hence as above, the simple action of $N_{\overline{K_r}}(Z)$ on $O_p(N_{\overline{K_r}}(Z))$ shows that $\overline{Q} = O_p(N_{\overline{K_1}}(Z))O_p(N_{\overline{K_2}}(Z))$. Moreover, $V/[V,Q]$ is a simple $N_M(Q)$-module, $[V,Q,Q] = Z$, and $Q$ does not act quadratically on $Y_L$. Thus by (2°) $Y_L \nleqslant Q^\bullet$, and again Theorem A(6) holds. This finishes (Case 2).

In all the remaining cases of Theorem J $V$ is a simple $M^\circ$-module. Suppose that $\overline{M^\circ} \leqslant \overline{L}$. Then $V$ is a simple $L$-module and so $[Y, M^\circ, O_p(\overline{L})] = [V, O_p(\overline{L})] = 1$. Also $[O_p(\overline{L}), \overline{M^\circ}] \leqslant O_p(\overline{M^\circ}) = 1$, and the Three Subgroups Lemma implies $[Y, O_p(\overline{L}), M^\circ] = 1$. Since $C_Y(M^\circ) = 1$ by $Q!$, we have $[Y, O_p(\overline{L})] = 1$ and so $O_p(\overline{L}) = 1$. Thus $Y = Y_L$. Moreover $\overline{L^\circ} = \overline{M^\circ}$ and so by 1.52(c),

$$L^\circ = (L^\circ C_M(Y))^\circ = (M^\circ C_M(Y))^\circ = M^\circ.$$

We conclude that one of the cases of Theorem J holds for $L$ in place of $M$, which gives the corresponding case for $L$ in Theorem A . Thus we may assume

3°. $\overline{M^\circ} \nleqslant \overline{L}$. *In particular, $\overline{L} \cap \overline{M^\circ}$ is a proper parabolic subgroup of $\overline{M^\circ}$.*

We first consider the case where $\overline{M^\circ}$ is a genuine group of Lie-Type in characteristic $p$. Then by (3°) $O_p(\overline{L} \cap \overline{M^\circ}) \neq 1$. Let $\Delta$ be the corresponding Dynkin diagram for $\overline{M^\circ}$. For any $\Psi \subseteq \Delta$ let $\overline{M_\Psi}$ be the Lie-parabolic subgroup of $\overline{M^\circ}$ with $\overline{S} \cap \overline{M^\circ} \leqslant \overline{M}_\Psi$ and Dynkin diagram $\Psi$. Put $\overline{R_\Psi} := O^{p'}(\overline{M_\Psi})$, and let $R_\Psi$ be the inverse image of $\overline{R}_\Psi$ in $M^\circ$.

By A.63 there exists a unique $\Lambda \subsetneq \Delta$ with

$$\overline{R}_\Lambda = O^{p'}(\overline{L} \cap \overline{M^\circ}) \leqslant \overline{L} \cap \overline{M^\circ} \leqslant \overline{M}_\Lambda.$$

Recall that $\mathbb{K} = End_{M^\circ}(V)$. Observe also that in all cases of Theorem J where $\overline{M^\circ}$ is a genuine group of Lie-type there exists a unique $\delta \in \Delta$ with $[Z, \overline{R}_\delta] \neq 1$. Moreover, $U_\delta := C_V(O_p(\overline{R}_\delta))$ is a natural $SL_2(\mathbb{K})$-module or the symmetric or unitary square of a natural $SL_2(\mathbb{K})$-module, i.e $U_\delta$ is a natural $SL_2(\mathbb{K})$-, $\Omega_3(\mathbb{K})$- or $\Omega_4^-(q)$-module for $\overline{R}_\delta$. By $Q!$, $Q \trianglelefteq R_\rho$ for all $\delta \neq \rho \in \Delta$. Since $Q \ntrianglelefteq L \cap M^\circ$ we have $\delta \in \Lambda$. Let $\Xi$ be the connected component of $\Lambda$ containing $\delta$. Then $R_{\Lambda \setminus \Xi}$ normalizes $Q$. We conclude that $\overline{L^\circ(S \cap M^\circ)} = \overline{R}_\Xi$. Smith's Lemma A.63, applied to $\overline{M}_\Lambda$ and $V$, shows that $U$ is a simple $\mathbb{K}\overline{R}_\Lambda$-module. Hence $R_{\Lambda \setminus \Xi}$ centralizes $U$, and $U$ is a semisimple $\mathbb{F}_pR_\Xi$-module. Since $U_\delta$ is a simple $\mathbb{F}_pR_\delta$-module we conclude that $U$ is a simple $\mathbb{F}_pR_\Xi$-module. Hence $U$ is a simple $\mathbb{F}_pL^\circ(S \cap M^\circ)$-module, and $U = V \cap Y_L$ by an earlier remark. Since each $R_\rho$, $\rho \in \Xi \setminus \{\delta\}$, centralizes

$Z$, the Ronan-Smith's Lemma A.64 implies that the isomorphism type of $U$ as an $R_\Xi$-module (and so as an $L^\circ(S \cap M^\circ)$-module) is uniquely determined by $\delta$ and the isomorphism type of $U_\delta$ as an $R_\delta$-module. We have proved:

4°. *Suppose that $\overline{M^\circ}$ is a genuine group of Lie type. Then*

(a) $\overline{L^\circ(S \cap M^\circ)} = \overline{R}_\Xi$.
(b) *$U$ is the simple $\mathbb{F}_p\overline{R}_\Xi$-module uniquely determined by $\delta$, $[Z, \overline{R}_\rho] = 1$ for $\rho \in \Xi \backslash \{\delta\}$, and the isomorphism type of $U_\delta$ as an $\overline{R}_\delta$-module.*

Next we show:

5°. *Suppose that $\overline{M^\circ}$ is a genuine group of Lie type, $V \nleqslant Q^\bullet$, $N_{\overline{M^\circ}}(Z)$ acts simply on $O_p(N_{\overline{M^\circ}}(Z))$, $[V, Q, Q] \leqslant Z$, and $[Y_L, Q, Q] \neq 1$. Then $Y_L \nleqslant Q^\bullet$.*

Since $N_{\overline{M^\circ}}(Z)$ acts simply on $O_p(N_{\overline{M^\circ}}(Z))$ we have $\overline{Q} = O_p(N_{\overline{M^\circ}}(Z))$. Hence Smith's Lemma A.63 applied to the dual of $V$ shows that $N_M(Q)$ acts simply on $V/[V, Q]$. From $[V, Q, Q] \leqslant Z$ we get $[V, Q, Q, Q] = 1$. Thus (2°) implies that $Y_L \nleqslant Q^\bullet$.

6°. *Suppose that $\overline{M^\circ}$ is a genuine group of Lie type. Then $Y_L = U$; in particular, $Y_L$ is the simple $\mathbb{F}_p\overline{R}_\Xi$-module uniquely determined by $\delta$, $[Z, \overline{R}_\rho] = 1$ for $\rho \in \Xi \backslash \{\delta\}$, and the isomorphism type of $U_\delta$ as an $\overline{R}_\delta$-module.*

Otherwise, $Y_L \nleqslant V$ and $V \neq Y$. Then Theorem J shows that one of the following holds

(A) $p = 2$. $|Y/V| = 2$ and $V$ is a natural $SL_3(2)$-module,
(B) $p = 2$, $\overline{M^\circ} \cong Sp_{2n}(q)$, $n \geqslant 2$ and $|Y/V| \leqslant q$.
(C) $p = 3$, $\overline{M^\circ} \cong \Omega_4^-(3)$, $V$ is the corresponding natural module, $|Y/V| = 3$, $Y$ is isomorphic to the 5-dimensional quotient of a 6-dimensional permutation module for $\overline{M^\circ} \cong Alt(6)$, and $V \nleqslant Q^\bullet$.
(D) $p = 3$, $\overline{M^\circ} \cong \Omega_5(3)$, $V$ is the corresponding natural module, $|Y/V| = 3$ and $V \nleqslant Q^\bullet$.
(E) $p = 2$, $\overline{M^\circ} \cong \Omega_6^+(2)$, $V$ is the corresponding natural module, and $|Y/V| = 2$.

Let $x \in Y_L \backslash U$. We discuss the cases (A) – (E) one by one.

Suppose that $\overline{M^\circ} \cong SL_3(2)$. Then $C_{\overline{S}}(x) = 1$, a contradiction to $1 \neq O_2(\overline{L})) \leqslant C_{\overline{S}}(x)$.

Suppose that $\overline{M^\circ} \cong Sp_{2n}(q)$. Let $s$ be an $M^\circ$-invariant non-degenerate symplectic form on $V$. Then $C_{\overline{M^\circ}}(x) \cong O_{2n}^\epsilon(q)$, and there exists a non-degenerate $C_{\overline{M^\circ}}(x)$-invariant quadratic form $t$ on $V$ with $s$ being the associate symmetric form. With respect to the symplectic form $s$ on $V$, the Lie-parabolic subgroups of $\overline{M^\circ}$ normalize a unique $s$-singular $\mathbb{K}$-subspace of $V$. Since $U$ is a simple $\overline{R}_\Xi$-module, we conclude that $U$ is the $s$-singular $\mathbb{K}$-subspace of $V$ corresponding to $\overline{M}_\Xi$, and $\dim_{\mathbb{K}} U \geqslant 2$ since $\delta \in \Xi$. In particular, $[U, R_\Xi] \neq 1$. Note that radical of $t$ on $U$ has codimension at most 1 in $U$ (see B.5), and so there exists $u \in U^\sharp$ with $t(u) = 0$.

Choose $\overline{g} \in \overline{M^\circ}$ with $[V, g] = \mathbb{K}u$. Then $g$ centralizes the hyperplane $u^\perp$ in the symplectic space $V$, and since $U$ is singular with respect to $s$, $U \leqslant C_V(g)$. Thus $g$ centralizes $U$, $V/U$ and $Y/V$. It follows that $\overline{g} \in \overline{S}$ and $\overline{g} \in O_2(\overline{M^\circ} \cap \overline{L}) \leqslant O_2(\overline{L})$. Hence $Y_L \leqslant C_Y(O_2(\overline{L})) \leqslant C_Y(x)$ and $\overline{g} \in C_{M^\circ}(x)$. Thus $\overline{g}$ leaves invariant the quadratic form $t$, a contradiction to $t(u) = 0$ (see B.9(c)).

Suppose that $\overline{M^\circ} \cong \Omega_4^-(3)$. Then $|\Delta| = 1$ a contradiction to $\varnothing \neq \Lambda \subsetneq \Delta$.

Suppose that $\overline{M_\circ} \cong \Omega_5(3)$. Then $U$ is natural $SL_2(3)$-module for $\widetilde{\overline{L^\circ}}$. Since $[Y_L, L^\circ] \leqslant U$ and $C_{Y_L}(L^\circ) = 1$ this gives $U = Y_L$.

Suppose that $\overline{M^\circ} \cong \Omega_6^+(2)$. Then $Y$ is isomorphic to the 7-dimensional quotient of the 8-dimensional permutation module for $\overline{M^\circ} \cong Alt(8)$. Moreover, since $L^\circ \nleqslant N_M(Z)$, there exists $\overline{R} \leqslant \overline{M^\circ}$ with $\overline{R} \sim 2^3 L_3(2)$ and $\overline{L \cap M^\circ} \leqslant R$. Moreover, $O_2(\overline{R})$ acts regularly on the eight objects, so $C_Y(O_2(R)) \leqslant V$. Then $O_2(\overline{R}) \leqslant O_2(L \cap M^\circ) \leqslant O_2(L)$ and $Y_L \leqslant C_Y(O_2(L)) \leqslant C_Y(O_2(R)) \leqslant V$. So $Y_L = Y_L \cap V = U$.

We have shown that $U = Y_L$ in all cases, and (6°) is proved.

Case 3. *Suppose that $V$ is a natural $SL_n(q)$, $Sp_{2n}(q)$ or $\Omega_n^\epsilon(q)$-module for $\overline{M^\circ}$ (with $p$ odd if $n$ is odd in the $\Omega_n^\epsilon(q)$-module case).*

Then $\delta$ is an end-node of $\Delta$, with $\delta$ being short in the $Sp_{2n}(q)$-case and long in the $\Omega_n^{\epsilon}(q)$-case. Since $\Xi$ is a proper connected subdiagram of $\Delta$ containing $\delta$, $\Xi$ is a Dynkin diagram of type $A_{m-1}$. Also $U_\delta$ is a natural $SL_2(q)$-module for $R_\delta$ and so by (6°) $Y_L$ is a natural $SL_m(q)$-module for $L^\circ$. Thus Theorem A(3) holds if $m = 2$, and Theorem A(1) holds if $m \geqslant 3$.

Case 4. *Suppose that $V$ is the exterior square of a natural $SL_n(q)$-module for $\overline{M^\circ}$, where $n \geqslant 5$.*

In this case $\delta$ is adjacent to an end-node of $\Delta$ and $U_\delta$ is the natural $SL_2(q)$-module for $R_\delta$. Hence by (6°) $Y_L$ is a natural $SL_m(q)$- or the exterior square of a natural $SL_m(q)$-module (with $m \geqslant 4$ in the second case). In the first case Theorem A(1) holds if $m \geqslant 3$, and Theorem A(3) holds if $m = 2$. So suppose that $Y_L$ is the exterior square of a natural $SL_m(q)$-module with $m \geqslant 4$. Note that $O^{p'}(C_{\overline{M^\circ}}(Z)) \sim q^{2(n-2)}(SL_2(q) \times SL_{n-2}(q))$ and so $\overline{Q} = O_p(C_{M^\circ}(Z))$. In particular, $[V, Q, Q] = Z$ and $Q$ does not act quadratically on $Y_L$. Thus by (5°). we see that $Y_L \nleqslant Q^\bullet$. If $m = 4$, then the exterior square of a natural $SL_m(q)$-module is the natural $\Omega_6^+(q)$-module and so Theorem A(5) holds. If $m \geqslant 5$, then Theorem A(7:1) holds.

Case 5. *Suppose that $V$ is the symmetric or unitary square of a natural $SL_n(q)$-module for $\overline{M^\circ}$.*

In this case $\delta$ is an end-note, and $U_\delta$ is the symmetric or unitary square of a natural $SL_2(q)$-module. Hence by (6°) $Y_L$ is the symmetric or unitary square of a natural $SL_m(q)$-module for $L^\circ$. Also $O^{p'}(C_{\overline{M^\circ}}(Z)) \sim q^{n-1}SL_{n-1}(q)$, $\overline{Q} = O_p(C_{\overline{M^\circ}}(Z)))$, $[V, Q, Q] \leqslant Z$, and $Q$ does not act quadratically on $Y_L$. Thus (5°) gives $Y_L \nleqslant Q^\bullet$ and Theorem A(7:2) or A(7:3) holds.

Case 6. *Suppose that $\overline{M^\circ} \cong Spin_{10}^+(q)$ and $V$ is the half-spin module.*

In this case $\delta$ is one of the end notes of $\Delta$ corresponding to an $SL_5(q)$-parabolic and so $\Xi$ is of type $A_{m-1}$, $2 \leqslant m \leqslant 5$, or $D_4$. Moreover, $U_\delta$ is a natural $SL_2(q)$-module for $R_\delta$. Thus by (6°) $Y_L$ is a natural $SL_m(q)$-module, $2 \leqslant m \leqslant 5$, or a natural $\Omega_8^+(q)$-module for $L^\circ$. In the $SL_m(q)$-case, Theorem A(1) holds if $m \geqslant 3$, and Theorem A(3) holds if $m = 2$. So suppose that $Y_L$ is a natural $\Omega_8^+(q)$-module. We have $O^{p'}(C_{\overline{M^\circ}}(Z)) \sim q^{10}SL_5(q)$ and so $\overline{Q} = O_p(C_{\overline{M^\circ}}(Z))$. Thus $[V, Q, Q] = Z$, and $Q$ does not act quadratically on $Y_L$. Hence by (2°) $Y_L \nleqslant Q^\bullet$, and Theorem A(5) holds.

Case 7. *Suppose that $\overline{M^\circ} \cong E_6(q)$ and $|V| = q^{27}$.*

In this case $\delta$ is one of the end nodes of $\Delta$ corresponding to an $\Omega_{10}^+(q)$-parabolic and so $\Xi$ is of type $A_{m-1}$, $2 \leqslant m \leqslant 6$, or $D_5$. Moreover, $U_\delta$ is a natural $SL_2(q)$-module for $R_\delta$. Hence by (6°) $Y_L$ is a natural $SL_m(q)$-module ($2 \leqslant m \leqslant 6$), or the natural $\Omega_{10}^+(q)$-module for $L^\circ$. In the $SL_m(q)$-case, Theorem A(1) holds if $m \geqslant 3$, and Theorem A(3) holds if $m = 2$. So suppose that $Y_L$ is the natural $\Omega_{10}^+(q)$-module. We have $O^{p'}(C_{\overline{M^\circ}}(Z)) \sim q^{16}Spin_{10}^+(q)$ and so $\overline{Q} = O_p(C_{\overline{M^\circ}}(Z))$. Hence $[V, Q, Q] = Z$, and $Q$ does not act quadratically on $Y_L$. Thus by (2°) $Y_L \nleqslant Q^\bullet$, and Theorem A(5) holds.

This concludes the discussion of the cases where $\overline{M^\circ}$ is a genuine group of Lie-type.

Case 8. *Suppose that $\overline{M^\circ} \cong Sp_4(2)'$ and $|V| = 2^4$.*

Then $\overline{L} \cap \overline{M^\circ} \cong Sym(4)$, $U$ is a natural $SL_2(2)$-module for $L^\circ$, $C_Y(O_2(\overline{L} \cap \overline{M})) \leqslant V$, and so $Y_L = Y_L \cap V = U$. Thus by 1.58 Theorem A(3) holds.

Case 9. *Suppose that $\overline{M} \sim 3{\cdot}Sym(6)$ and $|Y| = 2^6$.*

Then $\overline{L} \cong C_2 \times Sym(4)$, $Sym(3) \times Dih_8$ or $Sym(3) \times Sym(4)$. In the first two cases $Y_L$ is the natural $SL_2(2)$-module for $L$. Thus Theorem A(3) holds.

So suppose that $\overline{L} \cong Sym(3) \times Sym(4)$. Then $Y_L$ has order $2^4$ and is the tensor product of two natural $SL_2(2)$-modules. Since $\langle Y_L^{C_{M'}(Z)} \rangle = Y$ and $Y \nleqslant Q^\bullet$, we have $Y_L \nleqslant Q^\bullet$. If $\overline{M^\circ} \sim 3{\cdot}Alt(6)$, then $\overline{L^\circ} \cong Sym(4)$ and so $\widetilde{L^\circ} \cong SL_2(2)$; and if $\overline{M^\circ} \sim 3{\cdot}Sym(6)$, then $\overline{L} = \overline{L^\circ}$ and $\widetilde{L^\circ} \cong SL_2(2) \times SL_2(2)$. Thus Theorem A(6) holds.

Case 10. *Suppose that $p = 2$, $\overline{M^\circ} \cong Mat_{22}$, and $Y$ is the simple Golay-code module of $\mathbb{F}_2$-dimension* 10.

Then $Y = V$ and $C_{\overline{M^\circ}}(Z) \sim 2^4 Alt(6)$. For a description of the action of the maximal parabolic subgroups of $\overline{M^\circ}$ on (the dual of) $V$ see [**MSt**, 3.3]. It follows that $\overline{L} \cap \overline{M^\circ} \sim 2^4 \Gamma SL_2(4)$ and so $\overline{L^\circ} \sim 2^4 SL_2(4)$. Moreover, $Y_L = C_V(O_2(L^\circ))$ is a natural $\Omega_4^-(2)$-module for $L^\circ$, and so also the unitary square of natural $SL_2(4)$-module. Also $[V, Q, Q, Q] = 1$ an $[Y_L, Q, Q] \neq 1$ and so $(2^\circ)$(c) shows that $Y_L \not\leqslant Q^\bullet$. Thus Theorem A(7:3:3) holds.

Case 11. *Suppose that $p = 2$, $\overline{M^\circ} \cong Mat_{24}$, and $Y$ is the 11-dimensional simple Golay code module.*

Then $\overline{M} = \overline{M^\circ}$ and $Y = V$. For a description of the action of the maximal parabolic subgroups of $M$ on (the dual of) $V$ see [**MSt**, 3.5]. In particular, $C_{\overline{M}}(Z) \sim 2^4 SL_4(2)$.

Assume that $\overline{L}$ is a maximal subgroup of $\overline{M}$. Then $\overline{L} \sim 2^6.3.Sym(6)$ or $2^6.(SL_2(2) \times SL_3(2))$, and $U$ is a natural $Sp_4(2)$- or $SL_2(2)$-module, respectively. Thus $Y_L = U$. In the first case, since $[C_L(Y_L), Q] \leqslant O_p(L)$ and $3^\cdot Sym(6)$ acts non-trivially on $Z(3^\cdot Alt(6))$, $Y_L$ is a natural $Sp_4(2)'$-module for $L^\circ$. Moreover, as $[V, Q, Q, Q] = 1$ and $[Y_L, Q, Q] \neq 1$, $(2^\circ)$(c) gives $Y_L \not\leqslant Q^\bullet$, and so Theorem A(2) holds. In the second case Theorem A(3) holds.

If $\overline{L}$ is not a maximal subgroup, then $\overline{L}$ is contained in a maximal subgroup $\overline{P} \sim 2^6(SL_2(2) \times SL_3(2))$, $\widetilde{L} \cong SL_2(2)$, and $U$ is a natural $SL_2(2)$-module for $L$. Hence Theorem A(3) holds.

Case 12. *Suppose that $p = 2$, $\overline{M^\circ} \cong Mat_{24}$, and $V$ is the 11 dimensional simple Todd-module.*

Then $\overline{M} = \overline{M^\circ}$ and $|Y/V| \leqslant 2$. For a description of the action of the maximal parabolic subgroups of $M$ on $V$ see [**MSt**, 3.5]. In particular, $C_{\overline{M}}(Z) \sim 2^6.3^\cdot Sym(6)$ and $[V, Q, Q, Q] = 1$.

Assume that $\overline{L}$ is a maximal subgroup of $\overline{M}$. Then $\overline{L} \sim 2^4.L_4(2)$ or $2^6.(SL_2(2) \times SL_3(2))$, and $U$ is a natural $\Omega_6^+(2)$- or $SL_3(2)$-module, respectively. Thus $Y_L \cap V = U$.

Suppose that $U$ is a natural $\Omega_6^+(2)$-module. Then $Q$ does not act quadratically on $U$. So by $(2^\circ)$(c) $U \not\leqslant Q^\bullet$. If $Y = V$ then Theorem A(5) holds. Suppose $Y \neq V$ and let $x \in Y \backslash V$. Then $C_{\overline{M}}(x) \cong Mat_{23}$ or $L_3(4).Sym(3)$ and so $C_{\overline{M}}(x)$ contains a conjugate of $O_2(\overline{L})$. Thus $|Y_L/U| = 2$ and Theorem A(10:5) holds,

Suppose that $U$ is a natural $SL_3(2)$-module. Since for $x \in Y \backslash V$, $C_{\overline{M}}(x)$ does not contain an elementary abelian subgroup of order $2^6$, we get $C_Y(O_2(\overline{L})) \leqslant V$. Hence $Y_L = U$ and Theorem A(1) holds.

Assume that $\overline{L}$ is not a maximal subgroup of $\overline{M}$, then $\overline{L}$ is contained in one of the above maximal subgroups. Thus $U = Y_L$ and $Y_L$ is a natural $SL_2(2)$ or $SL_3(2)$-module for $L^\circ$. Hence Theorem A(3) or Theorem A(1) holds.

Case 13. *Suppose that $p = 3$, $\overline{M^\circ} \cong Mat_{11}$ and $Y$ is the 5-dimensional simple Golay-code module.*

Then $Y = V$ and $C_{\overline{M^\circ}}(Z) \sim 3^2 SDih_{16}$. It follows that $L^\circ \cong Alt(6)$, $Y_L = V$ and $[Y_L, L^\circ]$ is the natural $\Omega_4^-(3)$-module. Since $Y = \langle [Y, L^\circ]^{C_{\overline{M^\circ}}(Z)} \rangle$ and $Y \not\leqslant Q^\bullet$, we have $[Y_L, L^\circ] \not\leqslant Q^\bullet$, and Theorem A(7:3:2) holds.

Case 14. *Suppose that $p = 2$, $\overline{M^\circ} \cong Aut(Mat_{22})$, and $Y$ is the 10-dimensional simple Todd-module.*

Then $Y = V$ and $\overline{M} = \overline{M^\circ}$. For a description of the action of the maximal parabolic subgroups of $\overline{M^\circ}$ on $V$ see [**MSt**, 3.3]. In particular, $C_M(Z) \sim 2^{4+1}.Sym(5)$, and $\overline{Q} = O_2(C_M(Z))$, $[V, Q, Q, Q] = 1$.

Suppose first that $\overline{L^\circ}$ is a maximal subgroup of $\overline{M}$. Then $\overline{L^\circ} \sim 2^4.Sp_4(2)$, $[U, L^\circ]$ is a natural $Sp_4(2)$-module for $L^\circ$, $|U/[U, L^\circ]| = 2$ and $[[U, L^\circ], Q, Q] \neq 1$. It follows that $U = Y_L$ and by $(2^\circ)$(c) $[Y_L, L^\circ] \not\leqslant \overline{Q}$, and Theorem A(2:d:2) holds.

Suppose next that $\overline{L^\circ}$ is not maximal subgroup of $\overline{M}$. Then $\overline{L^\circ}$ is contained in the above maximal subgroup of shape $2^4.Sp_4(2)$ and we conclude that $Y_L$ is a natural $SL_2(2)$-module for $\overline{L^\circ}$. Hence Theorem A(3) holds.

Case 15. *Suppose that $p = 3$, $\overline{M^\circ} \cong 2^{\cdot}Mat_{12}$, and $Y$ is the 6-dimensional simple Golay-code module.*

Then $Y = V$, $C_{\overline{M^\circ}}(Z) \sim 3^2.GL_2(3)$ and $[V, Q, Q, Q] = 1$. It follows that $L^\circ \sim 3^2SL_2(3)$, $U$ is symmetric square of a natural $SL_2(3)$ for $L^\circ$, and $[U, Q, Q] \neq 1$. Hence $U = Y_L$ and by $(2^\circ)$(c) $Y_L \nleqslant \overline{Q}$. Thus Theorem A(7:3:4) holds.

## 10.3. Proof of the Corollary to the Local Structure Theorem

In this section we prove Corollary B. So as there let $G$ be a finite $\mathcal{K}_p$-group of local characteristic $p$, let $S \in Syl_p(G)$ and suppose that there exist $M, \widetilde{C} \in \mathcal{M}_G(S)$ such that the following hold for $Q := O_p(\widetilde{C})$:

(i) $N_G(\Omega_1 Z(S)) \leqslant \widetilde{C}$.
(ii) $C_G(x) \leqslant \widetilde{C}$ for every $1 \neq x \in Z(Q)$.
(iii) $M \neq \widetilde{C}$, and $M = L$ for every $L \in \mathcal{M}_G(S)$ with $M = (M \cap L)C_M(Y_M)$.
(iv) $Y_M \leqslant Q$.

It follows easily from (i) and (ii) that $Q$ is a weakly closed subgroup of $G$, see [**MSS**, 2.4.2(a)] for a proof. Since $\widetilde{C} \in \mathcal{M}_G(S)$ and $Q \trianglelefteq \widetilde{C}$ we have $N_G(Q) = \widetilde{C}$. Hence, by a Frattini argument and again (ii),

$$N_G(A) = C_G(A)(N_G(A) \cap N_G(Q)) \leqslant \widetilde{C}$$

for every $1 \neq A \leqslant Z(Q)$. Since $G$ is of local characteristic $p$, $\widetilde{C}$ is of characteristic $p$. So $C_G(Q) \leqslant Q$, and we get that $Q$ is a large subgroup of $G$. Note here that $Q = O_p(\widetilde{C}) = O_p(N_G(Q))$, so $Q = Q^\bullet$ in the notation of Theorem J.

By (iii) $M \neq \widetilde{C}$, and since $M \in \mathcal{M}_G(S)$, we conclude that $G \neq \widetilde{C} = N_G(Q)$. So $Q \ntrianglelefteq M$. By 1.56(a), applied with the roles of $M$ and $L$ reversed, there exist $L \in \mathfrak{M}_G(S)$ and $M^* \leqslant L$ with

$$S \leqslant M^*,\ Y_M = Y_{M*},\ MC_G(Y_M) = M^*C_G(Y_M), M^\circ = (M^*)^\circ. \text{ and } Q \ntrianglelefteq L.$$

Recall from the definition of $\mathfrak{M}_G(S)$ that $\mathcal{M}_G(L) = \{L^\dagger\}$ and $Y_L = Y_{L^\dagger}$. Also 2.2(b) gives $C_S(Y_L) = O_p(L)$. Since $M \in \mathcal{M}_G(S)$ and $Y_M = Y_{M*}$, we have $M^* \leqslant N_G(Y_{M*}) = N_G(Y_M) = M$. From $MC_G(Y_M) = M^*C_G(Y_M)$ we get

$$M = (M \cap L)C_G(Y_M) = (M \cap L^\dagger)C_G(Y_M).$$

As $L^\dagger \in \mathcal{M}_G(S)$, (iii) shows that $M = L^\dagger = LC_G(Y_L)$ and $Y_M = Y_{L^\dagger} = Y_L$. In particular, $M^\circ = (L^\dagger)^\circ = L^\circ$.

Since $G$ is of local characteristic $p$, $C_G(x)$ is of characteristic $p$ for all non-trivial $p$-elements $x$ of $G$, and in particular, for all $1 \neq x \in Y_L$. Thus Theorem J, the Structure Theorem for Maximal Local Parabolic Subgroups, applies to $L$. By (iv) $Y_M = Y_L \leqslant Q = Q^\bullet$, and so only the first three cases, namely the linear , symplectic and wreath product case, of Theorem J are relevant. Moreover, as $Y_L \leqslant Q^\bullet$ we have $Y_L = [Y_L, L^\circ]$ in the wreath product case. Hence one of the following holds, where $\overline{L^\dagger} := L^\dagger/C_{L^\dagger}(Y_L)$.

(I) $\overline{L^\circ} \cong SL_n(q)$, $n \geqslant 3$, $Sp_{2n}(q)$, $n \geqslant 2$, or $Sp_4(2)'$ (and $p = 2$) and $[Y_L, L^\circ]$ is a corresponding natural module for $\overline{L^\circ}$. Moreover, $Y_L = [Y_L, L^\circ]$ or $p = 2$ and $\overline{L^\circ} \cong Sp_{2n}(q)$, $n \geqslant 2$.
(II) There exists a unique $\overline{L}$-invariant set $\mathcal{K}$ of subgroups of $\overline{L}$ such that $Y_L$ is a natural $SL_2(q)$-wreath product module for $\overline{L}$ with respect to $\mathcal{K}$. Moreover, $\overline{L^\circ} = O^p(\langle\mathcal{K}\rangle)\overline{Q}$, and $Q$ acts transitively on $\mathcal{K}$.

Put $M_1 := \langle Q^M\rangle C_S(Y_M)$ and $P_1 := M_1S$. Note that $M_1 = M^\circ C_S(Y_L) = L^\circ O_p(L)$ and so $O^p(M_1) = O^p(P_1) = M_\circ$. In particular $P_1S = L^\circ S$.

Assume Case (II). Let $P_1^*$ be the inverse image of $\langle\mathcal{K}\rangle$ in $P_1$. We apply 1.58. By 1.58(c) $O^p(P_1) = O^p(P_1^*)$ and $\overline{P_1^*} \trianglelefteq \overline{L}$; in particular $P_1^* \trianglelefteq M$. Since $O^p(P_1) = O^p(M_1)$, we conclude that Corollary B(2:i) – (2:iii) hold. Moreover, 1.58(f) implies Corollary B(2:iv).

If $|\mathcal{K}| = 1$ then Corollary B(1) holds with $n = 2$ and $Y_M = [Y_M, M^\circ]$. If $|\mathcal{K}| > 1$ then $Q \nleqslant P^*$, and Corollary B(2) holds.

Assume that Case (I) holds. Note that $SL_n(q)$, $n \geqslant 3$, $Sp_{2n}(q)$,$n \geqslant 2$ and $Sp_4(2)'$ all are quasisimple, except for $Sp_4(2)$. As $\overline{M_1} \cong \overline{L^\circ}$ we conclude that $F^*(\overline{M_1}) = \overline{M_1}'$ and $[Y, M_1]$ is a natural $SL_n(q)$, $Sp_{2n}(q)$ or $Sp_4(2)'$-module for $M_1$. To show that Corollary B(1) holds, it remains to determine $C_{M_1}(Y_M)$.

Since $M_1 = L^\circ O_p(L) = M^\circ O_p(M_1)$, we have $C_{M_1}(Y_M) = C_{M^\circ}(Y_M)O_p(M_1)$. Also 1.52(c) gives $[M^\circ, C_M(Y_M)] \leqslant O_p(M^\circ) \leqslant O_p(M_1)$. Thus $M_1/O_p(M_1)$ is a central extension of $\overline{M_1} = \overline{L^\circ}$ by the $p'$-group $C_{M_1}(Y_M)/O_p(M_1)$. Since $M_1$ is generated by $p$-elements we conclude that $C_{M_1}(Y_M)/O_p(M_1) \leqslant \Phi(M_1/O_p(M_1))$ and therefore $C_{M_1}(Y_M)/O_p(M_1)$ embeds into the Schur multiplier of $\overline{M_1}$. By [**Gr1**] the $p'$-part of the Schur multiplier of $SL_n(q)$ and $Sp_{2n}(q)$ is trivial, while the $2'$-part of the Schur multiplier of $Sp_4(2)'$ has order 3. (Note here that $Sp_4(2)$ inverts the Schur multiplier of $Sp_4(2)'$). It follows that either $C_{M_1}(Y_M) = O_p(M)$ or $M_1/O_p(M_1) \cong 3{\cdot}Sp_4(2)'$. Thus Corollary B(1) holds.

APPENDIX A

# Module theoretic Definitions and Results

In this chapter we present the module-theoretic definitions used throughout this paper. Results based on these definitions can be found in [**MS1**], [**MS2**], [**MS3**], [**MS4**], [**MS5**], and [**MS6**]. Some of these results are used so often in various different places that we state them either in this or in one of the later appendices.

Throughout this appendix $H$ is always a finite group and all modules considered are finite dimensional.

## A.1. Module-theoretic Definitions

Definition A.1. Let $V$ be an $\mathbb{F}_pH$-module and $A \leqslant H$. Then $A$ acts

(1) *quadratically* on $V$ if $[V, A, A] = 0$,
(2) *cubically* on $V$ if $[V, A, A, A] = 0$,
(3) *nilpotently* on $V$ if $[U, A] < U$ for every non-zero $A$-submodule $U \leqslant V$,
(4) *nearly quadratically* on $V$ if $A$ acts cubically on $V$ and

$$[v, A] + C_V(A) = [V, A] + C_V(A) \text{ for every } v \in V \backslash [V, A] + C_V(A).$$

Moreover, $V$ is a *quadratic, cubic* or *nearly quadratic module* for $H$, if there exists a subgroup $A \leqslant H$ with $[V, A] \neq 0$ that acts quadratically, cubically and nearly quadratically on $V$, respectively.

Definition A.2. An $\mathbb{F}_pH$-module $V$ is

(1) *simple* if $V \neq 0$, and 0 and $V$ are the only $H$-submodules of $V$,
(2) *central* if $[V, H] = 0$,
(3) *p-reduced* if $O_p(H/C_H(V)) = 1$,
(4) *perfect* if $V \neq 0$ and $[V, H] = V$,
(5) *quasisimple* if $V$ is perfect and $p$-reduced, and $V/C_V(O^p(H))$ is a simple $\mathbb{F}_pH$-module.

Definition A.3. Let $V$ an be $\mathbb{F}_pH$-module and $S \in Syl_p(H)$.

(a) $rad_V(H)$ is the intersection of all maximal $H$-submodules of $V$.
(b) $P_H(S, V) := O^{p'}(C_H(C_V(S)))$ is the *point-stabilizer* of $H$ on $V$ with respect to $S$.

Definition A.4. Let $V$ be an $\mathbb{F}_pH$-module and let $A$ and $B$ be $p$-subgroups of $H$ with $A \leqslant B$. Then $V$ is a *minimal asymmetric $\mathbb{F}_pH$-module* with respect to $A \leqslant B$ provided that

(i) $A \trianglelefteq N_H(B)$, and $B$ is a weakly closed subgroup of $H$,
(ii) $[V, A, B] = [V, B, A] = 0$,
(iii) $\langle A^H \rangle$ does not act nilpotently on $V$,
(iv) $\langle A^F \rangle$ acts quadratically on $V$ for every proper subgroup $F$ of $H$ with $B \leqslant F$.

Definition A.5. Let $V$ be an $\mathbb{F}_pH$-module and $Q$ a $p$-subgroup of $H$. Then $V$ is a *$Q$!-module* for $H$ with respect to $Q$ if

(i) $Q$ is not normal in $H$, and
(ii) $N_H(A) \leqslant N_H(Q) \quad$ for every $0 \neq A \leqslant C_V(Q)$.

Definition A.6. Let $\mathcal{K}$ be a non-empty $H$-invariant set of subgroups of $H$. Then $V$ is a *wreath product module* for $H$ (with respect to $\mathcal{K}$) if

$$V = \bigoplus_{K \in \mathcal{K}} [V, K] \qquad \text{and} \qquad C_V(\langle \mathcal{K} \rangle) = 0.$$

DEFINITION A.7. Let $V$ be an $\mathbb{F}_pH$-module, and let $A$ be a subgroup of $H$ such that $A/C_A(V)$ is an elementary abelian $p$-group. Then

(1) $A$ is an *offender* on $V$ if $|V/C_V(A)| \leqslant |A/C_A(V)|$,
(2) $A$ is an *over-offender* if $|V/C_V(A)| < |A/C_A(V)|$,
(3) $A$ is a *best offender* on $V$ if
$$|B||C_V(B)| \leqslant |A||C_V(A)| \quad \text{for every } B \leqslant A,$$
(4) $A$ is a *strong offender* on $V$ if $A$ is an offender on $V$ and
$$C_V(A) = C_V(a) \quad \text{for every } a \in A\backslash C_A(V),$$
(5) $A$ is a *root offender* on $V$ if $A$ is an offender on $V$ and
$$C_V(A) = C_V(a) \quad \text{and} \quad [V,A] = [V,a] \qquad \text{for every } a \in A\backslash C_A(V),$$
(6) $A$ is a *strong dual offender* on $V$ if $A$ acts nilpotently on $V$ and
$$[V,A] = [v,A] \text{ for every } v \in V\backslash C_V(A).$$

By $J_H(V)$ we denote the normal subgroup of $H$ generated by the best offenders of $H$ on $V$. A non-trivial subgroup $K \leqslant J_H(V)$ with $K \nleqslant C_H(V)$ that is minimal with respect to $K = [K, J_H(V)]$ is a $J_H(V)$-*component* of $H$. By $\mathcal{J}_H(V)$ we denote the set of $J_H(V)$-components of $H$ and by $J_H^*(V)$ the normal subgroup generated by $\mathcal{J}_H(V)$.

## A.2. Naming Modules

In this section we assign names to certain modules.

Let $\mathbb{K}$ be a finite field of characteristic $p$ and let $V$ be a vector space of finite dimension $m$ over $\mathbb{K}$. Let $\Lambda_2(V)$, $S_2(V)$ and $U_2(V)$ be the set of symplectic, symmetric, and unitary forms on $V$, where in the last case we assume that $\mathbb{K}$ is a quadratic extension of a subfield $\mathbb{F}$ and so has a unique automorphism of order 2. Let $V^* := Hom_{\mathbb{K}}(V,\mathbb{K})$ be the dual of $V$. Then $\Lambda^2(V) := \Lambda_2(V^*)$ is the *exterior (or symplectic) square* of $V$, $S^2(V) := S_2(V^*)$ is the *symmetric square* of $V$, and $U^2(V) := U_2(V^*)$ is the *unitary square* of $V$. Note that $\Lambda^2(V)$ and $S^2(V)$ are vector spaces of dimension $\binom{m}{2}$ and $\binom{m+1}{2}$, respectively, over $\mathbb{K}$, and $U^2(V)$ is a vector space of dimension $m^2$ over $\mathbb{F}$. Also $\Lambda^2(V)$, $S^2(V)$ and $U^2(V)$ are $\mathbb{F}_pSL_{\mathbb{K}}(V)$-modules.

Suppose now that $V$ is an $\mathbb{F}_pH$-module. Let $K$ be a group and $W$ an $\mathbb{F}_pK$-module. Suppose that there exists a surjective homomorphism
$$\tau: \quad H \ \to K/C_K(W), \quad h \ \mapsto \ \tau_h,$$
and an $\mathbb{F}_p$-isomorphism $\phi: V \to W$, $v \mapsto \phi(v)$, such that
$$(*) \qquad \phi(v^h) = \phi(v)^{\tau_h} \quad \text{for all} \quad v \in V \text{ and } h \in H.$$

– If $K \leqslant GL_{\mathbb{F}_p}(W)$, then $V$ is a *natural* $K$-*module* for $H$[1]. In particular, if $(W,f,h)$ is a non-degenerate classical space over the field $\mathbb{K}$ and $K = Cl_{\mathbb{K}}(W)$,[2] then $V$ is a natural $Cl_{\mathbb{K}}(W)$-module for $V$.
– If $W_0$ is vector space over $\mathbb{K}$, $K = SL_{\mathbb{K}}(W_0)$, and $W$ is $\Lambda^2(W_0)$, $S^2(W_0)$ or $U^2(W_0)$, then $W$ is the *exterior, symmetric or unitary (respectively) square* of a natural $SL_{\mathbb{K}}(W_0)$-module for $H$.
– Let $I$ be a finite set and $K \leqslant Sym(I)$. View $\mathbb{F}_p^I$ as an $K$-module via $(w_i)_{i\in I}^{\pi} = (w_{i\pi^{-1}})_{i\in I}$.
 - If $W = \mathbb{F}_p^I$ then $V$ is an $\mathbb{F}_pK$-*permutation module* for $H$.
 - If $p = 2$ and $W = \{(w_i)_{i\in I} \in \mathbb{F}_p^I \mid \sum_{i\in I} w_i = 0\}$ then $V$ is an *even* $\mathbb{F}_pK$-*permutation module* for $H$.
 - If $K = Alt(I)$ or $Sym(I)$ and $W$ is the non-central simple section of $K$ on $\mathbb{F}_p^I$, then $V$ is a *natural* $\mathbb{F}_pK$-*module* for $H$.
– If $K = Sz(2^k)$ and $W$ is the simple $\mathbb{F}_2Sz(2^k)$-module of $\mathbb{F}_{2^k}$-dimension 4, then $V$ is a *natural* $Sz(2^k)$-*module.*

[1] Note here that $K$ is assumed to be subgroup of $GL_{\mathbb{F}_p}(W)$, not only isomorphic to a subgroup.
[2] Classical spaces and $Cl(W)$ as defined in Appendix B

- If $K = G_2(2^k)$ and $W$ is the simple $\mathbb{F}_2 G_2(2^k)$-module of $\mathbb{F}_{2^k}$-dimension 6, then $V$ is a *natural* $G_2(2^k)$-module.
- If $K = {}^3D_4(p^k)$ and $W$ is the simple $\mathbb{F}_p {}^3D_4(p^k)$-module of $\mathbb{F}_{p^{3k}}$-dimension 8, then $V$ is a *natural* ${}^3D_4(p^k)$*-module.*
- If $K = E_6(p^k)$ and $W$ is a simple $\mathbb{F}_p E_6(p^k)$-module of $\mathbb{F}_{p^k}$-dimension 27, then $V$ is a *natural* $E_6(p^k)$*-module.*
- Suppose that $(U, f, h)$ is a non-degenerate orthogonal space with Clifford algebra $C$ with grading $C = C_1 \oplus C_{-1}$.[3] Suppose also that $K = Spin(U)$ and note that $C$ is a $K$-module by right multiplication. If $W$ is a minimal right ideal of $C$, then $V$ is a *spin* $K$*-module* for $H$. If $W$ is a minimal right ideal of $C_1$, then $V$ is a *half-spin* $K$*-module* for $H$.
- Let $U \leqslant \mathbb{F}_2^{24}$ be the binary Golay-code of length 24, dimension 12 and minimum distance 8. Note that $M := Aut(U) = Mat_{24}$ and let $K$ be one of $M = Mat_{24}$, $C_M(24) = Mat_{23}$, $C_M(\{23, 24\}) = Mat_{22}$ or $N_M(\{23, 24\}) = Aut(Mat_{22})$. If $W$ is the non-central simple section of $K$ on $U$, then $V$ is a *Golay code* $K$*-module* for $H$. If $W$ is the non-central simple section of $K$ on $\mathbb{F}_2^{24}/U$, then $V$ is a *Todd* $K$*-module.*
- Let $U \leqslant \mathbb{F}_3^{12}$ be the ternary Golay code of length 12, dimension 6 and minimal distance 6. Let $M = Aut(U) \sim 2{\cdot}Mat_{12}$ and let $K$ be one of $M$ or $C_M(12)' = Mat_{11}$. If $W$ is the non-central simple section of $K$ in $U$, then $V$ is a *Golay code* $K$*-module* for $H$. If $W$ is the non-central simple section of $K$ on $\mathbb{F}_3^{12}/U$, then $V$ is a *Todd* $K$*-module.*

Note that the Todd $K$-module and Golay-code $K$-module are simple and dual to each other. The following table lists the order of some of the modules defined above.

| $V$ | $\lvert V\rvert$ |
|---|---|
| natural $Cl_{\mathbb{K}}(W)$ | $\lvert W\rvert$ |
| exterior square of natural $SL_n(q)$ | $q^{\binom{n}{2}}$ |
| symmetric square of natural $SL_n(q)$ | $q^{\binom{n+1}{2}}$ |
| unitary square of natural $SL_n(q_0^2)$ | $q_0^{n^2}$ |
| half-spin $Spin_{2n}^+(q)$ | $q^{2^{n-1}}$ |
| half-spin $Spin_{2n}^-(q)$ | $q^{2^n}$ |
| half-spin $Spin_{2n+1}(q)$ | $q^{2^n}$ |
| natural $\mathbb{F}_p Sym(n)$, natural $\mathbb{F}_p Alt(n)$ | $p^{n-1}$ if $p \nmid n$, $p^{n-2}$ if $p \mid n$ |
| natural $Sz(2^k)$ | $2^{4k}$ |
| natural $G_2(2^k)$ | $2^{6k}$ |
| natural ${}^3D_4(p^k)$ | $p^{24k}$ |
| natural $E_6(p^k)$ | $p^{27k}$ |
| Todd $Mat_{22}(.2)$, Golay-code $Mat_{22}(.2)$ | $2^{10}$ |
| Todd $Mat_{23}$, Golay-code $Mat_{23}$ | $2^{11}$ |
| Todd $Mat_{24}$, Golay-code $Mat_{24}$ | $2^{11}$ |
| Todd $Mat_{11}$, Golay-code $Mat_{11}$ | $3^5$ |
| Todd $2{\cdot}Mat_{12}$, Golay-code $2{\cdot}Mat_{12}$ | $3^6$ |

We remark that, for given $H$, $K$ and $W$, the $\mathbb{F}_pH$-module $V$ fulfilling $(*)$ might not be unique up to isomorphism. For example, if $H$ has two different normal subgroups $H_1$ and $H_2$ with $H/H_i \cong K/C_K(W)$ then there exist $\mathbb{F}_pH$-modules $V_1$ and $V_2$ fulfilling $(*)$ with $C_H(V_i) = H_i$, and so $V_1$ and $V_2$ are not isomorphic. Also if $K/C_K(W)$ has outer automorphisms which are not induced by elements of $GL_{\mathbb{F}_p}(W)$ there will exist non-isomorphic $V$'s with the same $C_H(V)$. We list some examples which occur in this paper:

- $SL_{\mathbb{K}}(V)$, $\dim_{\mathbb{K}} V \geqslant 3$, has two natural $SL_{\mathbb{K}}(V)$-modules, namely $V$ and its dual $V^*$.
- $SL_{\mathbb{K}}(V)$, $\dim_{\mathbb{K}} V \geqslant 5$, has two exterior squares of natural $SL_{\mathbb{K}}(V)$-modules, namely $\Lambda^2(V)$ and $\Lambda^2(V)^*$.
- $SL_{\mathbb{K}}(V)$, $\dim_{\mathbb{K}} V \geqslant 3$, $char\mathbb{K}$ odd, has two symmetric squares of natural $SL_{\mathbb{K}}(V)$-modules, namely $S^2(V)$ and $S^2(V)^*$.

[3]For the Clifford algebra see also Appendix B

– $SL_{\mathbb{K}}(V)$, $\dim_{\mathbb{K}} V \geqslant 3$, $\dim_{\mathbb{F}_p} \mathbb{K}$ even, has two unitary squares of natural $SL_{\mathbb{K}}(V)$-modules, namely $U^2(V)$ and $U^2(V)^*$.
– $O_4^+(2)$ has two natural $O_4^+(2)$-modules.
– $Sp_4(q)$, $q$ even, has two natural $Sp_4(q)$-modules. For $q = 2$, these are also natural $Sym(6)$-modules.
– $Sp_4(2)'$ has two natural $Sp_4(2)'$-modules, which are also natural $Alt(6)$-modules.
– $Spin_8^+(q)$ has three natural $\Omega_8^+(q)$-modules, all of which are also half-spin $Spin_8^+(q)$ -modules. For $q$ odd, these are distinguished by the kernel of the action.
– $Spin_{10}^+(q)$ has two half-spin $Spin_{10}^+(q)$-modules dual to each other.
– $E_6(q)$ has two natural $E_6(q)$-modules, dual to each other.

We also remark that a group $H$ can have the exterior, symmetric or unitary square of a natural $SL_{\mathbb{K}}(V)$-module without having a natural $SL_{\mathbb{K}}(V)$-module. For example, if $p$ is odd and $\dim_{\mathbb{K}} V = 2$, then $S^2(V)$, viewed as a module for $PSL_{\mathbb{K}}(V)$, is the symmetric square of a natural $SL_{\mathbb{K}}(V)$-module, but $PSL_{\mathbb{K}}(V)$ does not have a natural $SL_{\mathbb{K}}(V)$-module.

## A.3. p-Reduced Modules

In this section $H$ is a finite group, $p$ is a prime and $V$ is an $\mathbb{F}_p H$-module.

Definition A.8. (a) $C_H^*(V)$ is inverse image $O_p(H/C_H(V))$ in $H$.
(b) $Y_V(H)$ is the $H$-submodule of $V$ generated by all the $p$-reduced $H$-submodules of $V$.

Lemma A.9. *Let $L \trianglelefteq\trianglelefteq H$. Then*
(a) *$L$ acts nilpotently on $V$ if and only if $L \leqslant C_H^*(V)$.*
(b) *$C_L^*(V) = C_L^*([V, L]) = C_L^*([V, O^p(L)]) \leqslant C_H^*(V)$.*
(c) *If $V$ is $p$-reduced for $H$ then $[V, L]$ and $[V, O^p(L)]$ are $p$-reduced for $L$.*
(d) *If $V$ is $p$-reduced and faithful for $H$, then each of $V$, $[V, L]$ and $[V, O^p(L)]$ is $p$-reduced and faithful for $L$.*

Proof. (a): Without loss $V$ is a faithful $H$-module. Then $C_H^*(V) = O_p(H)$. Note that $L$ acts nilpotently on $V$ if and only if $L$ is a $p$-group and so if and only if $L \leqslant O_p(H)$.

(b): Put $X := C_L^*([V, O^p(L)])$. Observe that $X$, $C_L^*(V)$ and $C_L^*([V, L])$ are normal in $L$ and thus subnormal in $H$. By (a) $C_L^*(V)$ and $C_L^*([V, L])$ act nilpotently on $[V, L]$ and $[V, O^p(L)]$, so again by (a),

$$(*) \qquad C_L^*(V) \leqslant C_L^*([V, L]) \leqslant X.$$

Note that $X$ acts as a $p$-group on $V/[V, O^p(L)]$. Since $V$ is an $\mathbb{F}_p$-module, $X$ acts nilpotently on $V/[V, O^p(L)]$ and thus also nilpotently on $V$. Hence $X \leqslant C_L^*(V)$ and $X = C_L^*(V)$, and equality holds in $(*)$. Since $X \trianglelefteq\trianglelefteq H$, (a) shows that $X \leqslant C_H^*(V)$.

(c) and (d): These are direct consequences of (b). □

Lemma A.10. *The following are equivalent:*
(a) *$V$ is $p$-reduced for $H$.*
(b) *$C_H^*(V) = C_H(V)$.*
(c) *Any normal subgroup of $H$ which acts nilpotently on $V$ centralizes $V$.*
(d) *Any subnormal subgroup of $H$ which acts nilpotently on $V$ centralizes $V$.*

Proof. By definition $V$ is $p$-reduced for $H$ if and only if $O_p(H/C_H(V)) = 1$, that is, if and only if $C_H^*(V)/C_H(V) = 1$, that is, if and only if $C_H^*(V) = C_H(V)$. Also by A.9(a) a subnormal subgroup of $H$ acts nilpotently on $V$ if and only if it is contained in $C_H^*(V)$. It follows that both (c) and (d) are equivalent to (a). □

Lemma A.11. (a) *Let $\mathcal{W}$ be a set of $p$-reduced $H$-submodule of $V$. Then $\langle \mathcal{W} \rangle$ is a $p$-reduced $H$-module.*
(b) *$Y_V(H)$ is the unique maximal $p$-reduced $H$-submodule of $V$.*

Proof. (a): Let $K$ be a normal subgroup of $H$ acting nilpotently on $\langle \mathcal{W} \rangle$. Then $K$ acts nilpotently on each $W \in \mathcal{W}$ and so centralizes $W$ and $\langle \mathcal{W} \rangle$. Thus A.10 shows that $\langle \mathcal{W} \rangle$ is $p$-reduced.

(b): By definition, $Y_V(H)$ is the submodule of $V$ generated by all the $p$-reduced $H$-submodules of $V$, and by (a) $Y_V(H)$ is $p$-reduced. Hence (b) holds. $\square$

Lemma A.12. *Let $L$ be a parabolic subgroup of $H$ and $U$ a $p$-reduced $L$-submodule of $V$. Then $\langle U^H \rangle$ is $p$-reduced for $H$. In particular, $Y_V(L) \leqslant Y_V(H)$.*

Proof. Let $M$ be a normal subgroup of $H$ acting nilpotently on $W := \langle U^H \rangle$. Then $M \cap L$ is a normal subgroup of $L$ acting nilpotently on $U$. Thus $M \cap L \leqslant C_M(U)$. Since $M/C_M(W)$ is a $p$-group and normal in $H/C_H(W)$ and $L$ is a parabolic subgroup of $H$, $M = (M \cap L)C_M(W)$. Thus $[M, U] = 1$ and since $M \trianglelefteq H$, also $[M, \langle U^H \rangle] = 1$. Hence by A.10, $\langle U^H \rangle$ is $p$-reduced for $H$. $\square$

Lemma A.13. *Let $P, Q \leqslant H$ and suppose that $P/C_P(V)$ is a $p$-group and $[P, Q] \leqslant C_H(V)$. Then*

(a) *If $Q/C_Q(V)$ is a $p'$-group, $C_Q(C_V(P)) = C_Q(V)$.*
(b) *$C^*_Q(C_V(P)) = C^*_Q(V)$.*
(c) *If $V$ is $p$-reduced for $Q$, then $C^*_Q(C_V(P)) = C_Q(C_V(P)) = C_Q(V)$.*
(d) *Suppose that $V$ is a faithful $Q$-module and $O_p(Q) = 1$, then $C_V(P)$ is a faithful $Q$-module.*

Proof. We may assume that $V$ is a faithful $H$-module, so $[P, Q] = 1$ and $P$ is a $p$-group.
(a): This follows from the $P \times Q$-Lemma.

(b): Let $x$ be a $p'$- element in $C^*_Q(C_V(P))$. Then $x$ centralizes $C_V(P)$, and so by (a) applied with $Q = \langle x \rangle$, $x$ centralizes $V$. Thus $x = 1$ and $C^*_Q(C_V(P))$ is a $p$-group. Hence $C^*_Q(C_V(P))$ is a normal $p$-subgroup of $Q$ and so $C^*_Q(C_V(P)) \leqslant C^*_Q(V)$. The other inclusion is obvious.

(c): Since $V$ is $p$-reduced for $Q$, $C_Q(V) = C^*_Q(V)$. Thus using (b),

$$C_Q(C_V(P)) \leqslant C^*_Q(C_V(P)) = C^*_Q(V) = C_Q(V) \leqslant C_Q(C_V(P)),$$

and so (c) holds.

(d): Since $O_p(Q) = 1$ and $V$ is a faithful $Q$-module, $V$ is a $p$-reduced $Q$-module. Thus (c) gives $C_Q(C_V(P)) = C_Q(V) = 1$. $\square$

Lemma A.14. *Let $U$ and $W$ be $H$-submodules of $V$ with $C_H(W) = C_H(U)$. Then $C^*_H(U) = C^*_H(W)$. In particular, $U$ is $p$-reduced for $H$ if and only if $W$ is $p$-reduced for $H$.*

Proof. Just recall that, by definition, $C^*_H(U)$ and $C^*_H(W)$ are the preimages of $O_p(H/C_H(U))$ and of $O_p(H/C_H(W))$, respectively, in $H$. $\square$

Lemma A.15. *Let $L \trianglelefteq\trianglelefteq H$.*

(a) *If $V$ is $p$-reduced for $H$, $V$ is $p$-reduced for $L$.*
(b) *$Y_V(H) \leqslant Y_V(L)$ and $C_L(Y_V(H)) = C_L(Y_V(L))$.*

Proof. (a): By A.10 $V$ is $p$-reduced if and only if any subnormal subgroup of $H$ which acts nilpotently on $V$ centralizes $V$. As any subnormal subgroup of $L$ is subnormal in $H$, this gives (a).

(b): By (a) $Y_V(H)$ is $p$-reduced for $L$ and so $Y_V(H) \leqslant Y_V(L)$ and $C_L(Y_V(L)) \leqslant C_L(Y_V(H))$. It remains to show that $C_L(Y_V(H)) \leqslant C_L(Y_V(L))$. By induction on $H/L$ we may assume that $L \trianglelefteq H$. Hence $Y_V(L)$ is an $H$-submodule of $V$. To simplify notation we replace $V$ by $Y_V(L)$ and so $V$ is $p$-reduced for $L$. Let $W$ be an $H$-submodule of $V$ minimal with $C_L(W) = C_L(V)$. Since $V$ is $p$-reduced for $L$, A.14 shows that $W$ is also $p$-reduced for $L$. Let $P := C^*_H(W)$. Then $P/C_P(W)$ is a $p$-group and $[P, L] \leqslant C^*_L(W) = C_L(W)$. Thus A.13(c) shows that $C_L(C_W(P)) = C_L(W) = C_L(V)$. The minimal choice of $W$ implies that $W = C_W(P)$. Thus $P = C_H(W)$ and $W$ is $p$-reduced for $H$. Hence $W \leqslant Y_V(H)$ and so

$$C_L(Y_V(H)) \leqslant C_L(W) = C_L(V).$$

□

## A.4. Wreath Product Modules

In this section $H$ is a finite group and $V$ a finite $\mathbb{F}_pH$-module.

LEMMA A.16. *Let $\mathcal{K}$ be an $H$-invariant set of subgroups of $H$ and suppose that $V$ is a wreath product module for $H$ with respect to $\mathcal{K}$. Then for each $A \in \mathcal{K}$:*

(a) *If $[V, A] \neq 0$. then $N_H([V, A]) = N_H(A)$.*
(b) *$[V, B] \leqslant C_V(A)$ for all $B \in \mathcal{K}\backslash\{A\}$; in particular*

$$V = [V, A] \oplus C_V(A), \qquad C_V(A) = \left[V, \left\langle\mathcal{K}\backslash\{A\}\right\rangle\right], \quad \text{and} \quad [V, A] = C_V\left(\left\langle\mathcal{K}\backslash\{A\}\right\rangle\right).$$

(c) *$[V, A] = [V, A, A]$ and $C_A([V, A]) = C_A(V)$.*
(d) *$[A, B] \leqslant C_{\langle\mathcal{K}\rangle}(V)$ for all $B \in \mathcal{K}\backslash\{A\}$.*

PROOF. By the definition of a wreath product module

$$(*) \qquad V = \bigoplus_{K\in\mathcal{K}} [V, K] \quad \text{and} \quad C_V(\langle\mathcal{K}\rangle) = 0.$$

(a): Clearly $N_H(A) \leqslant N_H([V, A])$. Let $h \in H$ and assume that $[V, A] \neq 0$. Since $\mathcal{K}$ is $H$-invariant, $A^h \in \mathcal{K}$. Hence $(*)$ shows that either $A = A^h$ or $[V, A] \cap [V, A^h] = 0$. In the second case $[V, A] \neq [V, A]^h$ since $[V, A] \neq 0$. Thus also $N_H([V, A]) \leqslant N_H(A)$.

(b): Put $\mathcal{K}_A := \mathcal{K}\backslash\{A\}$ and $W := \sum_{B\in\mathcal{K}_A}[V, B]$. Note that $V = [V, A] \oplus W$ by $(*)$. Since $\mathcal{K}_A$ is $A$-invariant, also $W$ is $A$-invariant, and so $[W, A] \leqslant [V, A] \cap W = 0$. Hence $W \leqslant C_V(A)$ and $V = [V, A] + C_V(A)$.

Since this is true for all $A \in \mathcal{K}$, $[V, A]$ is centralized by each $B \in \mathcal{K}_A$. Hence $C_{[V,A]}(A) \leqslant C_V(\langle\mathcal{K}\rangle) = 0$ and $V = [V, A] \oplus C_V(A)$.

(c): By (b), $V = [V, A] + C_V(A)$, and (c) follows.

(d): Let $B \in \mathcal{K}_A$. By (b) $[V, A, B] = [V, B, A] = 0$, and the Three Subgroups Lemma gives $[A, B, V] = 0$. Thus $[A, B] \leqslant C_{\langle\mathcal{K}\rangle}(V)$. □

LEMMA A.17. *Let $\mathcal{K}$ be an $H$-invariant set of subgroups of $H$. Suppose that $V$ is a faithful $\langle\mathcal{K}\rangle$-module and a wreath product module for $H$ with respect to $\mathcal{K}$. Then for all $A \in \mathcal{K}$:*

(a) *$[V, A]$ is a faithful $A$-module.*
(b) *$\langle\mathcal{K}\rangle = \times_{K\in\mathcal{K}} K$.*

PROOF. (a): Since $\langle\mathcal{K}\rangle$ acts faithfully on $V$, $C_A(V) = 1$. By A.16(c) $C_A([V, A]) = C_A(V)$ and so (a) holds.

(b): Put $L_A := \langle\mathcal{K}\backslash\{A\}\rangle$. By A.16(b) $L_A$ centralizes $[V, A]$ and so by (a) $L_A \cap A = 1$. By A.16(d) $[A, B] \leqslant C_{\langle\mathcal{K}\rangle}(V)$. The faithful action of $\langle\mathcal{K}\rangle$ implies that $[A, B] = 1$ and thus $A \trianglelefteq \langle\mathcal{K}\rangle$.

We have proved that $A \trianglelefteq \langle\mathcal{K}\rangle$ and $A \cap L_A = 1$ for all $A \in \mathcal{K}$. Hence $\langle\mathcal{K}\rangle = \times_{K\in\mathcal{K}} K$. □

LEMMA A.18. *Let $\mathcal{K}$ be an $H$-invariant set of subgroups of $H$. Suppose that $V$ is a faithful $p$-reduced $\langle\mathcal{K}\rangle$-module and $[V, \langle\mathcal{K}\rangle]$ is a wreath product module for $H$ with respect to $\mathcal{K}$. Then $[V, K] = [V, \langle\mathcal{K}\rangle, K]$ for each $K \in \mathcal{K}$.*

PROOF. Put $R := \langle\mathcal{K}\rangle$ and $W := [V, R]$. Since $V$ is a faithful and $p$-reduced $R$-module, A.9(d) shows that $W$ is a faithful $R$-module. Thus by A.17 $R = \times_{K\in\mathcal{K}} K$.

Let $K \in \mathcal{K}$ and put $L := \langle\mathcal{K}\backslash\{K\}\rangle$. Note that $[L, K] = 1$ and $[V, L, K] \leqslant [W, K]$. The Three Subgroups Lemma gives $[V, K, L] \leqslant [W, K]$. On the other hand by A.16(b) $W = [W, K] \oplus C_V(K)$, $[W, K] = C_W(L)$ and $C_W(K) = [W, L]$. Hence $C_{[W,L]}(L) = 0$ and $[W, L]$ and $W/[W, K]$ are isomorphic $L$-modules. It follows that $C_{W/[W,K]}(L) = 0$, and since $[V, K, L] \leqslant [W, K]$ we get $[V, K] \leqslant [W, K]$. Thus $[V, K] = [W, K]$. □

Definition A.19. Let $\Delta$ be a set of non-zero subspaces of $V$. Then $\Delta$ is a system of imprimitivity for $H$ on $V$ if

$$\Delta \text{ is } H\text{-invariant}, \qquad |\Delta| > 1, \qquad \text{and } V = \bigoplus_{W \in \Delta} W.$$

Lemma A.20. *Let $\Delta$ be a system of imprimitivity for $H$ on $V$. Suppose that $E$ is a subgroup of $H$ that acts non-trivially on $\Delta$ and that $|[V,E]| \leqslant |W|$ for some $W \in \Delta \backslash C_\Delta(E)$. Then*

(a) $|W| = |[V,E]|$, $W \cap [V,E] = 0$ *and* $N_E(W) = C_E(V) = C_E(\Delta)$ *for all* $W \in \Delta\backslash C_\Delta(E)$,
(b) $|E/C_E(V)| = 2 = |\Delta\backslash C_\Delta(E)|$, *and*
(c) $[X,E] = 0$ *for all* $X \in C_\Delta(E)$.

Proof. Pick $e \in E$ with $W^e \neq W$ and put $\Lambda := \{w^e - w \mid w \in W\}$. Since $W \cap W^e = 0$ we get $\Lambda \cap W = 0$ and $|\Lambda| = |W|$. Now $|[V,E]| \leqslant |W|$ and $\Lambda \subseteq [V,E]$ imply $[V,E] = \Lambda$, and so $[V,E] = \Lambda \leqslant W + W^e$. It follows that $\langle W^E \rangle = W + [W,E] \leqslant W + W^e$. Since $V = \bigoplus_{D \in \Delta} D$ this gives $W^E = \{W, W^e\}$. Put $Y = \langle \Delta \backslash W^E \rangle$. Then $Y$ is $E$-invariant and so $[Y,E] \leqslant Y \cap [V,E] \leqslant Y \cap (W + W^e) = 0$. In particular, $\Delta\backslash C_\Delta(E) = \{W, W^e\}$ and $|E/C_E(\Delta)| = 2$. Moreover, $[W, C_E(\Delta)] \leqslant W \cap [V,E] = W \cap \Lambda = 0$ and since $C_E(\Delta) \trianglelefteqslant E$ also $[W^e, C_E(\Delta)] = 0$. Thus $C_E(\Delta) = C_E(V)$ and the lemma is proved. □

Lemma A.21. *Let $\mathcal{K}$ be a non-empty $H$-invariant set of subgroups of $H$. Suppose that*

(i) $[V,A,A] = [V,A]$ *and* $[V,A] \cap C_V(A) = 0$ *for all* $A \in \mathcal{K}$, *and*
(ii) $[A,B] \leqslant A$ *and* $[V,A] \cap [V,B] = 0$ *for all distinct* $A, B \in \mathcal{K}$.

*Then $[V, \langle \mathcal{K} \rangle]$ is a wreath product module for $H$ with respect to $\mathcal{K}$.*

Proof. Observe that (ii) shows that any two subgroups in $\mathcal{K}$ normalize each other. Let $A \in \mathcal{K}$ and put $W = \sum_{A \neq B \in \mathcal{K}} [V,B]$. Since $A$ normalizes $B$, $[V,B,A] \leqslant [V,B] \cap [V,A] = 0$ and so $[W,A] = 0$ and $[V,A] \cap W \leqslant [V,A] \cap C_V(A) = 0$.

Also $[V,\langle \mathcal{K}\rangle] = \sum_{A \in \mathcal{K}} [V,A]$ and so $[V,\langle \mathcal{K}\rangle] = \bigoplus_{A \in \mathcal{K}} [V,A]$ by the definition of an internal direct sum. From $W \leqslant C_V(A)$ and $[V,A] \cap C_V(A) = 0$ we conclude that $C_{[V,\langle \mathcal{K}\rangle]}(A) = W$. As this holds for all $A \in \mathcal{K}$ we get $C_{[V,\langle \mathcal{K}\rangle]}(\langle \mathcal{K}\rangle) = 0$.

Since $[V,A] = [V,A,A]$ we have $[[V,\langle \mathcal{K}\rangle],A] = [V,A]$, and so $[V,\langle \mathcal{K}\rangle]$ is a wreath product module for $H$ with respect to $\mathcal{K}$. □

Lemma A.22. *Let $\mathcal{K}$ be a non-empty $H$-invariant set of subnormal subgroups of $H$. Suppose that*

(i) $|K/C_K([V,K])| > 2$ *for all* $K \in \mathcal{K}$,
(ii) $[V,K] \cap C_V(K) = 0$ *and* $[V,K,K] = [V,K]$ *for all* $K \in \mathcal{K}$,
(iii) $[V,A] \cap [V,B] = 0$ *for all distinct* $A, B$ *in* $\mathcal{K}$ *with* $[A,B] \leqslant A \cap B$, *and*
(iv) $[V,B] \nleqslant [V,A]$ *for all distinct* $A, B$ *in* $\mathcal{K}$.

*Then $[V,\langle \mathcal{K}\rangle]$ is a wreath product module for $H$ with respect to $\mathcal{K}$.*

Proof. If $h \in N_H([V,K])$, then $[V,K] = [V,K^h]$ and by (iv) $K = K^h$. We have shown:

$$(*) \qquad N_H([V,K]) \leqslant N_H(K) \text{ for every } K \in \mathcal{K}.$$

Let $A, B$ be distinct elements of $\mathcal{K}$. In view of A.21, (ii) and (iii) it suffices to show that $A$ and $B$ normalizes each other. Put $R := \langle A, B\rangle$. If $R = A$, then $[V,B] \leqslant [V,R] = [V,A]$ a contradiction to (iv). Thus $A \neq R$ and so $A$ is a proper subnormal subgroup of $R$. Hence $\langle A^R \rangle \neq R$ and by induction on $|\langle \mathcal{K}\rangle|$, $[V,\langle A^R\rangle]$ is a wreath product module for $R$ with respect to $A^R$. By symmetry, also $B \neq R$, and $[V,\langle B^R\rangle]$ is a wreath product module for $R$ with respect to $B^R$.

We now assume without loss that $|[V,B]| \leqslant |[V,A]|$. Suppose for a contradiction that $B$ does not normalize $A$. Then by $(*)$ $B$ does not normalize $[V,A]$. Put $\Delta := [V,A]^R$. Since $U := [V,\langle A^R\rangle]$ is a wreath product module for $R$ with respect to $A^R$, $\Delta$ is a system of imprimitivity for $R$ on $U$. Since $|[V,B]| \leqslant |[V,A]|$, we can apply A.20 and conclude that $|B/C_B(U)| = 2$ and $|[U,B]| = |[V,A]|$. Since $|[V,B]| \leqslant |[V,A]|$ this gives $|[V,B]| = |[U,B]| = |[V,A]|$ and $[V,B] = [U,B] \leqslant U$. But then $C_B(U) \leqslant C_B([V,B])$ and $|B/C_B([V,B])| \leqslant |B/C_B(U)| \leqslant 2$, contrary to (i).

Thus $B$ normalize $A$; in particular $AB$ is a subgroup of $H$. Suppose for a contradiction that $A$ does not normalizes $B$ and pick $a \in A$ with $B^a \neq B$. Since $V$ is a wreath product module for $R$ with respect to $B^R$, A.16(c) shows that $[V, B, B^a] = 0$. Note that $B \leqslant BA = B^aA$. Also by (ii), $[V, B] = [V, B, B]$ and so

$$[V, B] = [V, B, B] \leqslant [V, B, B^aA] = [V, B, A] \leqslant [V, A],$$

a contradiction to (iv)

Hence $A$ and $B$ normalize each other, and the lemma is proved. □

Lemma A.23. *Let $\mathcal{K}$ be a non-empty $H$-invariant set of subnormal subgroups of $H$ and suppose that*

(i) $|A/C_K([V, A])| > 2$ *for all* $A \in \mathcal{K}$.
(ii) $[V, A]$ *is a simple $K$-module for all* $A \in \mathcal{K}$.
(iii) $[V, B] \nleqslant [V, A]$ *for all distinct $A$ and $B$ in* $\mathcal{K}$.

*Then $[V, \langle\mathcal{K}\rangle]$ is a wreath product module for $H$ with respect to $\mathcal{K}$.*

Proof. We will verify that the hypothesis of A.22 holds. Let $K \in \mathcal{K}$. Since $|K/C_K([V, K])| > 2$, $K$ does not centralize $[V, K]$, and since $[V, K]$ is a simple $K$-module, we conclude that $[V, K] = [V, K, K]$ and $[V, K] \cap C_V(K) = 0$. So A.22(i) and (ii) hold.

Now let $A, B$ be distinct elements of $\mathcal{K}$ with $[A, B] \leqslant A \cap B$. Then $B$ normalizes $A$ and $[V, A] \cap [V, B]$ is an $B$-submodule of $[V, B]$. Since $[V, B] \nleqslant [V, A]$, it is a proper $B$-submodule, and since $[V, B]$ is simple, we conclude that $[V, A] \cap [V, B] = 0$. Hence also A.22(iii) holds. Also, (iii) is the same as A.22(iv).

Thus, we can apply A.22, and $[V, \langle\mathcal{K}\rangle]$ is a wreath product module for $H$ with respect to $\mathcal{K}$. □

Definition A.24. Let $H$ be a finite group, $\mathcal{K}$ a non-empty $H$-invariant set of subgroups of $H$ and $\Xi$ a class of modules. Then $V$ is a $\Xi$-wreath product module for $H$ with respect to $\mathcal{K}$ provided that $V$ is wreath product module for $H$ with respect to $\mathcal{K}$ and for each $K \in \mathcal{K}$, $[V, K]$ is a $\Xi$-module for $K$.

Most important for our paper are faithful natural $SL_2(q)$-wreath product modules, that is, where $\Xi$ consists only of the natural $\mathbb{F}_pSL_2(q)$- modules and the action of $H$ is faithful. The next remark gives an explicit description of natural $SL_n(q)$-wreath product modules.

Remark A.25. Suppose that $V$ is a faithful $H$-module and $\mathcal{K}$ is non-empty $H$-invariant set of subgroups of $H$. Then $V$ is a natural $SL_n(q)$-wreath product module for $H$ with respect to $\mathcal{K}$ if and only if

$$V = \bigoplus_{K \in \mathcal{K}} [V, K] \qquad \text{and} \qquad \langle\mathcal{K}\rangle = \bigtimes_{K \in \mathcal{K}} K,$$

and for each $K \in \mathcal{K}$, $K \cong SL_n(q)$ and $[V, K]$ is a natural $SL_n(q)$-module for $K$.

Proof. Suppose that $V$ is a natural $SL_n(q)$-wreath product-module for $H$ with respect to $V$. Then by definition, $V = \bigoplus_{K \in \mathcal{K}}[V, K]$ and for each $K \in \mathcal{K}$, $[V, K]$ is a natural $SL_n(q)$-module for $K$. Since $V$ is faithful, A.17 shows that $[V, K]$ is a faithful $K$-module and $\langle\mathcal{K}\rangle = \bigtimes_{K \in \mathcal{K}} K$. So $[V, K]$ is a faithful natural $SL_n(q)$-module for $K$ and thus $K \cong SL_n(q)$.

The converse should be obvious. □

Lemma A.26. *Let $\mathcal{K}$ be a non-empty $H$-invariant set of subnormal subgroups of $H$.*

(a) *Suppose that for all $K \in \mathcal{K}$, $K$ is quasisimple and $[V, K]$ is a simple $K$-module. Then $[V, \langle\mathcal{K}\rangle]$ is a wreath product module for $H$ with respect to $\mathcal{K}$.*
(b) *Let $q$ be a power of $p$ and $n \geqslant 2$. Suppose that $V$ is a faithful $H$-module and*
  (i) *for all $K \in \mathcal{K}$, $K \cong SL_n(q)$ and $[V, K]$ is a natural $SL_n(q)$-module for $K$, or*
  (ii) *for all $K \in \mathcal{K}$, $K \cong SL_2(q)'$ and $[V, K]$ is a natural $SL_2(q)'$-module for $K$.*

  *Then $[V, \langle\mathcal{K}\rangle]$ is a natural $SL_n(q)$- or natural $SL_2(q)'$-wreath product module for $H$ with respect to $\langle\mathcal{K}\rangle$.*

PROOF. We will prove (a) and (b) simultaneously by verifying the hypothesis of A.23. Let $K \in \mathcal{K}$. Observe that in both cases $[V, K]$ is a non-central simple $K$-module. Also $|K/C_K([V,K])| > 2$ since in (a) $K$ is quasisimple and in (b) $|K/C_K([V,K])| \geqslant |SL_2(q)'| \geqslant 3$. Hence A.23(i) and (ii) hold and it remains to verify that $K = E$ for all $E, K \in \mathcal{K}$ with $[V, E] \leqslant [V, K]$.

Put $W := [V, K]$, so $[V, E] \leqslant W$. Since $W$ is a simple $K$-module, $End_{\mathbb{F}_p H}(W)$ is a finite division ring by Schur's Lemma, and so is commutative by Wedderburn's theorem. We get

$$(*) \qquad [W, [E, K]] \neq 0 \quad \text{or} \quad E/C_E(W) \text{ is abelian.}$$

Suppose first that $K$ and $E$ are quasisimple. Then $K$ and $E$ are components of $H$, so $K = E$ or $[K, E] = 1$. By $(*)$ either $[W, [E, K]] \neq 0$ or $E/C_E(W)$ is abelian. In the first case $[E, K] \neq 1$ and so $E = K$. In the second case $E = C_E(W)$ since $E$ is quasisimple. But then $[V, E] = [V, E, E] \leqslant [W, E] = 0$, and $[V, E]$ is central $E$-module, a contradiction.

Suppose next that one of $K$ and $E$ is not quasisimple. Then we are in case (b) and $K \cong E \cong SL_2(p)$ or $SL_2(p)'$ with $p = 2$ or $3$. In particular, $O_{p'}(K) \neq 1$ and $W = [V, E] = [W, O_{p'}(K)]$. Put $F := \langle E, K\rangle$ and $R := O_{p'}(F)$. Since $K$ is subnormal in $F$, $1 \neq O_{p'}(K) \leqslant R$. Hence $[V, R] = W = [V, F]$, and coprime action shows that $V = W \oplus C_V(R)$. Since $F$ normalizes $R$, we conclude that $[C_V(R), F] \leqslant C_V(R) \cap W = 0$. Thus $V = W \oplus C_V(F)$. Since $V$ is a faithful $H$-module, $F$ acts faithfully on $W$. Note that $|W| = p^2$ and $Aut(W) = GL_2(p)$ has a unique subgroup isomorphic to $K$. So $E$ and $K$ have the same image in $Aut(W)$ and since $C_F(W) = 1$, $E = K$. □

LEMMA A.27. *Let $P \leqslant H$ and let $\mathcal{K}$ be a non-empty $P$-invariant set of subgroups of $P$. Suppose that*

(i) $O^p(\langle\mathcal{K}\rangle) \trianglelefteq H$,
(ii) *$V$ is natural $SL_2(q)$-wreath product module for $P$ with respect to $\mathcal{K}$, $q = p^n$,*
(iii) *$V$ is a faithful $H$-module.*

*Then*

(a) *If $E \leqslant H$ such that $[V, E]$ is a faithful natural $SL_2(q)$-module for $E$, then $E \in \mathcal{K}$.*
(b) *$V$ is a natural $SL_2(q)$-wreath product module for $H$ with respect to $\mathcal{K}$.*
(c) *$\mathcal{K}$ is the unique $H$-invariant set of subgroups of $H$ such that $V$ is a natural $SL_2(q)$-wreath product module for $H$ with respect to $\mathcal{K}$.*

PROOF. (a): Put $R := O^p(\langle\mathcal{K}\rangle)$. Since $V$ is a natural $SL_2(q)$-wreath product module for $P$ with respect to $\mathcal{K}$ and by (iii) $V$ is a faithful $P$-module, the definition of a wreath product module and A.17(b) give

$$V = \bigoplus_{K\in\mathcal{K}} [V, K] \quad \text{and} \quad \langle\mathcal{K}\rangle = \bigtimes_{K\in\mathcal{K}} K,$$

and $[V, K]$ is a natural $SL_2(q)$ modules for each $K \in \mathcal{K}$. By A.17(a) $[V, K]$ is a faithful $K$-module and so $K \cong SL_2(q)$. In particular, $[V, K]$ is a natural $SL_2(q)'$ module for $O^p(K)$ and for $R$. It follows that $[V, K]$, $K \in \mathcal{K}$, are pairwise non-isomorphic simple $R$-submodules of $V$ and so $\Delta := \{[V, K] \mid K \in \mathcal{K}\}$ is the set of Wedderburn components for $R$ on $V$. Since $R \trianglelefteq H$ we conclude that $H$ acts on $\Delta$.

Note that $|[V, K]| = q^2 = |[V, E]|$. Suppose that $E$ acts non-trivially on $\Delta$. Then $|\Delta| \geqslant 2$ and $\Delta$ is a system on imprimitivity for $H$ on $V$. Hence A.20(b) implies that $|E/C_E(V)| = 2$, a contradiction. Thus $E$ acts trivially on $\Delta$. In particular, $[V, E] = \bigoplus_{W\in\Delta}[W, E]$, and since $[V, E]$ is a simple $E$ module, there exists a unique $W \in \Delta$ with $[W, E] \neq 0$. Let $K$ be the unique element of $\mathcal{K}$ with $W = [V, K]$.

Put $F := \langle K, E\rangle$ and $U := \sum_{X\in\Delta\setminus\{W\}} X$. Then $V = W \oplus U$ and $F$ centralizes $U$. So $F$ acts faithfully on $W$. Moreover, since $R \trianglelefteq H$, $F$ normalizes $C_R(U) = O^p(K)$. Put $\mathbb{F} := End_K(W)$ and observe that $\mathbb{F} \cong \mathbb{F}_q$.

We claim that $F$ acts $\mathbb{F}$-linearly on $W$. If $p = q$, then $\mathbb{F} = \mathbb{F}_q$ and this is obvious. If $q > p$, then $K = O^p(K)$ and $F$ normalizes $K$. Since $F = EK$ is perfect and $Aut(\mathbb{F})$ is abelian, we again conclude that $F$ acts $\mathbb{F}$-linearly.

Note that $GL_{\mathbb{F}}(W)/SL_{\mathbb{F}}(W)$ is a $p'$-group and $SL_2(q)$ is generated by $p$-elements, so $SL_{\mathbb{F}}(W) \cong SL_2(q)$ is the unique subgroup of $GL_{\mathbb{F}}(W)$ isomorphic to $SL_2(q)$. Since $F$ acts faithfully on $W$ and both $E$ and $K$ are isomorphic to $SL_2(q)$, this gives $E = F = K$. So $E \in \mathcal{K}$.

(b): From (a) we conclude that $\mathcal{K}$ is $H$-invariant and so (b) holds.

(c) follows immediately from (a). □

LEMMA A.28. *Let $K \leqslant H$ and $S \in Syl_p(H)$, and suppose that $V$ is a faithful $H$-module. Put $\mathcal{K} := K^H$ and $R := \langle \mathcal{K} \rangle$. Suppose that $O^p(H) \leqslant R$ and $V$ is a natural $SL_2(q)$-wreath product module for $H$ with respect to $\mathcal{K}$, $q = p^n$. Then the following hold:*

(a) *$H = RS$, and $S$ is transitive on $\mathcal{K}$.*
(b) *$H$ is $p$-minimal, and $N_H(R \cap S)$ is the unique maximal subgroup of $H$ containing $S$.*
(c) *$V$ is a simple $H$-module. In particular, $V$ is a $p$-reduced $H$-module and $O_p(H) = 1$.*
(d) *Up to conjugation in $H$, $K$ is the unique subgroup of $H$ such that $K \cong SL_2(q)$ and $[V, K]$ is a natural $SL_2(q)$-module for $K$.*
(e) *Let $\mathcal{S} := \{v \in V \mid [v, F] \neq 0$ for all $F \in \mathcal{K}\}$. Then $R$ is transitive on $\mathcal{S}$, and $C_V(T)^\sharp \subseteq \mathcal{S}$ for every $T \leqslant H$ that is transitive on $\mathcal{K}$.*

PROOF. (a): Since $O^p(H) \leqslant R$, $H = RS$, and since $R$ normalizes $K$, $K^H = K^{RS} = K^S$.

(b): Let $L$ be a maximal subgroup of $H$ containing $S$. Since $K \cong SL_2(q)$, and $N_K(K \cap S)$ is the only maximal subgroup of $K$ containing $K \cap S$, it follows that $K \leqslant L$ or $K \cap L \leqslant N_K(K \cap S)$. In the first case $R \leqslant L$ since $S$ is transitive on $\mathcal{K}$, and so $L = H$, a contradiction. Hence $K \cap L \leqslant N_K(K \cap S)$; in particular $K \cap S = O_p(K \cap L)$. Since $K \cap L \trianglelefteq R \cap L$, we conclude that $R \cap L \trianglelefteq N_R(K \cap S)$. Now again the transitivity of $S$ on $\mathcal{K}$ gives $R \cap S = \langle (K \cap S)^S \rangle$ and $R \cap L \leqslant N_R(R \cap S)$ and so $L \leqslant N_H(R \cap S)$. Since $N_H(R \cap S)$ is a proper subgroup of $H$, $L = N_H(R \cap S)$ follows.

(c): Let $W$ be a non-zero $H$-submodule of $V$. By definition of a wreath product module, $C_V(\langle \mathcal{K} \rangle) = 0$ and so $[W, A] \neq 0$ for some $A \in \mathcal{K}$. Thus $W \cap [V, A] \neq 0$. Since $[V, A]$ is a natural $SL_2(q)$-module for $A$, $[V, A]$ is simple $A$-module and so $[V, A] \leqslant W$. As $H$ acts transitively on $\mathcal{K}$, this gives $[V, A] \leqslant W$ for all $A \in \mathcal{K}$. By definition of wreath product module, $V = \bigoplus_{A \in \mathcal{K}} [V, A]$ and so $V = W$.

(d): By A.27, applied with $(H, H, \mathcal{K})$ in place of $(P, H, \mathcal{K})$, any $E \leqslant H$ such that $[V, E]$ is a faithful natural $SL_2(q)$-module for $E$ is contained in $\mathcal{K}$ and so is conjugate to $K$.

(e): Let $v \in C_V(T) \backslash \mathcal{S}$. Then $[v, F] = 0$ for some $F \in \mathcal{K}$. Since $T$ acts transitively on $\mathcal{K}$, this gives $v \in C_V(\langle \mathcal{K} \rangle) = 0$. Thus $C_V(T)^\sharp \subseteq \mathcal{S}$. Since $K$ is transitive on $[V, K]$, $R$ is transitive on $\mathcal{S}$. □

## A.5. Offenders

In this section $p$ is a prime, $H$ is a finite group, $V$ is an $\mathbb{F}_p H$-module and $V^* := Hom_{\mathbb{F}_p}(V, \mathbb{F}_p)$ is the dual of $V$.

LEMMA A.29 (Chermak-Delgado Measuring Argument). *Let $\alpha$ be a positive real number and $X \leqslant H$.*

(a) *Suppose that $|V/C_V(B)| \leqslant |B/C_B(V)|^\alpha$ for some $B \leqslant X$ with $[V, B] \neq 1$. Then there exists a $N_H(X)$-invariant subgroup $D$ of $X$ such that $[V, D] \neq 1$, $|V/C_V(D)| \leqslant |D/C_D(V)|^\alpha$ and $|A|^\alpha |C_V(A)| \leqslant |D|^\alpha |C_V(D)|$ for all $A \leqslant X$.*
(b) *Suppose that $X/C_X(V)$ is elementary abelian and $X$ contains a non-trivial offender on $V$. Then $X$ contains an $N_H(X)$-invariant non-trivial best offender $D$ on $V$ with $|A||C_V(A)| \leqslant |D||C_V(D)|$ for all $A \leqslant X$.*

PROOF. Replacing $H$ be $H/C_H(V)$ we may assume that $V$ is a faithful $H$-module.
(a): Since $B \neq 1$ also $X \neq 1$ and we can define
$$m_\alpha = \max \left\{ \, |A|^\alpha |C_V(A)| \;\middle|\; 1 \neq A \leqslant X \, \right\}$$
and
$$\alpha\mathcal{M} = \left\{ \, 1 \neq A \leqslant X \;\middle|\; |A|^\alpha |C_V(A)| = m_\alpha \, \right\}.$$
Observe that $\alpha\mathcal{M} \neq \varnothing$. Since $|V/C_V(B)| \leqslant |B/C_B(V)|^\alpha = |B|^\alpha$, we have
$$m_\alpha \geqslant |B|^\alpha |C_V(B)| \geqslant |V|.$$

Thus [**CD**, 1.2] shows that $\alpha\mathcal{M}$ has a unique maximal element $D$. Since $D$ is unique, $D$ is $N_H(X)$-invariant. Also

$$|D|^\alpha |C_V(D)| = m_\alpha \geqslant |V|,$$

and so $|V/C_V(D)| \leqslant |D|^\alpha = |D/C_D(V)|^\alpha$.

By the definition of $\alpha\mathcal{M}$, $D \neq 1$ and so $[V, D] \neq 1$, and by the definition of $m_\alpha$,

$$|A|^\alpha |C_V(A)| \leqslant m_\alpha = |D|^\alpha |C_D(V)|$$

for all $1 \neq A \leqslant X$. Since $m_\alpha \geqslant |V|$, this also holds for $A = 1$.

(b): Let $D$ be as in (a) for $\alpha = 1$. Since $X/C_X(V)$ is elementary abelian, also $D/C_D(V)$ is elementary abelian. Thus (a) shows that (b) holds. □

LEMMA A.30. *Suppose that $H$ does not contain any over-offenders on $V$. Then every offender in $H$ on $V$ is a best offender.*

PROOF. Let $A \leqslant H$ be an offender on $V$ and let $B \leqslant A$. Since $A$ is an offender, $|V/C_V(A)| \leqslant |A/C_A(V)|$ and so $|V||C_A(V)| \leqslant |A||C_V(A)|$. By hypothesis, $B$ is not an over-offender and so $|V/C_V(B)| \geqslant |B/C_B(V)|$. Thus

$$|B||C_V(B)| \leqslant |V||C_B(V)| \leqslant |V||C_A(V)| \leqslant |A||C_V(A)|$$

and $A$ is a best offender on $V$. □

LEMMA A.31 ([**MS5**, 1.2]). *Let $A \leqslant H$. Then $A$ is a best offender on $V$ if and only if $A$ is an offender on every $A$-submodule of $V$.*

LEMMA A.32 ([**MS5**, 1.5]). *Let $A$ be a strong dual offender on $V$. Then the following hold:*

(a) *$A$ is quadratic on $V$.*
(b) *$A$ is a strong dual offender on every $A$-submodule of $V$ and $V^*$.*
(c) *$A$ is best offender on $V$ and on $V^*$.*
(d) *If $|[V, A]| = |A/C_A(V)|$, then $A$ is a strong offender on $V$.*

LEMMA A.33. *Let $A \leqslant H$, $\mathbb{F}$ a finite field and $V$ an $\mathbb{F}A$-module. Suppose that $A$ is an offender on $V$ and $[V, A]$ is 1-dimensional over $\mathbb{F}$. Then*

(a) $|V/C_V(A)| = |A/C_A(V)|$.
(b) *The canonical commutator map $A/C_A(V) \to Hom_{\mathbb{F}}\big(V/C_V(A), [V, A]\big)$ is an isomorphism.*
(c) *$A$ is a strong dual offender and a best offender on every $A$-submodule of $V$.*

PROOF. For (a) and (b) see [**MS6**, 3.4]. Note that (b) implies $[v, A] = [V, A]$ for all $v \in V \backslash C_V(A)$. Thus $A$ is a strong dual offender on $V$ and on every $A$-submodule of $V$. Hence by A.32(c) $A$ is also a best offender on every $A$-submodule of $V$. □

LEMMA A.34 ([**MS5**, 1.6]). *Let $A$ be a strong offender on $V$. Then $A$ is a quadratic best offender on $V$.*

LEMMA A.35. *Let $A \leqslant H$ be a strong offender on $V$. Then the following statements are equivalent:*

(a) *$A$ is a root offender on $V$.*
(b) $|[V, A]| = |V/C_V(A)|$.
(c) *$[V, A] = [V, a]$ for some $a \in A$.*
(d) *$[V, A] = [V, a]$ for some $1 \neq a \in A$.*
(e) *$C_{V^*}(A) = C_{V^*}(a)$ for some $a \in A$.*
(f) *$C_{V^*}(A) = C_{V^*}(a)$ for all $1 \neq a \in A$*
(g) *$A$ is a strong offender on $V^*$.*

PROOF. Without loss $V$ is a faithful $A$-module. If $A = 1$, the statements of the lemma are obvious. So suppose that $A \neq 1$.

(a)$\iff$ (b)$\iff$ (c)$\iff$(d): Let $1 \neq a \in A$. Since $A$ is a strong offender, $C_V(a) = C_V(A)$.

Thus

$$|V/C_V(A)| = |V/C_V(a)| = |[V,a]| \leqslant |[V,A]|.$$

Hence

$$|V/C_V(A)| = |[V,A]| \quad \iff \quad |[V,a]| = |[V,A]| \quad \iff \quad [V,a] = [V,A].$$

It follows that $|V/C_V(A)| = |[V,A]|$ iff $[V,a] = [V,A]$ for some $a \in A$ iff $[V,a] = [V,A]$ for all $1 \neq a \in A$. Since $A$ is a strong offender, the latter condition holds if and only if $A$ is a root offender on $V$. Thus (a), (b) , (d) and (c) are equivalent.

(c) $\iff$ (e) and (d)$\iff$(f): Since $C_{V^*}(A) = [V,A]^\perp$ and $C_{V^*}(a) = [V,a]^\perp$, (c) and (e) are equivalent, and also (d) and (f) are equivalent

(f) $\implies$ (g): Suppose that (f) holds. Then also (a) holds. Since $A$ is an offender we get $|[V,A]| = |V/C_V(A)| \leqslant |A|$. As $|V^*/C_{V^*}(A)| = |V^*/[V,A]^\perp| = |[V,A]|$ we conclude that $A$ is an offender on $V^*$. Together with (f) this shows that $A$ is a strong offender on $V^*$.

(g) $\implies$ (f): If $A$ is a strong offender on $V^*$, then by definition $C_{V^*}(A) = C_{V^*}(a)$ for all $1 \neq a \in A$. So (e) holds. □

LEMMA A.36. *Let $A \leqslant H$. Then the following are equivalent*

(a) *$A$ is a root offender on $V$.*
(b) *$A$ is a strong offender on $V$ and a strong offender on $V^*$.*
(c) *$A$ is a root offender on $V^*$.*

PROOF. Note that any root offender is a strong offender. By A.35 a strong offender is a root offender on $V$ if and only if it is a strong offender on $V^*$. So (a) and (b) are equivalent. This equivalence applied to $V^*$ in place of $V$ shows that (b) and (c) are equivalent. □

LEMMA A.37. *Let $A \leqslant H$ be a root offender on $V$. Then*

(a) $|V/C_V(A)| = |[V,A]| = |A/C_A(V)|$.
(b) *$A$ is strong dual offender on $V$.*
(c) *$A$ is quadratic on $V$.*

PROOF. (a) and (b): Let $v \in V \backslash C_V(A)$. By definition of a root offender, $C_V(a) = C_V(A)$ for all $1 \neq a \in A$. So $[v,a] \neq 1$ for all such $a$, and $C_A(v) = C_A(V)$. Thus

$$|A/C_A(V)| = |A/C_A(v)| \leqslant |[v,A]| \leqslant |[V,A]|.$$

By definition any root offender is a strong offender. So we can apply A.35 and conclude that $|[V,A]| = |V/C_V(A)|$, and since $A$ is an offender, $|[V,A]| \leqslant |A/C_A(V)|$. Hence equality holds in all these inequalities. In particular, (a) holds and $[v,A] = [V,A]$. So $A$ is strong dual offender, and (b) holds.

(c): By A.32(a) all strong dual offenders are quadratic, and so (c) follows from (b). □

LEMMA A.38. *Let $A$ be a subgroup of $H$. Suppose that $V$ is selfdual as an $\mathbb{F}_pA$-module. Then the following statements are equivalent:*

(a) *$A$ is a root offender on $V$.*
(b) *$A$ is a strong offender on $V$.*
(c) *$|V/C_V(A)| = |A|$ and $A$ is a strong dual offender.*

PROOF. (a) $\iff$ (b) : Since $V$ is selfdual, $A$ is a strong offender on $V$ if and only if $A$ is a strong offender on $V$ and $V^*$. By A.36 this is the case if and only if $A$ is a root offender on $V$.

(b) $\iff$ (c) : This is [**MS5**, 1.7]. □

LEMMA A.39 ([**MS5**, 1.3]). *Suppose that $B$ is a minimal offender on $V$ and $W$ is a $B$-submodule of $V$. Then $B$ is a quadratic best offender on $W$. In particular, every non-trivial offender on $V$ contains a non-trivial quadratic best offender on $V$.*

LEMMA A.40. *Let $Y$ be an elementary abelian normal subgroup of $H$ and $A$ an elementary abelian $p$-subgroup of maximal order in $H$. Suppose that $[Y, A] \neq 1$. Then $A$ acts as a non-trivial best offender on $Y$. Moreover, $C_A([Y, A])$ is a non-trivial quadratic best offender on $Y$.*

PROOF. Pick $B \leqslant A$. By the maximality of $|A|$,
$$|B||C_Y(B)||B \cap Y|^{-1} = |BC_Y(B)| \leqslant |A|.$$
Hence
$$|B||C_Y(B)| \leqslant |A||B \cap Y| \leqslant |A||A \cap Y| \leqslant |A||C_Y(A)|.$$
This shows that $A$ acts as a best offender on $Y$. The second statement now follows from Timmesfeld's replacement theorem [**KS**, 9.2.3]. □

LEMMA A.41 ([**MS5**, 2.2]). *Suppose that $V$ is a faithful $p$-reduced $\mathbb{F}_pH$-module and $J := J_H(V) \neq 1$. Put $\mathcal{J} := \mathcal{J}_H(V)$. Let $\mathcal{K}$ be the set of non-solvable members of $\mathcal{J}$ and put*
$$\mathcal{I} := \mathcal{J} \backslash \mathcal{K},\ E := \langle \mathcal{K} \rangle,\ I := \langle \mathcal{I} \rangle.$$
*Then the following hold:*

(a) $C_H(J/Z(J)) = C_H(J)$.
(b) *Let $N$ be a $J$-invariant subgroup of $H$ with $[N, J] \neq 1$. Then there exists $K \in \mathcal{J}$ with $K \leqslant N$.*
(c) $\mathcal{J} \neq \varnothing$, $\mathcal{J} = \mathcal{I} \cup \mathcal{K}$, *and $\mathcal{K}$ is the set of components of $J$.*
(d) *Let $K \in \mathcal{I}$. Then either $p = 2$, $K \cong C_3 \cong SL_2(2)'$, and $[V, K] \cong \mathbb{F}_2^2$, or $p = 3$, $K \cong Q_8 \cong SL_2(3)'$, and $[V, K] \cong \mathbb{F}_3^2$.*
(e) $[W, K] = [W, K, K]$ *for every $K \in \mathcal{J}$ and every $K$-submodule $W$ of $V$.*
(f) $[K, F] = 1$ *and* $[V, K, F] = 0$ *for every $K, F \in \mathcal{J}$ with $K \neq F$.*
(g) $C_J(IE) = Z(J)$, *or $p = 2$ and $C_J(IE) = Z(J)I$. So in both cases $C_J(IE)$ is an abelian $p'$-group.*
(h) *Let $U \leqslant H$ and $K \in \mathcal{J}$. Then either $[K, U] = 1$ or $[W, K] \leqslant [W, [K, U]]$ for every $K$-submodule $W \leqslant V$.*

LEMMA A.42. *Suppose $V$ is faithful and $p$-reduced for $H$, and let $L \trianglelefteq\trianglelefteq H$. Then $\mathcal{J}_L(V) = \{E \in \mathcal{J}_H(V) \mid [E, J_L(V)] \neq 1\}$. In particular, $\mathcal{J}_L(V) \subseteq \mathcal{J}_H(V)$.*

PROOF. Since $V$ is faithful $p$-reduced $H$-module and $L \trianglelefteq\trianglelefteq H$, $V$ is also a faithful $p$-reduced $L$-module, seeA.9(d). In particular, we can apply A.41 to $H$ and to $L$.

Let $E \in \mathcal{J}_H(V)$. Observe that both, $J_L(V)$ and $E$, are subnormal in $H$. Also observe that $J_L(V) \leqslant J_H(V)$, so $J_L(V) \trianglelefteq\trianglelefteq J_H(V)$. By definition of a $J_H(V)$-component, $E = [E, J_H(V)]$ and so $J_L(V)$ normalizes $E$.

1°. *Either $E \in \mathcal{J}_L(V)$ or $[E, J_L(V)] = 1$.*

Assume that $E$ is a component of $J_H(V)$. Since $J_L(V) \trianglelefteq\trianglelefteq J_H(V)$, [**KS**, 6.5.2] implies that $[E, J_L(V)] = 1$ or $E \leqslant J_L(V)$. In the latter case $E$ is a component of $J_L(V)$, and A.41(c) shows that $E \in \mathcal{J}_L(V)$.

Assume that $E$ is not a component of $J_H(V)$. Then by A.41(c) $E$ is solvable, and by A.41(d)
$$(*) \qquad E \cong C_3 \text{ and } p = 2 \qquad \text{or} \qquad E \cong Q_8 \text{ and } p = 3.$$
In particular, $E$ is a $p'$-group. Since $J_L(V)$ is generated by best offenders, $J_L(V) = O^{p'}(J_L(V))$, and since $E$ and $J_L(V)$ are both subnormal in $H$, we conclude that $J_L(V) = O^{p'}(J_L(V)E)$ and $E$ normalizes $J_L(V)$, see 1.23. Thus $[E, J_L(V)] \leqslant E \cap J_L(V)$.

By $(*)$ $E \cong C_3$ or $Q_8$, and coprime action implies that either $[E, J_L(V)] = 1$ or $E = [E, J_L(V)] \leqslant J_L(V)$. In the latter case $E$ is minimal in $J_L(V)$ with $1 \neq E = [E, J_L(V)]$, and so $E \in \mathcal{J}_L(V)$. Hence (1°) also holds in this case.

$2^\circ$. *Suppose that $[V, E, J_L(V)] \neq 0$, then $[E, J_L(V)] \neq 1$.*

Since $[V, E, J_L(V)] \neq 0$, there exists a best offender $B$ on $V$ in $J_L(V)$ such that $[V, E, B] \neq 0$. By A.41(e), $[V, E] = [V, E, E]$ and so $[V, E]$ is a perfect $E$-submodule of $V$. Hence by [**MS5**, 2.7] $[E, B] \neq 1$ and so $[E, J_L(V)] \neq 1$.

We are now able to prove the assertion. From ($1^\circ$) we get that

$$\{E \in \mathcal{J}_H(V) \mid [E, J_L(V)] \neq 1\} \subseteq \mathcal{J}_L(V).$$

Now let $K \in \mathcal{J}_L(V)$. It remains to show that $K \in \mathcal{J}_H(V)$ and $[K, J_L(V)] \neq 1$. Put $R := \langle \mathcal{J}_H(V) \rangle$ and $J := J_H(V)$. Since $R$ is normal in $H$ and $V$ is faithful $p$-reduced $H$-module, $[V, R]$ is a faithful $R$-module, see A.9(d). Hence $C_R([V, R]) = 1$.

Suppose for a contradiction that $[V, R, K] = 0$. Then $[R, K] \leqslant C_R([V, R]) = 1$. Note that $K \leqslant J_L(V) \leqslant J_H(V) = J$ and by A.41(g) $C_J(R) \leqslant Z(J)R$. Hence $K \leqslant Z(J)R$. By the definition of a $J_L(V)$-component we have $[K, J_L(V)] = K \neq 1$ and so $K \leqslant [K, J] \leqslant [Z(J)R, J] \leqslant R$. But then $K \leqslant C_R([V, R]) = 1$, a contradiction.

We have proved that $[V, R, K] \neq 1$ and so there exists $E \in \mathcal{J}_H(V)$ such that $[V, E, K] \neq 0$. Then also $[V, E, J_L(V)] \neq 0$, and ($2^\circ$) shows that $[E, J_L(V)] \neq 1$. Thus ($1^\circ$) implies $E \in \mathcal{J}_L(V)$. Hence $E$ and $K$ are $J_L(V)$-components with $[V, E, K] \neq 0$. Now A.41(f) gives $K = E$ and so $K \in \mathcal{J}_H(V)$. □

Lemma A.43 ([**MS5**, 2.4]). *Let $K \in \mathcal{J}_H(V)$ and let $A$ be a subgroup of $M$ such that $[V, A, A] = 0$ and $[K, A] \neq 1$. Suppose that $X$ is a perfect $K$-submodule of $V$ and $\overline{X}$ is a non-zero $K$-factor module of $X$. Then*

$$C_A(X) = C_A(K) = C_A(\overline{X}).$$

Lemma A.44 ([**MS5**, 2.8]). *Suppose that $V$ is a faithful $p$-reduced $\mathbb{F}_pH$-module. Let $K \in \mathcal{J}_H(V)$ and $X$ be a perfect $K$-submodule of $V$. Then $J_H(V)$ normalizes $X$.*

Lemma A.45 ([**MS6**, 2.12]). *Let $R := [O_p(H), O^p(H)]$ and $T \in Syl_p(H)$, and let $Y$ be a $T$-submodule of $V$ with $V = \langle Y^H \rangle \neq Y$. Then one of the following holds:*

(1) *$[V, R] = 0$ and $C_{O_p(H)}(Y) \trianglelefteq H$.*
(2) *$R$ is a non-trivial strong dual offender on $Y$.*
(3) *There exist $O_p(H)O^p(H)$-submodules $Z_1 \leqslant X_1 \leqslant Z_2 \leqslant X_2$ such that for $i = 1, 2$, $X_i/Z_i$ is a non-central simple $O^p(H)$-module and $X_i \cap Y \nleqslant Z_i$.*

## A.6. Nearly Quadratic Modules

In this section $A$ is a group, $\mathbb{F}$ is a field and $V$ is an $\mathbb{F}A$-module. Since quadratic action is a special case of nearly quadratic action, the results in this section also apply to quadratically acting groups. Recall the definition of a system of imprimitivity on $V$ from Definition A.19.

Definition A.46. Let $\mathbb{K}$ be a field extension of $\mathbb{F}$ such that $V$ is also a $\mathbb{K}$-vector space.

(a) Let $a \in A$ and $\sigma \in Aut(\mathbb{K})$. Then $a$ acts *$\sigma$-semilinearly* on $V$ if $(kv)^a = k^\sigma v^a$ for all $k \in \mathbb{K}$ and $v \in V$.
(b) Let $\sigma : A \to Aut(\mathbb{K})$, $a \mapsto \sigma_a$, be a homomorphism. Then $V$ is a *$\sigma$-semilinear* $\mathbb{K}A$-module provided that each $a \in A$ acts $\sigma_a$-semilinearly on $V$. Set $A_{\mathbb{K}} := \ker \sigma$ and $\mathbb{K}_A := C_{\mathbb{K}}(Im\sigma)$.

Lemma A.47 ([**MS3**, 2.4]). *Let $V$ be a nearly quadratic $\mathbb{F}A$-module and $W$ be an $\mathbb{F}A$-submodule of $V$. Then $W$ and $V/W$ are nearly quadratic $\mathbb{F}A$-modules.*

Lemma A.48 ([**MS3**, 2.13]). *Let $V$ be a nearly quadratic $\mathbb{F}A$-module, and let $\Delta$ be a system of imprimitivity of $\mathbb{F}$-subspaces for $A$ in $V$. Then one of the following holds:*

(1) *$A$ acts trivially on $\Delta$ and there exists at most one $W \in \Delta$ with $[W, A] \neq 0$.*
(2) *$A$ acts trivially on $\Delta$ and quadratically on $V$.*
(3) *$A$ acts quadratically on $V$, $char\mathbb{F} = 2$, and $|A/C_A(W)| \leqslant 2$ for every $W \in \Delta \backslash C_\Delta(A)$.*

(4) *$A$ does not act quadratically on $V$, $A/C_A(V)$ is elementary abelian and there exists a unique $A$-orbit $W^A \subseteq \Delta$ with $[W, A] \neq 0$. Moreover, $B := N_A(W)$ acts quadratically on $V$, $B = C_A(\Delta)$ and one of the following holds:*
  (1) *char*$\mathbb{F} = 2$, $|W^A| = 4$, $\dim_{\mathbb{F}} W = 1$, $B = C_A(V)$, *and* $A/C_A(V) \cong C_2 \times C_2$.
  (2) *char*$\mathbb{F} = 3$, $|W^A| = 3$, $\dim_{\mathbb{F}} W = 1$, $B = C_A(V)$, *and* $A/C_A(V) \cong C_3$.
  (3) *char*$\mathbb{F} = 2$, $|W^A| = 2$, *and* $C_A(W) = C_A(V)$. *Moreover,* $\dim_{\mathbb{F}} W/C_W(B) = 1$ *and* $C_W(B) = [W, B]$.

LEMMA A.49 ([**MS3**, 6.3]). *Suppose that $V$ is a semilinear but not linear $\mathbb{K}A$-module for some field extension $\mathbb{K}$ of $\mathbb{F}$ and that $V$ is a nearly quadratic $\mathbb{F}A$-module. Then $A/C_A(V)$ is elementary abelian and one of the following holds:*

(1) $[V, A, A] = 0$, $[V, A_{\mathbb{K}}] = 0$, *and char*$\mathbb{K} = 2 = |A/A_{\mathbb{K}}|$.
(2) $[V, A, A] \neq 0$, $[V, A_{\mathbb{K}}] = C_V(A_{\mathbb{K}})$, $\dim_{\mathbb{K}} V/C_V(A_{\mathbb{K}}) = 1$, $\mathbb{F} = \mathbb{K}_A$, *and char*$\mathbb{K} = 2 = |A/A_{\mathbb{K}}| = \dim_{\mathbb{F}} \mathbb{K}$.
(3) $[V, A, A] \neq 0$, $[V, A_{\mathbb{K}}] = 0$, $\mathbb{F} = \mathbb{K}_A$, $\dim_{\mathbb{K}} V = 1$, *and char*$\mathbb{F} = 3 = |A/A_{\mathbb{K}}| = \dim_{\mathbb{F}} \mathbb{K}$.
(4) $[V, A, A] \neq 0$, $[V, A_{\mathbb{K}}] = 0$, $\mathbb{F} = \mathbb{K}_A$, $\dim_{\mathbb{K}} V = 1$, *char*$\mathbb{F} = 2$, $A/A_{\mathbb{K}} \cong C_2 \times C_2$, $\dim_{\mathbb{F}} \mathbb{K} = 4$, *and $\mathbb{F}$ is infinite.*

## A.7. $Q!$-Modules

In this section $H$ is a finite group, $Q$ is a $p$-subgroup of $H$, and $V$ is a finite $Q!$-module for $\mathbb{F}_p H$ with respect to $Q$. By A.50(b) below $Q$ is a weakly closed subgroup of $H$. Hence, the results in Section 1.5 apply to $Q$ and $H$. In particular, we will use the $^\circ$-notion introduced there, so for $L \leqslant H$,

$$L^\circ = \langle P \in Q^H \mid P \leqslant L \rangle \qquad \text{and} \qquad L_\circ = O^p(L^\circ).$$

LEMMA A.50. *Let $V$ be a non-zero $Q!$-module for $H$ with respect to $Q$.*

(a) *$N_H(T) \leqslant N_H(Q)$ for every $p$-subgroup $T$ of $H$ with $Q \leqslant T$.*
(b) *$Q$ is a weakly closed subgroup of $H$.*
(c) *$C_V(Q) \cap C_V(Q^g) = 0$ for all $g \in H \backslash N_H(Q)$; in particular $N_H(Q) = N_H(C_V(Q))$.*
(d) *Let $K$ be a subgroup of $H$ acting transitively on $V$. Then $H^\circ = \langle Q^K \rangle$.*

PROOF. (a): Let $Q \leqslant T$, $T$ a $p$-subgroup of $H$. Then $0 \neq C_V(T) \leqslant C_V(Q)$ and $Q!$ implies

$$N_H(T) \leqslant N_H(C_V(T)) \leqslant N_H(Q).$$

(b): By 1.45 the condition in (a) is equivalent to $Q$ being a weakly closed subgroup of $H$.

(c) Let $g \in H$ with $C_V(Q) \cap C_V(Q)^g \neq 0$. By $Q!$, $Q$ and $Q^g$ are normal in $N_H(C_V(Q) \cap C_V(Q^g))$. Since $Q$ is a weakly closed subgroup of $H$, this gives $Q = Q^g$ and thus $g \in N_H(Q)$.

(d) Let $0 \neq v \in C_V(Q)$. By a Frattini argument, $H = C_H(v)K$ and by $Q!$, $C_H(v) \leqslant N_H(Q)$. Thus $Q^H = Q^K$ and so $H^\circ = \langle Q^H \rangle = \langle Q^K \rangle$. □

LEMMA A.51. *Let $V$ be a non-zero $Q!$-module for $H$ with respect to $Q$. Then $V$ is a $Q!$-module for $H/C_H(V)$ with respect to $QC_H(V)/C_H(V)$.*

PROOF. Put $\overline{H} = H/C_H(V)$. Since $N_H(A) \leqslant N_H(Q)$ for all $1 \neq A \leqslant C_V(Q)$, $N_{\overline{H}}(A) \leqslant N_{\overline{H}}(\overline{Q})$ for all $1 \neq A \leqslant C_V(\overline{Q})$. Since $V \neq 0$ also $C_V(\overline{Q}) \neq 0$. Thus

$$N_H(\overline{Q}) \leqslant N_H(C_V(\overline{Q})) \leqslant N_H(Q).$$

Since $Q \ntrianglelefteq H$ this implies $\overline{Q} \ntrianglelefteq \overline{H}$, and the lemma is proved. □

LEMMA A.52 ([**MS6**, 4.2]). *Let $V$ be a faithful $Q!$-module for $H$ with respect to $Q$.*

(a) *$H^\circ = \langle Q^h \mid h \in H^\circ \rangle$.*
(b) *$C_H(H^\circ/Z(H^\circ)) = C_H(H^\circ)$.*
(c) *Let $H^\circ \leqslant L \leqslant H$ and $W$ be a non-zero $L$-submodule of $V$. Then $C_L(W) \leqslant C_L(H^\circ)$. In particular $C_{H^\circ}(W)$ is a $p'$-group.*
(d) *$C_V(H^\circ) = 0$.*

(e) *Let $Q \leqslant L \leqslant H$ with $Q \ntrianglelefteq L$. Then $V$ is $Q!$-module for $\mathbb{F}_p L$ with respect to $Q$.*
(f) *Let $L \trianglelefteq\trianglelefteq H$ with $[L, Q] \neq 1$. Then $C_V(\langle L^Q \rangle) = 0$.*

Lemma A.53. *Let $V$ be a $Q!$-module for $H$ with respect to $Q$, and let $N$ be a $Q$-invariant subgroup of $H$ with $N \nleqslant N_H(Q)$. Then $C_H(W) \leqslant N_H(Q)$ for every non-trivial $NQ$-submodule $W$ of $V$. In particular $C_V(N) = 0$ and $C_H([V, N]) \leqslant N_H(Q)$.*

Proof. Let $W \neq 0$ be an $NQ$-submodule of $V$. Then $C_W(Q) \neq 0$, and the $Q!$-property of $V$ implies

$$C_H(W) \leqslant C_H(C_W(Q)) \leqslant N_H(Q).$$

In particular, we get that $C_V(N) = 0$ since $N \nleqslant N_H(Q)$. Then $[V, N]$ is a non-trivial $NQ$-submodule of $V$, and the last claim also follows. □

Lemma A.54. *Let $V$ be a faithful $p$-reduced $Q!$-module for $H$ with respect to $Q$ and let $K \trianglelefteq\trianglelefteq H$.*
(a) *$V \neq 0$.*
(b) *If $K \leqslant N_H(Q)$, then $[K, Q] = 1$.*
(c) *If $C_V(K) \cap C_V(Q) \neq 0$, then $[K, Q] = 1$.*
(d) *If $K \neq 1$, then $[V, K, Q] \neq 1$.*
(e) *If $K \leqslant N_H(Q)$, then $K$ acts faithfully on each of $[V, Q]$, $C_V(Q)$ and $C_V(Q) \cap [V, Q]$.*
(f) *$C_H([V, H_\circ]) = 1$.*
(g) *Let $E \trianglelefteq H_\circ$. Then $C_H([V, E]) \cap C_H(V/[V, E]) = 1$.*

Proof. Note first that $O_p(H) = 1$ since $V$ is faithful and $p$-reduced.
(a): If $V = 0$ then $Q = 1$ since $V$ is faithful. But then $Q \trianglelefteq H$, a contradiction to the definition of a $Q!$-module.

(b): Suppose that $K \leqslant N_H(Q)$ and put $L := \langle K^Q \rangle$. Then $L$ is subnormal in $H$ and so $O_p(L) \leqslant O_p(H) = 1$. Also $L \leqslant N_H(Q)$ and so $L \cap Q$ is a normal $p$-subgroup of $L$. Thus $[L, Q] \leqslant L \cap Q \leqslant O_p(L) = 1$.

(c): If $C_V(K) \cap C_V(Q) \neq 0$, then $Q!$ implies $K \leqslant N_H(C_V(K) \cap C_V(Q)) \leqslant N_H(Q)$ and (b) gives $[K, Q] = 1$.

(d): Suppose that $K \neq 1$ but $[V, K, Q] = 1$. Replacing $K$ by $\langle K^{N_H(Q)} \rangle$ we may assume that $N_H(Q) \leqslant N_H(K)$. Since $V$ is faithful, $1 \neq [V, K] \leqslant C_V(Q)$ and $Q!$ implies $N_H(K) \leqslant N_H([V, K]) \leqslant N_H(Q)$. Hence $N_H(K) = N_H(Q)$. Let $L$ be the largest subnormal subgroup of $H$ contained in $N_H(Q)$. Then $K \leqslant L$ and (b) gives $[L, Q] = 1$. Let $M$ be the largest subnormal subgroup of $H$ contained in $N_H(L)$. Since $Q$ centralizes $L$, $Q$ normalizes $M$ and $\langle Q^M \rangle \leqslant C_H(L) \leqslant C_H(K)$. Note that $[M, Q] \trianglelefteq M \trianglelefteq\trianglelefteq H$ and so $[M, Q]$ is a subnormal subgroup of $H$ contained in $C_H(K)$. In particular, $[M, Q] \leqslant N_H(K) = N_G(Q)$ and the maximal choice of $L$ gives $[M, Q] \leqslant L$. Thus $[M, Q, Q] \leqslant [L, Q] = 1$ and since $O_p(M) \leqslant O_p(H) = 1$, 1.9 gives $[M, Q] = 1$. Hence $M \leqslant N_H(Q)$ and so $M = L$ by maximality of $L$. Since $L \trianglelefteq\trianglelefteq H$ this implies $H = L \leqslant N_H(Q)$, a contradiction since $Q \ntrianglelefteq H$ by definition of a $Q!$-module.

(e): By (b), $[K, Q] = 1$. Put $K_0 := C_K([V, Q])$. Then $[V, Q, K_0] = 1$ and the Three Subgroups Lemma gives $[V, K_0, Q] = 1$. Hence (d) implies $K_0 = 1$ and so $K$ acts faithfully on $[V, Q]$. Since $[K, Q] = 1$ the $P \times Q$- Lemma shows

$$O^p(C_K([V, Q] \cap C_V(Q))) \leqslant C_K([V, Q]) = K_0 = 1,$$

and so $C_K([V, Q] \cap C_V(Q)) \leqslant O_p(K) = 1$.

(f): By A.52(d), $C_V(H^\circ) = 0$ and so also $C_V(H_\circ) = 0$ and $[V, H_\circ] \neq 0$. Hence by A.52(c) $E := C_H([V, H_\circ]) \leqslant C_H(H^\circ)$. Thus $[E, H_\circ] = 1$ and $[V, H_\circ, E] = 1$. The Three Subgroups Lemma implies $[V, E, H_\circ] = 1$. Since $C_V(H_\circ) = 0$ this gives $[V, E] = 0$, and as $V$ is faithful, $E = 1$.

(g): Put $C := C_H([V, E]) \cap C_H(V/[V, E])$. Then $C$ acts nilpotently on $V$ and $H_\circ$ normalizes $C$. Hence $[C, H_\circ] \leqslant C \cap H_\circ \trianglelefteq H_\circ$, so $C \cap H_\circ$ is subnormal in $H$. Since $V$ is $p$-reduced and faithful, $C \cap H_\circ = 1$, and thus $[C, H_\circ] = 1$. Since $E \leqslant H_\circ$, $C_H(H_\circ)$ normalizes $E$ and $C$. It follows that $C \trianglelefteq C_H(H_\circ) \trianglelefteq H$, and again since $V$ is $p$-reduced and faithful, $C = 1$. □

LEMMA A.55. *Let $V$ be a faithful $p$-reduced $Q$!-module for $H$ with respect to $Q$. Put $N := \bigcap_{g\in H} N_H(Q^g)$.*

(a) *$[N, H^\circ] = 1$ and $C_V(Q)$ is a faithful $p$-reduced $N$-module.*
(b) *Let $1 \neq t \in H$ with $|[V,t]| < |C_V(Q)|$. Then $t \in N$ and $[C_V(Q), t] \neq 1$.*
(c) *$|C_V(Q)| \leqslant |[V,t]|$ for all $1 \neq t \in H$ with $[C_V(Q), t] = 1$.*
(d) *$C_V(Q) = [V,t]$ for all $1 \neq t \in H$ with $[V,t] \leqslant C_V(Q)$.*

PROOF. (a): Note that $N \trianglelefteq H$ and $N \leqslant N_H(Q)$. Hence by A.54(b),(e) $[N,Q] = 1$ and $C_V(Q)$ is a faithful $N$-module. As $H^\circ = \langle Q^H\rangle$, this gives $[N, H^\circ] = 1$. Since $N \trianglelefteq H$ and $V$ is faithful and $p$-reduced for $H$, $O_p(N) \leqslant O_p(H) = 1$. Hence, since $C_V(Q)$ is a faithful $N$-module, $C_V(Q)$ is $p$-reduced for $N$.

(b): Let $t \in H$ with $|C_V(Q)| > |[V,t]|$, and let $g \in H$. Note that

$$|V/C_V(t)| = |[V,t]| < |C_V(Q)| = |C_V(Q^g)|.$$

Thus $A := C_V(Q^g) \cap C_V(t) \neq 0$ and $t \in C_H(A) \leqslant N_H(Q^g)$. Hence $t \in N$. By (a), $C_V(Q)$ is a faithful $N$-module and so $[C_V(Q), t] \neq 1$.

(c) follows immediately from (b).

(d): Suppose that $[V,t] \leqslant C_V(Q)$ but $[V,t] \neq C_V(Q)$. Let $g \in H$. Then $|[V,t]| < |C_V(Q)| = |C_V(Q^g)|$ and so by (b), $t \in N$ and

$$1 \neq [C_V(Q^g), t] \leqslant C_V(Q^g) \cap [V,t] \leqslant C_V(Q^g) \cap C_V(Q).$$

Thus $Q = Q^g$ and $Q \trianglelefteq H$, contrary to the definition of a $Q$!-module. □

LEMMA A.56. *Let $V$ be a faithful $p$-reduced $Q$!-module for $H$ with respect to $Q$, and let $\mathcal{K}$ be a non-empty $H$-invariant set of non-trivial subgroups of $H$. Put $R := \langle \mathcal{K}\rangle$. Suppose that $[V,R]$ is a wreath product module for $H$ with respect to $\mathcal{K}$ such that one of the following holds:*

(1) *$[V,K]$ is a simple $K$-module for all $K \in \mathcal{K}$, or*
(2) *$[R, H^\circ] \neq 1$.*

*Then*

(a) *$Q$ is transitive on $\mathcal{K}$ and $C_V(R) = 0$.*
(b) *Suppose that $|\mathcal{K}| > 1$. Let $K \in \mathcal{K}$ and put $T := N_Q(K)$. Then $C_K(z)$ is a $p$-group for all $0 \neq z \in C_{[V,K]}(T)$.*

PROOF. Put $W := [V,R]$. Since $V$ is $p$-reduced and faithful, A.9 shows that $W = [V,R]$ is a faithful $p$-reduced $R$-module. Since $W$ is faithful wreath product module, A.17 shows that

$$(*) \qquad R = \bigtimes_{K\in\mathcal{K}} K \qquad \text{and} \qquad W = \bigoplus_{K\in\mathcal{K}} [W,K].$$

(a): Let $\mathcal{K}_0$ be an orbit of $Q$ on $\mathcal{K}$ and put $R_0 := \langle \mathcal{K}_0\rangle$. Since $\mathcal{K}$ is $H$-invariant, $R \trianglelefteq H$, and since $\mathcal{K}$ is a non-empty set of non-trivial subgroups, $R \neq 1$. As $V$ is $p$-reduced, $O_p(R) = 1$. If $K \in \mathcal{K}\setminus\mathcal{K}_0$ then $(*)$ shows that $[W,K] \leqslant C_{[W,R]}(R_0)$. Thus either $\mathcal{K}_0 = \mathcal{K}$ or $C_{[W,R]}(R_0) \neq 0$.

Assume that $C_V(R_0) \neq 0$. Then also $C_V(R_0) \cap C_V(Q) \neq 0$ and so by A.54(c) $[R_0, Q] = 1$. Since either $Q$ acts transitively on $\mathcal{K}$ or $C_V(\langle K^Q\rangle) \neq 0$ for all $K \in \mathcal{K}$, we get $[R,Q] = 1$ and so $[R, H^\circ] = 1$. Hence $[V,K]$ is a simple $K$-module for all $K \in \mathcal{K}$. Since $Q$ centralizes $K$, $[V,K,Q] = 0$, and since $K \neq 1$, this contradicts A.54(d).

Thus $C_V(R_0) = 0$. It follows that $Q$ acts transitively on $\mathcal{K}$ and $C_V(R) = 0$.

(b): Let $0 \neq z \in C_{[V,K]}(T)$. By A.18 $[V,K] = [W,K]$. Thus $z \in [W,K]$ and since $[z, N_Q(K)] = [z,T] = 0$, the conjugates $z^Q$ are in distinct submodules $[W,F]$, $F \in \mathcal{K}$. Hence $z_0 := \sum_{z_1\in z^Q} z_1 \neq 0$, $[z_0, Q] = 1$ and $C_K(z) = C_K(z_0)$. By $Q$! $C_K(z_0) \leqslant N_H(Q)$ and so

$$[C_K(z_0), Q] \leqslant Q \cap C_R(z_0) \leqslant O_p(C_R(z_0)).$$

By $(*)$ $K \trianglelefteq R$ and so $C_K(z_0) \trianglelefteq C_R(z_0)$. Hence $O^p(C_K(z_0)) = O^p\big(C_K(z_0)O_p(C_R(z_0))\big)$, so $O^p(C_K(z_0))$ is $Q$-invariant. Thus $(*)$ shows that either $O^p(C_K(z_0)) = 1$ or $K$ is $Q$-invariant. In the first case $C_K(z_0) = C_K(z)$ is a $p$-group. In the second case the transitivity of $Q$ on $\mathcal{K}$ shows $|\mathcal{K}| = 1$. $\square$

Lemma A.57. *Suppose that $O_p(H) = 1$. Let $V$ be a faithful $Q!$-module for $H$ with respect to $Q$, and let $Y$ be a $p$-subgroup of $H$ with $C_Y([V,Y]) \neq 1$ and $[H^\circ, Y] \neq 1$. Then $C_Y(H^\circ) = 1$.*

Proof. See [**MS6**, 4.4]. $\square$

## A.8. Genuine Groups of Lie Type

Definition A.58. (a) A *genuine group of Lie-type* in characteristic $p$ is a group isomorphic to $O^{p'}(C_{\overline{K}}(\sigma))$, where $\overline{K}$ is a semisimple $\overline{\mathbb{F}_p}$-algebraic group, $\overline{\mathbb{F}_p}$ is the algebraic closure of $\mathbb{F}_p$, and $\sigma$ is a Steinberg endomorphism of $\overline{K}$, see [**GLS3**, Definition 2.2.2] for details.
(b) Let $K$ be a genuine group of Lie-type. Let $\Sigma$ be the root system, $d$ the order of the graph automorphism and $q$ the order of the fixed field of the field automorphism used to define $K$. Then we say that $K$ is a *version* of ${}^d\Sigma(q)$, see [**GLS3**, Definition 2.2.4] for the details.

Note that a given symbol ${}^d\Sigma(q)$ can have many non-isomorphic versions. Nevertheless, we will write $K = {}^d\Sigma(q)$ to indicated that $K$ is a version of ${}^d\Sigma(q)$. We will use $\Sigma(q)$ for ${}^1\Sigma(q)$.

Lemma A.59 ([**GLS3**, 2.2.6]). (a) *For each symbol ${}^d\Sigma(q)$, there is up to isomorphism a unique largest version $K_u$ (called the* universal version*) and a unique smallest version $K_a$ (called the* adjoint version*).*
(b) *For any version $K$ of a symbol as in (a), there are surjective homomorphisms $K_u \to K \to K_a$, whose kernels are central. In particular, if $K$ is simple, then $K \cong K_a$.*
(c) *$Z(K_a) = 1$, and $K/Z(K) \cong K_u/Z(K_u) \cong K_a$.*
(d) *The versions of a given symbol, up to isomorphism, are the groups $K_u/Z$ as $Z$ ranges over all subgroups of $Z(K_u)$.*

Lemma A.60. *Let $K = {}^d\Sigma(q)$ be an adjoint group or universal group of Lie-type with Dynkin diagram $\Delta$. Then there exist subgroups $Diag$ and $\Phi$ and a subset $\Gamma$ of $Out(K)$ such that*
(a) *$\Phi\Gamma$ is a subgroup of $Out(K)$, $\Phi \trianglelefteq \Phi\Gamma$, $Out(K) = Diag\Phi\Gamma$, $Diag \trianglelefteq Out(K)$, and $Diag \cap \Phi\Gamma = 1$.*
(b) *$Diag$ has order dividing $q-1$, $q+1$ or $\gcd(q-1,2)^2$. In particular, $Diag$ is a $p'$-group.*
(c) *$\Phi \cong Aut(\mathbb{F}_{q^d})$. In particular, $\Phi$ is cyclic.*
(d) *$C_{Diag\Phi\Gamma}(\Delta) = Diag\Phi$.*
(e) *One of the following holds:*
  (1) *$d = 1$, $\Delta$ has only single bonds, $\Gamma$ is a subgroup of $\Phi\Gamma$, $\Phi\Gamma = \Phi \times \Gamma$, and $\Gamma$ is the group of symmetries of $\Delta$.*
  (2) *$d = 1$, $\Delta$ has double or triple bonds, and*
    (i) *if $p = 2$ and $\Delta$ is of type $B_2$ or $F_4$, or $p = 3$ and $\Delta$ is of type $G_2$, then $\Gamma = \{1, \psi\}$, $\psi$ acts non-trivially on $\Delta$ and $\Phi = \langle \psi^2 \rangle$,*
    (ii) *otherwise $\Gamma = 1$.*
  *In particular, $\Phi\Gamma$ is cyclic.*
  (3) *$d \neq 1$ and $\Gamma = 1$. In particular, $\Phi\Gamma = \Phi$ is cyclic.*

Proof. See [**GLS3**, section 2.5]; in particular Theorem 2.5.12. $\square$

Corollary A.61. *Let $K = {}^d\Sigma(q)$ be an adjoint group or universal group of Lie-type with Dynkin diagram $\Delta$, and let $\Gamma$ and $Diag$ be as in A.60. Suppose that $Diag\Gamma$ is not abelian. Then $K = D_4(q)$, $\Gamma \cong Sym(3)$ and $(\Gamma Diag)' = \Gamma' \cong C_3$.*

Proof. Since $\Gamma Diag$ is not abelian, $\Gamma Diag$ is not cyclic. This rules out the last two cases in A.60(e). Hence A.60(e:1) holds and so $d = 1$, $\Delta$ has only single bonds, $\Gamma$ is the group of symmetries on $\Delta$, and $\Phi\Gamma = \Phi \times \Gamma$. By A.60(b), $\Phi$ is cyclic. So $(\Phi\Gamma)' = \Gamma'$ and $\Gamma$ is not abelian. Thus $\Sigma = D_4$, $\Gamma \cong Sym(3)$ and $\Gamma' \cong C_3$. $\square$

LEMMA A.62 (**Steinberg's Lemma,** [**MS5**, 4.1]). *Let $M$ be a genuine group of Lie-type defined over a finite field of characteristic $p$. Let $V$ be a simple $\mathbb{F}_pM$-module, $S \in Syl_p(M)$, and $B := N_M(S)$. Put $\mathbb{K} := End_M(V)$. Then $C_V(S)$ is 1-dimensional over $\mathbb{K}$, $\mathbb{K}$ is isomorphic to the subring of $End_{\mathbb{F}_p}(C_V(S))$ generated by the image of $B$, and $C_V(S)$ is a simple $\mathbb{F}_pB$-module.*

THEOREM A.63 (**Smith's Lemma,** [**MS5**, 4.2]). *Let $M$ be a genuine group of Lie-type defined over a finite field of characteristic $p$. Let $V$ be a simple $\mathbb{F}_pM$-module, $\mathbb{K} := End_M(V)$, $E$ a parabolic subgroup of $M$, $L := O^{p'}(E)$ and $P = N_M(L)$. Then $L = O^{p'}(P)$, $O_p(E) = O_p(P) = O_p(L)$ and $P$ is a Lie-parabolic subgroup of $M$. Moreover, $C_V(O_p(P))$ is a simple $\mathbb{F}_pP$-module, an absolutely simple $\mathbb{K}L$-module, and an absolutely simple $\mathbb{K}E$-module*

Let $\mathbb{F}$ be a finite field of characteristic $p$, $M$ a finite group, $V$ a simple $\mathbb{F}M$-module and $W$ a simple $\mathbb{F}_pM$-submodule. Recall that the field $\mathbb{K} := End_M(W)$ is called the field of definition of the $\mathbb{F}M$-module $W$.

THEOREM A.64 (**Ronan-Smith's Lemma,** [**MS5**, 4.3]). *Let $M$ be a universal group of Lie-type defined over a finite field of characteristic $p$, $S$ a Sylow $p$-subgroup of $M$, $P_1, P_2, \dots, P_n$ the minimal Lie-parabolic subgroups of $M$ containing $S$, and $L_i = O^{p'}(P_i)$. Let $\mathcal{V}$ be the class of all tuples $(\mathbb{K}, V_1, V_2, \dots, V_n)$ such that*

(i) *$\mathbb{K}$ is a finite field of characteristic $p$.*
(ii) *Each $V_i$ is an absolutely simple $\mathbb{K}L_i$-module.*
(iii) *$\mathbb{K} = \langle \mathbb{K}_i \mid 1 \leqslant i \leqslant n\rangle$, where $\mathbb{K}_i$ is the field of definition of the $\mathbb{K}L_i$-module $V_i$.*

*Define two elements $(\mathbb{K}, V_1, V_2, \dots, V_n)$ and $(\widetilde{\mathbb{K}}, \widetilde{V}_1, \widetilde{V}_2, \dots, \widetilde{V}_n)$ of $\mathcal{V}$ to be isomorphic if there exists a field isomorphism $\sigma : \widetilde{\mathbb{K}} \to \mathbb{K}$ such that $V_i \cong \widetilde{V}_i^{\sigma}$ as an $\mathbb{K}L_i$-module for all $1 \leqslant i \leqslant n$. Then the map*

$$V \to \big(End_M(V), C_V(O_p(L_i)), \dots C_V(O_p(L_n))\big) \quad (V \text{ a simple } \mathbb{F}_pM\text{-module})$$

*induces a bijection between the isomorphism classes of simple $\mathbb{F}_pM$-modules and the isomorphism classes of $\mathcal{V}$.*

LEMMA A.65. *Let $K = {}^d\Sigma(q)$ be a universal group of Lie-type with Dynkin diagram $\Delta$. Define $\tau \in \Gamma$[4] as follows:*

(1) *If $K = A_n(q)$, $n \geqslant 2$, $K = D_{2n+1}(q)$, $n \geqslant 2$,[5] or $E_6(q)$, then $\tau$ induces the unique non-trivial graph automorphism on $\Delta$;*
(2) *otherwise $\tau = 1$.*

*Then $\tau^2 = 1$ and $V^* \cong V^\tau$ for all simple $\mathbb{F}_pK$-modules $V$.*

PROOF. See [**St**, Lemma 73]. □

LEMMA A.66. *Let $M$ be a genuine group of Lie-type defined over a finite field of characteristic $p$, $S$ a Sylow $p$-subgroup of $M$, $P_1, P_2, \dots, P_n$ the minimal Lie-parabolic subgroups of $M$ containing $S$, $L_i = O^{p'}(P_i)$ and $B = N_M(S)$. Suppose that $V$ is a simple $\mathbb{F}_pM$-module such that $[C_V(S), B] = 0$ and $[C_V(S), L_i] \neq 0$ for all $1 \leqslant i \leqslant n$. Then $V$ is the Steinberg module for $M$ over $\mathbb{F}_p$ of $\mathbb{F}_p$-dimension $|S|$. Moreover, as an $\mathbb{F}_pS$-module $V$ is isomorphic to the regular permutation module $\mathbb{F}_p[S]$.*

PROOF. We may assume without loss that $M$ is universal. Let $\mathbb{F}$ be the algebraic closure of $\mathbb{F}_p$ and $St$ the Steinberg module for $M$ over $\mathbb{F}$. Then by [**GLS3**, 2.8.7] $St$ is a simple $\mathbb{F}M$ module of dimension $|S|$. It is well-known and also follows from the weight of $St$ as given in [**GLS3**, 2.8.7(b)] that $[C_{St}(S), B] = 0$ and $[C_V(S), L_i] \neq 0$ for all $1 \leqslant i \leqslant n$.

Put $\mathbb{K} = End_M(V)$ and let $\mathbb{F}$ be the algebraic closure of $\mathbb{K}$. By [**As**, 25.8] $V$ is an absolutely simple $\mathbb{K}M$-module and so $\overline{V} := \mathbb{F} \otimes_{\mathbb{K}} V$ is a simple $\mathbb{F}M$-module. By A.62 $\mathbb{K}$ is isomorphic to the subring of $End_{\mathbb{F}_p}(C_V(S))$ generated by the image of $B$. Since $[C_V(S), B] = 0$ this gives $\mathbb{K} = \mathbb{F}_p$. We will now show that $\overline{V}$ is uniquely determined and so $\overline{V} \cong St$.

[4]see A.60 for the definition of $\Gamma$

[5]Note here that $A_3 = D_3$

Suppose first that $n = 1$. By [**St**, Theorem 46] a simple $\mathbb{F}M$-module $W$ is uniquely determined by the action of $B$ on $C_W(S)$ and a parameter $\mu \in \{0, -1\}$. In particular, there are (up to isomorphism) at most two simple $\mathbb{F}M$-modules $W$ with $[C_W(S), B] = 0$. Hence $St$ is the unique non-central $\mathbb{F}M$-module with $[C_{St}(S), B] = 0$.

In the general case, put $U_i = C_{\overline{V}}(O_p(P_i))$. By Smith's Lemma A.63 $U_i$ is a simple $L_i$-module and the $n = 1$ case applied to $L_i/O_p(P_i)$ shows that $U_i$ is the Steinberg-module for $L_i/O_p(P_i)$ over $\mathbb{F}$. This uniquely determines the parameters $\mu_i, 1 \leqslant i \leqslant n$ in [**St**, Theorem 46] and so $\overline{V}$ is uniquely determined.

Thus $\overline{V}$ is the Steinberg-module $St$. In particular, $\dim_{\mathbb{F}_p} V = \dim_{\mathbb{K}} V = \dim_{\mathbb{F}} \overline{V} = |S|$. By [**St**, Theorem 46] the conjugates of $C_{\overline{V}}(S)$ under the opposite Sylow $p$-subgroup $S^-$ span $\overline{V}$. Let $0 \neq v \in C_V(S)$. Then $\langle v^{S^-} \rangle = V$ and since $|S^-| = |S|$ we conclude that $v^{S^-}$ is an $\mathbb{F}_p$-basis of $V$ regularly permuted by $S^-$. Hence $V \cong \mathbb{F}_p[S]$ as an $\mathbb{F}_pS$-module. $\square$

APPENDIX B

# Classical Spaces and Classical Groups

In this appendix $\mathbb{K}$ is a finite field, $p := char\mathbb{K}$, $V$ is a finite dimensional dimensional vector space over $\mathbb{K}$, $\alpha \in Aut(\mathbb{K})$ with $\alpha^2 = id_{\mathbb{K}}$ and $\mathbb{F}$ is the fixed field of $\alpha$.

Definition B.1. Let $f : V \times V \to \mathbb{K}$ and $h : V \to \mathbb{F}$ be functions.

(i) $(V, f, h)$ is a *linear space* if $\alpha = id_{\mathbb{K}}$, $f = 0$ and $h = 0$.

(ii) $(V, f, h)$ is a *symplectic space* if $\alpha = id_{\mathbb{K}}$, $f$ is $\mathbb{K}$-bilinear and for all $v \in V$,

$$h(v) = f(v, v) = 0.$$

(iii) $(V, f, h)$ is a *unitary space* if $\alpha \neq id_{\mathbb{K}}$, $f$ is $\mathbb{K}$-linear in the first component and for all $v, w \in V$,

$$f(v, w) = f(w, v)^\alpha \qquad \text{and} \qquad h(v) = f(v, v).$$

(iv) $(V, f, h)$ is an *orthogonal space,* if $\alpha = id_{\mathbb{K}}$, $f$ is $\mathbb{K}$-bilinear, and for all $v, w \in V$ and $k \in \mathbb{K}$

$$h(kv) = k^2 h(v) \qquad \text{and} \qquad h(v + w) = h(v) + f(v, w) + h(w).$$

(v) $(V, f, h)$ is a *classical space* (of linear, symplectic, unitary or orthogonal type), if it is a linear, symplectic, unitary or orthogonal space.

Let $(V, f, h)$ be a classical space. Abusing notion we will often just say that $V$ is a classical space.

Assume that $V$ is an orthogonal space. Then $f(v, w) = h(v + w) - h(v) - h(w)$ and so $f$ is symmetric, that is $f(v, w) = f(w, v)$. Also

$$4h(v) = h(2v) = h(v + v) = h(v) + f(v, v) + h(v),$$

and so $f(v, v) = 2h(v)$. In particular, $f$ is a symplectic form if $p = 2$.

Definition B.2. Let $V$ be a classical space, and $v, w \in V$, and let $U$ and $W$ be $\mathbb{K}$-subspaces of $V$.

(a) $v$ and $w$ are *isometric* if $h(v) = h(w)$.[1]

(b) $v$ and $w$ are *perpendicular,* and we write $v \perp w$, if $f(v, w) = 0$. We write $U \perp W$ if $u \perp w$ for all $u \in U$, $w \in W$. We write $V = U \oplus W$ if $V = U \oplus W$ and $U \perp W$.

(c) $v$ is *isotropic* if $f(v, v) = 0$; and $U$ is isotropic if $f\,|_{U \times U} = 0$.

(d) $v$ is *singular* if $h(v) = 0$; and $U$ is *singular* if $U$ is isotropic and all its elements are singular.

(e) $U^\perp = \{v \in V \mid f(v, u) = 0 \text{ for all } u \in U\}$.

(f) $rad(U) = \{u \in U \cap U^\perp \mid h(u) = 0\}$.

(g) $U$ is *non-degenerate* if $rad(U) = 0$.

(h) $\mathcal{S}(U)$ is the set of 1-dimensional singular subspaces of $U$.

(i) The *Witt index* of $V$ is the maximum of the dimensions of the singular subspaces of $V$.

Definition B.3. Let $(V, f, h)$ and $(V', f', h')$ be classical spaces over $\mathbb{K}$ and $\phi : V \to V'$ a bijection.

(a) $\phi$ is an *isometry* if $\phi$ is $\mathbb{K}$-linear and for all $v, w \in V$,

$$h(v^\phi) = h(v) \qquad \text{and} \qquad f(v^\phi, w^\phi) = f(v, w).$$

We also will say that $h$ and $f$ are *$\phi$-invariant* if these equations hold.

[1] Note that this implies $f(v, v) = f(w, w)$.

(b) $\phi$ is a *similarity* if $\phi$ is $\mathbb{K}$-linear and there exists $k \in \mathbb{F}^\sharp$ such that for all $v, w \in V$,

$$h(v^\phi) = kh(v) \qquad \text{and} \qquad f(v^\phi, w^\phi) = kf(v, w).$$

(c) $\phi$ is a *semisimilarity* if there exist $\sigma \in Aut(\mathbb{K})$ and $k \in \mathbb{F}^\sharp$ such that $\phi$ is $\sigma$-semilinear[2] and for all $v, w \in V$,

$$h(v^\phi) = kh(v)^\sigma \qquad \text{and} \qquad f(v^\phi, w^\phi) = kf(v, w)^\sigma.$$

We denote the group of isometries of $V$ by $Cl_{\mathbb{K}}(V, f, h)$, by $Cl_{\mathbb{K}}(V)$ or by $Cl(V)$. We will also use the notation $GL(V)$, $Sp(V)$, $GU(V)$ and $O(V)$ for $Cl(V)$, if $V$ is a linear, symplectic, unitary and orthogonal space, respectively.

For the remainder of this appendix $(V, f, h)$ is a non-degenerate or linear classical space and $H = Cl(V)$. If $V$ is linear we define $R(V) := 0$, otherwise $R(V) := V^\perp$. So $R(V) = 0$ unless $V$ is an orthogonal space, $p = 2$ and $\dim_{\mathbb{K}} V$ is odd.

Note that (by B.18 below) $V$ is uniquely determined, up to similarity, by its type and dimension, except in the case of an orthogonal space of even dimension. We sometimes use the notation $Cl_m(\mathbb{F})$ or $Cl_m(q)$, where $m := \dim_{\mathbb{K}} V$ and $q := |\mathbb{F}|$.

For an orthogonal space $V$ of dimension $2n$ we write $O^+(V)$ or $O^+_{2n}(\mathbb{K})$ if $V$ has Witt index $n$, and $O^-(V)$ or $O^-_{2n}(\mathbb{K})$ if $V$ has Witt index $n-1$.

Notation B.4. For $Z \in \mathcal{S}(V)$ define

$$Q_Z := C_H(Z) \cap C_H(Z^\perp/Z),\ Cl^\diamond(V) := H^\diamond := \langle Q_Z \mid Z \in \mathcal{S}(V)\rangle \text{ and } D_Z := C_H(Z^\perp) \cap C_H(V/Z).$$

We remark that $Q_Z$ is a weakly closed subgroup of $H$, so the notation $H^\diamond$ is analogue to the $\circ$-notation 1.44 for weakly closed subgroup.

Note that we have one of the following cases:

| $Cl^\diamond(V)$ | Type of $V$ | $V^\perp$ | $R(V)$ | Remark |
|---|---|---|---|---|
| $SL(V)$ | linear | $V$ | 0 | — |
| $Sp(V)$ | symplectic | 0 | 0 | — |
| $SU(V)$ | unitary | 0 | 0 | — |
| 1 | orthogonal | 0 | 0 | $\dim V \leqslant 2$, $\dim V$ even or $p$ odd |
| $\Omega(V)$ | orthogonal | 0 | 0 | $\dim V \geqslant 3$, $\dim V$ even or $p$ odd |
| $O(V) = \Omega(V)$ | orthogonal | 1-dim | $V^\perp$ | $\dim V$ odd and $p$=2 |

## B.1. Elementary Properties

Lemma B.5 ([**MS5**, 3.1]). *Let $U$ be an isotropic but not singular $\mathbb{K}$-subspace of $V$. Let $U_0$ be the set of singular vectors in $U$. Then $V$ is orthogonal, $p = 2$, $U_0$ is a $\mathbb{K}$-subspace of $U$, and* $\dim_{\mathbb{K}} U/U_0 = 1$. *In particular,* $\dim_{\mathbb{K}} V^\perp \leqslant 1$.

Lemma B.6 ([**MS5**, 3.2]). *Let $U$ be a $\mathbb{K}$-subspace of $V$, and let $A$ be a subgroup of $H$. Suppose that $V$ is not a linear space.*

(a) *$V/U^\perp$ and $(U/U \cap V^\perp)^*$ are isomorphic $\mathbb{F}N_H(U)$-modules. In particular, if $V^\perp = 0$, then $V$ and $V^*$ are isomorphic $\mathbb{F}H$-modules.*
(b) $C_{V/V^\perp}(A) = C_V(A)/V^\perp$.
(c) $C_V(A) = [V, A]^\perp$.
(d) *$C_H(V/U) \leqslant C_H(U^\perp)$; in particular $C_H(V/U) \leqslant C_H(U)$ if $U$ is isotropic.*
(e) *If $A$ acts quadratically on $V/V^\perp$, then $A$ acts quadratically on $V$ and $[V, A]$ is an isotropic subspace of $V$.*

This is [**MS5**, 3.2], except we corrected a misprint in statement (a).

[2] for the definition see A.46

LEMMA B.7 ([**MS5**, 1.9]). *Let $L$ be a finite group and $N \trianglelefteq L$, and let $\mathbb{F}$ be a finite field of characteristic $p$ and $V$ a finite dimensional $\mathbb{F}L$-module. Put $\mathbb{K} := End_{\mathbb{F}N}(V)$ and suppose that $V$ is a selfdual simple $\mathbb{F}N$-module. Then the following hold:*

(a) *There exists an $N$-invariant non-degenerate symmetric, symplectic or unitary $\mathbb{K}$-form $s$ on $V$.*
(b) *There exists a homomorphism $\rho: L \to Aut_{\mathbb{F}}(\mathbb{K})$, $h \mapsto \rho_h$, such that $L$ acts $\rho$-semilinearly on $V$.*
(c) *There exists a map $\lambda : L \to \mathbb{K}^\sharp$, $h \mapsto \lambda_h$, such that the map $L \to \mathbb{K}^\sharp \rtimes Aut_{\mathbb{F}}(K)$, $h \mapsto (\lambda_h, \rho_h)$, is a homomorphism and*
$$s(v^h, w^h) = \lambda_h s(v, w)^{\rho_h}$$
*for all $v, w \in V$, $h \in H$.*
(d) *Let $U$ be a $\mathbb{K}$-subspace of $V$ and put $U^\perp = \{v \in V \mid s(u, v) = 0 \text{ for all } u \in U\}$. Then $U^\perp$ is $N_L(U)$-invariant.*
(e) *Let $U$ be a non-zero $\mathbb{K}$-subspace of $V$ such that $C_L(U)$ acts simply on $V/U^\perp$. Then $U$ is 1-dimensional over $\mathbb{K}$.*
(f) *Put $L_0 = \ker \rho$. Then $s$ is $O^{p'}(L_0)N$-invariant.*

LEMMA B.8. *Let $H = GL(V)$, $V^*$ the dual of $V$ and $D, E \leqslant H$. Then $[V, E, D] = 0$ if and only if $[V^*, D, E] = 0$.*

PROOF. For $\alpha \in End(V)$ let $\alpha^* \in End(V^*)$ be the dual homomorphism. Note that $End(V) \to End(V^*), \alpha \mapsto \alpha^*$, is an anti-isomorphism of rings. Hence $[V, E, D] = 0$ iff $(e-1)(d-1) = 0$ for all $d \in D, e \in E$, iff $(d^*-1)(e^*-1) = 0$ for all $d \in D, e \in E$, iff $[V^*, D, E] = 0$. $\square$

LEMMA B.9. (a) *Suppose that $V$ is an orthogonal space. Let $v \in V$ and $a \in H$. Then $h([v, a]) = -f(v, [v, a])$. In particular, $[v, a]$ is singular if and only if $v \perp [v, a]$.*
(b) *Suppose that $V$ is an orthogonal space. Let $a \in H$ such that $[V, a]$ is 1-dimensional, and let $0 \neq w \in [V, a]$. Then $h(w) \neq 0$, and $a$ is the reflection associated to $w$, that is,*
$$v^a = v - h(w)^{-1} f(v, w) w \qquad \text{for all } v \in V.$$
*In particular, $C_H(V/[V, a]) = \{1, a\}$, $|a| = 2$ and $[V, a]$ is not singular.*
(c) *Suppose that $V$ is an orthogonal space. Let $X \leqslant H$ such that $[V, X, X] = 0$ and $[V, X]$ is 1-dimensional. Then $p = 2$, $|X| = 2$, $X$ is generated by a reflection, and $[V, X]$ is isotropic and not singular.*
(d) *Let $A \leqslant H$ and suppose that $U$ is an $A$-invariant subspace of $V$ with $[U, A] \leqslant U^\perp$. Then $[U, A]$ is singular.*
(e) *Let $A \leqslant H$ and suppose that $A$ acts cubically on $V$. Then $[V, A, A]$ is singular.*

PROOF. (a): We have $h(v^a) = h(v + [v, a]) = h(v) + f(v, [v, a]) + h([v, a])$. Since $h(v^a) = h(v)$, this gives $h([v, a]) = -f(v, [v, a])$.

(b): Let $r$ be any element of $H$ with $[V, r] = [V, a]$. Define $\alpha : V \to \mathbb{K}$ by $[v, r] = \alpha(v) w$ for $v \in V$. By B.6(c), $C_V(r) = [V, r]^\perp = w^\perp$. Thus $\ker \alpha = w^\perp$ and so there exists $0 \neq k \in \mathbb{K}$ with $\alpha(v) = k f(v, w)$ for all $v \in V$.

$$(*) \qquad [v, r] = k f(v, w) w \qquad \text{and} \qquad v^r = v + k f(v, w) w$$

for all $v \in V$. Since $w^\perp = C_V(r) \neq V$ we can choose $u \in V$ with $f(u, w) = 1$. Then $[u, r] = k f(u, w) w = k w$. Thus

$$h([u, r]) = h(kw) = k^2 h(w) \qquad \text{and} \qquad f(u, [u, r]) = f(u, kw) = k f(u, w) = k.$$

By (a) $h([u, r]) = -f(u, [u, r])$ and so $k^2 h(w) = -k$. Recall that $k \neq 0$. So $h(w) = -k^{-1} \neq 0e$. Together with the second equation in $(*)$ this shows that

$$(**) \qquad v^r = v - h(w)^{-1} f(v, w) w \qquad \text{for all } v \in V.$$

Obviously, the action of $r$ is uniquely determined by $(**)$, and the right hand side of the equation is independent of $r$. Since also $a$ satisfies $(**)$ in place of $r$, we get $r = a$, and the equation in (b) holds. Moreover, $C_H(V/[V,a]) = \{1,a\}$ and $|a| = 2$. As $h(w) \neq 0$, $[V,a]$ is not singular. Hence (b) holds.

(c) Let $1 \neq a \in X$. Then $X \subseteq C_H(V/[V,X]) = C_H(V/[V,a])$. Thus by (b), $X \subseteq \{1,a\}$ and $[V,X] = [V,a]$ is not singular. In particular, $|X| = 2$. Since $X$ acts quadratically on $V$, B.6(e) shows that $[V,X]$ is isotropic. As $[V,X]$ is not singular, B.5 implies that $p = 2$.

(d): Since $U$ is $A$-invariant, $[U,A] \leqslant U$ and since $[U,A] \leqslant U^\perp$ we conclude that $[U,A]$ is isotropic. Let $u \in U$ and $a \in A$. Then $[u,a] \in [U,A] \leqslant U^\perp \leqslant u^\perp$ and so by (a) $[u,a]$ is singular. Since $[U,A]$ is isotropic, the singular vectors in $[U,A]$ form a subspace of $[U,A]$ (seeB.5). Since $[U,A]$ is generated by the singular vectors $[u,a]$, $u \in U, a \in A$, we conclude that $[U,A]$ is singular.

(e): Since $A$ acts cubically on $V$, $[V,A,A] \leqslant C_V(A)$ and so by B.6(c), $[V,A,A] \leqslant [V,A]^\perp$. Thus (e) follows from (d) applied with $U = [V,A]$. □

Lemma B.10. *Let $V_1$ and $V_2$ be $\mathbb{K}$-subspaces of $V$ with $V = V_1 + V_2$. Let $a \in GL_{\mathbb{K}}(V)$ and define $a_i : V_i \to V_i^a, v \mapsto v^a$. Then $a$ is an isometry on $V$ if and only if $a_1$ and $a_2$ are isometries and $f(v_1^a, v_2^a) = f(v_1, v_2)$ for all $v_1 \in V_1$ and $v_2 \in V_2$.*

Proof. The forward direction is obvious. So suppose that $a_1$ and $a_2$ are isometries and $f(v_1^a, v_2^a) = f(v_1, v_2)$ for all $v_1 \in V_1$ and $v_2 \in V_2$. Since $f$ is $\mathbb{F}$-bilinear we conclude that $f$ is $a$-invariant. Since $h(v_1 + v_2) = h(v_1) + f(v_1, v_2) + h(v_2)$, in the case of an orthogonal space, also $h$ is $a$-invariant. □

Lemma B.11. *Suppose that $V$ is not a linear space. Let $Z \in \mathcal{S}(V)$ and $v \in V \backslash Z^\perp$. Let $i_v$ be the number of elements $w \in v + Z$ isometric to $v$, and for $\lambda \in \mathbb{F}$ let $s_\lambda$ be the number of elements $w \in v + Z$ with $h(w) = \lambda$.*

(a) *If $V$ is a symplectic space then $i_v = s_0 = |v + Z| = |\mathbb{K}|$.*
(b) *If $V$ is a unitary space then $i_v = s_\lambda = |\mathbb{F}|$ for all $\lambda \in \mathbb{F}$.*
(c) *If $V$ is an orthogonal space then $i_v = s_\lambda = 1$ for all $\lambda \in \mathbb{K}$.*

Proof. Choose $z \in Z$ with $f(z,v) = -1$ and let $k \in \mathbb{K}$.

(a): In a symplectic space all elements are singular. Hence (a) holds.

(b): Suppose that $f$ is unitary, so $h(x) = f(x,x)$ for $x \in V$. Then

$$h(v + kz) = f(v + kz, v + kz) = f(v,v) + kf(z,v) + k^\alpha f(z,v) + kk^\alpha f(z,z) = h(v) - (k + k^\alpha).$$

Thus $h(v + kz) = \lambda$ if and only if $k + k^a = h(v) - \lambda$. Since the function $\mathbb{K} \to \mathbb{F}$, $k \mapsto k + k^\alpha$, is $\mathbb{F}$-linear and surjective, we conclude that for a given $\lambda \in \mathbb{F}$ there are exactly $|\mathbb{F}|$ elements $k \in \mathbb{K}$ with $k + k^\alpha = h(v) - \lambda$. So (b) holds.

(c): Suppose $H = O(V)$. Then

$$h(v + kz) = h(v) + f(v, kz) + k^2 h(z) = h(v) - k.$$

Hence $h(v + kz) = \lambda$ if and only if $k = h(v) - \lambda$. This gives (c). □

Corollary B.12. *Suppose that $\mathcal{S}(V) \neq \varnothing$ and that $V$ is not a 2-dimensional orthogonal space.*

(a) *The number of singular 1-spaces in $V$ is congruent to 1 modulo $p$.*
(b) *Then number of non-zero singular vectors in $V$ is congruent to $|\mathbb{K}^\sharp|$ modulo $p$.*
(c) *Any $p$-group of semilinear similarities of $V$ fixes a singular vector and a singular 1-space in $V$.*

Proof. (a): Let $Z \in \mathcal{S}(V)$. Put $m := \dim_{\mathbb{K}} V$. If $V$ is a linear, symplectic or unitary space put $t = |\mathbb{F}|$. If $V$ is an orthogonal space put $t = 1$. Let $E$ be a 2-dimensional subspace of $V$ with $Z \leqslant E$
Suppose that $E \leqslant Z^\perp$. Note that one of the following holds:

– $E$ is singular, $E/Z \in \mathcal{S}(Z^\perp/Z)$, and all $|\mathbb{K}|$ 1-spaces of $E$ distinct from $Z$ are in $\mathcal{S}(V)$.
– $E$ is not singular, $E/Z \notin \mathcal{S}(Z^\perp/Z)$, and $Z$ is the only singular 1-space of $E$.

It follows that

$$|S(Z^\perp)\backslash\{Z\}| = |\mathbb{K}||\mathcal{S}(Z^\perp/Z)|.$$

Suppose next that $E \nleqslant Z^\perp$. Then B.11 shows that $E$ contains exactly $t$ elements of $\mathcal{S}(V)$ distinct from $Z$. Note also that there are $|\mathbb{K}|^{m-2}$ 2-dimensional subspaces $E$ of $V$ with $Z \leqslant E \nleqslant Z^\perp$. Thus

$$|\mathcal{S}(V)\backslash\mathcal{S}(Z^\perp)| = |\mathbb{K}|^{m-2}t.$$

Hence

$$|\mathcal{S}(V)| = 1 + |\mathbb{K}||\mathcal{S}(Z^\perp/Z)| + |\mathbb{K}|^{m-2}t.$$

Since $p \mid |\mathbb{K}|$, we conclude that either $|\mathcal{S}(V)| \equiv 1 \pmod p$ or $m = 2$ and $p \nmid t$. In the latter case, since $p \mid |\mathbb{F}|$ we get $t = 1$ and $V$ is an orthogonal space. Since 2-dimensional orthogonal spaces are excluded by the hypothesis of the corollary, (a) is proved.

(b): Note that every singular 1-space contains exactly $|\mathbb{K}^\sharp|$ non-zero singular vectors. So (b) follows from (a).

(c): By (a) and (b), neither the number of singular 1-spaces nor the number of non-zero singular vectors in $V$ is divisible by $p$. This gives (c). □

LEMMA B.13. *Let $U$ be a $\mathbb{K}$-subspace of $V$ and put $W = \langle\mathcal{S}(U)\rangle$. Then $W = U$ or $W = rad(U)$. In particular, $V = \langle\mathcal{S}(V)\rangle$ or $\mathcal{S}(V) = \varnothing$.*

PROOF. Let $Y$ be a 1-dimensional subspace of $U$. If $Y \nleqslant W^\perp$, then there exists $Z \in \mathcal{S}(V)$ with $Y \nleqslant Z^\perp$. By B.11 $Z + Y$ contains a singular 1-space $X \neq Z$. Thus $Y \leqslant Z + Y = Z + X \leqslant W$.

We have proved that $U \subseteq W \cup W^\perp$, and so $U \leqslant W^\perp$ or $U \leqslant W$. In the second case $U = W$ and we are done. So suppose $U \leqslant W^\perp$. Then $W \leqslant U \cap U^\perp$, $W$ is singular and $W \leqslant rad(U)$. Clearly $rad(U) \leqslant W$ and so $W = rad(U)$.

Either $V$ is linear or $rad(V) = 0$. In the first case $\langle\mathcal{S}(V)\rangle = V$ is obvious, in the second case either $\langle\mathcal{S}(V)\rangle = V$ or $\langle\mathcal{S}(V)\rangle = rad(V) = 0$ and $\mathcal{S}(V) = \varnothing$. □

LEMMA B.14. *Suppose that $V$ is a symplectic space and $p = 2$. Let $V' := V \times \mathbb{K}$ as a set. Define an addition and scalar multiplication on $V'$ by*

$$(v,k) + (w,l) := (v + w, k + l + f(v,w)) \qquad \text{and} \qquad l(v,k) := (lv, l^2k)$$

*for all $v, w \in V$, $k, l \in \mathbb{K}$. Define*

$$h' : V' \to \mathbb{K},\ (v,k) \mapsto k, \quad \text{and} \quad f' : V' \times V' \to \mathbb{K},\ \big((v,k),(w,l)\big) \mapsto f(v,w).$$

(a) *$(V', f', h')$ is a non-degenerate orthogonal space with $V'^\perp = \{(0,k) \mid k \in \mathbb{K}\}$.*
(b) *The function $V \to V'/V'^\perp, v \mapsto (v,0) + V'^\perp$ an isometry of symplectic spaces.*
(c) *Let $a \in Sp(V)$ and define*

$$a' : \quad V' \to V', \quad (v,k) \mapsto (v^a, k).$$

*Then $a' \in O(V')$.*

(d) *The function*

$$Sp(V) \to O(V'), \qquad a \mapsto a',$$

*is an isomorphism.*

(e) *Suppose in addition that $V = V_1/V_1^\perp$ where $(V_1, f_1, h_1)$ is a non-degenerate orthogonal space with $V_1^\perp \neq 0$. Then the function*

$$V_1 \to V',\ w \to \big(w + V_1^\perp, h_1(w)\big)$$

*is an isometry.*

PROOF. (a): Let $u, v, w \in V$ and $j, k, l \in \mathbb{K}$. The addition is clearly commutative, $(0,0)$ is an additive identity and $(v,k)$ is its own inverse. Also

$$(u,j)+\big((v,k)+(w,l)\big) = \big(u+v+w, j+k+l+f(u,v)+f(u,w)+f(v,w)\big) = \big((u,j)+(v,k)\big)+(w,l)$$

and so $V'$ is an abelian group. Note that

$$j\big((v,k)+(w,l)\big) = (jv+jw, j^2k+j^2l+j^2f(v,w)) = j(v,k)+j(w,l),$$

$$(j+k)(w,l) = (jw+kw, j^2l+k^2l) = j(w,l)+k(w,l) \text{ and } (jk)(w,l) = (jkw, j^2k^2l) = j(k(w,l)).$$

Thus $V'$ is a vector space over $\mathbb{K}$. Since $f$ is a symmetric form, so is $f'$. Moreover,

$$h'\big(k(w,l)\big) = h'(kw, k^2l) = k^2l = k^2h'(w,l)$$

and

$$h'\big((v,k)+(w,l)\big) = h'\big(v+w, k+l+f(v,w)\big) = k+l+f(v,w) = h'(v,k)+f'(v,w)+h'(w,l))$$

and so $(V', f', h')$ is an orthogonal space. Note that $(v,k) \in V'^{\perp}$ if and only if $v \in V^{\perp} = 0$. Also $h'(v,k) = 0$ if and only if $k = 0$. Thus $rad(V') = 0$ and $V'$ is non-degenerate.

(b) and (c) should be obvious.

(d): Let $b \in O(V)$. Then $b$ induces an isometry on the symplectic space $V'/V'^{\perp}$. Together with (b) we conclude that there exists a unique $a \in Sp(V)$ such that $(v,k)^b + V'^{\perp} = (v^a, 0) + V'^{\perp}$ for all $v \in V$, $k \in \mathbb{K}$. Since $b$ is an isometry, $h'\big((v,k)^b\big) = h'(v,k) = k$ and we conclude that $(v,k)^b = (v^a, k)$. Thus $b = a'$ and (d) holds.

(e) is readily verified. □

LEMMA B.15 (Witt's Lemma). *Let $U$ and $W$ be $\mathbb{K}$-subspaces of $V$ suppose that $\beta : U \to W$ is an isometry with $(U \cap R(V))^\beta = W \cap R(V)$. Then $\beta$ extends to an isometry of $V$.*

PROOF. If $V$ is a linear space, this is obvious. So suppose $V$ is a symplectic, orthogonal or unitary space. If $V^\perp = 0$, this is Witt's Lemma on page 81 of [**As**, 20].

It remains to treat the case where $V$ is an orthogonal space with $V^\perp \neq 0$. Then $R(V) = V^\perp$. Since $(U \cap R(V))^\beta = W \cap R(V)$ we conclude that $\beta$ induces an isometry of symplectic spaces

$$b : U + V^\perp/V^\perp \to W + V^\perp/V^\perp.$$

According to the already treated symplectic case, $b$ extends to an isometry $a$ of the symplectic space $V/V^\perp$. By B.14(c), there exists an isometry $a'$ of $V$ with $v^{a'} + V^\perp = (v + V^\perp)^a$ for all $v \in V$. Let $u \in U$. Then $u^\beta + V^\perp = u^{a'} + V^\perp$ and since both $\beta$ and $a'$ are isometries, $h(u^\beta) = h(u^{a'})$. It follows that $u^\beta = u^{a'}$, and so the lemma also holds for an orthogonal space with $V^\perp \neq 0$. □

LEMMA B.16. *Let $v$ and $w$ be isometric elements in $V \backslash R(V)$. Then there exists $a \in H$ with $w^a = v$. In particular, $H$ acts transitively on the set of non-zero singular vectors.*

PROOF. Since $v$ and $w$ are isometric, the function $\beta : \mathbb{K}v \to \mathbb{K}w$, $kv \mapsto kw$ is an isometry. Also $\mathbb{K}v \cap R(V) = 0 = \mathbb{K}w \cap R(V)$, and so by Witt's Lemma $\beta$ extends to an isometry $a$ of $V$. Then $v^a = v^\beta = w$. □

## B.2. The Classification of Classical Spaces

DEFINITION B.17. Let $(v_i)_{i=1}^n$ be a family of vectors in $V$.

(a) $(v_i)_{i=1}^n$ is *orthogonal* if $f(v_i, v_j) = 0$ for all $1 \leqslant i, j \leqslant n$ with $i \neq j$.
(b) $(v_i)_{i=1}^n$ is *orthonormal* if it is orthogonal and $h(v_i) = 1$ for all $1 \leqslant i \leqslant n$.
(c) $(v_i)_{i=1}^n$ is *hyperbolic* if $n = 2l$ is even, $h(v_i) = 0$ for all $1 \leqslant i \leqslant n$, and $f(v_i, v_{n+1-i}) = 1$ for all $1 \leqslant i \leqslant l$, and $f(v_i, v_j) = 0$ for all $1 \leqslant i, j \leqslant n$ with $i + j \neq n + 1$.
(d) $V$ is *hyperbolic* if $V$ has a hyperbolic basis.
(e) $V$ is *definite,* if $V$ has no non-zero singular vectors.

LEMMA B.18. *Let* $\dim V =: m =: 2n + \epsilon$, $\epsilon \in \{0, 1\}$.

(a) *Suppose $V$ is a symplectic space.*
  (a) *$V$ has a hyperbolic basis. In particular, $m$ is even.*
  (b) *$V$ has Witt index $n$.*
  (c) *Up to isometry, $V$ is uniquely determined by $m$.*
(b) *Suppose $V$ is a unitary space.*
  (a) *$V$ has an orthonormal basis.*
  (b) *$V$ has a basis $(v_i)_{i=1}^m$ such that $(v_i)_{i=1}^{2n}$ is hyperbolic, and if $m$ is odd, $f(v_i, v_m) = 0$ for all $1 \leqslant i \leqslant 2n$ and $h(v_m) = 1$.*
  (c) *$V$ has Witt index $n$.*
  (d) *Up to isometry, $V$ is uniquely determined by $m$.*
(c) *Suppose $V$ is an orthogonal space and $p$ is odd. Then $V$ has an orthogonal basis $(v_i)_{i=1}^m$ such that $h(v_i) = 1$ for $1 \leqslant i \leqslant m - 1$.*
(d) *Suppose $V$ is an orthogonal space and $m$ is odd.*
  (a) *$V$ has a basis $(u_i)_{i=1}^m$ such that $(u_i)_{i=1}^{2n}$ is hyperbolic, $f(u_i, u_m) = 0$ for all $1 \leqslant i \leqslant m - 1$, and $h(u_m) \neq 0$.*
  (b) *$V$ has Witt index $n$.*
  (c) *Up to similarity, $V$ is uniquely determined by $m$.*
  (d) *If $p = 2$ then $V$ is uniquely determined up to isometry by $m$.*
  (e) *If $p$ is odd then $V$ is uniquely determined up to isometry by $m$ and the coset $h(u_m)\mathbb{K}^{\sharp 2}$.*
(e) *Suppose $V$ is an orthogonal space and $m$ is even.*
  (a) *Either $V$ has Witt index $n$ and a hyperbolic basis, or $V$ has Witt index $n - 1$ and a basis $(v_i)_{i=1}^m$ such that $(v_i)_{i=3}^m$ is hyperbolic, $f(v_i, v_j) = 0$ for $1 \leqslant i \leqslant 2$ and $3 \leqslant j \leqslant m$, $h(v_1) = h(v_2) = 1$ and the polynomial $x^2 - f(v_1, v_2)x + 1$ has no roots in $\mathbb{K}$.*
  (b) *Up to isometry, $V$ is uniquely determined by $m$ and its Witt index.*

PROOF. Suppose that $V$ is a symplectic, unitary or orthogonal space. Let $U$ be a singular subspace of $V$, so $U \leqslant U^\perp$. Then $U \cap V^\perp = 0$ and so $\dim_{\mathbb{K}} V/U^\perp = \dim_{\mathbb{K}} U$. In particular, $2\dim_{\mathbb{K}} U \leqslant \dim_{\mathbb{K}} V$ and thus $V$ has Witt index at most $n$.

(a): By [**Hu**, II.9.6(b)] $V$ has a hyperbolic basis, $\dim V$ is even and, up to isometry, $V$ is uniquely determined by its dimension. Let $(v_i)_{i=1}^{2n}$ be a hyperbolic basis. Then $\mathbb{K}\langle v_1, \ldots, v_n\rangle$ is a singular subspace of dimension $n$. Thus $V$ has Witt index at least $n$, and (a) holds.

(b): By [**Hu**, II.10.4a ] $V$ has an orthonormal basis $(v_i)_{i=1}^m$ and up to isometry is uniquely determined by its dimension. Put $W = \mathbb{K}\langle v_1, \ldots, v_{2d}\rangle$. Then by [**Hu**, I0.4b] $W$ has a hyperbolic basis $(u_i)_{i=1}^{2n}$. Then $W^\perp$ is 1-dimensional with orthonormal basis say $u_m$. Also $\mathbb{K}\langle u_1, \ldots, u_n\rangle$ is a singular subspace of dimension $n$, and so (b) holds.

(c): [**Hu**, II.10.9b].

(d): Suppose first that $p$ is odd. Then by [**As**, 21.3] $V$ has a hyperbolic hyperplane $W$. Note that $W$ has Witt index $n$ and so also $V$ has Witt index $n$. As in [**As**] choose $0 \neq x \in W^\perp$ and a generator $c$ of $\mathbb{K}^\sharp$, and define $sgn(V) = +1$ if $h(x)$ is a square in $\mathbb{K}$ and $sgn(V) = -1$ if not. By [**As**, 21.4], up to isometry, there are exactly two $m$-dimensional orthogonal spaces, namely $(V, f, h)$ and $(V, cf, ch)$. Moreover, $(V, f, h)$ and $(V, cf, ch)$ are similar, and one has $sgn$ equal to $+1$ and the other equal to $-1$. So (d) holds if $p$ is odd.

If $p = 2$ then $V^\perp \neq 0$ and by (a), the symplectic space $V/V^\perp$ has a hyperbolic basis $(v_i + V^\perp)_{i=1}^{2n}$. Since $h(V^\perp) = \mathbb{K}$, $v_i + V^\perp$ contains a singular vector and we may choose $v_i$ to be singular. Then $\mathbb{K}\langle (v_i)_{i=1}^n \rangle$ is an $n$-dimensional singular subspace of $V$ and so $V$ has Witt index $n$. Since $\mathbb{K}^2 = \mathbb{K}$ we can choose $v_m \in V^\perp$ with $h(v_m) = 1$. In particular, up to isometry, $V$ is uniquely determined by its dimension, and so (d) also holds if $p = 2$.

(e): By [**As**, 21.6] $V$ is isometric to $D^n$ or $D^{n-1}Q$, where $D$ and $Q$ are 2-dimensional orthogonal spaces with $D$ hyperbolic and $Q$ definite. Moreover, $D^n$ has Witt index $n$, while $D^{n-1}Q$ has Witt index $n-1$. Let $\mathbb{E}$ be an extension field of $\mathbb{K}$ with $\dim_\mathbb{K} \mathbb{E} = 2$ and let $id_\mathbb{K} \neq \sigma \in Aut_\mathbb{K}(\mathbb{E})$. Define

$$T_\sigma : \ \mathbb{E} \times \mathbb{E} \to \mathbb{K}, \ (a,b) \ \mapsto \ a^\sigma b + b^\sigma a, \qquad \text{and} \qquad N_\sigma : \ \mathbb{E} \to \mathbb{K}, \ a \ \mapsto \ a^\sigma a.$$

Then by [**As**, 21.9] $(\mathbb{E}, T_\sigma, N_\sigma)$ is a definite orthogonal space and isometric to $Q$. Let $\lambda$ be any element of $\mathbb{K}$ such that $x^2 - \lambda x + 1$ has no root in $\mathbb{K}$. Then $x^2 - \lambda x + 1$ has a root $\xi \in \mathbb{E}$. It follows that $\xi + \xi^\sigma = \lambda$ and $\xi^\sigma \xi = 1$. Since $\xi \notin \mathbb{K}$, $(1, \xi)$ is $\mathbb{K}$-basis for $\mathbb{E}$ and

$$N_\sigma(1) = 1^\sigma 1 = 1, \qquad T_\sigma(1, \xi) = 1^\sigma \xi + \xi^\sigma 1 = \lambda, \qquad N_\sigma(\xi) = \xi^\sigma \xi = 1.$$

Thus (e) holds. □

LEMMA B.19. *Suppose that $f \neq 0$ and let $d$ be the Witt index of $V$.*

(a) *$H$ acts transitively on the maximal singular subspace of $V$. In particular, all maximal singular subspace of $V$ have dimension $d$.*
(b) *Let $W$ be a maximal hyperbolic subspace of $V$. Then $V = W \oplus W^\perp$ and $W^\perp$ is definite.*
(c) *If $V$ is definite, then $\dim_\mathbb{K} V \leqslant 1$ or $H = O^-(V)$ and $\dim_\mathbb{K} V = 2$.*

PROOF. (a): See [**As**, 20.8].

(b): See [**As**, 19.5].

(c) follows from B.18. □

### B.3. The Clifford Algebra

In this section $(V, f, h)$ is an orthogonal space. We will define a normal subgroup $\Omega(V)$ of $O(V)$ via the Spinor norm and Dickson invariant. In our definition the Spinor norm $S$ is defined for all $a \in O(V)$ and not only for products of reflections. This allows to define $\Omega(V)$ also in the case of a four dimensional orthogonal space of Witt index two over $\mathbb{F}_2$.

We remark that $\Omega(V)$ is often defined to be $O(V)'$. With our definition $\Omega(V) = O(V)'$ with two exceptions: $\Omega_4^+(2) \cong SL_2(2) \times SL_2(2)$, while $O_4^+(2)' \sim 3^2.2$ and $\Omega_5(2) = O_5(2) \cong Sym(6)$, while $O_5(2)' \cong Alt(6)$.

We will also prove that $C_{\Omega(V)}(W^\perp) \cong \Omega(W)$ if $V^\perp = 0$ and $W$ is a $\mathbb{K}$-subspace of $V$ with $W \cap W^\perp = 0$. This of course is well-known, but the case, where $W$ is a four dimensional orthogonal space of Witt index two over $\mathbb{F}_2$, is often ignored.

As a byproduct we obtain that the elements of $Q_Z, Z \in \mathcal{S}(V)$, naturally correspond to elements of the Clifford algebra of $V$, see the elements $\omega_{1+bc}$ below.

Let $C := C(V, f, h)$ be the *Clifford algebra* of the orthogonal space $(V, f, h)$. So $C$ is an associative $\mathbb{K}$-algebra with identity generated by the $\mathbb{K}$-space $V$ subject to the relations $v^2 = h(v)$ for $v \in V$.

Let $v, w \in V$. Then

$$h(v) + f(v,w) + h(w) = h(v+w) = (v+w)^2 = v^2 + vw + wv + w^2 = h(v) + vw + wv + h(w),$$

and so

$$vw + wv = f(v,w) \in \mathbb{K} \qquad \text{and} \qquad vw = -wv + f(v,w).$$

Note that the opposite algebra fulfills the same relations and so there exists a unique anti-automorphism of $\mathbb{K}$-algebras

$$\theta : \qquad C \to C \quad \text{with} \quad v^\theta = v \quad \text{for } v \in V.$$

Then for $x, y \in V$, $h(x) = x^2 = x^\theta x$ and $f(x,y) = xy + yx = x^\theta y + y x^\theta$. We extend $h$ and $f$ as follows:

$$h: \quad C \to C,\ x \ \mapsto\ x^\theta x \qquad \text{and} \qquad f: \quad C \times C \to C, (x,y) \ \mapsto\ x^\theta y + y^\theta x$$

Note that $f$ is $\mathbb{K}$-bilinear and for all $x, y \in C$ :

$$h(x+y) = (x+y)^\theta (x+y) = (x^\theta + y^\theta)(x+y) = x^\theta x + x^\theta y + y^\theta x + y^\theta y = h(x) + f(x,y) + h(y).$$

Put $E := \{1, -1\} \subseteq \mathbb{Z}$ (so $|E| = 2$ even if $p = 2$). For $i \in E$ define

$$C_i := C_i(V, f, h) := \mathbb{K}\langle v_1 \cdots v_n \mid n \in \mathbb{N}, v_1, \ldots, v_n \in V,\ (-1)^n = i \rangle,$$

where $v_1 \cdots v_n = 1$ if $n = 0$.

Then $(C_1, C_{-1})$ is an $E$-grading of $C$, that is $C = C_1 \oplus C_{-1}$ and $C_i C_j \subseteq C_{ij}$ for all $i, j \in E$. Define

$$Cliff(V) := \{x \in C_1 \cup C_{-1} \mid 0 \neq h(x) \in \mathbb{K}, xV = Vx\}.$$

Let $x \in Cliff(V)$ and $y \in C$. Then $0 \neq x^\theta x = h(x) \in \mathbb{K}$ and so $x$ is invertible with inverse $h(x)^{-1} x^\theta$. We compute

$$h(xy) = (xy)^\theta xy = y^\theta x^\theta x y = y^\theta h(x) y = h(x) y^\theta y = h(x) h(y).$$

For $y = x^{-1}$ this shows that $h(x^{-1}) = h(x)^{-1} \in \mathbb{K}^\sharp$. For $y \in Cliff(V)$ we get $0 \neq h(xy) \in \mathbb{K}$. It follows that $Cliff(V)$ is a multiplicative subgroup of $C$ and the restriction of $h$ to $Cliff(V)$ is a multiplicative homomorphism from $Cliff(V)$ to $\mathbb{K}^\sharp$.

For $x \in Cliff(V)$ and $y \in C$ define $y^x := x^{-1} y x$. Then, since $h(x) \in \mathbb{K} \subseteq Z(C)$,

$$h(y^x) = h(x^{-1} y x) = h(x^{-1})(h(y) h(x)) = h(x)^{-1} h(y) h(x) = h(y).$$

So $h$ and thus also $f$ is invariant under conjugation by $x \in Cliff(V)$.

Let $d(x)$ be the unique element of $E$ with $x \in C_{d(x)}$. Then

$$d: \quad Cliff(V) \to E, \quad x \mapsto d(x),$$

is a group homomorphism.

Since $x$ is invertible the condition $xV = Vx$ is equivalent to $V^x = V$ (where $V^x := x^{-1} V x$). Define

$$\omega_x: \quad V \to V, \quad v \ \mapsto\ d(x) v^x.$$

Since $d(x) = \pm 1$, $h$ and $f$ are invariant under multiplication by $d(x)$, and, as seen above, also under conjugation by $x$. Hence $\omega_x \in O(V)$ and

$$\omega: \quad Cliff(V) \to O(V), \quad x \mapsto \omega_x,$$

is a homomorphism.

Let $a \in V$ with $h(a) \neq 0$, so $a$ is invertible with inverse $h(a)^{-1} a$. Let $v \in V$. Observe that $d(a) = -1$, $va = -av + f(v,a)$ and $a^{-1} = h(a)^{-1} a$. Thus

$$d(a) a^{-1} v a = -\big(a^{-1}(-av + f(v,a))\big) = v - f(v,a) a^{-1} = v - h(a)^{-1} f(v,a) a.$$

In particular, $a \in Cliff(V)$, and $\omega_a$ is the reflection associated to $a$.

Next let $b, c \in V$ such that $b$ is singular and $c \perp b$. Note that $b^2 = h(b) = 0$ and $bc = -cb + f(b,c) = -cb$. Hence

$$(*) \qquad bcb = -bbc = h(b)c = 0, \qquad \text{and} \qquad (bc)^2 = bcbc = 0$$

Put $x := 1 + bc$. Then $x^\theta = 1 + cb = 1 - bc$ and

$$h(x) = h(1+bc) = \theta(1+bc)(1+bc) = (1-bc)(1+bc) = 1 - bc + bc - (bc)^2 = 1.$$

In particular, $x = 1 + bc$ is invertible with inverse $1 - bc$. Recall that $1 \in C_1$ and so also $x \in C_1$ and $d(x) = 1$. Thus $\omega_x$ is conjugation by $x$.

We compute:

$$(**) \ \ vbc = (-bv + f(v,b))c = -bvc + f(v,b)c = -b(-cv + f(v,c)) + f(v,b)c = bcv - f(v,c)b + f(v,b)c$$

and

$$\begin{aligned} x^{-1}vx &= (1-bc)v(1+bc) = (1-bc)(v+vbc) = (1-bc)v + (1-bc)vbc \\ &\overset{(**)}{=} v - bcv + (1-bc)\big(bcv - f(v,c)b + f(v,b)c\big) \\ &= v - bcv + bcv - f(v,c)b + f(v,b)c - bcbcv + f(v,c)bcb - f(v,b)bcc \\ &\overset{(*)}{=} v - f(v,c)b + f(v,b)c - f(v,b)bh(c) \\ &= v - f(v,c)b + f(v,b)\big(c - h(c)b\big). \end{aligned}$$

In particular, $x^{-1}Vx = V$. Hence $x = 1 + bc \in Cliff(V)$ and

$$\omega_{1+bc}: \quad V \to V, \quad v \ \mapsto \ v - f(v,c)b + f(v,b)\big(c - h(c)b\big)$$

is an isometry of $V$.

We claim that

(I) $$O(V) = \langle \omega_a, \omega_{1+bc} \mid a, b, c \in V, h(a) \neq 0, h(b) = 0, f(b,c) = 0 \rangle.$$

Indeed, by [**As**, 22.7] $O(V)$ is generated by the reflections $\omega_a$, unless $O(V) = O_4^+(2)$. In the latter case the group generated by reflections has index two and does not contain $\omega_{1+bc}$ for $b, c \in V^\sharp$ with $h(b) = h(c) = 0$, $f(b,c) = 0$ and $b \neq c$. So (I) holds.

In particular, $\omega$ defined above is surjective. Put $Z := \ker \omega$. So $Cliff(V)/Z \cong O(V)$, and $h$ and $d$ induce well-defined homomorphisms

$$S: \ O(V) \to \mathbb{K}^\sharp/h(Z), \ \ \omega_x \ \mapsto \ h(x)h(Z) \qquad \text{and} \qquad D: \ \ O(V) \to E/d(Z), \ \omega_x \ \mapsto \ d(x)d(Z),$$

where $x$ runs through the elements of $Cliff(V)$. $S(x)$ is called the *Spinor norm* of $x$, and $D(x)$ is called the *Dickson invariant* of $x$. We define

$$\Omega(V) := \ker S \cap \ker D = \{z \in O(V) \mid S(x) = 1 \text{ and } D(x) = 1\}.$$

Since $h(1+bc) = 1$ and $d(1+bc) = 1$ we have $S(\omega_{1+bc}) = 1$ and $D(\omega_{1+bc}) = 1$. So

$$O^\diamond(V) := \langle \omega_{1+bc} \mid b, c \in V, h(b) = 0, f(b,c) = 0 \rangle \leqslant \Omega(V).$$

We will determine now $O(V)/\Omega(V)$. By (I), $O(V) = O^\diamond(V)\langle \omega_a \mid a \in V, h(a) \neq 0 \rangle$, and since $O^\diamond(V) \leqslant \Omega(V)$, we conclude that

$$O(V)/\Omega(V) \cong \Big\langle \big( S(\omega_a), D(\omega_a) \big) \mid a \in V, h(a) \neq 0 \Big\rangle.$$

Put $m := \dim \mathbb{K}$ and note that

$$Z(C) = \begin{cases} \mathbb{K} & \text{if } m \text{ is even,} \\ \mathbb{K} + \mathbb{K}(v_1 \cdots v_m) & \text{if } m \text{ is odd, } p \text{ is odd and } (v_i)_{i=1}^m \text{ is an orthogonal basis for } V, \\ \mathbb{K} + V^\perp & \text{if } m \text{ is odd and } p = 2. \end{cases}$$

In particular, $Z(C) \cap C_1 = \mathbb{K}$. We claim that

(II) $$Z = \mathbb{K}^\sharp \cup V^{\perp\sharp}.$$

Let $x \in Cliff(V)$. Then $x \in Z$ if and only $v^x = d(x)v$ for all $v \in V$. If $p = 2$ this just means $x \in Z(C)$, and so $x \in C_1 \cap Z(C) = \mathbb{K}^\sharp$ or $x \in C_{-1} \cap Z(C) = V^{\perp\sharp}$, where $m$ is odd in the latter case. So (II) holds for $p = 2$.

So suppose $p$ is odd. Assume $x \in C_{-1}$. Then $d(x) = -1$ and so $v^x = -v$ for all $v \in V$. It follows that $w^x = -w$ for all $w \in C_{-1}$ and so also $x^x = -x$, a contradiction since $x^x = x$, $p$ is odd and $x \neq 0$. Hence $x \in C_1$, $d(x) = 1$ and $x \in Z(C) \cap C_1 = \mathbb{K}$. So $x \in \mathbb{K}^\sharp$. Since $p$ is odd, $V^\perp = 0$, and so (II) also holds for odd $p$.

If $V^\perp \neq 0$, then $p = 2$, $\mathbb{K}^{\sharp 2} = \mathbb{K}^\sharp$, and so $h(Z) = \mathbb{K}^\sharp$ and $d(Z) = E$. It follows that $O(V) = \ker D = \ker S = \Omega(V)$ in this case.

So suppose $V^\perp = 0$. Then (II) shows that $h(Z) = \mathbb{K}^{\sharp 2}$ and $d(Z) = 1$. Hence $S$ and $D$ are given by

$$S: \ O(V) \to \mathbb{K}^\sharp/\mathbb{K}^{\sharp 2}, \ \omega_x \mapsto h(x)\mathbb{K}^{\sharp 2}, \qquad \text{and} \qquad D: \ O(V) \to E, \ \omega_x \mapsto d(x).$$

Hence $S(\omega_a) = 1$ if and only if $a$ is a square in $\mathbb{K}^2$. Moreover, $D(\omega_a) = -1$.

Suppose that $p = 2$. Then $\mathbb{K}^\sharp = \mathbb{K}^{\sharp 2}$, and so $S(\omega_a) = 1$ for all $a \in V$ with $h(a) \neq 0$. Thus $\ker S = O(V)$, $\Omega(V) = \ker D$ and $O(V)/\Omega(V) \cong C_2$.

Suppose now that $p$ is odd and $\dim V \geqslant 2$. Then $\mathbb{K}^\sharp/\mathbb{K}^{\sharp 2} \cong C_2$, and for each $k \in \mathbb{K}^\sharp$ there exists $a \in V$ with $h(a) = k$. If $k$ is a square $S(\omega_a) = 1$ and $D(\omega_a) = -1$, and if $k$ is not a square, $S(\omega_a) \neq 1$ and $D(\omega_a) = -1$. Thus $O(V)/\Omega(V) \cong C_2 \times C_2$.

Suppose that $p$ is odd and $\dim V = 1$. Then (for example by B.9(b)) $O(V) = \{1, \omega_a\}$, where $a \in V^\sharp$. Thus $\Omega(V) = 1$ and $O(V)/\Omega(V) \cong C_2$.

Suppose $p$ is odd. Then we can identify $e \in E$ with $e1_{\mathbb{K}}$ in $\mathbb{K}$. It follows that $D(\omega_a) = -1 = \det(\omega_a)$ and $D(\omega_{1+bc}) = 1 = \det(\omega_{1+bc})$. By (I) the elements $\omega_a$ and $\omega_{1+bc}$ generate $O(V)$. Thus $D(z) = \det(z)$ for all $z \in O(V)$. In particular, $\Omega(V) = \{z \in SO(V) \mid S(z) = 1\}$.

The following table summarize the preceding results:

| $\dim V$ | $p$ | $O(V)/\Omega(V)$ | $\Omega(V)$ |
|---|---|---|---|
| odd | 2 | 1 | $O(V)$ |
| even | 2 | $C_2$ | $\{z \in O(V) \mid D(z) = 1\}$ |
| $\geqslant 2$ | odd | $C_2 \times C_2$ | $\{z \in SO(V) \mid S(z) = 1\}$ |
| 1 | odd | $C_2$ | 1 |

We define

$$Spin(V) := \langle 1 + bc \mid b, c \in V, h(b) = 0, f(b,c) = 0 \rangle \leqslant Cliff(V) \cap C_1.$$

Note that $\omega(Spin(V)) = \Omega(V)$ if $\dim_{\mathbb{K}} V \geqslant 3$.

Lemma B.20. *Suppose that $V$ is an orthogonal space with $V^\perp = 0$ and let $W$ be a $\mathbb{K}$-space of $V$ with $W \cap W^\perp = 0$.*

(a) *The restriction function $\tau : C_{O(V)}(W^\perp) \to O(W)$, $t \mapsto t|_W$, is an isomorphism.*
(b) *Let $S^W$ and $D^W$ be the Spinor norm and Dickson invariant for the orthogonal space $W$. Then $S(t) = S^W(t|_W)$ and $D(t) = D^W(t|_W)$ for all $t \in C_{O(V)}(W^\perp)$.*
(c) *$\tau$ induces an isomorphism from $C_{\Omega(V)}(W^\perp)$ to $\Omega(W)$.*

Proof. (a): Since $V^\perp = 0$ and $W \cap W^\perp = 0$, $V = W \oplus W^\perp$. Hence (a) follows from B.10.

(b): Let $t \in C_{O(V)}(W^\perp)$. Then $t|_W$ is a product of elements of the form $\omega_a^W$, $\omega_{1+bc}^W$, $a, b, c \in W$, $h(a) \neq 0$, $h(b) = 0$, $f(b,c) = 0$. Since $W^\perp \subseteq a^\perp \cap b^\perp \cap c^\perp$ , $\omega_a$ and $\omega_{1+bc}$ centralize $W^\perp$, we get

$$\omega_a|_W = \omega_a^W \qquad \text{and} \qquad \omega_{1+bc}|_W = \omega_{1+bc}^W.$$

Thus $t$ is the corresponding product of the elements of the form $\omega_a$, $\omega_{1+bc}$. Also

$$\begin{aligned} S(\omega_a) &= h(a)\mathbb{K}^{\sharp 2} = S^W(\omega_{1+bc}^W), & D(\omega_a) &= -1 = D^W(\omega_a^W), \\ S(\omega_{1+bc}) &= \quad 1 \quad = S^W(\omega_{1+bc}^W), & D(\omega_{1+bc}) &= \;\; 1 \;\; = D^W(\omega_{1+bc}^W). \end{aligned}$$

So indeed $S(t) = S^W(t|_W)$ and $D(t) = D^W(t|_W)$.

(c): Let $t \in C_{O(V)}(W^\perp)$. Then $t \in \Omega(V)$ if and only if $S(t) = D(t) = 1$ and so by (b) if and only if $S^W(t|_W) = D^W(t|_W) = 1$ and thus if and only if $t|_W \in \Omega(W)$. Hence (c) follows from (a). □

## B.4. Normalizers of Singular Subspaces

Lemma B.21. *Let $U$ be an $k$-dimensional isotropic subspace of $V$ and $E := C_H(U) \cap C_H(V/U)$.*

(a) *Suppose $V$ is not a linear space. Then $E = C_H(V/U)$.*
(b) *Suppose $V^\perp = 0$. Then $E = C_H(V/U) = C_H(U^\perp)$.*
(c) *Suppose that $V$ is a linear space. Then $E \cong U \otimes_{\mathbb{K}} (V/U)^*$, $|E| = |\mathbb{K}|^{k(n-k)}$ and $|V/C_V(E)| = |\mathbb{K}|^{n-k}$.*
(d) *Suppose that $V$ is a symplectic space. Then $E \cong S^2(U)$, $|E| = |\mathbb{K}|^{\frac{k(k+1)}{2}}$ and $|V/C_V(E)| = |\mathbb{K}|^k$.*
(e) *Suppose that $V$ is a unitary space. Then $E \cong U^2(U)$, $|E| = |\mathbb{F}|^{k^2}$ and $|V/C_V(E)| = |\mathbb{F}|^{2k}$.*

(f) *Suppose that* $V$ *is an orthogonal space and* $U$ *is singular. Then* $E \cong \Lambda^2(U)$, $|E| = |\mathbb{K}|^{\frac{k(k-1)}{2}}$, $|V/C_V(E)| = |\mathbb{K}|^k$,

(g) *Suppose that* $V$ *is an orthogonal space and* $U$ *is not singular. Put* $U_0 := \{u \in U \mid h(u) = 0\}$, $E_0 := C_E(V/U_0)$, *and* $E_1 := E \cap \Omega(V)$. *Then* $p = 2$, $E_0 \leqslant E_1 \leqslant E$, $E_1/E_0 \cong U_0$, $E_0 \cong \Lambda^2(U_0)$, *and* $|E_1| = |\mathbb{K}|^{\frac{k(k-1)}{2}}$. *If* $V^\perp \cap U \neq 0$ *then* $|V/C_V(E)| = |\mathbb{K}|^{k-1}$ *and* $E = E_1$. *If* $V^\perp \cap U = 0$ *then* $|V/C_V(E)| = |\mathbb{K}|^k$ *and* $|E/E_1| = 2$.

*Here all the isomorphisms are* $\mathbb{Z}N_H(U)$*-module isomorphisms.*

Proof. (a): By B.6(d) $C_H(V/U) \leqslant C_H(U)$ and so $C_H(V/U) = C_H(V/U) \cap C_H(U) = E$.

(b): By B.6(a) $V/U^{\perp\perp}$ is dual to $U^\perp/U^\perp \cap V^\perp$ as an $\mathbb{F}N_H(U)$ module. Since $V^\perp = 0$ we have $U^{\perp\perp} = U$ and so $V/U$ is dual to $U^\perp$. Hence $C_H(V/U) = C_H(U)$. By (c) $C_H(V/U) = E$ and so (b) is proved.

The remaining statements are [**MS5**, 3.4]. □

Lemma B.22 ([**MS5**, 3.5]). *Let* $U$ *be a* $k$*-dimensional isotropic subspace of* $V$. *Let* $U_0$ *be the subspace of all singular elements of* $U$ *and put* $k := \dim_{\mathbb{K}} U_0$. *Suppose that* $k \geqslant 2$. *Put* $E := C_H(U) \cap C_H(V/U)$, *and* $P := O^{p'}(N_{H'}(U))$.

(a) *If* $V$ *is a linear or unitary space, then* $E$ *is a simple* $\mathbb{F}_pP$*-module.*
(b) *If* $V$ *is a symplectic space and* $p$ *is odd, then* $E$ *is a simple* $\mathbb{F}_pP$ *module.*
(c) *If* $V$ *is an orthogonal space and* $U$ *is singular, then one of the following holds:*
  (1) $k \geqslant 3$ *and* $E$ *is a simple* $\mathbb{F}_pP$*-module.*
  (2) $k = 2$, $P$ *centralizes* $E$ *and* $E$ *is a simple* $\mathbb{F}_pN_{H'}(U)$*-module.*
(d) *Suppose that* $V$ *is a symplectic space and* $p = 2$ *or an orthogonal space and* $U$ *is not singular. Then* $p = 2$. *Let* $E_0$ *be the sum of the simple* $\mathbb{F}_2P$*-submodules of* $E$. *Then one of the following holds:*
  (1) $k \geqslant 3$, $E_0$ *is a simple* $\mathbb{F}_2P$*-module, and* $E_0 \cong \bigwedge_2 U_0^*$.
  (2) $k = 2$, $|\mathbb{K}| > 2$ *or* $V^\perp \nleqslant U$, $E_0 = C_E(P)$. $|E_0| = |\mathbb{K}|$ *and* $N_{H'}(U)$ *acts simply on* $E_0$.
  (3) $k = 2$, $|\mathbb{K}| = 2$, $V$ *is symplectic or* $V^\perp \leqslant U$, *and* $E$ *is the direct sum of simple* $\mathbb{F}_2P$*-modules of order* 2 *and* 4.

## B.5. Point-Stabilizers

Lemma B.23. *Suppose that* $V$ *is not a linear space. Let* $Z \in \mathcal{S}(V)$, $0 \neq z \in Z$ *and* $v \in V$ *with* $f(z, v) = -1$. *Let* $\mathcal{T}$ *be the set of all* $a$ *in* $Z^\perp$ *such that* $v$ *and* $v + a$ *are isometric. For* $a \in \mathcal{T}$ *let* $\gamma_a$ *be the unique element of* $GL_{\mathbb{K}}(V)$ *with*

$$v^{\gamma_a} = v + a \qquad \textit{and} \qquad u^{\gamma_a} = u + f(u, a)z \quad \textit{for all } u \in Z^\perp.$$

(a) $\gamma_a \in Q_Z$ *for all* $a \in \mathcal{T}$, $[v, q] \in \mathcal{T}$ *for all* $q \in Q_z$, *and the function* $\mathcal{T} \to Q_z, a \to \gamma_a$ *is a bijection with inverse* $Q_z \to \mathcal{T}, q \to [v, q]$.
(b) *Let* $a \in \mathcal{T}$. *Then* $\gamma_a \in D_Z$ *if and only if* $a \in Z$.
(c) *For each* $w \in V \backslash Z^\perp$, $Q_Z$ *acts regularly on the set of elements in* $w + Z^\perp$ *isometric to* $w$.
(d) *For each* $w \in V \backslash Z^\perp$, $D_Z$ *acts regularly on the set of elements in* $w + Z$ *isometric to* $w$.
(e) *Let* $a, b \in \mathcal{T}$. *Then*

$$\gamma_a\gamma_b = \gamma_{a+b+f(a,b)z}, \qquad [\gamma_a, \gamma_b] = \gamma_{(f(b,a)-f(a,b))z}, \qquad \textit{and} \qquad \gamma_a^p = \gamma_{-f(a,a)z}.$$

(f) *The function* $Q_Z/D_Z \to Z^\perp/Z$, $qD_Z \mapsto [v, q] + Z$, *is an* $\mathbb{F}_pC_H(Z)$*-isomorphism.*
(g) *If* $Q_Z \neq 1$, *then* $C_V(Q_Z) = V^\perp + Z$.

Proof. (a): Let $a \in Z^\perp$ and $\tau \in Hom_{\mathbb{K}}(Z^\perp, \mathbb{K})$ with $Z\tau = 0$. Define $\gamma_{a,\tau} \in GL_{\mathbb{K}}(V)$ by

$$v^{\gamma_{a,\tau}} = v + a \qquad \text{and} \qquad u^{\gamma_{a,\tau}} = u + (u\tau)z \text{ for } u \in Z^\perp.$$

Then $\gamma_{\alpha,\tau}$ centralizes $Z$, $Z^\perp/Z$ and $V/Z^\perp$.

Now let $\gamma \in GL_{\mathbb{K}}(V)$ such that $\gamma$ centralizes $Z$, $Z^\perp$, and $V/Z^\perp$. Then there exists a unique $a \in Z^\perp$ and $\tau \in Hom_{\mathbb{K}}(Z^\perp, \mathbb{K})$ with $Z\tau = 0$ such that $\gamma = \gamma_{a,\tau}$, namely $a = [v, \gamma]$ and $\tau$ is defined by $[u, \gamma] = (u\tau)z$ for $u \in Z^\perp$.

Observe that $\gamma|_{Z^\perp}$ is an isometry of $Z^\perp$ and that $\gamma|_{\mathbb{K}v} : \mathbb{K}v \to \mathbb{K}(v+a)$ is an isometry if and only if $v+a$ is isometric to $v$, that is, if and only if $a \in \mathcal{T}$.

For $u \in Z^\perp$ we have

$$f(u^\gamma, v^\gamma) = f(u+(u\tau)z, v+a) = f(u,v)+f(u,a)+(u\tau)f(z,v)+(u\tau)f(z,a) = f(u,v)+f(u,a)-u\tau.$$

Hence B.10 shows that $\gamma$ is an isometry if and only if $a \in \mathcal{T}$ and $u\tau = f(u,a)$ for all $u \in Z^\perp$, and so if and only if $a \in \mathcal{T}$ and $\gamma = \gamma_a$.

(b): If $a \in \mathcal{T} \cap Z$, then $f(u,a) = 0$ for all $u \in Z^\perp$. Thus $\gamma_a$ centralizes $Z^\perp$ and $V/Z$ and so $\gamma_a \in D_Z$. Conversely, if $\gamma_a \in D_Z$, then $a = [v, \gamma_a] \in Z$.

(c) and (d): Without loss $w = v$. Let $v' \in v + Z^\perp$ be isometric to $v$ and put $a := v' - v$. Then $a \in \mathcal{T}$ and by (a) $\gamma_a$ is the unique element of $Q_Z$ with $[v, \gamma_a] = a$ and so also the unique element of $Q_Z$ with $v^{\gamma_a} = v'$. Thus (c) holds. Note that $v' \in v + Z$ if and only if $a \in Z$ and so by (b) if and only if $\gamma_a \in D_Z$. Hence also (d) holds.

(e): Let $a, b \in \mathcal{T}$. Then $\gamma_a\gamma_b$ is an isometry on $V$, so $v^{\gamma_a\gamma_b} = v + [v, \gamma_a\gamma_b]$ is isometric to $v$ and $[v, \gamma_a\gamma_b] \in \mathcal{T}$. Since

$$v^{\gamma_a\gamma_b} = (v+a)^{\gamma_b} = v + b + a + f(a,b)z$$

we conclude that $[v, \gamma_a\gamma_b] = b + a + f(a,b)z \in \mathcal{T}$ and

$$\gamma_a\gamma_b = \gamma_{a+b+f(a,b)z}.$$

It follows that

$$\gamma_a\gamma_b = \gamma_b\gamma_a\gamma_{(f(b,a)-f(a,b))z}, \quad [\gamma_a, \gamma_b] = \gamma_{(f(b,a)-f(a,b))z} \quad \text{and} \quad \gamma_a^p = \gamma_{pa+(p-1)f(a,a)z} = \gamma_{-f(a,a)z}.$$

(f): Define $\Phi : Q_Z \to Z^\perp/Z$, $q \mapsto [v,q] + Z$. Since $Q_Z$ centralizes $Z^\perp/Z$, $\Phi$ is a homomorphism. Let $q \in Q_Z$. Note that $q \in D_Z$ if and only if $[v,q] \in Z$ and so $\ker\Phi = D_Z$. Let $u \in Z^\perp$. By B.11 there exists $v' \in v + u + Z$ with $h(v') = h(v)$. Hence by (b), $v^q = v'$ for some $q \in Q_Z$ and so $[v,q] = v' - v \in u + Z$, So $\Phi$ is surjective, and (f) is proved.

(g): By (f) $[V, Q_Z] + Z = Z^\perp$ and so B.6(c) gives

$$C_V(Q_Z) \cap Z^\perp = [V, Q_Z]^\perp \cap Z^\perp = ([V,Q_Z] + Z)^\perp = Z^{\perp\perp} = Z + V^\perp.$$

By (c) all orbits of $Q_Z$ on $V \backslash Z$ are regular. So if $Q_Z \neq 1$, $C_V(Q_Z) \leqslant Z^\perp$, and (g) holds. □

LEMMA B.24. *Let $Z \in \mathcal{S}(V)$.*

(a) *Let $\tau \in Hom_{\mathbb{K}}(Z^\perp, Z)$ with $Z + R(V) \leqslant \ker\tau$. Then there exists $q \in Q_Z$ with $u^q = u + u\tau$ for all $u \in Z^\perp$.*
(b) *Let $z \in Z$ and $w \in Z^\perp$ with $w \notin Z + R(V)$. Then there exists $q \in Q_Z$ with $w^q = w + z$.*

PROOF. (a): Suppose that $V$ is a linear space. Then $Z^\perp = V$. Define $q : V \to V, u \to u + u\tau$. Then $q \in Q_Z$ and (a) holds.

So suppose that $V$ is not a linear space. Pick $0 \neq z \in Z$ and $v \in V$ such that $f(z,v) = -1$. Since $Z + V^\perp = Z + R(V) \leqslant \ker\tau$, there exists $a \in z^\perp$ with $u\tau = f(u,a)z$ for all $u \in U^\perp$. By B.11 $v + a + Z$ contains an element isometric to $v$ and so we may assume that $v + a$ is isometric to $v$. Let $\gamma_a$ be the element of $Q_Z$ defined in B.23. Then for all $u \in Z^\perp$

$$u^{\gamma_a} = u + f(u,a)z = u + u\tau.$$

(b): Since $w \notin Z + R(V)$, there exists $\tau \in Hom(Z^\perp, Z)$ with $Z + R(V) \leqslant \ker\tau$ and $w^\tau = z$. Now (b) follows from (a). □

LEMMA B.25. *Suppose that $V$ is not a linear space. Let $Y, Z \in \mathcal{S}(V)$ with $Y \nleqslant Z^\perp$. Put*

$$K := C_H(Z) \cap C_H(Y), \quad K^* := N_H(Z) \cap N_H(Y) \quad \textit{and} \quad C := C_{K^*}(Z^\perp \cap Y^\perp).$$

(a) *$V = Z + \langle Y^{Q_Z} \rangle$.*
(b) *$Q_Z$ acts regularly on $\mathcal{S}(V) \backslash \mathcal{S}(Z^\perp)$.*
(c) *$C_H(Z) = Q_Z K$, $Q_Z \cap K = 1$, $K \cong Cl(Z^\perp/Z)$, and $Q_Z/D_Z$ is a natural $Cl(Z^\perp/Z)$-module for $C_H(Z)$ and for $K$.*

(d) $N_H(Z) = Q_Z K^*$, $Q_Z \cap K^* = 1$, *$N_H(Z)$ acts transitively on $Z$ and on $V/Z^\perp$*, $K^* = C \times K$ *and* $Q_Z/D_Z \cong V/Z^\perp \otimes_{\mathbb{K}} Z/Z^\perp$ *as an* $\mathbb{F}_p(C \times K)$*-module and as an* $\mathbb{F}_p N_H(Z)$*-module.*

PROOF. Let $0 \neq y \in Y$ and choose $z \in Z$ with $f(z, y) = -1$.

(a): By B.23(f) the function $Q_Z/D_Z \to Z^\perp/Z$, $qD_Z \mapsto [y, q] + Z$, is an isomorphism. Thus

$$V = Y + Z^\perp = Y + [y, Q_Z] + Z \leqslant \langle Y^{Q_Z} \rangle + Z.$$

(b): Let $X \in \mathcal{S}(V)$ with $X \nleqslant Z^\perp$. Then $V = X + Z^\perp$ and we can choose $x \in X$ with $f(z, x) = -1$. Hence $x \in y + Z^\perp$. Note that $x$ and $y$ are both singular and so isometric. By B.23(c), $Q_Z$ acts regularly on the elements in $y + Z^\perp$ isometric to $y$ and so $x \in y^{Q_Z}$. Hence also $X \in Y^{Q_Z}$.

(c) and (d): Since $Q_Z$ acts regularly on the elements in $y + Z^\perp$ isometric to $y$, a Frattini argument shows that $C_H(Z) = Q_Z(C_H(Z) \cap K)$ and $Q_Z \cap K = 1$. Similarly, as $Q_Z$ acts regularly on $\mathcal{S}(V) \backslash \mathcal{S}(Z^\perp)$ we have $N_H(Z) = Q_Z K^*$ and $Q_Z \cap K^* = 1$. Put $W := Z^\perp \cap Y^\perp$. Then, as a module for $K^*$,

$$V = (Z \oplus Y) \ominus W.$$

Let $k, l \in \mathbb{K}^\sharp$ and $b \in GL_{\mathbb{K}}(W)$. Define $a \in GL_{\mathbb{K}}(V)$ by $z^a = kz$, $y^a = ly$ and $w^a = w^b$ for all $w \in W$. By B.10 $a$ is an isometry if and only if $a|_{Z+Y}$ and $b$ are isometries. Since $Y$ and $Z$ are singular, $a|_Y$ and $a|_Z$ are isometries, and another application of B.10 shows that $a|_{Z+Y}$ is an isometry if and only if $f(z^a, y^a) = -1$. This holds if and only if $kl^\alpha = 1$ , that is if and only if $k = l^{-\alpha}$. Thus $C \cong \mathbb{K}^\sharp$ is cyclic and acts transitively on $Z$ and $V/Z^\perp$. Also $K = K^* \cap C_K(Z) \cong Cl(W) \cong Cl(Z^\perp/Z)$. Since the function

$$\tau : Q_Z/D_Z \to Z^\perp/Z,\ qD_Z \mapsto [y, q] + Z$$

is a $C_H(Z)$-isomorphism, we conclude that $Q_Z/D_Z$ is a natural $Cl(Z^\perp/Z)$-module for $C_H(Z)$ and for $K$. Let $q \in Q_Z$ and $c \in C$ with $y^c = ly$. Then $\tau(q^c) = [y^c, q] + Z = l\tau(q)$. Hence $Q_Z/D_Z \cong V/Z^\perp \otimes_{\mathbb{K}} Z/Z^\perp$ as an $\mathbb{F}_p(C \times K)$-module and so also as an $\mathbb{F}_p N_H(Z)$-module. □

LEMMA B.26. *Suppose that $V$ symplectic space with* $\dim_{\mathbb{K}} V \geqslant 2$, *a unitary space with* $\dim_{\mathbb{K}} V \geqslant 2$, *or an $V$ is an orthogonal space with* $\dim_{\mathbb{K}} V \geqslant 3$. *Let $Y, Z \in \mathcal{S}(V)$ with $Y \nleqslant Z^\perp$.*

(a) $\langle Q_Y, Q_Z \rangle = H^\diamond$.

(b) *$H^\diamond$ acts transitively on $\mathcal{S}(V)$.*

PROOF. Put $L = \langle Q_Y, Q_Z \rangle$. We claim that $L$ acts transitively on $\mathcal{S}(V)$. Let $X \in \mathcal{S}(V)$. By B.25(b) $Q_Z$ acts transitively on $\mathcal{S}(V) \backslash \mathcal{S}(Z^\perp)$. If $X \nleqslant Z^\perp$ this gives $X \in Y^{Q_Z} \subseteq Y^L$.

Suppose next that $X \leqslant Z^\perp$. Note that $X \nleqslant V^\perp$ and, by B.25(a), $V = Z + \langle Y^{Q_Z} \rangle$. Thus there exists $a \in Q_Z$ with $X \nleqslant Y^{a\perp}$. Note that also $Z \nleqslant Y^{a\perp}$ and so B.25(b) applied with $Y^a$ in place of $Z$ shows that $X \in Z^{Q_{Y^a}} \subseteq Z^L$. We proved that $\mathcal{S}(V) \backslash \mathcal{S}(Z^\perp) \subseteq Y^L$ and $\mathcal{S}(Z^\perp) \subseteq Z^L$.

Suppose for a contradiction that $L$ does not acts transitively on $\mathcal{S}(V)$. Then $L$ has two orbits on $\mathcal{S}(V)$, namely $Z^L = \mathcal{S}(Z^\perp)$ and $Y^L = \mathcal{S}(V) \backslash \mathcal{S}(Z^\perp)$. By symmetry, $Y^L = \mathcal{S}(Y^\perp)$. Hence $U := \langle Y^L \rangle \leqslant Y^\perp$ and so also $U \leqslant U^\perp$ and $U$ is singular. Note that

$$\mathcal{S}(U \cap Z^\perp) \subseteq \mathcal{S}(Y^\perp) \cap \mathcal{S}(Z^\perp) = Y^L \cap Z^L = \varnothing.$$

Since $U \cap Z^\perp$ is singular, this gives $U \cap Z^\perp = 0$ and $U = Y + (U \cap Z^\perp) = Y$. Thus $Y^L = \{Y\}$ and by symmetry $Z^L = \{Z\}$. It follows that $\mathcal{S}(V) = \{X, Y\}$. By B.13 $V = \langle \mathcal{S}(V) \rangle$ and so $V = X + Y$. On the other hand, by B.11 $|\mathcal{S}(X + Y) \backslash \{Y\}| = |\mathbb{K}|$, $|\mathbb{F}|$ or 2 if $V$ is a symplectic, unitary or orthogonal space, respectively. Thus $V$ is an orthogonal space. Since $\dim_{\mathbb{K}} V = 2$ this contradicts the hypothesis of the Lemma.

Hence $L$ acts transitively on $\mathcal{S}(V)$. In particular, $H^\diamond$ acts transitively on $\mathcal{S}(V)$ and

$$H^\diamond = \langle Q_X \mid X \in \mathcal{S}(V) \rangle = \langle Q_Z^L \rangle \leqslant L.$$

□

LEMMA B.27. *Suppose that $V$ contains a 2-dimensional singular subspace. Let $v$ and $w$ be isometric elements in $V \backslash R(V)$. Then there exists $a \in H^\diamond$ with $w^a = v$. In particular, $H^\diamond$ acts transitively on the set of non-zero singular vectors.*

Proof. If $V$ is a linear space $H^\diamond = SL_{\mathbb{K}}(V)$ acts transitively on $V$ and the lemma holds. So suppose that $V$ is not a linear space.

Let $E$ be a 2-dimensional singular subspace of $V$ and $Z \in \mathcal{S}(E)$. By B.26(b) $H^\diamond$ acts transitively on $\mathcal{S}(V)$ and by B.13, $V = \langle \mathcal{S}(V)\rangle$. Thus $V = \langle Z^{H^\diamond}\rangle$ and so $Z^b \nleqslant v^\perp$ for some $b \in H^\diamond$. Replacing $v$ by $v^{b^{-1}}$ we may assume that $v \notin Z^\perp$. Similarly we may assume that $w \notin Z^\perp$. Choose $z \in Z$ with $f(z,v) = -1$. Note that $X := E \cap w^\perp \in \mathcal{S}(w^\perp)$. Also $V = Z + w^\perp$ and since $Z \leqslant X^\perp \neq V$, we get $V = X^\perp + w^\perp$ and $w^\perp \nleqslant X^\perp$. Thus $X \nleqslant w^{\perp\perp}$ and so $X \nleqslant rad(w^\perp)$. In particular, $\langle \mathcal{S}(w^\perp)\rangle \neq rad(w^\perp)$, and B.13 shows that $\langle \mathcal{S}(w^\perp)\rangle = w^\perp$. As $Z \leqslant Z^\perp \neq V$ and $V = Z + w^\perp$, we have $w^\perp \nleqslant Z^\perp$ and so there exists $U \in \mathcal{S}(w^\perp)$ with $U \nleqslant Z^\perp$.

We claim that there exists $Y \in \mathcal{S}(w^\perp)$ with $Y \nleqslant Z^\perp$ and $Y \nleqslant \mathbb{K}w + V^\perp$. If $U \nleqslant \mathbb{K}w + V^\perp$ we can choose $Y = U$. So suppose $U \leqslant \mathbb{K}w + V^\perp$. Since $w \perp X$ this gives $U \perp X$. Thus $U + X$ is singular. Since $X \leqslant Z^\perp$ and $U \nleqslant Z^\perp$, we have $U \neq X$ and $(U + X) \cap Z^\perp = X$. Let $Y$ be any 1-dimensional subspace $U + X$ with $Y \neq X$ and $Y \neq U$. Then $Y \nleqslant Z^\perp$ and $Y \in \mathcal{S}(w^\perp)$. If $Y \leqslant \mathbb{K}w + V^\perp$, then $U + Y \leqslant \mathbb{K}w + V^\perp$ and so $0 \neq V^\perp \leqslant U + Y$, a contradiction, since $V^\perp$ contains no non-zero singular vectors. Thus the claim is proved.

Choose $Y$ as in the claim. From $Y \nleqslant \mathbb{K}w + V^\perp$ and $w \notin V^\perp$ get $w \notin Y + V^\perp$. Note that $Y + Z^\perp = V$, so we can choose $y \in Y$ with $f(z, w + y) = -1$. In particular $f(z,v) = f(z, w + y)$ and $w + y \in v + Z^\perp$. Recall that $Y \in \mathcal{S}(w^\perp)$, $w \notin Y + V^\perp$ and $V^\perp = R(V)$. Thus $w \in Y^\perp \backslash (Y + R(V))$, and B.24(b) shows that there exists $c \in Q_Y$ with $w^c = w + y$. In particular, $w + y, w, v$ are all isometric. As $w + y \in v + Z^\perp$, we conclude from B.23(d) that there exists $d \in Q_Z$ with $(w + y)^d = v$. So $w^{cd} = v$, and the lemma is proved. □

Lemma B.28. *Let $Z \in \mathcal{S}(V)$. Suppose that* $\dim_{\mathbb{K}} V \geqslant 3$. *Put* $P := C_{H^\diamond}(Z)$.

- (a) *Suppose that $V$ is a linear space.*
  - (a) $D_Z = 1$ *and $Q_Z$ is elementary abelian.*
  - (b) $P/Q_Z \cong SL(V/Z)$ *and $Q_Z$ is the corresponding natural module for $P$ dual to $V/Z$.*
- (b) *Suppose that $V$ is a symplectic space.*
  - (a) $|D_Z| = |\mathbb{K}|$, $D_Z = Q_Z' = \Phi(Q_Z) = Z(Q_Z)$ *if $p$ is odd, and $Q_Z$ is elementary abelian if* $p = 2$.
  - (b) $P/Q_Z \cong Sp(Z^\perp/Z)$ *and $Q_Z/D_Z$ is the corresponding natural module for $P$.*
- (c) *Suppose that $V$ is a unitary space.*
  - (a) $|D_Z| = |\mathbb{F}|$ *and* $D_Z = Q_Z' = \Phi(Q_Z) = Z(Q_Z)$.
  - (b) $P/Q_Z \cong SU(Z^\perp/Z)$ *and $Q_Z/D_Z$ is the corresponding natural module for $P$.*
- (d) *Suppose that $V$ is an orthogonal space.*
  - (a) $D_Z = 1$ *and $Q_Z$ is elementary abelian.*
  - (b) $P/Q_Z \cong \Omega(Z^\perp/Z)$ *and $Q_Z$ is the corresponding natural module for $P$.*

Proof. Suppose first that $f = 0$, that is $H = GL(V)$. Then $V^\perp = V$ and so $D_Z = 0$. By B.21(c) $Q_Z \cong Z \otimes_{\mathbb{K}} (V/Z)^*$ as an $\mathbb{F}_pP$-module. Since $P$ centralizes $Z$ and is 1-dimensional over $\mathbb{K}$ this shows $Z \otimes_{\mathbb{K}} (V/Z)^* \cong (V/Z)^*$. Note that $P$ induces $SL(V/Z)$ on $V/Z$ and so also on $(V/Z)^*$. Hence (a) holds.

Suppose now that $f \neq 0$. We will use the description of $Q_Z$ given in B.23. So let $v, z, \mathcal{T}$, and $\gamma_a, a \in \mathcal{T}$, be as there. By B.23(d), $D_Z$ acts regularly on the set of elements in $v + Z$ isometric to $v$. By B.11 the number of such elements is $|\mathbb{K}|$ if $H = Sp(V)$, $|\mathbb{F}|$ if $H = GU(V)$, and 1 if $H = O(V)$. So also $|D_Z| = |\mathbb{K}|$, $|\mathbb{F}|$ and 1, respectively.

Let $a, b \in \mathcal{T}$. Then by B.23(e):

$$[\gamma_a, \gamma_b] = \gamma_{(f(b,a) - f(a,b))z} \qquad \text{and} \qquad \gamma_a^p = \gamma_{-f(a,a)z}.$$

If either $H = Sp(V)$ and $p = 2$ or $H = O(V)$, we conclude that $Q_Z$ is elementary abelian. If either $H = Sp(V)$ and $p$ is odd or $H = GU(V)$, we conclude that $\Phi(Q_Z) = D_Z = Q_Z' = Z(Q_Z)$.

Put $P^* = C_H(Z)$, $K^* = C_{P^*}(v)$ and $K = C_P(v)$. Note that $Q_Z$ act regularly on $v + \mathcal{T}$. Since $v + \mathcal{T}$ is the set of singular vectors in $v + Z^\perp$, $v + \mathcal{T}$ is $P^*$ invariant and a Frattini argument gives $P^* = K^*Q_Z$ and $P = KQ_Z$. Put $W = Z^\perp \cap v^\perp$ and note that $V = W \oplus (\mathbb{K}v + Z)$. Since $K^*$

centralizes $\mathbb{K}v + Z$, we conclude from B.10 that $K^*$ is (isomorphic to) the group of isometries of $W$. Note that $K = K^* \cap H^\diamond$. If $H = Sp(V)$ then $H = H^\diamond$ and $K = K^* = Sp(W)$. If $H = GU(V)$, then $H^\diamond = SU(V)$ and so $K = SU(W)$. If $H = O(V)$ and $V^\perp = 0$, then $H^\diamond = \Omega(V)$. Thus $K = C_{\Omega(V)}(W^\perp)$, and B.20 shows that $K = \Omega(W)$. If $H = O(V)$ and $V^\perp \neq 0$, then $H = H^\diamond$ and $K = K^* = O(W) = \Omega(W)$.

Since $P/Q_Z = KQ_Z/Q_Z \cong K$ and $Z^\perp/Z \cong W$ this shows that $P/Q_Z \cong Sp(Z^\perp/Z)$, $SU(Z^\perp/Z)$ and $\Omega(Z^\perp/Z)$, respectively. By B.23(f) $Q_Z/D_Z \cong Z^\perp/Z$ as an $\mathbb{F}_pP$-module and so all parts of the lemma are proved. □

## B.6. Simplicity of the Natural Module

LEMMA B.29. *Suppose that* $\dim_{\mathbb{K}} V \geqslant 3$ *if* $V$ *is an orthogonal space, and* $\dim_{\mathbb{K}} V \geqslant 2$ *otherwise. Suppose that there exists a proper* $\mathbb{F}_pH^\diamond$*-submodule* $W$ *of* $V$ *with* $W \nleqslant R(V)$*. Then one of the following holds:*

(a) *$V$ is a unitary space, $\dim_{\mathbb{K}} V = 2$, $H^\diamond \cong SL_2(\mathbb{F})$, $W$ is an $\mathbb{F}$-subspace of $V$, $W$ is a natural $SL_2(\mathbb{F})$-module for $H^\diamond$, $V$ is the direct sum of two natural $SL_2(\mathbb{F})$-module for $H^\diamond$, and $H$ acts transitively on the $|\mathbb{F}| + 1$ simple $H^\diamond$-submodules of $V$. In particular, $V$ is a simple $H$-module and a simple $\mathbb{K}H^\diamond$-module.*

(b) *$V$ is an orthogonal space, $\dim_{\mathbb{K}} V = 3$, $|\mathbb{K}| = 2$, $H = H^\diamond \cong SL_2(2)$, $W = [V, H]$ is a natural $SL_2(2)$-module for $H$ and $V = V^\perp \ominus W$.*

PROOF. If $V$ is linear, then $H^\diamond = SL_{\mathbb{K}}(V)$ and $H^\diamond$ acts transitively on $V$. Thus $V$ is a simple $H$-module, and no proper $H^\diamond$-submodule of $V$ exists.

Hence $V$ is not linear and so $R(V) = V^\perp$. Note that the hypothesis on $\dim_{\mathbb{K}} V$ ensure that there exists $Z \in \mathcal{S}(V)$ (see B.19(c)). By B.13, $V = \langle \mathcal{S}(V) \rangle$. Since $W \nleqslant V^\perp$ we conclude that $W \nleqslant Z^\perp$ for some $Z \in \mathcal{S}(V)$. Let $w \in W \backslash Z^\perp$. By B.23(e) the function $Q_Z/D_Z \to Z^\perp/Z, qD_Z \to [w,q] + Z$ is an isomorphism. Thus $Z^\perp \leqslant [W, Q_Z] + Z$ and so $[Z^\perp, Q_Z] = [W, Q_Z, Q_Z] \leqslant W$.

Suppose that $Z^\perp \neq Z + V^\perp$ or $H = Sp(V)$. In the first case B.24(b) shows that $Z = [Z^\perp, Q_Z] \leqslant W$, and in the second case B.28(b:a) shows that $D_Z \neq 1$ and so $[W, D_Z] = Z \leqslant W$. Thus $Z^\perp = [W, Q_Z] + Z \leqslant W$, $\langle \mathbb{K}W \rangle = V$, $W^\perp = V^\perp$ and $Z_1 \in W$ for all $Z_1 \in \mathcal{S}(V)$, a contradiction since $W \neq V$ and $V = \langle \mathcal{S}(V) \rangle$.

It follows that $Z^\perp = Z + V^\perp$ and $H \neq Sp(V)$. Hence $\dim V/V^\perp = 2$ and either $H = GU(V)$, $V^\perp = 0$ and $\dim V = 2$, or $p = 2$, $H = O(V)$, $\dim V^\perp = 1$.

Suppose that $H = GU(V)$. For $i = 1, 2$ let $U_i \in \mathcal{S}(V)$ with $U_1 \neq U_2$. Then $V = U_1 + U_2$. We can choose the following further notation:

$$0 \neq t \in \mathbb{K} \text{ with } t + t^\alpha = 0, \quad 0 \neq u_1 \in U_1, \quad u_2 \in U_2 \text{ with } f(u_1, u_2) = t.$$

Let $X$ be the $\mathbb{F}$-subspace of $V$ spanned by $u_1$ and $u_2$, and let $\lambda_i \in \mathbb{F}$. Then

$$h(\lambda_1 u_1 + \lambda_2 u_2) = \lambda_1 \lambda_2^\alpha t + \lambda_1^\alpha \lambda_2 t = \lambda_1 \lambda_2 (t + t^\alpha) = 0.$$

Thus all elements in $X$ are singular. Let $0 \neq x \in X$ and choose $y \in X$ with $f(x, y) \neq 0$. By B.11(b) $\mathbb{K}x + y$ contains exactly $|\mathbb{F}|$ singular vectors. It follows that $\mathbb{F}x + y$ is the set of singular vectors in $\mathbb{K}x + y$ and so $Q_{\mathbb{K}x}$ normalizes $\mathbb{F}x + y$. Hence $Q_{\mathbb{K}x}$ normalizes $X$. As $\mathcal{S}(V) = |\mathbb{F}| + 1$ and $X$ has $|\mathbb{F}| + 1$ 1-dimensional subspaces each $U \in \mathcal{S}(V)$ intersects $X$ in 1-dimensional $\mathbb{F}$-space and so $Q_U$ normalizes $X$. Hence $X$ is an $\mathbb{F}H^\diamond$ submodule of $V$. Observe that $X$ is natural $SL_2(\mathbb{F})$-module for $H^\diamond$. Since $u_1$ was an arbitrary non-zero element on $U_1$, each of the $|\mathbb{F}|+1$ 1-dimensional $\mathbb{F}$-subspaces of $U_1$ lies in a 2-dimensional $\mathbb{F}H^\diamond$-submodule of $V$. It follows that $V$ is a direct sum of two natural $SL_2(\mathbb{F})$-modules for $H^\diamond$. In particular, there exists exactly $|\mathbb{F}|+1$ non-zero proper $\mathbb{F}_pH^\diamond$ submodules of $V$. As $N_{H^\diamond}(U_1)$ acts transitively on $U_1$, we conclude that $N_{H^\diamond}(U_1)$ also acts transitively on the set of non-zero proper $F_pH^\diamond$-submodules of $V$. Thus (a) holds.

So suppose that $H = O(V)$. Then $H^\diamond$ induces $Sp(V/V^\perp) = Sp_2(\mathbb{K}) = SL_2(\mathbb{K})$ on $V/V^\perp$ and so $V = W + V^\perp$; in particular $[V, H^\diamond] \leqslant W$. Thus $[V, Q_Z]$ is 1-dimensional and so (for example by B.11) $Z_0 + [V, Q_Z]$ contains exactly two singular 1-spaces. Since $\langle Z_0^{Q_Z} \rangle \leqslant Z_0 + [V, Q_Z]$ this shows that $|\mathbb{K}| = |Q_Z| = |Z_0^{Q_Z}| = 2$, and (b) holds. □

LEMMA B.30. *Let $Z \in \mathcal{S}(V)$, and put $P := C_{H^\diamond}(Z)$. Suppose that there exists a proper $P$-invariant subgroup $T$ of $Q_Z$ with $T \nleqslant D_Z$.*

(a) *Suppose that $V$ is a linear space and $\dim_{\mathbb{K}} V \geqslant 2$. Then $\dim_{\mathbb{K}} V = 2$.*
(b) *Suppose that $V$ is a symplectic space and $\dim_{\mathbb{K}} V \geqslant 2$. Then $|\mathbb{K}| = 2$, $\dim_{\mathbb{K}} V = 4$, $T = [Q_Z, P] = H' \cap Q_Z$, and $T$ has order 4.*
(c) *Suppose that $V$ is a unitary space and $\dim_{\mathbb{K}} V \geqslant 4$. Then $\dim_{\mathbb{K}} V = 4$, $D_Z \leqslant T$, $T/D_Z$ is a natural $SL_2(\mathbb{F})$-module for $P$, and $T$ is not invariant under $C_{H^\diamond}(D_Z)$. In particular, $N_{H^\diamond}(Z)$ acts simply on $Q_Z/D_Z$.*
(d) *Suppose that $V$ is an orthogonal space and $V^\perp = 0$.*[3] *Then $\dim_{\mathbb{K}} V \leqslant 4$.*

PROOF. (a): By B.28(a) $Q_Z$ is a natural $SL(V/Z)$-module for $P$ and so, if $\dim V/Z \geqslant 2$, a simple $P$-module. This gives (a).

(b): Note first that $T \nleqslant D_Z$ gives $Q_Z \neq D_Z$, and so $\dim_{\mathbb{K}} V \geqslant 4$. By B.28(b:b) $Q_Z/D_Z$ is a natural $Sp(Z^\perp/Z)$-module for $P$ and so simple. Thus $TD_Z = Q_Z$. If $p$ is odd, then B.28(b:a) implies $D_Z = \Phi(Q_Z)$ and so $T = Q_Z$.

Hence $p = 2$. By B.14 $V \cong W/W^\perp$ and $H \cong O(W)$ for some non-degenerate orthogonal space $W$. Without loss $V = W/W^\perp$. Then the inverse image of $Z$ in $W$ contains a unique 1-dimensional singular subspace $Z_0$, $Q_Z = Q_{Z_0}$ and $P = C_H(Z_0)$. Now B.28(d:b) shows that $Q_Z$ is a natural $\Omega(Z_0^\perp/Z_0)$-module for $P$. Thus by B.29, $|\mathbb{K}| = 2$, $\dim Z_0^\perp/Z_0 = 3$ and $T = [Q_Z, P]$ has order 4. Hence (b) holds.

(c): By B.28(c) $Q_Z/D_Z$ is a natural $SU(Z^\perp/Z)$-module for $P$ and $D_Z = \Phi(Q_Z)$. It follows that $Q_Z \neq TD_Z$. We now apply B.29(a) to $Z^\perp/Z$ and $GU(Z^\perp/Z)$. Then $\dim Z^\perp/Z = 2$, and $TD_Z/D_Z$ is a natural $SL_2(\mathbb{F})$-module for $P$.

Let $\lambda \in \mathbb{K}$ be of multiplicative order $|\mathbb{F}| + 1$. Then $\lambda\lambda^\alpha = 1$. By B.18(b:a) $V$ has a hyperbolic basis $(v_1, v_2, v_3, v_4)$. Then $f(v_1, v_4) = f(v_2, v_3) = 1$ and $f(v_i, v_j) = 0$ for all other $1 \leqslant i \leqslant j \leqslant 4$. Since $H$ acts transitive on $\mathcal{S}(V)$, we may assume that $v_1 \in Z$. Define $\phi \in GL_{\mathbb{K}}(V)$ by $v_i\phi = \lambda v_i$ for $i = 1, 4$ and $v_i\phi = \lambda^{-1}v_i$ for $i = 2, 3$. Observe that $\phi \in SU(V) = H^\diamond$ and that $\phi$ normalizes $Z$. Since $\phi$ acts as scalar multiplication by $\lambda$ on $V/Z^\perp$ and $Z$, $\phi$ centralizes $D_Z$. As $\phi$ acts as scalar multiplication by $\lambda^{-1}$ on $Z^\perp/Z$, $\phi$ centralizes $P/Q_Z$. It follows that $\phi$ does not normalizes $TD_Z/D_Z$ and so $Q_Z = TT^\phi D_Z$. Since $T$ is $P$ invariant, $[T, Q_Z] \leqslant T \cap D_Z$, and since $\phi$ centralizes $D_Z$ this gives $[T^\phi, Q_Z] \leqslant T^\phi \cap D_Z = T \cap D_Z$. Thus $D_Z = Q_Z' = [TT^\phi D_Z, Q_Z] \leqslant T \cap D_Z \leqslant T$ and (c) holds.

(d): By B.28(d:b) $Q_Z$ is a natural $\Omega(Z^\perp/Z)$-module for $P$. If $V^\perp = 0$ and $\dim_{\mathbb{K}}(Z^\perp/Z) \geqslant 3$, we conclude from B.29(b) that $Q_Z$ is a simple $P$-module. Thus (d) holds. □

## B.7. Normalizers of Classical Groups

In this section we view $\mathbb{K}$ as a subring of $End_{\mathbb{F}_pH}(V)$.

LEMMA B.31. *Suppose that $\dim_{\mathbb{K}} V \geqslant 3$ if $V$ is an orthogonal space, and $\dim_{\mathbb{K}} V \geqslant 2$ otherwise. Then $End_{\mathbb{F}_pH^\diamond}(V) = \mathbb{K}$, unless $H = O(V) = O_3(2)$ or $H = GU(V) = GU_2(\mathbb{F})$.*

PROOF. Suppose first that $V^\perp = 0$ and $H \neq GU_2(\mathbb{K})$. Then by B.29, $V$ is a simple $\mathbb{F}_pH^\diamond$-module and so $End_{\mathbb{F}_pH^\diamond}(V)$ is a division ring. Let $Z \in \mathcal{S}(V)$. By B.23(g), $C_V(Q_Z) = V^\perp + Z = Z$ and so $C_V(Q_Z)$ is 1-dimensional over $\mathbb{K}$. This gives $End_{\mathbb{F}_pH^\diamond}(V) = \mathbb{K}$.

Suppose next that $V^\perp \neq 0$ and $H \neq O_3(2)$. Then B.29 shows that $V = [V, H^\diamond]$. Let $\beta \in End_{\mathbb{F}_pH^\diamond}(V)$ with $V\beta \leqslant V^\perp$. Then $[V\beta, H^\diamond] = 0$ and so also $[V, H^\diamond]\beta = 0$. As $V = [V, H^\diamond]$ we get that $\beta = 0$, and $End_{\mathbb{F}_pH^\diamond}(V)$ acts faithfully on $V/V^\perp$. Since $H^\diamond$ induces $Sp(V/V^\perp)$ on $V$ we conclude from the previous case that $End_{\mathbb{F}_pH^\diamond}(V) = \mathbb{K}$. □

LEMMA B.32. *Suppose that $V$ is a linear space and put $H^* = N_{GL_{\mathbb{F}_p}(V)}(H^\diamond)$. Suppose that $\dim_{\mathbb{K}} V \geqslant 2$.*

[3] If $V^\perp \neq 0$, then $p = 2$, $V/V^\perp$ is a non-degenerate symplectic space, $H \cong Sp(V/V^\perp)$, and (b) applies

(a) *$H^* = \Gamma GL_{\mathbb{K}}(V)$, that is, $g \in GL_{\mathbb{F}_p}(V)$ normalizes $H^\diamond$ if and only if there exists $\sigma \in Aut(\mathbb{K})$ such that $g$ acts $\sigma$-semilinearly on $V$.*[4]

(b) *There exists a homomorphism $\rho : H^* \to Aut(\mathbb{K}), g \mapsto \rho_g$, such that each $g \in H^*$ acts $\rho_g$-semilinearly on $\mathbb{K}$.*

(c) *$\ker \rho = H$ and $\rho$ is surjective. In particular, $H^*/H \cong Aut(\mathbb{K})$.*

(d) *Let $T$ be $p$-subgroup of $H^*$ acting $\mathbb{K}$-linearly on $V$. Then $T \leqslant H^\diamond$.*

Proof. (a) and (b): Let $b, c \in GL_{\mathbb{F}_p}(V)$ acting $\beta$- and $\gamma$-semilinearly on $\mathbb{K}$, respectively. Then $bc$ acts $\beta\gamma$ semilinearly on $V$ and $b^{-1}$ acts $\beta^{-1}$-semilinearly on $V$. In particular, if $c$ acts $\mathbb{K}$-linearly, so does $c^b$. Hence $b$ normalizes $H$ and thus also $H^\diamond$.

By B.31 $\mathbb{K} = End_{\mathbb{F}_p H^\diamond}(V)$. Hence $H^*$ acts on $\mathbb{K}$ by conjugation and we obtain a homomorphism $\rho : H^* \to Aut(\mathbb{K})$, $g \mapsto \rho_g$, such that $g^{-1}kg = k^{\rho_g}$. It follows that $g \in H^*$ acts $\rho_g$-semilinearly on $V$.

(c): Clearly $\ker \rho = H$. To show that $\rho$ is surjective let $(v_i)_{i=1}^n$ be a $\mathbb{K}$-basis and $\sigma \in Aut(\mathbb{K})$. Define $g \in GL_{\mathbb{F}_p}(V)$ by $(kv_i)^g = k^\sigma v_i$. Then $g$ acts $\sigma$-linearly on $V$, and by (a) $g \in H^*$. Hence $\rho(g) = \sigma$.

(d): Since $T$ acts $\mathbb{K}$-linearly on $V$, $T \leqslant H$ and since $H/H^\diamond = GL_{\mathbb{K}}(V)/SL_{\mathbb{K}}(V)$ is a $p'$ group, $T \leqslant H^\diamond$. □

Lemma B.33. *Let $k \in \mathbb{F}^\sharp$ and $\sigma \in Aut(\mathbb{K})$. Define*

$$\widetilde{f} := f_{k,\sigma} : V \times V \to \mathbb{K},\ (v, w) \mapsto kf(v, w)^\sigma, \qquad \widetilde{h} := h_{k,\sigma} : V \to \mathbb{F}, v \mapsto kh(v)^\sigma.$$

*Let $V_\sigma$ be the $\mathbb{K}$-space with $V_\sigma = V$ as abelian group and scalar multiplication*

$$\cdot_\sigma : \mathbb{K} \times V \to V, (l, v) \mapsto l^{\sigma^{-1}} v.$$

*Then*

(a) *$(V^\sigma, \widetilde{f}, \widetilde{h})$ is a classical space of the same type as $(V, f, h)$.*

(b) *The $\mathbb{K}$-subspaces of $V$ are the same as the $\mathbb{K}$-subspaces of $V_\sigma$.*

(c) *A $\mathbb{K}$-subspace of $V$ is singular with respect to $(f, h)$ if and only if it is singular with respect to $(\widetilde{f}, \widetilde{h})$.*

(d) *$H$ is the isometry group of $(V, \widetilde{f}, \widetilde{h})$.*

(e) *$(V^\sigma, \widetilde{f}, \widetilde{h})$ is not isometric to $(V, f, h)$ if and only if $H = O(V)$, $p$ is odd, $\dim_{\mathbb{K}} V$ is odd, and $k$ is not a square in $\mathbb{K}$.*

Proof. (a) is readily verified, and (b) should be obvious.

(c): Just observe that $kf(v, w)^\sigma$ and $kh(v)^\sigma$ are 0 if and only if $f(v, w)$ and $h(v)$ are 0.

(d): Let $g \in GL_{\mathbb{K}}(V)$. Then $(f, h)$ is $g$-invariant if and only if $(\widetilde{f}, \widetilde{h})$ is.

(e): By B.18 any two linear spaces, any two symplectic spaces, and any two unitary spaces of the same dimension are isometric.

Note that $U$ is a singular subspace of $V$ if and only if $U_\sigma$ is a singular subspace of $V_\sigma$. Hence $V$ and $V_\sigma$ have the same Witt index. Any two orthogonal spaces of the same even dimension are isometric if and only if they have the same Witt index. Also if $p = 2$ then any two orthogonal spaces of the same odd dimension are isometric.

So it remains to consider the case $H = O(V)$, $\dim_{\mathbb{K}} V$ odd and $p$ odd. Let $Y$ be a maximal hyperbolic subspace of $V$ and put $X = Y^\perp$. Then by B.19 $V = X \oplus Y$ and $\dim_{\mathbb{K}} X = 1$. Let $0 \neq x \in X$ and observe that, by the even dimensional orthogonal case, $Y_\sigma$ is hyperbolic. Also $V_\sigma = X_\sigma \oplus Y_\sigma$ and $X_\sigma \perp Y_\sigma$. Hence by B.18(d:e) $(V, f, h)$ and $(V_\sigma,\ f,\ h)$ are isometric if and only if $\widetilde{h}(x)h(x)^{-1}$ is a square in $\mathbb{K}$. Note that $\widetilde{h}(x)h(x)^{-1} = kh(x)^\sigma h(x)^{-1}$ and that $h(x)^\sigma h(x)^{-1}$ is a square in $\mathbb{K}$. Thus $(V, f, h)$ and $(V^\sigma, \widetilde{f}, \widetilde{h})$ are isometric if and only if $k$ is a square. □

[4]For the definition of $\sigma$-semilinear see A.46

LEMMA B.34. *Let $g \in GL_{\mathbb{F}_p}(V)$. Define*

$$f_g : V \times V \to \mathbb{K}, (v, w) \mapsto f(v^g, w^g), \quad \text{and} \quad h_g : V \to \mathbb{F}, v \mapsto h(v^g).$$

*Let $V_g$ be the $\mathbb{K}$-space with $V_g = V$ as abelian group and scalar multiplication*

$$\cdot_g : \mathbb{K} \times V \to V, (k, v) \mapsto \left(kv^g\right)^{g^{-1}}.$$

*Then*

(a) *$g$ is an isometry from $(V_g, f_g, h_g)$ to $(V, f, h)$.*
(b) *$(V_g, f_g, h_g)$ is a classical space of the same type as $(V, f, h)$.*
(c) *$H^{g^{-1}}$ is the isometry group of $(V_g, f_g, h_g)$.*

PROOF. (a): Let $k \in \mathbb{K}$ and $v \in V$. Then

$$(k \cdot_g v)^g = \left(\left(kv^g\right)^{g^{-1}}\right)^g = kv^g,$$

so $g : V_g \to V$ is an isomorphism of $\mathbb{K}$ spaces.

By the definition of $f_g$ and $h_g$

$$f_g(v, w) = f(v^g, w^g) \qquad \text{and} \qquad h_g(v) = h(v^g)$$

for all $v, w \in V_g$, and so (a) holds.

(b) and (c) both follow from (a). □

LEMMA B.35. *Suppose that $V$ is not a linear space and $\dim_{\mathbb{K}} V \geqslant 3$. Put $H^* := N_{GL_{\mathbb{F}_p}(V)}(H^\diamond)$.*

(a) *Let $g \in GL_{\mathbb{K}}(V)$ Then $g \in H^*$ if and only there exist $k \in \mathbb{F}^\sharp$ and $\sigma \in Aut(\mathbb{K})$ such that $g$ is a $(k, \sigma)$-semisimilarity of $V$.*
(b) *Let $g \in H^*$. Then the elements $k \in \mathbb{F}^\sharp$ and $\sigma \in Aut(\mathbb{K})$ in (a) are uniquely determined. Moreover, if we denote $k$ by $\lambda_g$ and $\sigma$ by $\rho_g$, then the function*

$$(\lambda, \rho) : \quad H^* \;\to\; \mathbb{F}^\sharp \rtimes Aut(\mathbb{K}), \quad g \;\mapsto\; (\lambda_g, \rho_g),$$

*is a homomorphism.*
(c) *Put $S := \{k^2 \mid k \in \mathbb{K}^\sharp\}$ if $V$ is an orthogonal space with $\dim_{\mathbb{K}} V$ odd and $p$ odd; put $S := \mathbb{F}^\sharp$ otherwise. Then $\ker(\lambda, \rho) = H$ and $Im(\lambda, \rho) = S \rtimes Aut(\mathbb{K})$. In particular, $H^*/H \cong S \rtimes Aut(\mathbb{K})$.*
(d) *Let $T$ be a $p$-subgroup of $H^*$ acting $\mathbb{K}$-linearly on $V$. Then $T \leqslant H$. Moreover, $T \leqslant H^\diamond$, unless $p = 2$, $V$ is an orthogonal space and $V^\perp = 0$.*

PROOF. Suppose first that $H = O(V) = O_3(2)$. Then $H = H^\diamond$, $V = V^\perp \oplus [V, H]$ and $H$ induces $GL_2(2)$ on $[V, H]$. It follows that $H^* = H$. Also $\mathbb{K}^\sharp = \{1\}$ and $Aut(\mathbb{K}) = 1$, and so the lemma holds in this case. So we may assume from now on that $H \neq O_3(2)$.

(a) and (b): Suppose first that there exists $k \in \mathbb{F}^\sharp$ and $\sigma \in Aut(\mathbb{K})$ such that $g$ is a $(k, \sigma)$-semisimilarity of $V$. Then

$$f(v^g, w^g) = kf(v, w)^\sigma \text{ and } h(v^g) = kh(v)^\sigma$$

for all $v, w \in V$. In the notation of B.33 and B.34 this just says that $f_g = f_{k,\sigma}$ and $h_\sigma = h_{k,\sigma}$. Since $g$ is $\sigma$-semilinear, $(lv^g)^{g^{-1}} = l^{\sigma^{-1}} v$ and so $V_g = V_\sigma$. The isometry group of $(V_g, f_g, h_g)$ is $H^{g^{-1}}$ and the isometry group of $(V_\sigma, f_{\sigma,k}, h_{\sigma,k})$ is $H$. So $H^g = H$ and thus also $(H^\diamond)^g = H^\diamond$. Therefore, $g \in H^*$.

Since $\dim_{\mathbb{K}} V \geqslant 3$ and we exclude the $O_3(2)$-case, B.31 shows that $\mathbb{K} = End_{\mathbb{F}_p H^\diamond}(V)$. Hence $H^*$ acts on $\mathbb{K}$ by conjugation and we obtain a homomorphism $\rho : H^* \to Aut(\mathbb{K}), g \mapsto \rho_g$, such $g^{-1}kg = k^{\rho_g}$. It follows that $g \in H^*$ acts $g_\rho$-semilinearly on $V$. By B.29 $V/V^\perp$ is a simple $H^\diamond$-module and by B.31, $End_{\mathbb{F}_p H^\diamond}(V/V^\perp) = \mathbb{K}$. Since $f$ induces a non-degenerate $\mathbb{K}$-sesquilinear form $\overline{f}$ on $V/V^\perp$, $V/V^\perp$ is selfdual as an $\mathbb{F}_p H^\diamond$-module. So we can apply B.7 and conclude that there exists a function $\lambda : H^* \to \mathbb{K}^\sharp, g \mapsto \lambda_g$, such that the function

$$H^* \to \mathbb{K}^\sharp \rtimes Aut(\mathbb{K}), g \mapsto (\lambda_g, \rho_g)$$

is a homomorphism and

$$\overline{f}(\overline{v}^g, \overline{w}^g) = \lambda_g \overline{f}(\overline{v}, \overline{w})^{\rho_g}$$

for all $\overline{v}, \overline{w} \in V/V^\perp$. Hence also

$$(*) \qquad f(v^g, w^g) = \lambda_g f(v, w)^{\rho_g}$$

for all $v, w \in V$.

We claim that $\lambda_g \in \mathbb{F}$. If $\mathbb{F} = \mathbb{K}$ there is nothing to prove. So suppose $H = GU(V)$ and choose $v, w \in V$ with $f(v, w) = 1$. Then also $f(w, v) = 1^\alpha = 1$ and

$$\lambda_g = \lambda_g f(v, w)^{\sigma_g} = f(v^g, w^g) = f(w^g, v^g)^\alpha = \left(\lambda_g f(w, v)^{\sigma_g}\right)^\alpha = \lambda_g^\alpha,$$

and so indeed $\lambda_g \in \mathbb{F}$.

It remains to show that $h(v^g) = \lambda_g h(v)^{\rho_g}$ for all $v \in V$. If $H = Sp(V)$ or $GU(V)$, then $h(v) = f(v, v)$ and this follows from $(*)$. So assume $H = O(V)$.

Fix $g \in H^*$. Put

$$k := \lambda_g, \qquad \sigma := \rho_g, \qquad \widetilde{f} := f_{k,\sigma}, \qquad \widetilde{h} := h_{k,\sigma}.$$

Then $(*)$ says that $f_g = \widetilde{f}$ and we need to show that $h_g = \widetilde{h}$. Since $g$ is $\sigma$-semilinear, $V_g = V_\sigma$. Note that both $(V_g, f_g, h_g)$ and $(V_\sigma, \widetilde{f}, \widetilde{g})$ are orthogonal spaces. Also $H^\diamond$ is contained in their isometry group since $g$ normalizes $H^\diamond$. In particular,

$$h_g(v + w) = h_g(v) + f_g(v, w) + h_g(w) \quad \text{and} \quad \widetilde{h}(v + w) = \widetilde{h}(v) + \widetilde{f}(v, w) + \widetilde{h}(w)$$

for all $v, w \in V$. Since $f_g = \widetilde{f}$ we conclude that the function $r : V \to \mathbb{K}$, $v \mapsto h_g(v) - \widetilde{h}(v)$, is $\mathbb{F}_p$-linear.

Since both $h_g$ and $\widetilde{h}$ are $H^\diamond$-invariant, so is $r$. Thus $\ker r$ is an $\mathbb{F}_p$-submodule of $V$. As $|Imr| \leqslant |\mathbb{K}|$ and $|V| \geqslant |\mathbb{K}|^3$ we have $|\ker r| \geqslant |\mathbb{K}|^2 > |\mathbb{K}| \geqslant |V^\perp|$ and so $\ker r \nleqslant V^\perp$. Recall that we excluded the $O_3(2)$-case and so B.29 shows that $\ker r = V$. Thus $r = 0$, $h = \widetilde{h}$, and (a) and (b) are proved.

(c): Let $g \in H^*$. Then $g \in H$ if and only if $f$ and $h$ are $g$-invariant and if and only if $\lambda_g = 1$ and $\rho_g = 1$. So $\ker(\lambda, \rho) = H^*$.

To compute the image of $(\lambda, \rho)$, let $k \in \mathbb{F}^\sharp$ and $\sigma \in Aut(\mathbb{K})$ and put $\widetilde{f} = f_{k,\sigma}$ and $\widetilde{h} = h_{k,\sigma}$. Note that $(k, \sigma)$ is in the image of $(\lambda, \rho)$ if and only if $\widetilde{f} = f_g$ and $\widetilde{h} = h_g$ for some $g \in GL_{\mathbb{F}_p}(V)$. This in turn holds if and only if $(V_\sigma, \widetilde{f}, \widetilde{h})$ is isometric to $(V, f, h)$. By B.33(e) $(V_\sigma, \widetilde{f}, \widetilde{h})$ is not isometric to $(V, f, h)$ if and only if $\mathbb{K} = O(V)$, $\dim_{\mathbb{K}} V$ is odd, $p$ is odd, and $k$ is not a square in $\mathbb{K}$. Hence $(k, \sigma) \in Im(\lambda, \rho)$ if and only if $k \in S$. This gives (c).

(d): Since $T$ acts $\mathbb{K}$-linearly on $V$, $\rho(T) = 1$. Since $S \leqslant \mathbb{K}^\sharp$ is a $p'$-group we conclude that $T \leqslant \ker(\lambda, \rho)$ and so $T \leqslant H$. Since $GL_{\mathbb{K}}(V)/SL_{\mathbb{K}}(V)$ is a $p'$-group, $T \leqslant SL_{\mathbb{K}}(V) \cap H$. Note that either $SL_{\mathbb{K}}(V) \cap H = H^\diamond$, or $H = O(V)$ and $\dim_{\mathbb{K}} V$ is even if $p = 2$. In the later case $H^\diamond$ has index two in $SL_{\mathbb{K}}(V) \cap H$. So either $T \leqslant H^\diamond$ or $H = O(V)$, $p = 2$, and $\dim_{\mathbb{K}} V$ is even. □

## B.8. $Q$-Uniqueness in Classical Groups

LEMMA B.36. *Let $H^\diamond \trianglelefteq L \leqslant GL_{\mathbb{F}_p}(V)$ and let $Q$ be a $p$-subgroup of $L$. Suppose that $V$ is a $Q!$-module for $L$ with respect to $Q$ and that $V$ contains a 2-dimensional singular subspace. Then $\langle Q^L \rangle = \langle Q^{H^\diamond} \rangle$. In particular, $H^\diamond Q \trianglelefteq L$ and $O^p(\langle Q^L \rangle) \leqslant H^\diamond$.*

PROOF. By B.35(a) $L$ is contained in the group of semisimilarities of $V$ and so acts on the set of non-zero singular vectors.

By B.27 $H^\diamond$ acts transitively on this set. By B.12(c) there exists a non-zero singular vector centralized by $Q$. By a Frattini argument $L = C_L(v)H^\diamond$ and by $Q!$, $C_L(v) \leqslant N_L(Q)$. Thus $\langle Q^L \rangle = \langle Q^{H^\diamond} \rangle$. □

LEMMA B.37. *Suppose that* $\dim_{\mathbb{K}} V \geqslant 3$ *if* $V$ *is a linear space, and* $\dim_{\mathbb{K}} V \geqslant 4$ *otherwise. Let* $Q$ *be a* $p$*-subgroup of* $GL_{\mathbb{F}_p}(V)$ *normalizing* $H^\diamond$ *and suppose that* $V$ *is a* $Q!$*-module for* $H^\diamond Q$ *with respect to* $Q$*. Put* $X := C_V(Q)$*. Then* $C_V(H^\diamond) = 0$*; in particular, the case* $V$ *orthogonal,* $p = 2$ *and* $\dim_{\mathbb{K}} V$ *odd does not occur. Moreover, one of the following holds:*

(1) $X \in \mathcal{S}(V)$, $Q = Q_X$ *and* $\langle Q^{H^\diamond}\rangle = H^\diamond$.
(2) $H = H^\diamond = Sp(V) = Sp_4(2)$, $X \in \mathcal{S}(V)$, $Q = Q_X \cap H'$ *and* $\langle Q^{H^\diamond}\rangle = H' = Sp_4(2)'$.
(3) $H = GU(V) = GU_4(\mathbb{F})$, $p = 2$, $|X| = |\mathbb{F}|$, $\widehat{X} := \langle \mathbb{K}X\rangle \in \mathcal{S}(V)$, $H^\diamond Q = \langle Q^{H^\diamond}\rangle \cong O_6^-(\mathbb{F})$, $|Q/Q \cap Q_{\widehat{X}}| = 2$ *and either* $Q_{\widehat{X}} \leqslant Q$ *or* $D_{\widehat{X}} \leqslant Q$ *and* $Q \cap Q_{\widehat{X}}/D_{\widehat{X}}$ *is a natural* $SL_2(\mathbb{F})$*-module for* $C_{H^\diamond}(\widehat{X})$.
(4) $H = O(V) = O_4^+(\mathbb{K})$, $X$ *is a* 2*-dimensional singular subspace of* $V$, $Q = C_{H^\diamond}(X)$ *and* $\langle Q^{H^\diamond}\rangle \cong SL_2(\mathbb{K})$.
(5) $H = H^\diamond Q = O(V) = O_4^+(2)$, $X \in \mathcal{S}(V)$ *and either* $Q \cong D_8$ *and* $\langle Q^{H^\diamond}\rangle = H$ *or* $Q \cong C_4$ *and* $\langle Q^{H^\diamond}\rangle \sim 3^2C_4$.
(6) $H = O(V) = O_4^+(4)$, $H^\diamond Q = \langle Q^{H^\diamond}\rangle \cong O_4^+(4)$, $|X| = 2$, $\widehat{X} := \langle \mathbb{K}X\rangle \in \mathcal{S}(V)$, $|Q/Q \cap Q_{\widehat{X}}| = 2$, $QQ_{\widehat{X}} \in Syl_2(H^\diamond Q)$ *and either* $Q_{\widehat{X}} \leqslant Q$ *or* $Q$ *is the unique maximal elementary abelian subgroup of order* 8 *in* $Q_{\widehat{X}}Q$.

PROOF. Since $V$ is a $Q!$-module, $C_V(H^\diamond Q) = 0$ (see A.53) and so also $C_V(H^\diamond) = 0$. So the first statement holds. Moreover, by B.12(c) there exists a $Q$-invariant $Z \in \mathcal{S}(V)$. Put $P^* := N_{H^\diamond}(Z)$ and $P := C_{H^\diamond}(Z)$.

Note that $C_Q(\mathbb{K})$ is a $p$-group acting $\mathbb{K}$-linearly on $V$ and normalizing $H^\diamond$. Thus B.32(d) (if $V$ is linear) and B.35(d) (if $V$ is not linear) show that $C_Q(\mathbb{K}) \leqslant H$. Put $Q^* := C_Q(Z^\perp/Z)$. Then $Q^*$ acts $\mathbb{K}$-linearly on $V$ and so $Q^* \leqslant C_Q(\mathbb{K}) \leqslant H$. Since $Z$ is 1-dimensional, $Q^*$ centralizes $V/Z^\perp$, $Z^\perp/Z$ and $Z$. Thus $Q^* \leqslant Q_Z$.

Suppose first that $Q$ centralizes $Z^\perp/Z$. Then $Q = Q^* \leqslant Q_Z$. Since $Q$ centralizes $Z$, $Q!$ implies that $Q \trianglelefteq P^*$.

Assume for a contradiction that $Q \leqslant D_Z$. Then $D_Z \neq 1$ and so by B.28 $H = Sp(V)$ or $SU(V)$. Since $Z^\perp/Z$ is at least 2-dimensional, it has a 1-dimensional singular subspace (see B.19(c)). Hence there exists $Y \in \mathcal{S}(Z^\perp)$ with $Y \neq Z$. Since $Q \leqslant D_Z$ we get $[Y, Q] = 0$, and $Q$!-shows that $N_{H^\diamond}(Y)$ normalizes $Q$ and so also $[V, Q] = Z$. Thus $[Z, Q_Y] = 0$, a contradiction as $C_V(Q_Y) = Y + V^\perp = Y$ by B.23(g).

Thus $Q \nleqslant D_Z$. If $Q = Q_Z$, then $C_V(Q) = C_V(Q_Z) = Z$. Hence (1) holds in this case.

So suppose that $Q \neq Q_Z$. Since $Q$ is $P^*$-invariant, B.30 implies that either $H = Sp(V)$, $|\mathbb{K}| = 2$, $\dim_{\mathbb{K}} V = 4$ and $Q = Q_Z \cap H'$ or $H = O(V)$ and $\dim V = 4$. In the first case (2) holds.

So suppose $H = O(V)$. If the quadratic form $h$ is of $-$-type, then $H^\diamond \cong L_2(|\mathbb{K}|^2)$ and so $P^*$ acts simply on $Q_Z$. Thus $h$ is of $+$-type and $H^\diamond = H_1H_2$ with $H_i \cong SL_2(\mathbb{K})$ and $[H_1, H_2] = 1$. Note that $P^* \cap H_i$ acts simply on $Q_Z \cap H_i$. If $|\mathbb{K}| > 3$, then $[Q_Z, P^* \cap H_i] = Q_Z \cap H_i$ and we conclude that $Q = H_i \cap Q_Z$ for some $i \in \{1, 2\}$. Thus (4) holds in this case. If $|\mathbb{K}| \leqslant 3$, then $|Q_Z| = p^2$, $|Q| = p$ and and since $[Z^\perp, Q] \leqslant Z$, $X$ is 2-dimensional over $\mathbb{K}$. If $Q = Q_Z \cap H_i$ for some $1 \leqslant i \leqslant 2$, then again (4) holds. Otherwise $X$ is non-singular and contains a non-singular 1-space $Y$. By $Q!$, $Q \trianglelefteq C_{H^\diamond}(Y)$, a contradiction since $C_{H^\diamond}(Y) \cong \Omega_3(\mathbb{K}) \cong L_2(\mathbb{K})$ and so does not have any non-trivial normal $p$-subgroups. This completes the case where $Q$ centralizes $Z^\perp/Z$.

Suppose now that $Q$ does not centralizes $Z^\perp/Z$. Since $C_Z(Q) \neq 0$, $Q!$ implies that $P$ normalizes $Q$. In particular, $[Q, P] \leqslant Q \cap P \leqslant Q_Z$ and $P$ does not act simply on $Z^\perp/Z$. By B.28 $P$ induces $SL(V/Z)$, $Sp(Z^\perp/Z)$, $SU(Z^\perp/Z)$ and $\Omega(Z^\perp/Z)$, respectively, on $Z^\perp/Z$. Moreover, B.29 shows that $\dim_{\mathbb{K}} V = 4$ and $H = GU(V)$ or $H = O(V)$. In either case since $Z^\perp/Z \cong Q_Z/D_Z$ as an $P$-module, $[Q, Q_Z] \nleqslant D_Z$. Note also that $[Q, Q_Z] \neq Q_Z$.

Suppose $H = GU(V)$. Then B.30 (with $T := [Q, Q_Z]$) shows that $D_Z \leqslant [Q, Q_Z]$, $[Q, Q_Z]/D_Z$ is a natural $SL_2(\mathbb{F})$-module for $P$ and either $Q \cap Q_Z = [Q, Q_Z]$ or $Q \cap Q_Z = Q_Z$. By B.29(a) all $P$-submodules of $Z^\perp/Z$ are $\mathbb{F}$-subspaces. It follows that $[Z^\perp/Z, Q]$ is a non-trivial $\mathbb{F}$-subspace centralized by $Q$, and so $Q$ acts $\mathbb{F}$-linearly on $\mathbb{K}$. Hence $|Q/C_Q(\mathbb{K})| \leqslant 2$. By B.29(a) $Z^\perp/Z$ is a simple $\mathbb{K}P$-module. Thus $C_Q(\mathbb{K})$ centralizes $Z^\perp/Z$. As seen above $Q^* \leqslant Q_Z$, and we conclude that $C_Q(\mathbb{K}) = Q \cap Q_Z$. Together with $Q \nleqslant Q_Z$ this gives $|Q/Q \cap Q_Z| = 2$ and so $p = 2$ and

$H^\diamond Q \cong O_6^-(\mathbb{F})$. Since $C_V(Q \cap Q_Z)$ is an $\mathbb{K}$-subspace normalized by $P$ and $Z^\perp/Z$ is a simple $\mathbb{K}P$-module, $C_V(Q \cap Q_Z) = Z$. Thus $Z = \langle \mathbb{K}C_V(Q)\rangle$ and (3) holds.

Suppose $H = O(V)$. If $h$ is of $-$-type then $\Omega(Z^\perp/Z)$ has order $|\mathbb{K}|+1$ or $(|\mathbb{K}|+1)/2$ depending on $p = 2$ or $p$ odd. It follows that $Z^\perp/Z$ is a simple $P$-module unless $|\mathbb{K}| = 3$. In the latter case $Q$ acts $\mathbb{K}$-linearly on $V$. Hence $Q \leqslant H$. As $Q_Z \in Syl_p(H)$ this gives $Q \leqslant Q_Z$, a contradiction.

Thus $h$ is of $+$-type and $H^\diamond = H_1H_2$ with $H_i \cong SL_2(\mathbb{K})$ and $[H_1, H_2] = 1$.

For $i \in \{1,2\}$ define $Z_i := C_V(Q_Z \cap H_i)$. Let $z_k$, $0 \leqslant k \leqslant 3$, be non-zero singular vectors in $V$ such that $z_0 \in Z$, $z_1 \in Z_1\backslash Z, z_2 \in Z_2\backslash Z$, $f(z_0,z_3) = f(z_1,z_2) = 1$ and $f(z_k,z_l) = 0$ for all other $0 \leqslant k \leqslant l \leqslant 3$. For $\lambda \in \mathbb{K}^\sharp$ define $a_\lambda \in GL_{\mathbb{K}}(V)$ by

$$z_0^{a_\lambda} = z_0, \quad z_1^{a_\lambda} = \lambda z_1, \quad z_2^{a_\lambda} = \lambda^{-1}z_2, \quad z_3^{a_\lambda} = z_3.$$

Observe that $a_\lambda$ is an isometry, and since $H/H^\diamond$ is elementary abelian, $a_\lambda^2 \in H^\diamond$ and $a_\lambda^2 \in P$. It follows that $Z_i/Z$ is a simple $\mathbb{F}_pP$-module. Since $P$ normalizes $N_Q(H_i)$, we conclude that $N_Q(H_i)$ centralizes $Z_i/Z$. Together with $N_Q(H_1) = N_Q(H_2)$ this shows that $N_Q(H_1)$ centralizes $Z^\perp/Z$ and so $N_Q(H_i) \leqslant Q^* \leqslant Q_Z$. Thus $N_Q(H_i) = Q \cap Q_z$, $|Q/Q_\cap Q_Z| = 2$ and $p = 2$.

Suppose that $|\mathbb{K}| = 2$. Then $H^\diamond Q = H$. Since $1 \neq [Q_Z, Q] \leqslant Q_Z \cap Q$ we have $|Q| \geqslant 4$. Let $y$ be a non-singular vector of $V$. Then $C_H(y) \cong C_2 \times SL_2(2)$. Thus $O_2(C_H(y)| = 2$ and $Q \not\trianglelefteq C_H(y)$. Hence $Q!$ implies $y \notin C_V(Q)$, and so $C_V(Q)$ is singular. It follows $C_V(Q) = Z$ and $Q \cong C_4$ or $D_8$, and (5) holds.

Suppose that $|\mathbb{K}| \geqslant 4$. Then $a_\lambda$ has odd order and so $a_\lambda \in P$ for all $\lambda \in \mathbb{K}^\sharp$. Note that $a_\lambda$ acts as a scalar multiplication by $\lambda$ and $\lambda^{-1}$ on $Z_1/Z$ and $Z_2/Z$, respectively. Since $|\mathbb{K}| \geqslant 4$ there exists $\lambda \in \mathbb{K}$ with $\lambda \neq \lambda^{-1}$. Thus $Z_1/Z$ and $Z_2/Z$ are non-isomorphic as $\mathbb{K}P/Q_Z$-module. Since $Q \neq Q \cap Q_Z = N_Q(H_1)$ we have $Z_1^Q = \{Z_1, Z_2\}$. We conclude that $Z^\perp/Z$ is a simple $\mathbb{K}PQ$-module. It follows that $C_Q(\mathbb{K}) \leqslant Q^* \leqslant Q_Z$ and so $Q$ does not act $\mathbb{K}$-linearly on $V$. Moreover, as $Z^\perp/Z$ is not a simple $\mathbb{F}_2PQ$-module, we infer that $Z_1/Z$ and $Z_2/Z$ are isomorphic as $\mathbb{F}_2P$-modules, and so there exists an $\mathbb{F}_2P$-isomorphism $\phi$ from $Z_1/Z$ to $Z_2/Z$. The action of $a_\lambda$ on $Z_1/Z$ and $Z_2/Z$ shows that $\mathbb{K} = End_{\mathbb{F}_2P}(Z_i/Z)$, and $\phi$ induces a field automorphism $\sigma$ on $\mathbb{K}$ with $\lambda^\sigma = \lambda^{-1}$ for all $\lambda$ in $\mathbb{K}^\sharp$. It follows that $\sigma^2 = 1$, $C_{\mathbb{K}}(\sigma) = \mathbb{F}_2$ and $\mathbb{K} = \mathbb{F}_4$.

Let $\lambda \in \mathbb{K}\backslash\mathbb{F}_2$ and put $d := a_\lambda$. Then $d$ is an element of order three in $P$, and $d$ acts fixed-point freely on $Q_Z$ and so also on $Q_Z \cap Q$. Since $|Q_Z/Q_Z \cap Q| = 2$ we conclude that there exists $t \in Q$ with $Q = (Q_Z \cap Q)\langle t\rangle$, $[t,d] = 1$ and $|t| = 2$. Now $Q \cap Q_Z = N_Q(H_1)$ implies $H_1^t = H_2$. Hence $H^\diamond Q = H^\diamond\langle t\rangle \cong SL_2(4) \wr C_2 \cong H$. As $Q$ does not act $\mathbb{K}$-linearly $H^\diamond Q \neq H$. Note that $[Q_Z, t] \leqslant Q$ and so by the action of $d$ either $Q = Q_Z\langle t\rangle$ or $Q = [Q_Z, t]\langle t\rangle = C_{Q_Z}(t)\langle t\rangle$. In either case, $Q!$ or equally well the action of $C_{H^\diamond}(t)$ on $V$ show that $C_V(Q) = C_Z(Q)$ has order 2, and so (6) holds. □

Lemma B.38. *Let $Q$ be a $p$-subgroup of $GL_{\mathbb{F}_p}(V)$ normalizing $H^\diamond$ and $U$ a $\mathbb{K}$-subspace of $V$. Put*

$$E_U := \langle Q^g \mid g \in H^\diamond, C_U(Q^g) \neq 0\rangle, \quad F_U := \langle (Q \cap H^\diamond)^g \mid g \in H^\diamond, C_U(Q^g) \neq 0\rangle \text{ and } W := \langle \mathcal{S}(U)\rangle.$$

*Suppose that*

(i) $\dim_{\mathbb{K}} V \geqslant 3$ *if $V$ is a linear space, $dim_{\mathbb{K}}V \geqslant 4$ if $V$ is a symplectic or unitary space, and* $\dim_{\mathbb{K}} V \geqslant 5$ *if $V$ is an orthogonal space.*

(ii) *$V$ is a $Q!$-module for $H^\diamond Q$ with respect to $Q$.*

*If $W$ is not singular, then $E_U = \langle Q^{H^\diamond}\rangle$ and $V = \langle U^{E_U}\rangle$. If $W$ is singular, then each of the following hold:*

(a) *$W \leqslant U \leqslant W^\perp$ and $F_U$ normalizes $U$ and $W$.*

(b) *$F_U$ centralizes $W^\perp/W$.*

(c) *$W$ is a natural $SL_{\mathbb{K}}(W)$-module for $F_U$.*

(d) *Let $T$ be a proper, non-zero $\mathbb{K}$-subspace of $V$. Then $F_U$ normalizes $T$ if and only if $W \leqslant T \leqslant W^\perp$.*

(e) *If $E_U \neq F_U$, then $H^\diamond = SU_4(\mathbb{F})$, $E_U$ normalizes $W$, and either* $\dim_{\mathbb{K}} W = 1$, *$F_U = Q_W = C_{E_U}(W)$ and $E_U/F_U \cong O_2^-(\mathbb{F})$, or* $\dim_{\mathbb{K}} W = 2$, *$U = W$, $|E_U/F_U| = 2$ and $E_U/C_{E_U}(W) \cong O_4^-(\mathbb{F})$.*

Proof. Put $X := \langle \mathbb{K}C_V(Q)\rangle$. Note that we can apply B.37; in particular, $V^\perp = 0$ if $V$ is an orthogonal space. In the last three cases of B.37 $V$ is a 4-dimensional orthogonal space, a contradiction to Hypothesis (i). In the other cases $X \in \mathcal{S}(V)$, and one of the following holds:

(A) $Q = Q_X$.

(B) $H = H^\diamond = Sp_4(2)$ and $Q = Q_X \cap H'$.

(C) $H^\diamond = SU_4(\mathbb{F})$, $H^\diamond Q \cong O_6^-(\mathbb{F})$, $|Q/Q \cap Q_X| = 2$ and $D_X < Q \cap Q_X$.

If $V$ is a linear space then $X = C_V(Q_X)$, and if $V$ is not a linear space then by B.23(g) $C_V(Q_X) = X + V^\perp$ and again $C_V(Q_X) = X$ since $V^\perp = 0$.

Note that $Q \leqslant E_X$, $Q \cap Q_X \leqslant F_X$ and $N_{H^\diamond}(X)$ normalizes $F_X$. Let $g \in H^\diamond$. If $C_X(Q^g) \neq 0$, then $X = \langle \mathbb{K}C_V(Q^g)\rangle$ and so $Q^g$ normalizes $X$. If $Q^g$ normalizes $X$, the $C_X(Q^g) \neq 0$. This shows that

$$E_X = \langle Q^g \mid g \in H^\diamond, Q^g \leqslant N_H(X)\rangle \quad \text{and} \quad F_X = \langle Q^g \cap H^\diamond \mid g \in H^\diamond, Q^g \leqslant N_{H^\diamond}(X)\rangle.$$

In case (A), $Q = Q_X \leqslant H^\diamond$ and so $E_X = F_X = Q_X$. In particular, $C_V(F_X) = C_V(Q_X) = X$ and $\langle Q^{H^\diamond}\rangle = H^\diamond$.

In case (B), $Q = Q_X \cap H' \leqslant H^\diamond$ and so $E_X = F_X = Q_X \cap H'$. Observe that $D_X \nleqslant H'$, so $Q_X = F_X D_X$. Since $Q_X$ acts regularly on $V \backslash Z^\perp$, $C_V(F_X) \leqslant Z^\perp = C_V(D_X)$. Thus $C_V(F_X) = C_V(F_X D_X) = C_V(Q_X) = X$. Since $Sp_4(2)'$ is simple, $\langle Q^{H^\diamond}\rangle = H'$ and so $D_X\langle Q^{H^\diamond}\rangle = D_X H' = H = H^\diamond$.

In case (C) $Q \nleqslant H^\diamond$ and $|Q/Q \cap Q_X| = 2$. Thus $Q \cap H^\diamond = Q \cap Q_X$. By B.30(c) $Q_X/D_X$ is a simple $N_{H^\diamond}(X)$-module. As $D_X < Q \cap H^\diamond \leqslant Q_X$ and $N_{H^\diamond}(X)$ normalizes $F_X$, this gives $F_X = Q_X$. Thus $C_V(F_X) = C_V(Q_X) = X$, $H^\diamond = \langle F_X^{H^\diamond}\rangle$ and $H^\diamond Q = \langle Q^{H^\diamond}\rangle$.

In each case we have proved:

$$(*) \qquad H^\diamond Q = D_X\langle Q^{H^\diamond}\rangle, \qquad Q_X = D_X F_X \qquad \text{and} \qquad C_V(F_X) = X.$$

Let $Z \in \mathcal{S}(V)$. By B.26(b) $H^\diamond$ acts transitively on $\mathcal{S}(V)$. Thus $(*)$ holds for $Z$ in place of $X$.

Let $g \in H^\diamond$. Since $U$ is an $\mathbb{K}$-subspace and $\langle \mathbb{K}C_V(Q^g)\rangle = X^g$ is a 1-dimensional singular subspace of $V$, we see that $C_U(Q^g) \neq 0$ if and only if $X^g \leqslant U$ and if and only if $C_Z(Q^g) \neq 0$ for some $Z \in \mathcal{S}(U)$. It follows that

$$E_U = \langle E_Z \mid Z \in \mathcal{S}(U)\rangle \qquad \text{and} \qquad F_U = \langle F_Z \mid Z \in \mathcal{S}(U)\rangle.$$

Observe that $\mathcal{S}(U) = \mathcal{S}(W)$. Thus

$$(**) \qquad E_U = \langle E_Z \mid Z \in \mathcal{S}(W)\rangle = E_W \qquad \text{and} \qquad F_U = \langle F_Z \mid Z \in \mathcal{S}(W)\rangle = F_W$$

If $W = 0$ then $E_U = E_W = 1$ and $F_U = F_W = 1$, and the lemma holds in this case. So suppose that $W \neq 0$ and without loss that $X \leqslant W$.

Suppose first that $W$ is not singular and choose $Y, Z \in \mathcal{S}(U)$ with $Z \nleqslant Y^\perp$. Then by B.26 $\langle Q_Y, Q_Z\rangle = H^\diamond$. Since $[V, D_Y] \leqslant Y$, $D_Y$ normalizes $Y + Z$ and so also $E_{Y+Z}$. Since $F_Y \leqslant E_{Y+Z}$ and $Q_Y = D_Y F_Y$, we conclude that $Q_Y$ normalizes $E_{Y+Z}$. By symmetry $Q_Z$ normalizes $E_{Y+Z}$. Pick $g \in H^\diamond$ with $X^g = Z$. Then $Q^g \leqslant E_{Y+Z}$, and we get

$$E_{Y+Z} \trianglelefteq \langle Q_Y, Q_Z\rangle Q^g = H^\diamond Q^g = H^\diamond Q.$$

Thus $\langle Q^{H^\diamond}\rangle = E_{X+Y} = E_U$. By $(*)$ $H^\diamond Q = D_X\langle Q^{H^\diamond}\rangle = D_X E_U$, so

$$V = \langle \mathcal{S}(V)\rangle = \langle X^{H^\diamond}\rangle \leqslant \langle X^{D_X E(U)}\rangle = \langle X^{E_U}\rangle \leqslant \langle U^{E_U}\rangle.$$

Hence $V = \langle U^{E_U}\rangle$, and the lemma holds in this case.

Suppose now that $W$ is singular.

(a) and (b): By B.13 $W = U$ or $W = rad(U)$. In either case $W \leqslant U \leqslant W^\perp$. Let $Z \in \mathcal{S}(W)$. Then $Z \leqslant W \leqslant W^\perp \leqslant Z^\perp$. Since $F_Z \leqslant Q_Z$ and $Q_Z$ centralizes $Z^\perp/Z$ we conclude that $F_Z$ normalizes $W$ and centralizes $W^\perp/W$. By $(**)$, $F_U = \langle F_Z \mid Z \in \mathcal{S}(W)\rangle$. Hence also $F_U$ normalize $W$ and centralizes $W^\perp/W$. Thus (a) and (b) hold.

(c): Let $Z \in \mathcal{S}(W)$. Then $W \leqslant Z^\perp \leqslant C_V(D_Z)$. As by $(*)$ $Q_Z = D_Z F_Z$, this gives $Q_Z = C_{Q_Z}(W) F_Z$. Let $\beta \in Hom_{\mathbb{K}}(W, Z)$ with $Z \leqslant \ker\beta$. Since $W \leqslant Z^\perp$ and $W \cap R(V) = 0$ we can choose $\tau \in Hom_{\mathbb{K}}(Z^\perp, Z)$ with $Z + R(V) \leqslant \ker\tau$ and $\tau\,|_W = \beta$. By B.24 there exists $q \in Q_Z$ with

$u^q = u + u\tau$ for all $u \in Z^\perp$ and so $w^q = w + w\beta$ for all $w \in W$. Hence $Q_Z$ induces all possible transvections with center $Z$ on $W$. Note that this also holds for $F_Z$ in place of $Q_Z$, since, as seen above, $Q_Z = F_Z C_{Q_Z}(W)$.

Since $Z \in \mathcal{S}(W)$ was arbitrary and $SL_{\mathbb{K}}(W)$ is generated by its transvections, we conclude that $F_U$ induces $SL_{\mathbb{K}}(W)$ on $W$. Thus (c) is proved.

(d): Let $T$ be any proper non-zero $\mathbb{K}$-subspace of $V$ and $Z \in \mathcal{S}(W)$. If $W \leqslant T \leqslant W^\perp$ then $F_U$ normalizes $T$ since by (b) $F_U$ centralizes $W^\perp/W$. Conversely, suppose $F_U$ normalizes $T$. Then $C_T(F_Z) \neq 0$. By $(*)$ $C_V(F_Z) = Z$. Since $Z$ is 1-dimensional we conclude that $Z \leqslant T$. As $Z \in \mathcal{S}(W)$ was arbitrary, we conclude that $W \leqslant T$. It remains to show that $T \leqslant W^\perp$. This is obvious if $W^\perp = V$. Thus, we may assume that $V$ is not a linear space. Then also $T^\perp$ is a proper, non-zero $F_U$-invariant $\mathbb{K}$-subspace of $V$ and thus $W \leqslant T^\perp$. Hence $T \leqslant W^\perp$.

(e): Since $E_U \neq F_U$ we have $Q \nleqslant H^\diamond$. Thus (C) holds. In particular, $\dim_{\mathbb{K}} V = 4$ and so $\dim_{\mathbb{K}} W \leqslant 2$. Let $L$ be the largest subgroup of $H^\diamond Q$ normalizing $X$ and acting trivially on $\mathcal{S}(X^\perp/X)$. We claim that $Q \leqslant L$. Since $X = \langle \mathbb{K}C_V(Q)\rangle$, $Q$ normalizes $X$. By B.12(c) applied to $X^\perp/X$, $Q$ fixes at least one element of $\mathcal{S}(X^\perp/X)$. Also $C_{H^\diamond}(X)$ induces $SU(X^\perp/X)$ on $X^\perp/X$ and so acts transitively on $\mathcal{S}(X^\perp/X)$. By $Q!$, $C_{H^\diamond}(X)$ normalizes $Q$. Thus $Q$ acts trivially on $\mathcal{S}(X^\perp/X)$ and so $Q \leqslant L$. It follows that $E_X \trianglelefteq L$.

Suppose that $\dim_{\mathbb{K}} W = 1$. Then $W = X$ (but not necessarily $W = U$). Let $l \in L \cap H^\diamond$. Then $l$ acts $\mathbb{K}$-linearly on $X^\perp/X$ and fixes the $|\mathbb{F}| + 1$ elements of $\mathcal{S}(X^\perp/X)$. It follows that there exists $\lambda \in \mathbb{K}^\sharp$ such that $l$ acts as scalar multiplication by $\lambda$ on $X^\perp/X$. Let $\mu \in \mathbb{K}^\sharp$ such that $l$ acts as scalar multiplication by $\mu$ on $X$. Since $f$ is $l$-invariant, $\lambda^\alpha\lambda = 1$ and $l$ acts as scalar multiplication by $\mu^{-\alpha}$ on $V/X^\perp$. Since $l \in H^\diamond = SU_{\mathbb{K}}(V)$, $\det l = 1$ and so $\lambda^2 = \mu^{-1}\mu^\alpha$. Conversely, since $p = 2$, for any $\mu \in \mathbb{K}^\sharp$ there exists a unique $\lambda \in \mathbb{K}^\sharp$ with $\lambda^2 = \mu^{-1}\mu^\alpha$, and then, since $\alpha^2 = 1$,

$$(\lambda^\alpha\lambda)^2 = (\mu^{-1}\mu^\alpha)^\alpha(\mu^{-1}\mu^\alpha) = 1 \qquad \text{and} \quad \lambda^\alpha\lambda = 1.$$

In particular, if $\mu = 1$ then $\lambda = 1$. Hence $C_L(X) = C_L(X) \cap C_L(X^\perp/X) = Q_X$ and

$$(L \cap H^\diamond)/Q_X \cong \mathbb{K}^\sharp \cong C_{q^2-1} \cong C_{q-1} \times C_{q+1}.$$

Note that the elements in $L \backslash H^\diamond$ act $\alpha$-semilinear on $V$ and so centralize the first factor and invert the second one of the above decomposition of $(L \cap H^\diamond)/Q_X$. Recall that $F_X = Q_X$ in case (C). So $F_U = F_W = F_X = Q_X$ and $E_U = E_W = E_X \trianglelefteq L$. From $Q \leqslant L \leqslant H^\diamond Q$ we get $L = (L \cap H^\diamond)Q$. Since $Q/Q \cap Q_X$ has order 2 we conclude that $L/Q_X \cong C_{q-1} \times Dih_{q+1}$. Moreover, $E_U/Q_X$ is a normal subgroup of $L/Q_X$ generated by involutions. Thus $E_U/Q_X \cong Dih_{2(q+1)} \cong O_2^-(\mathbb{F})$, and (e) holds in this case.

Suppose next that $\dim_{\mathbb{K}} W = 2$. Then $W = W^\perp$. By (a) $W \leqslant U \leqslant W^\perp$ and so $W = U$. Note that $W/X \in \mathcal{S}(X^\perp/X)$. Since $Q \leqslant L$, we conclude that $Q$ normalizes $W$. Thus $E_U$ normalizes $W$. By (c), $W$ is a natural $SL_2(\mathbb{K})$-module for $F_U$. So $F_U$ acts transitively on $W$, and $Q!$ implies $E_U = \langle Q^{F_U}\rangle = QF_U$, see A.50(d). Thus $|E_U/F_U| = |Q/Q \cap F_U| = |Q/Q \cap Q_X| = 2$. As the elements of $Q\backslash Q_X$ act $\alpha$-semilinearly on $W$, we conclude that $E_U/C_{E_U}(W) \cong O_4^-(\mathbb{F})$ and so (e) also holds in this case. $\square$

APPENDIX C

# FF-Module Theorems and Related Results

## C.1. FF-Module Theorems

Definition C.1. A finite group $M$ is *$\mathcal{CK}$-group* if each composition factor of $M$ is one of the known finite simple groups.

Theorem C.2 (**General FF-Module Theorem,** [**MS5**]). *Let $M$ be a finite $\mathcal{CK}$-group with $O_p(M) = 1$ and $V$ be a faithful finite dimensional $\mathbb{F}_pM$-module. Suppose that $J := J_M(V) \neq 1$. Then for $\mathcal{J} := \mathcal{J}_M(V)$, $W := [V, \mathcal{J}] + C_V(\mathcal{J})/C_V(\mathcal{J})$, $K \in \mathcal{J}$ and $\overline{J} := J/C_J([W, K])$ the following hold:*

(a) *$K$ is either quasisimple, or $p = 2$ or $3$ and $K \cong SL_2(p)'$.*
(b) *$[V, K, L] = 0$ for all $K \neq L \in \mathcal{J}$, and $W = \bigoplus_{K\in\mathcal{J}}[W, K]$.*
(c) *$J^pJ' = O^p(J) = F^*(J) = \times_{K\in\mathcal{K}} K$.*
(d) *$W$ is a faithful semisimple $\mathbb{F}_pJ$-module.*
(e) *If $A \leqslant M$ is a best offender on $V$, then $A$ is a best offender on $W$.*
(f) *$\overline{K} = \overline{F^*(J)} = O^p(\overline{J})$ and $C_J([W, K]) = C_J([V, K])$.*
(g) *Either $[W, K]$ is a simple $\mathbb{F}_pK$-module, or one of the following holds, where $q$ is a power of $p$:*
    (1) *$\overline{J} \cong SL_n(q)$, $n \geqslant 3$, and $[W, K] \cong N^r \oplus N^{*s}$, where $N$ is a natural $SL_n(q)$-module, $H$ its dual, and $r, s$ are integers with $0 \leqslant r, s < n$ and $\sqrt{r} + \sqrt{s} \leqslant \sqrt{n}$.*
    (2) *$\overline{J} \cong Sp_{2m}(q)$, $m \geqslant 3$, and $[W, K] \cong N^r$, where $N$ is a natural $Sp_{2m}(q)$-module and $r$ is a positive integer with $2r \leqslant m + 1$.*
    (3) *$\overline{J} \cong SU_n(q)$, $n \geqslant 8$, and $[W, K] \cong N^r$, where $N$ is a natural $SU_n(q)$-module and $r$ is a positive integer with $4r \leqslant n$.*
    (4) *$\overline{J} \cong \Omega_n^\epsilon(q)$ with $p$ odd if $n$ is odd, or $\overline{J} \cong O_n^\epsilon(q)$ with $p = 2$ and $n$ even.*[1] *Moreover, $n \geqslant 10$ and $[W, K] \cong N^r$, where $N$ is a natural $\Omega_n^\epsilon(q)$-module and $r$ is a positive integer with $4r \leqslant n - 2$.*
(h) *If $[W, K]$ is not a homogeneous $\mathbb{F}_pK$ module, then (g:1) holds with $r \neq 0 \neq s$ and $n \geqslant 4$.*

Theorem C.3 (**FF-Module Theorem,** [**MS5**]). *Let $M \neq 1$ be a finite $\mathcal{CK}$-group and $V$ be a faithful $\mathbb{F}_pM$-module. Put*

$\mathcal{D} := \{A \leqslant M \mid$ *there exists* $1 \neq B \leqslant A$ *such that* $[V, B, A] = 0$ *and* $A$ *and* $B$ *are offenders on* $V\}$.[2]

*Suppose that $V$ is a simple $\mathbb{F}_pJ_M(V)$-module and $M = \langle\mathcal{D}\rangle$. Then one of the following holds, where $q$ is a power of $p$:*

(1) *$M \cong SL_n(q)$, $n \geqslant 2$, and $V$ is a natural $SL_n(q)$-module.*
(2) *$M \cong Sp_{2n}(q)$, $n \geqslant 1$, and $V$ is a natural $Sp_{2n}(q)$-module.*
(3) *$M \cong SU_n(q)$, $n \geqslant 4$, and $V$ is a natural $SU_n(q)$-module.*
(4) *$M \cong \Omega_{2n}^+(q)$ for $2n \geqslant 6$, $M \cong \Omega_{2n}^-(q)$ for $p = 2$ and $2n \geqslant 6$, $M \cong \Omega_{2n}^-(q)$ for $p$ odd and $2n \geqslant 8$, $M \cong \Omega_{2n+1}(q)$ for $p$ odd and $2n + 1 \geqslant 7$, $M \cong O_4^-(2)$, or $M \cong O_{2n}^\epsilon(q)$ for $p = 2$ and $2n \geqslant 6$, and $V$ is a corresponding natural module.*
(5) *$M \cong G_2(q)$, $p = 2$, and $V$ is a natural $G_2(q)$-module (of order $q^6$).*
(6) *$M \cong SL_n(q)/\langle -id^{n-1}\rangle$, $n \geqslant 5$, and $V$ is the exterior square of a natural $SL_n(q)$-module.*
(7) *$M \cong Spin_7(q)$, and $V$ is a spin module of order $q^8$.*

[1] The odd-dimensional orthogonal groups in characteristic 2 are covered in case (g:2).

[2] Note here that $\mathcal{D}$ contains all quadratic offenders and by the Timmesfeld Replacement Theorem [**KS**, 9.2.3], also all best offenders in $M$ on $V$.

(8) *$M \cong Spin^+_{10}(q)$, and $V$ is a half-spin module of order $q^{16}$.*
(9) *$M \cong 3{\cdot}Alt(6)$, $p = 2$ and $|V| = 2^6$.*
(10) *$M \cong Alt(7)$, $p = 2$, and $|V| = 2^4$.*
(11) *$M \cong Sym(n)$, $p = 2$, $n$ odd, $n \geqslant 3$, and $V$ is a natural $Sym(n)$-module.*
(12) *$M \cong Alt(n)$ or $Sym(n)$, $p = 2$, $n$ is even, $n \geqslant 6$, and $V$ is a corresponding natural module.*

THEOREM C.4 (**Best Offender Theorem,** [**MS5**]). *Let $M \neq 1$ be a finite group, $T \in Syl_p(M)$, and $V$ be a faithful $\mathbb{F}_pM$-module, and let $A \leqslant T$ be a non-trivial offender on $V$.*

(a) *Suppose that $M \cong G_2(q)$, $p = 2$, and $V$ is a natural $G_2(q)$-module. Then $N_M(A)$ is a maximal Lie-parabolic subgroup, $|A| = |V/C_V(A)| = q^3$, $[V, A] = C_V(A)$, and $C_T(A) = A$.*
(b) *Suppose that $M \cong SL_n(q)/\langle -id^{n-1}\rangle$, $n \geqslant 5$, and $V$ is the exterior square of a natural $SL_n(q)$-module $W$. Let $U$ be the (unique) $T$-invariant $\mathbb{F}_q$-hyperplane of $W$. Then $A = C_M(U)$. In particular, $A$ is uniquely determined in $T$, $C_T(A) = A$, $[V, A] = C_V(A)$ and $|V/C_V(A)| = |A| = q^{n-1}$.*
(c) *Suppose that $M \cong Spin_7(q)$, and $V$ is a spin module of order $q^8$. Then $C_V(A) = [V, A]$, $|V/C_V(A)| = q^4 \leqslant |A| \leqslant q^5$, and if $A$ is maximal, then $|A| = q^5$, $C_T(A) = A$, $O^{p'}(N_M(A))/A \cong Sp_4(q)$, and $A$ is uniquely determined in $T$.*
(d) *Suppose that $M \cong Spin^+_{10}(q)$, and $V$ is a half-spin module of order $q^{16}$. Then $[V, A] = C_V(A)$, $q^8 = |A| = |V/C_V(A)|$, $O^{p'}(N_M(A)/A) \cong Spin^+_8(q)$, and $A$ is uniquely determined in $T$.*
(e) *Suppose that $M \cong 3{\cdot}Alt(6)$, $p = 2$ and $|V| = 2^6$. Then $[V, A] = C_V(A)$, $|[V, A]| = |C_V(A)| = 16$, $|V/C_V(A)| = |A| = 4$, and $A$ is uniquely determined in $T$.*
(f) *Suppose that $M \cong Alt(7)$, $p = 2$ and $|V| = 2^4$. Then $[V, A] = C_V(A)$, $|[V, A]| = |C_V(A)| = 4$, $|V/C_V(A)| = |A| = 4$, and $A$ is uniquely determined in $T$.*
(g) *Suppose that $M \cong Sym(n)$, $p = 2$, $n$ odd, and $V$ is a natural $Sym(n)$-module. Then every offender on $V$ is a quadratic best offender, $A$ is generated by commuting transpositions and $|V/C_V(A)| = |[V, A]| = |A|$.*
(h) *Suppose that $M \cong Alt(n)$ or $Sym(n)$, $p = 2$, $n$ is even, $n \geqslant 6$, and $V$ is a corresponding natural module. Then every offender on $V$ is a best offender, and there exists a set of pairwise commuting transpositions $t_1, \dots, t_k$ such that one of the following holds:*
    (1) *$A = \langle t_1, \dots, t_k\rangle$, and either $n \neq 2k$, $[V, A] \leqslant C_V(A)$ and $|[V, A]| = |V/C_V(A)| = |A|$ or $n = 2k$, $[V, A] = C_V(A)$ and $2|V/C_V(A)| = |A|$.*
    (2) *$n = 2k$ and $A = \langle t_1t_2, t_2t_3 \dots, t_{l-1}t_l, t_{l+1}, t_{l+2}, \dots, t_k\rangle$ for some $2 \leqslant l \leqslant k$, $[V, A] = C_V(A)$ and $|V/C_V(A)| = |A|$.*
    (3) *$n = 2k$ and $A = \langle t_1t_2, s_1s_2, t_3, t_4 \dots, t_k\rangle$, where $s_1, s_2$ are transpositions distinct from $t_1$ and $t_2$ and $s_1s_2$ moves the same four symbols as $t_1t_2$, $A$ is not quadratic and $|[V, A]| = |V/C_V(A)| = |A|$.*
    (4) *$n = 8 = |A|$, $A$ acts regularly on $\{1, 2, \dots, 8\}$, $[V, A] = C_V(A)$ and $|V/C_V(A)| = |A|$.*

    *In particular, if $A \leqslant Alt(n)$ and $n \neq 8$, then $n = 2k$ and $A = \langle t_1t_2, t_2t_3, \dots, t_{k-1}t_k\rangle$.*

The next result essentially is [**MS6**, 3.1]. We just use a slightly different hypothesis.

THEOREM C.5 (**Strong Dual FF-Module Theorem,** [**MS6**, 3.1]). *Let $M$ be a finite $\mathcal{CK}$-group, and let $V$ be a faithful $\mathbb{F}_pM$-module. Let $\mathcal{A}$ be the set of strong dual offenders in $M$ on $V$. Suppose that $M = \langle \mathcal{A}\rangle$ and that*

(i) *$V$ is a simple $M$-module, or*
(ii) *$C_V(M) = 0$, $V = [V, M]$, and there exists $B \in \mathcal{A}$ with $M = \langle B^M\rangle$.*

*Then $V$ is a simple $M$-module, and one of the following holds, where $q$ is a power of $p$.*

(1) *$M \cong SL_n(q)$, $n \geqslant 2$, or $Sp_{2n}(q)$, $n \geqslant 2$, and $V$ is a corresponding natural module.*
(2) *$p = 2$, $M \cong Alt(6)$ or $Alt(7)$, $V$ is a spin-module of order $2^4$, and $A \cong \langle (12)(34), (13)(24)\rangle$ for all $A \in \mathcal{A}$.*[3]
(3) *$p = 2$, $M \cong O^\epsilon_{2n}(2)$, $n \geqslant 3$, or $Sym(n)$, $n = 5$ or $n \geqslant 7$, $V$ is a corresponding natural module, and $|A| = 2$ for all $A \in \mathcal{A}$.*

[3] Note that in the $Alt(6)$-case, $V$ might also be viewed as a natural $Alt(6)$-module with $A \cong \langle (12)(34), (34)(56)\rangle$).

Proof. By A.32(c) strong dual offenders are best offender. Thus

1°. *$A$ is a best offender for every $A \in \mathcal{A}$.*

It follows that $\langle \mathcal{A} \rangle \leqslant J_M(V)$, and $M = \langle \mathcal{A} \rangle$ gives

2°. $M = J_M(V)$.

Now let $W$ be a non-zero $M$-submodule of $V$. If (i) holds, $V$ is a simple $M$-module and so $W = V$. Assume that (ii) holds. Then there exists $B \in \mathcal{A}$ such that $M = \langle B^M \rangle$. Hence $C_V(M) = 0$ implies $[W, B] \neq 0$. Since $B$ is a strong dual offender, this gives $[V, B] = [W, B] \leqslant W$, and so $[V, \langle B^M \rangle] = [V, M] \leqslant W$. Now $[V, M] = V$ yields $V = W$. We have shown that always $W = V$ and so

3°. *$V$ is a simple $M$-module.*

Observe that (2°) now shows that $V$ is a simple $J_M(V)$-module. Hence we can apply C.3 and get

4°. *Either $F^*(M)$ is quasisimple and $|M/F^*(M)| \leqslant 2$, or $M \cong SL_2(q)$, $q = 2$ or $3$, and $V$ is a natural $SL_2(q)$-module for $M$.*

In the second case of (4°), (1) holds. Thus we may assume the first case in (4°). Since $M = \langle \mathcal{A} \rangle$ there exists $B \in \mathcal{A}$ such that $M = F^*(M)B$. Then, for any such $B$, $M = \langle B^M \rangle$ and the hypothesis of [**MS6**, 3.1] is fulfilled for $M$ and $B$. Thus one of the following holds:

(A) $M \cong SL_n(q)$, $n \geqslant 2$, or $Sp_{2n}(q)$, $n \geqslant 2$, and $V$ is a corresponding natural module.
(B) $p = 2$, $M \cong Alt(6)$ or $Alt(7)$, $V$ is a spin-module of order $2^4$, and $B \cong \langle (12)(34), (13)(24) \rangle$.
(C) $p = 2$, $M \cong O^{\epsilon}_{2n}(2)$, $n \geqslant 3$, or $Sym(n)$, $n = 5$ or $n \geqslant 7$, $V$ is a corresponding natural module, and $|B| = 2$.

In case (A), (1) holds. In case (B), $M$ is simple and so $M = F^*(M)A$ for all $A \in \mathcal{A}$ and so (2) holds.

So suppose (C) holds and let $A \in \mathcal{A}$. If $A \leqslant F^*(M)$, then $F^*(M) = \langle A^{F^*(M)} \rangle$ and we can apply [**MS6**, 3.1] to $F^*(M)$ and $V$ and so one of (A)–(C) holds for $F^*(M)$ in place of $M$. But since (C) holds for $M$, $F^*(M) \cong \Omega^{\epsilon}_{2n}(2)$, $n \geqslant 3$, or $Alt(n)$, $n = 5$ or $n \geqslant 7$, and $V$ is a corresponding natural module, a contradiction. Thus $A \nleqslant F^*(M)$, $F^*(M)A = M$, and (3) holds. □

Theorem C.6 (**Strong FF-Module Theorem,** [**MS6**, 3.2]). *Let $M$ be a finite $\mathcal{CK}$-group such that $K := F^*(M)$ is quasisimple, and let $V$ be a faithful simple $\mathbb{F}_pK$-module. Suppose that $A \leqslant M$ is a strong offender on $V$ and $M = \langle A^M \rangle$. Then one of the following holds, where $q$ is a power of $p$:*

(1) *$M \cong SL_n(q)$ or $Sp_{2n}(q)$ and $V$ is a corresponding natural module.*
(2) *$p = 2$, $M \cong Alt(6)$, $3{}^{\cdot}Alt(6)$ or $Alt(7)$, $|V| = 2^4$, $2^6$ or $2^4$, respectively, and $|A| = 4$.*
(3) *$p = 2$, $M \cong O^{\epsilon}_{2n}(2)$ or $Sym(n)$, $V$ is a corresponding natural module, and $|A| = 2$.*

Definition C.7. Let $M$ be a finite group and $V$ a faithful $M$-module. Recall the definition of a point-stabilizer of $M$ on $V$ from A.3. By $\mathcal{AP}_M(V)$ we denote the set of non-trivial best offenders $A$ of $M$ on $V$ such that $A \leqslant O_p(P)$ for some point-stabilizer $P$ of $M$ on $V$.

Theorem C.8 (**Point-Stabilizer Theorem,** [**MS6**, 3.5]). *Let $M$ be a finite $\mathcal{CK}$-group with $O_p(M) = 1$ and let $V$ be a faithful $\mathbb{F}_pM$-module. Suppose that $M = \langle \mathcal{AP}_M(V) \rangle$ and that there exists a $J_M(V)$-component $K$ with $V = [V, K]$ and $C_V(K) = 0$. Let $A \in \mathcal{AP}_M(V)$ and let $P$ be a point-stabilizer for $M$ on $V$ with $A \leqslant O_p(P)$. Then the following hold:*

(a) *$M \cong SL_n(q)$, $Sp_{2n}(q)$, $G_2(q)$ or $Sym(n)$, $q$ a power of $p$, where $p = 2$ in the last two cases, and $n \equiv 2, 3 \pmod 4$ in the last case.*
(b) *$V$ is a corresponding natural module.*
(c) *Put $\mathbb{F} := End_M(V)$, $q := |\mathbb{F}|$ and $Z := C_V(P)$. Then $Z$ is 1-dimensional over $\mathbb{F}$, and one of the following holds:*
    (1) *$M \cong SL_n(q)$, $[V, A] = Z$, and $A = C_M(C_V(A)) \cap C_M(V/Z)$.*
    (2) *$M \cong Sp_{2n}(q)$, $Z \leqslant [V, A] \leqslant Z^{\perp}$, and $A = C_M(C_V(A)) \cap C_M(Z^{\perp}/Z)$.*

(3) $M \cong G_2(q)$, $[V, A] = C_V(A)$, $|V/C_V(A)| = |A| = q^3$, *and* $A \trianglelefteq P$.
(4) $M \cong Sym(n)$, $n \equiv 2, 3 \pmod 4$, $n > 6$, $|A| = 2$, *and* $A \trianglelefteq P$.

(d) $|V/C_V(A)| = |A|$, *and* $V$ *is a simple* $\mathbb{F}_pK$*-module.*

THEOREM C.9 (**General Point-Stabilizer Theorem,** [**MS6**, 3.6]). *Let* $M$ *be a finite* $\mathcal{CK}$*-group with* $O_p(M) = 1$ *and let* $V$ *be a faithful* $\mathbb{F}_pM$*-module. Put* $\mathcal{AP} := \mathcal{AP}_M(V)$ *and suppose that* $\mathcal{AP} \neq \varnothing$*. Then there exists an* $M$*-invariant set* $\mathcal{N}$ *of subnormal subgroups of* $M$ *such that the following hold:*

(a) $\langle \mathcal{AP} \rangle = \times_{N \in \mathcal{N}} N$, *and* $N = \langle A \in \mathcal{AP} \mid A \leqslant N \rangle$ *for all* $N \in \mathcal{N}$.
(b) *For all* $N_1 \neq N_2 \in \mathcal{N}$, $[V, N_1, N_2] = 0$.
(c) *Put* $\overline{V} = V/C_V(\mathcal{N}))$*. Then* $[\overline{V}, \mathcal{N}] = \bigoplus_{N \in \mathcal{N}} [\overline{V}, N]$.
(d) *Let* $N \in \mathcal{N}$*. Then* $(N, [\overline{V}, N])$ *satisfies the hypothesis of C.8 in place of* $(M, V)$.
(e) *For all* $N \in \mathcal{N}$, $C_V(N) = C_V(O^p(N))$ *and* $[V, O^p(N)] = [V, N]$.
(f) *Let* $A \in \mathcal{AP}$*. Then*
  (a) $|V/C_V(A)| = |A|$,
  (b) $A = \times_{N \in \mathcal{N}} A \cap N$,
  (c) $A \cap N \in \mathcal{AP}$ *for all* $N \in \mathcal{N}$ *with* $A \cap N \neq 1$.

LEMMA C.10. *Let* $L$ *be a finite* $\mathcal{CK}$*-group of characteristic* $p$*. Suppose that*

(i) $C_L(Z_L)$ *is* $p$*-closed,*
(ii) $P$ *is a point-stabilizer of* $L$ *on* $Z_L$,[4]
(iii) $O_p(L) \leqslant R \leqslant O_p(P)$,
(iv) $A$ *and* $Y$ *are elementary abelian subgroup of* $R$*, and*
(v) $A$ *normalizes* $Y$, $Z_L \leqslant Y$*, and* $A \cap O_p(L)$ *centralizes* $Y$.

*Then the following hold:*

(a) $C_A(Y) = A \cap O_p(L) = C_A(Z_L)$*. In particular, if* $A \not\leqslant O_p(L)$ *then* $[Y, A] \neq 1$.
(b) *Suppose that* $A$ *is a best offender on* $Y$*. Then*
  (a) $|A/A \cap O_p(L)| = |Z_L/C_{Z_L}(A)| = |Y/C_Y(A)|$,
  (b) $Y = C_Y(A)Z_L$.
(c) *Suppose that* $A \in \mathcal{A}_R$*. Then*
  (a) $A$ *is a best offender on* $Y$ *and on* $Z_L$,
  (b) $Z_L(A \cap O_p(L)) \in \mathcal{A}_R \cap \mathcal{A}_{O_p(L)}$,
  (c) $Y = (A \cap Y)Z_L$.
(d) $\mathcal{A}_{O_p(L)} \subseteq \mathcal{A}_R$*. In particular,* $J(O_p(L)) \leqslant J(R)$.
(e) $\Omega_1 Z(J(R)) \leqslant \Omega_1 Z(J(O_p(L)))$.
(f) $[\Omega_1 Z(J(R)), \langle J(R)^L \rangle] \leqslant [\Omega_1 Z(J(O_p(L))), \langle J(R)^L \rangle] \leqslant Z_L$.

PROOF. (a): Note that $O_p(L)$ centralizes $Z_L$. As $C_L(Z_L)$ is $p$-closed we get that $O^{p'}(C_L(Z_L)) = O_p(L)$. Thus $C_A(Z_L) = A \cap O_p(L)$. Now $Z_L \leqslant Y$ and $[Y, A \cap O_p(L)] = 1$ give $A \cap O_p(L) \leqslant C_A(Y) \leqslant C_A(Z_L) = A \cap O_p(L)$, and so (a) holds.

(b): Since $A$ is a best offender on $Y$, A.31 shows that $A$ is a best offender on $Z_L$. By 1.24(i), $Z_L$ is $p$-reduced for $L$ and thus $O_p(L/C_L(Z_L)) = 1$. Also $A \leqslant R \leqslant O_p(P)$, and so C.9 shows that $|Z_L/C_{Z_L}(A)| = |A/C_A(Z_L)|$. Thus using (a) and that $A$ is an offender on $Z_L$:

$$\begin{aligned}|A/A \cap O_p(L)| = |A/C_A(Z_L)| = |Z_L/C_{Z_L}(A)| = |Z_L/Z_L \cap C_Y(A)| &= |Z_L C_Y(A)/C_Y(A)| \\ \leqslant |Y/C_Y(A)| \leqslant |A/C_A(Y)| &= |A/A \cap O_p(L)|,\end{aligned}$$

and so (b) holds.

(c:a) follows from A.40.

(c:b): Note that $Z_L(A \cap O_p(L))$ is an elementary abelian subgroup of $R$. Since $A$ is a maximal elementary abelian subgroup of $R$, $C_{Z_L}(A) = Z_L \cap A$. Using (b)

$$|Z_L(A \cap O_p(L))| = |Z_L/A \cap Z_L||A \cap O_p(L)| = |A/A \cap O_p(L)||A \cap O_p(L)| = |A|.$$

[4] See 1.1(c) for the definition of $Z_L$

Thus $Z_L(A \cap O_p(L)) \in \mathcal{A}_R$ and so also $Z_L(A \cap O_p(L)) \in \mathcal{A}_{O_p(L)}$.

(c:c): Since $A$ is a maximal elementary abelian subgroup of $R$, $C_Y(A) = Y \cap A$. By (c:a) $A$ is a best offender on $Y$. Hence we can apply (b:b), and so $Y = C_Y(A)Z_L = (Y \cap A)Z_L$. Thus (c:c) is proved.

Let $D \in \mathcal{A}_{O_p(L)}$ and $A \in \mathcal{A}_R$.

(d): By (c:b) $|D| = |Z_L(A \cap O_p(L))| = |A|$ and so $D \in \mathcal{A}_R$.

(e): By (d) $J(O_p(L)) \leqslant J(R)$ and so both $D$ and $J(O_p(L))$ centralize $\Omega_1 Z(J(R))$. Also by (d) $D \in \mathcal{A}_R$ and so the maximality of $D$ gives $\Omega_1 Z(J(R)) \leqslant D \leqslant J(O_p(L))$. Hence

$$\Omega_1 Z(J(R)) \leqslant C_R(J(O_p(L))) \cap J(O_p(L)) = Z(J(O_p(L))),$$

and (e) follows.

(f): Put $Y := \Omega_1 Z(J(O_p(L)))$. By (c:b) $Z_L(A \cap O_p(L)) \in \mathcal{A}_{O_p(L)}$ and so $[Y, A \cap O_p(L)] = 1$. Thus by (c:c), $Y = (Y \cap A)Z_L$ and so $[Y, A] \leqslant Z_L$. Hence $[\Omega_1 Z(J(O_p(L))), J(R)] \leqslant Z_L$. Since $\Omega_1 Z(J(O_p(L)))$ and $Z_L$ are normal in $L$, the second inclusion in (f) holds. The first inclusion follows from (e). $\square$

THEOREM C.11 ([**Gl1**, Theorem 2]). *Let $L$ be a finite group, $A$ a non-trivial abelian $p$-subgroup of $L$ and $V$ a faithful $p$-reduced $\mathbb{F}_pL$-module. Suppose that $A$ is a quadratic offender on $V$, $L$ is $A$-minimal and $C_V(L) = 0$. Then $L \cong SL_2(q)$, $V$ is a natural $SL_2(q)$-module for $L$ and $A \in Syl_p(L)$; in particular $q = |A|$.*

PROOF. This is [**Gl1**, Theorem 2] just that the hypothesis and conclusion are stated differently:
Since $V$ is a vector space over $\mathbb{F}_p$, $V$ is an abelian $p$-group, and since $V$ is a faithful $L$-module, we may view $L$ as a subgroup of $Aut(V)$. Since $L$ is $A$-minimal, $A$ is contained in a unique maximal subgroup $M$ of $L$. Let $S \in Syl_p(M)$ with $A \leqslant S$. Since $V$ is faithful and $p$-reduced, $O_p(L) = 1$. As $A$ is quadratic on $V$, $[V, A, A] = 0$. The uniqueness of $M$ shows that $\langle A, g\rangle = L$ for all $g \in L \backslash M$. So Hypothesis I in [**Gl1**] holds.
By assumption $C_V(L) = 0$, and since $A$ is an offender on $V$, $|V/C_V(A)| \leqslant |A|$. Hence the Hypothesis of Theorem 2 in [**Gl1**] holds. Thus, there exists a field $\mathbb{K}$ of endomorphisms of $V$ such that $|\mathbb{K}| = |A|$, $\dim_{\mathbb{K}} V = 2$ and $L = SL_{\mathbb{K}}(V)$. In particular, the Sylow $p$-subgroups of $L$ have order $|\mathbb{K}| = |A|$, and $A \in Syl_p(L)$. $\square$

LEMMA C.12. *Let $p$ be prime, $M$ be a finite group, $V$ a faithful $\mathbb{F}_pM$ module and $\mathcal{D}$ a non-empty $M$-invariant set of subgroups of $M$. Suppose that*

(i) *Each $A \in \mathcal{D}$ is a non-trivial root offender on $V$.*
(ii) *$C_V(A) \cap [V, B] = 0$ for all $A, B \in \mathcal{D}$ with $[V, A] \neq [V, B]$.*
(iii) *$M = \langle \mathcal{D}\rangle$ and $V = [V, M]$.*

*Then $M \cong SL_2(q)$ and $V$ is a natural $SL_2(q)$-module for $M$, where $q = |A|$. In particular, $A \in Syl_p(M)$.*

PROOF. For $X \leqslant V$ put

$$\mathcal{T}_X := \{[V, D] \mid D \in \mathcal{D}, [V, D] \leqslant X\}.$$

$1^\circ$. *Let $D, E \in \mathcal{D}$. Then $[V, D] \leqslant C_V(D)$, and either $[V, D] = [V, E]$ or $[V, D] \cap [V, E] = C_V(D) \cap [V, E] = 0$.*

By A.37(c) $D$ acts quadratically on $V$, so $[V, D] \leqslant C_V(D)$. By (ii) $[V, D] = [V, E]$ or $C_V(D) \cap [V, E] = 0$. In the latter case also $[V, D] \cap [V, E] = 0$ since $[V, D] \leqslant C_V(D)$.

$2^\circ$. *Let $D \in \mathcal{D}$. Then $|D| = |V/C_V(D)| = |[V, D]|$, $[v, D] = [V, D]$, and $v^D = v + [V, D]$ for every $v \in V \backslash C_V(D)$.*

As $D$ is a root offender on $V$, A.37(a) gives $|D/C_D(V)| = |V/C_V(A)| = |[V,A]| = q$, and $C_D(V) = 1$ since $V$ is faithful. Moreover, A.37(b) shows that $D$ is a strong dual offender on $V$ and so $[v,D] = [V,D]$ for $v \in V \backslash C_V(D)$. Thus also $v^D = v + [V,D]$, and (2°) holds.

Since $\mathcal{D} \neq \varnothing$ and $M = \langle \mathcal{D} \rangle$, $M \neq 1$, and since $V = [V,M]$, $M$ does not act nilpotently on $V$. Hence, there exist $A, B \in \mathcal{D}$ with $[V,A] \neq [V,B]$. Let

$$Y := [V,A], \qquad Z := [V,B], \qquad X := Y + Z, \qquad L := \langle A, B \rangle.$$

3°. *$X = Y \oplus Z$, $C_X(A) = Y$, $\mathcal{T}_X$ is a partition of $X$, $\mathcal{T}_X = \{Y\} \cup Z^A$, and $L$ acts doubly transitively on $\mathcal{T}_X$.*

Note that $\langle A, B \rangle$ normalizes $X$. Since $[V,A] \neq [V,B]$, (1°) shows that $C_V(A) \cap [V,B] = Y \cap Z = 0$. Hence $X = Y \oplus Z$, and since by (1°) $Y \leqslant C_V(A)$, $C_X(A) = Y$.

Pick $0 \neq z \in Z$. Then $z \notin Y = C_X(A)$ and by (2°) $z + Y = z^A \subseteq \bigcup Z^A$. Since $X = Y + Z$ this shows that $X = Y + \bigcup Z^A$. Now (1°) implies that $\{Y\} \cup Z^A$ forms a partition of $X$ and $\mathcal{T}_X = \{Y\} \cup Z^A$. By symmetry also $\mathcal{T}_X = \{Z\} \cup Y^B$, and $L = \langle A, B \rangle$ acts doubly transitively on $\mathcal{T}_X$.

4°. *$M$ is transitive on $\mathcal{T}_V$ and $V = X$.*

Let $D \in \mathcal{D}$. If $[V,D] \neq [V,A]$ then (3°) (with $D$ in place of $B$) shows $[V,A]$ and $[V,D]$ are conjugate under $\langle A, D \rangle$. Hence $M$ is transitive on $\mathcal{T}_V$. In particular, there exists $q$ such that $|[V,D]| = q$ for every $D \in \mathcal{D}$.

By (2°) $|V/C_V(D)| = |[V,D]| = q$ while by (3°) $|X| = q^2$. Hence $C_V(D) \cap X \neq 0$. Let $0 \neq w \in C_X(D)$. By (3°) $\mathcal{T}_X$ is a partition of $X$ and so there exists $E \in \mathcal{D}$ with $w \in [V,E] \leqslant X$. Then $[V,E] \cap C_V(D) \neq 0$. Now (ii) yields $[V,D] = [V,E] \leqslant X$.

We have shown that $\mathcal{T}_V = \mathcal{T}_X$. Hence by (iii) $V = [V,M] = [V, \langle \mathcal{D} \rangle] \leqslant X$ and $X = V$.

5°. *$M$ acts transitively on $V$. In particular, $V$ is a simple $M$-module.*

Let $0 \neq y \in Y$ and $0 \neq z \in Z$. Since by (2°) $z^A = z + Y$, $z + y \in z^M$. By symmetry, $(z+y)^B = y + Z$ and so $y + Z \subseteq z^M$. As $V = X = Y + Z$ by (4°), this gives $V \backslash Z \subseteq z^M$. In particular, $y \in Y^\sharp \subseteq z^M$. By symmetry also $Z^\sharp \subseteq y^M \subseteq z^M$, and so $z^M = V^\sharp$.

6°. *$A = C_M(Y) \cap C_M(V/Y)$, $N_M(A) = N_M(Y)$, and $\mathcal{D} = \{A\} \cup B^A$. In particular $M = L$.*

Let $E := C_M(Y) \cap C_M(V/Y)$. Clearly $N_M(A) \leqslant N_M(Y) \leqslant N_M(E)$. Moreover, by the quadratic action of $A$ on $V$, $\langle A^{N_M(Y)} \rangle \leqslant E$. Thus, if $A = E$, then also $N_M(A) = N_M(Y) = N_M(E)$.

Note that by (4°) $V = X$ and so by (3°) $V = Y \oplus Z$ and $\mathcal{T}_V = \{Y\} \cup Z^A$. Hence $E$ acts on $Z^A$. A Frattini argument gives $E = AN_E(Z)$. Thus $[Z, N_E(Z)] \leqslant Y \cap Z = 0$. By the definition of $E$, $N_E(Z)$ also centralizes $Y$. Since $V = Y + Z$, $N_E(Z)$ centralizes $V$, and since $V$ is faithful, we get $N_E(Z) = 1$ and $E = AN_E(Z) = A$. Thus $A = E$ and $N_M(A) = N_M(Y)$.

We have shown that $C_M([V,D]) \cap C_M(V/[V,D]) \mapsto [V,D]$ induces a bijection from $\mathcal{D}$ to $\mathcal{P}_V = \{Y\} \cup Z^A$. It follows that $\mathcal{D} = \{A\} \cap B^A$. In particular $\langle \mathcal{D} \rangle = M \leqslant L$.

7°. *$M = \langle A, A^g \rangle$ for all $g \in M$ with $A \neq A^g$. In particular, $N_M(A)$ is the unique maximal subgroup of $M$ containing $A$, and $M$ is $A$-minimal.*

Pick $g \in M \backslash N_M(A)$. By (6°) there exists $a \in A$ such that $A^g = B^a$. Hence $\langle A, A^g \rangle = \langle A, B^a \rangle = L$, and again by (6°) $\langle A, A^g \rangle = M$. Hence $N_M(A)$ is the unique maximal subgroup of $M$ containing $A$, and $M$ is $A$-minimal.

We are now able to prove the lemma. By (5°), $V$ is a simple $M$-module. In particular, $V$ is $p$-reduced and $C_V(M) = 0$. By assumption $A$ is a non-trivial root offender and so a non-trivial quadratic offender. By (7°), $M$ is $A$-minimal. Hence C.11 shows that $M \cong SL_2(q)$, $V$ is a natural $SL_2(q)$-module, and $A \in Syl_p(M)$. □

Theorem C.13 ([**MS5**, 8.1]). *Let $p$ be a prime, $M$ be a finite $p$-minimal group, $V$ a faithful $p$-reduced $\mathbb{F}_pM$-module and $T \in Syl_p(M)$. Set $J := J_M(V)$ and $\mathcal{J} := \mathcal{J}_M(V)$. Then there exist subgroups $E_1, \dots, E_r$ such that the following hold:*

(a) $J = E_1 \times \cdots \times E_r$ *and* $\mathcal{J} = \{E_1', \dots, E_r'\}$.
(b) $V = C_V(J) + \sum_{i=1}^r [V, E_i]$ *and* $[V, E_i, E_j] = 0$ *for* $i \neq j$.
(c) $[C_V(T), O^p(M)] \neq 0$.
(d) $T$ *is transitive on* $E_1, \dots, E_r$.
(e) *There are no over-offenders on* $V$ *in* $M$.
(f) $E_i \cong SL_2(q)$, $q = p^n$, *and* $[V, E_i]/C_{[V,E_i]}(E_i)$ *is a natural* $SL_2(q)$*-module for* $E_i$, *or* $p = 2$, $E_i \cong Sym(2^n + 1)$, *and* $[V, E_i]$ *is a natural* $Sym(2^n + 1)$*-module for* $E_i$.
(g) *If* $A \leqslant M$ *is an offender on* $V$, *then* $A = (A \cap E_1) \times \ldots \times (A \cap E_r)$, *and each* $A \cap E_i$ *is an offender on* $V$.

The following lemma is an easy consequence of the Quadratic $L$-lemma [**MS6**, Lemma 2.9], in fact its proof is hidden in the proof of the Quadratic $L$-Lemma [**MS6**, Lemma 2.9]. But since the Quadratic $L$-Lemma was proved under a $\mathcal{CK}$-group assumption we prefer to reproduced the proof.

Lemma C.14. *Let $L$ be a $p$-minimal finite group, and for $i = 1, 2$ let $V_i$ be a natural $SL_2(q_i)$-module for $L$, where $q_i$ be a power of $p$. Then $q_1 = q_2$, $L/C_L(V_1 \oplus V_2) \cong SL_2(q_i)$ and $V_1$ and $V_2$ are isomorphic $L$-modules.*

Proof. Put $V := V_1 \oplus V_2$. Replacing $L$ by $L/C_M(V)$ we may assume that $V$ is faithful $L$-module. In particular, $O_p(L) = 1$. Let $A \in Syl_p(L)$ and let $L_0$ be the unique maximal subgroup of $L$ containing $A$. For $i = 1, 2$ put $C_i := C_L(V_i)$. Then $C_1 \cap C_2 = C_L(V) = 1$. If $C_1 = C_2$, then $C_1 = C_2 = 1$, $L \cong SL_2(q_1) \cong SL_2(q_2)$ and $q_1 = q_2$. Since $SL_2(q_1)$ has a unique natural $SL_2(q_i)$-module, the lemma holds in this case.

So we may assume for a contradiction that $C_1 \nleqslant C_2$. Note that $AC_1 \neq L$ and so $AC_1 \leqslant L_0$ and $C_1 \leqslant \bigcap L_0^L$. Since $O_p(L) = 1$, 1.42(d) shows that $\bigcap L_0^L = \Phi(L)$. Thus $C_1 \leqslant \Phi(L)$ and so

$$1 \neq C_1 \cong C_1C_2/C_2 \leqslant \Phi(L/C_2).$$

Note that $\Phi(SL_2(q_2)) = Z(SL_2(q_2))$. It follows that $p$ is odd and $|C_1| = 2$. In particular, $C_1 \leqslant Z(L)$. Since $L$ is $p$-minimal, $L = \langle A^L \rangle = L'A$. So $L/L'$ is a $p$-group and $C_1 \leqslant L'$. Thus the 2-part of the Schur–multiplier of $L$ is non-trivial.

Suppose that $q_1 > 3$. Then $L/C_1 \cong SL_2(q_1)$ is quasisimple. By [**Hu**, V.25.7] the 2-part of Schur multiplier of $SL_2(q_1)$ is trivial, a contradiction. Thus $q_1 = 3$ and $L'/C_1 \cong Q_8$ By [**Hu**, V.25.3] the Schur multiplier of $Q_8$ is trivial, so $C_1 \nleqslant L''$. Note that $L'/L''$ is a 2-group and coprime action shows that $C_1 \nleqslant [L', A]$. But then also

$$C_1 \nleqslant [L', A]A = \langle A^{L'} \rangle = \langle A^{L'A} \rangle = \langle A^L \rangle = L,$$

a contradiction. □

Theorem C.15 ([**MS6**, 2.10]). *Let $L \cong SL_2(q)$ or $Sz(q)$, $q = p^k$, where $p = 2$ in the latter case, and let $V$ be a non-central simple $\mathbb{F}_pL$-module. Suppose that $L$ is $A$-minimal for some $A \leqslant L$ with $[V, A, A] = 0$. Then $V$ is a corresponding natural module.*

Lemma C.16. *Let $M$ be a finite group, $K \trianglelefteq H$, $A \leqslant K$ and $V$ a faithful $\mathbb{F}_2H$-module. Suppose that $K \cong 3^{\cdot}Alt(6)$, $A$ is a non-trivial offender on $V$ and $|V| = 2^6$. Put $K_2 := C_M(V/C_V(A))$ and let $\mathcal{V}$ be the set of 3-dimensional $\mathbb{F}_2K_2$-submodules of $V$. Then*

(a) *Either* $H = K \cong 3^{\cdot}Alt(6)$ *or* $|H/K| = 2$ *and* $H \cong 3^{\cdot}Sym(6)$.
(b) $|A| = 4 = |V/C_V(A)|$ *and* $[V, A] = C_V(A)$.
(c) $N_M(A) = N_M(C_V(A))$ *is a maximal* 2*-parabolic subgroup of* $M$, *and* $N_K(A) \cong C_3 \times Sym(4)$.
(d) $K_2 = O^{2'}(N_K(A)) \cong Sym(4)$.
(e) $\mathcal{V} = \{V_1, V_2, V_3\}$ *has size three, and both,* $N_M(A)$ *and* $Z(K)$, *act transitively on* $\mathcal{V}$.
(f) $V = V_i \oplus V_j$ *for all* $1 \leqslant i < j \leqslant 3$.
(g) $C_{V_i}(A)$ *is a natural* $SL_2(2)$*-module for* $K_2$.

PROOF. Let $\mathbb{K} := End_K(V)$. Since $\mathbb{K}$ contains the image of $Z(K)$ in $End_{\mathbb{F}_2}(V)$, $\mathbb{K}$ is a field of order 4. Put $M_2 := N_M(C_V(A))$.

(a): Note that $K$ has orbits of length 15 and 6 on the 1-dimensional $\mathbb{K}$-subspaces of $V$. Since $K \trianglelefteq M$, $M$ acts on the orbit of length 6. The kernel of this action centralizes $K$ and, since $|\mathbb{K}^\sharp| = 3$, is equal to $Z(K)$. Thus $M/Z(K)$ is isomorphic to a subgroup of $Sym(6)$ containing $Alt(6)(\cong K/Z(K))$. So (a) holds.

(b) follows from Part (e) of the Offender Theorem C.4.

(c): Observe that $N_M(A) = N_M(AZ(K))$. Since $A$ is an elementary abelian subgroup of order 4 with $A \leqslant K$ we conclude that $N_M(A)/Z(K) \cong Sym(4)$ if $M/Z(K) \cong Alt(6)$ and $N_M(A)/Z(K) \cong C_2 \times Sym(4)$ if $M/Z(K) \cong Sym(6)$. Thus $N_M(A)$ a maximal 2-parabolic subgroup of $M$ and $N_K(A) \cong C_3 \times Sym(4)$. As $N_M(A)$ is a maximal subgroup of $M$ and $N_M(A) \leqslant N_M(C_V(A))$ we have $N_M(A) = N_M(C_V(A)) = M_2$.

(d): Since $K_2$ centralizes the $\mathbb{K}$-space $V/C_V(A)$, $K_2$ acts $\mathbb{K}$-linearly on $V$ and so $K_2 \leqslant K$. By (c), $M_2 \cap K = N_M(A) \cap K = N_K(A)$ and $N_K(A) \cong C_3 \times Sym(4)$. As $Z(K)$ acts transitively on $V/C_V(A)$, we get $M_2 \cap K = Z(K) \times K_2$. Thus $K_2 = O^{2'}(M_2 \cap K) \cong Sym(4)$.

(e)-(g): Let $D_2 \in Syl_3(K_2)$ and $D_2^* := N_{K_2}(D_2)$, so $D_2 \cong C_3$, $K_2^* \cong Sym(3)$ and $K_2 = AK_2^*$. Then $D_2$ acts fixed-point freely on $C_V(A)$ and centralizes $V/C_V(A)$. It follows that $V = C_V(D_2) \oplus C_V(A)$ and $C_V(D_2) = C_V(D_2^*)$.

Let $v_1, v_2, v_3$ be the three nontrivial elements of $C_V(D_2^*)$ and define $V_i := \langle v_i^{K_2}\rangle$. Since $K_2 = AD_2^*$, we get $V_i = \langle v_i^A\rangle = \langle v_i\rangle[v_i, A]$. Note that $C_V(A) = C_V(a)$ for all $1 \neq a \in A$. Thus $C_A(v_i) = 1$ and $|v_i^A| = 4$. Since $A$ act quadratically on $V$, this gives $|[v_i, A]| = 4$, and so $V_i$ is an $K_2$-submodule of order 8.

Let $1 \leqslant i < j \leqslant 3$. Then $\langle v_i, v_j\rangle C_V(A) = V$ and so $[v_i, A] + [v_j, A] = [V, A] = C_V(A)$. Hence $V = V_i + V_j = V_i \oplus V_j$ and $C_{V_i}(A) = [v_i, A]$ as order 4. As $D_2$ acts fixed-point freely on $C_V(A)$, this shows that $C_{V_i}(A)$ is a natural $SL_2(2)$-module for $K_2$.

Let $U$ be any $K_2$-submodule of $V$ of order 8. Then $C_U(D_2) \neq 0$. Thus $v_i \in U$ for some $1 \leqslant i \leqslant 3$ and so $V_i = \langle v_i^{K_2}\rangle \leqslant U$ and $U = V_i$. It follows that $\mathcal{V} = \{V_1, V_2, V_3\}$. Observe that $N_M(A)$ normalizes $K_2$ and so acts on $\mathcal{V}$. In particular, $Z(K)$ acts on $\mathcal{V}$ since $Z(K) \leqslant N_M(A)$. As $Z(K)$ does not normalize any of the $V_i$ and $|\mathcal{V}| = 3$, we conclude that that $Z(K)$ acts transitively on $\mathcal{V}$. □

## C.2. $H^1$- and $H^2$-Results

LEMMA C.17 (Gaschütz). *Let $T \in Syl_p(H)$, let $V$ be an $\mathbb{F}_pH$ module, and let $W$ an $\mathbb{F}_pH$-submodule of $V$ with $[V, O^p(H)] \leqslant W$. Then $C_V(T)+W = C_V(H)+W$. In particular, if $C_V(H) = 0$, then $C_V(T) \leqslant W$.*

PROOF. Note that $H = O^p(H)T$. Since $[V, O^p(H)] \leqslant W$, we conclude that $[C_V(T), H] \leqslant W$. Thus, $Y := C_V(T) + W$ is an $H$-submodule of $V$ and $[Y, H] \leqslant W$.

Let $X := Y \rtimes H$ be the semidirect product of $Y$ with $H$ and let $Y_0$ be a complement to $C_W(T)$ in $C_V(T)$. Then $Y_0T$ is a complement to $W$ in $YT$. Note that $YT$ is a Sylow $p$-subgroup of $X$ and so Gaschütz' Theorem [**KS**, 3.3.2] gives a complement $X_0$ to $W$ in $X$. Then $X = X_0W$ and since $W \leqslant Y$, $Y = (Y \cap X_0)W$. Hence $Y \cap X_0$ is an $H$-invariant complement to $W$ in $Y$. Since $[Y, H] \leqslant W$ we get $[Y \cap X_0, H] \leqslant (Y \cap X_0) \cap W = 0$ and so $Y \cap X_0 \leqslant C_Y(H)$. Hence $Y = (Y \cap X_0) + W \leqslant C_Y(H) + W$. As $C_V(H) \leqslant C_V(T) \leqslant Y$ this gives $Y = C_V(H) + W$. □

THEOREM C.18 ([**MS5**, 6.1]). *Let $H$ be a finite group, $V$ an $\mathbb{F}_pH$-module, and $\mathbb{K} := End_H(V)$. Table 1 lists the dimension $d := \dim_{\mathbb{K}}(H^1(H, V))$ for various pairs $(H, V)$.*

LEMMA C.19. *Let $V$ be an $\mathbb{F}_pH$-module, and let $K_1$ and $K_2$ be subgroups of $H$. Suppose that*

(i) *$[K_1, K_2] = 1$,*
(ii) *$K_2$ has no central composition factor on $[V, K_1]$,*
(iii) *$C_V(K_1) = 0$.*

TABLE 1. H1 for common modules

| $H$ | $p$ | $V$ | Conditions | $d$ |
|---|---|---|---|---|
| $\Omega_n^\epsilon(p^k), n \geqslant 3$ | $p$ | $V_{nat}^*$ | $n = 3, p^k = 2$ | 1 |
| " | " | " | $n = 3, p^k = 5$ | 1 |
| " | " | " | $n = 4, \epsilon = -, p^k = 3$ | 2 |
| " | " | " | $n = 5, p^k = 3$ | 1 |
| " | " | " | $n = 6, \epsilon = +, p^k = 2$ | 1 |
| " | " | " | all others | 0 |
| $Sp_{2n}(p^k)$ | $p$ | $V_{nat}$ | $p = 2, (2n, p^k) \neq (2, 2)$ | 1 |
| " | " | " | all others | 0 |
| $SL_n(p^k)$ | $p$ | $V_{nat}$ | $n = 2, p = 2, k > 1$ | 1 |
| " | " | " | $n = 3, p = 2, k = 1$ | 1 |
| " | " | " | all others | 0 |
| $SU_n(p^k), n \geqslant 3$ | $p$ | $V_{nat}$ | $n = 4, p^k = 2$ | 1 |
| " | " | " | all others | 0 |
| $G_2(2^k)'$ | 2 | $\mathbb{K}^6$ | – | 1 |
| $G_2(p^k)'$ | $p \neq 2$ | $\mathbb{K}^7$ | – | 0 |
| ${}^3D_4(p^k)$ | $p$ | $\mathbb{K}^8$ | – | 0 |
| $Spin_n^\epsilon(p^k)$ | $p$ | (Half)-Spin | $n \geqslant 7$ | 0 |
| $3{\cdot}Alt(6)$ | 2 | $\mathbb{K}^3$ | – | 0 |
| $Alt(n), n \geqslant 5$ | 2 | $V_{nat}$ | $n$ even | 1 |
| " | " | " | $n$ odd | 0 |
| $SL_n(p^k), n \geqslant 5$ | $p$ | $\Lambda^2(V_{nat})$ | – | 0 |
| $SL_n(p^k), n \geqslant 3$ | odd | $Sym^2(V_{nat})$ | – | 0 |
| $SL_n(p^{2k}), n \geqslant 3$ | $p$ | $V_{nat} \otimes V_{nat}^{p^k}$ | $n = 3, p^{2k} = 4$ | 2 |
| " | " | " | all others | 0 |
| $E_6(p^k)$ | $p$ | $\mathbb{K}^{27}$ | – | 0 |
| $Mat_n, 22 \leqslant n \leqslant 24$ | 2 | Todd | $n = 24$ | 1 |
| " | " | " | $n = 22, 23$ | 0 |
| $Mat_n, 22 \leqslant n \leqslant 24$ | 2 | Golay | $n = 22$ | 1 |
| $Mat_n, 22 \leqslant n \leqslant 24$ | 2 | Golay | $n = 23, 24$ | 0 |
| $3.Mat_{22}$ | 2 | $\mathbb{F}_4^6$ | – | 0 |
| $Mat_{11}$ | 3 | Todd | – | 0 |
| $Mat_{11}$ | 3 | Golay | – | 1 |
| $2.Mat_{12}$ | 3 | Todd | – | 0 |
| $2.Mat_{12}$ | 3 | Golay | – | 0 |

*Then $K_2$ has no central composition factor on $V$. In particular, $V = [V, K_2]$ and $C_V(K_2) = 1$.*

PROOF. Let $g \in K_1$. Since $K_2$ centralizes $g$, $V/C_V(g) \cong [V, g]$ as an $K_2$-module. Since $[V, g] \leqslant [V, K_1]$, (ii) implies that $K_2$ has no central composition factor on $V/C_V(g)$. As $0 = C_V(K_1) = \bigcap_{g \in K_1} C_V(g)$, we conclude that $K_2$ has no central composition factor on $V$. □

LEMMA C.20. *Let $V$ be an $\mathbb{F}_pH$-module. Suppose that $I := [V, O^p(H)]$ is a natural $Sp_{2m}(q)$- or $Sp_{2m}(q)'$-module for $H$ with $m \geqslant 1$ and $q$ a power of $p$. If $C_V(O^p(H)) = 1$ and $C_H(V) = C_H(I)$, then $[V, D] = [I, D]$ for all $D \leqslant H$.*

PROOF. We may assume that $V$ is a faithful $H$-module and $V \neq I$. Then C.18 shows that $p = 2$ and $V$ is as an $O^2(H)$-module isomorphic to a submodule of the dual of a natural $\Omega_{2m+1}(q)'$-module for $O^2(H)$. In particular, $H^1(O^2(H), I)$ is 1-dimensional over $\mathbb{F}_q$ and since $H$ acts $\mathbb{F}_q$-linearly on $I$, $[V, H] \leqslant I$. So we can choose a natural $\Omega_{2m+1}(q)$ or $\Omega_{2m+1}(q)'$-module $U$ for $H$ with $V \leqslant U^*$, where $U^*$ is the $\mathbb{F}_q$-dual of $U$. Since $[U^*, H] = [U^*, H, H]$, $I = [V, H] = [U^*, H]$.

By B.6(b) $C_{U/U^\perp}(D) = C_U(D)/U^\perp$ for all $D \leqslant H$. In particular, $U^\perp = C_U(H)$. So if $0 \leqslant U_1 \leqslant U_2 \leqslant U$ such that $H$ centralizes $U_1$ and $D$ centralizes $U_2/U_1$, then $D$ centralizes $U_2$. For $U^*$ this means if $U^* \geqslant W_1 \geqslant W_2 \geqslant 0$ such that $H$ centralizes $U^*/W_1$ and $D$ centralizes $W_1/W_2$, then $D$ centralizes $U^*/W_2$. Hence $[U^*, H, D] = [U^*, D]$. Thus

$$[I, D] \leqslant [V, D] \leqslant [U^*, D] = [U^*, H, D] = [I, D],$$

and $[V, D] = [I, D]$. □

Lemma C.21. *Let $H$ be a finite group and $V$ a natural $SL_n(q)$-module for $H$, $q$ a power of $p$ and $n \geqslant 2$, and let $V_1$ be an $\mathbb{F}_q$-hyperplane of $V$. Suppose that $C_H(V) \leqslant Z(H)$ and that there exists a $N_H(V_1) \cap C_H(V/V_1)$-invariant complement to $C_H(V)$ in $C_H(V_1)$. Then there exists a complement $K$ to $C_H(V)$ in $H$. In particular, if $C_H(V) \leqslant H'$, then $C_H(V) = 1$.*

Proof. Put $Z := C_H(V)$, $Z_0 := O_{p'}(Z)$ and $H_1 := N_H(V_1) \cap C_H(V/V_1)$, and let $B$ be an $H_1$-invariant complement to $C_H(V)$ in $C_H(V_1)$ . Then $H/Z \cong SL_n(q)$ and $Z \leqslant Z(H)$. By [**Gr1**] the Schur Multiplier of $SL_n(q)$ is a $p$-group, so $H' \cap Z$ is a $p$-group and $H' \cap Z_0 = 1$.

Suppose that $n = 2$. Then $BZ/Z \in Syl_p(H/Z)$, and by Gaschütz' Theorem [**KS**, 3.3.2] there exists a complement $L/Z_0$ to $Z/Z_0$ in $H/Z_0$. It follows that $H = LZ$, $H' = L'$ and $H' \cap Z \leqslant Z_0$, so $H' \cap Z = 1$. If $q \geqslant 4$, then $H/Z$ is perfect and we can choose $K = H'$. If $q \leqslant 3$, then $H'$ is a $p'$-group, $|B| = p$ and we can choose $K = H'B$.

Suppose now that $n \geqslant 3$. Then $H = H'Z$ and $H'$ is perfect. Note that $V_1$ is a natural $SL_{n-1}(q)$-module for $H_1$, $C_{H_1}(V_1) = C_H(V_1) = Z \times B$, and $B \cong C_H(V_1)/Z$ is isomorphic to $V_1$ as an $\mathbb{F}_pH_1$-module. In particular, $B = [B, H_1] \leqslant H'$ and replacing $H$ be $H'$ we may assume that $H$ is perfect. Thus $Z$ is a $p$-group.

Let $X$ be a 1-subspace of $V_1$ and $\widehat{V}$ a hyperplane of $V$ with $V = X \oplus \widehat{V}$. Define

$$\widehat{H} := C_H(X) \cap N_H(\widehat{V}), \quad \widehat{V}_1 := V_1 \cap \widehat{V}, \quad \widehat{H}_1 := N_{\widehat{H}}(\widehat{V}_1) \cap C_{\widehat{H}}(\widehat{V}/\widehat{V}_1), \quad \widehat{B} := B \cap C_{\widehat{H}}(\widehat{V}_1).$$

Then $\widehat{V}$ is a natural $SL_{n-1}(q)$-module for $\widehat{H}$, $\widehat{V}_1$ is a hyperplane of $\widehat{V}$ and $V_1 = X \oplus \widehat{V}_1$. Thus $\widehat{H}_1 \leqslant N_H(V_1) \cap C_H(V/V_1) = H_1$.

Since $C_H(V) \leqslant \widehat{H}$ and $C_{\widehat{H}}(\widehat{V}) \leqslant C_H(X \oplus \widehat{V}) = C_H(V)$ we have $C_H(V) = C_{\widehat{H}}(\widehat{V})$. Also $C_{\widehat{H}}(\widehat{V}_1) \leqslant C_H(X \oplus \widehat{V}_1) = C_H(V_1)$ and so

$$C_H(V) = C_{\widehat{H}}(\widehat{V}) \leqslant C_{\widehat{H}}(\widehat{V}_1) \leqslant C_H(V_1) = C_H(V) \times B.$$

Thus $\widehat{B} = B \cap C_{\widehat{H}}(\widehat{V}_1)$ is a complement to $C_H(V) = C_{\widehat{H}}(\widehat{V})$ in $C_{\widehat{H}}(\widehat{V}_1)$. Since $\widehat{H}_1 \leqslant H_1 \cap \widehat{H}$, $\widehat{H}_1$ normalizes $\widehat{B}$. Recall that $2 \leqslant n-1$, so by induction there exists a complement $\widehat{K}$ to $C_{\widehat{H}}(\widehat{V})$ in $\widehat{H}$. Then $\widehat{K} \cong SL_{n-1}(q)$ acts faithfully on $\widehat{V}$.

Pick $g \in H$ with $\widehat{V}^g = V_1$. Then $V = X^g \oplus V_1$, $\widehat{K}^g$ normalizes $V_1$, and $\widehat{K}^g$ centralizes $X^g$ and $V/V_1$. So $\widehat{K}^g \leqslant H_1$, $H_1 = C_H(V_1)\widehat{K}^g$ and $\widehat{K}^g \cap C_H(V_1) = 1$. Since $B$ is a complement to $C_H(V)$ in $C_H(V_1)$, and $\widehat{K}^g$ normalizes $B$ we conclude that $BK^g$ is a complement to $C_H(V)$ in $H_1$.

Since $C_H(V)$ is an abelian $p$-group and $H_1$ contains a Sylow $p$-subgroup of $H$, Gaschütz' Theorem shows that there exists a complement $K$ to $C_H(V)$ in $H$.

If $K$ is any complement to $Z$ in $H$, then $H = KZ$ and $K' = H'$, so $H' \cap Z = 1$. In particular $Z = 1$ if $Z \leqslant H'$. □

Theorem C.22 ([**MS5**, 8.4]). *Let $M$ be a finite $\mathcal{CK}$-group with $O_p(M) = 1$ and $V$ a faithful $\mathbb{F}_pM$-module. Suppose that*

(i) *$M = J_M(V)$ and there exists a unique $J_M(V)$-component $K$,*
(ii) *$C_V(K) \leqslant [V, K]$ and either $C_V(K) \neq 0$ or $V \neq [V, K]$.*

*Let $A \leqslant M$ be a best offender on $V$ and put $W := [V, K]$ and $\overline{V} := V/C_V(K)$. Then $p = 2$, and one of the following holds:*

(a) *$M = K \cong SL_3(2)$, $V = W$, $|C_V(K)| = 2$, $\overline{V}$ is a natural $SL_3(2)$-module, $|A| = 4$, $[\overline{V}, A]| = 2$ and $C_V(A) = [V, A]$ has order 4.*

(b) $M = K \cong SL_3(2)$, $|V/W| = 2$, $C_V(K) = 0$, $W$ *is a natural* $SL_3(2)$*-module,* $|A| = 4 = |C_W(A)|$ *and* $C_V(A) = [V, A] = C_W(A)$.
(c) $M = K \cong SU_4(2)$, $V = W$, $2 \leqslant |C_V(K)| \leqslant 4$, $\overline{V}$ *is a natural* $SU_4(2)$*-module,* $A$ *is the centralizer of a singular* 2*-subspace of* $\overline{V}$, *and* $C_V(A) = [V, A]$.
(d) $M \cong G_2(q)$, $q = 2^k$, $V = W$, $2 \leqslant |C_V(K)| \leqslant q$, $\overline{V}$ *is a natural* $G_2(q)$*-module,* $|A| = q^3$, *and* $C_V(A) = [V, A]$.
(e) $K \cong Alt(2m)$ *and* $M \cong Sym(2m)$ *or* $Alt(2m)$. *For* $\Omega = \{1, 2, \dots, 2m\}$ *let* $N = \{n_\Sigma \mid \Sigma \subseteq \Omega\}$ *be the* $2m$*-dimensional natural permutation module and* $\widetilde{N}$ *be the* $\mathbb{F}_2M$*-module defined by* $\widetilde{N} = N$ *as an* $\mathbb{F}_2$*-space and*

$$n_\Sigma^g = n_{\Sigma^g} \text{ if } |\Sigma| \text{ is even or } g \in Alt(\Omega), \text{ and } n_\Sigma^g = n_{\Sigma^g} + n_\Omega \text{ if } |\Sigma| \text{ is odd and } g \notin Alt(\Omega).$$

*Then one of the following holds, where* $t_1, t_2, \dots, t_m$ *is a maximal set of commuting transpositions:*
  (1) $M = Sym(n)$, $V$ *is isomorphic to* $N$ *or* $N/C_N(K)$, *and* $A = \langle t_1, t_2, \dots, t_k\rangle$ *for some* $1 \leqslant k \leqslant m$.
  (2) $M = Sym(n)$, $V \cong \widetilde{N}$ *and* $A = \langle t_1, t_2, \dots, t_m\rangle$.
  (3) $V \cong [N, K]$ *and* $A$ *fulfills one of the cases (h:1) – (h:3) of Theorem C.4.*
(f) $M = K \cong Sp_{2m}(q)$, $m \geqslant 1$, $q = 2^k$, $(m, q) \neq (1, 2), (2, 2)$[5], *and* $\overline{W}$ *is the direct sum of* $r$ *natural* $Sp_{2n}(q)$*-modules.*[6] *Moreover, the following hold:*
  (a) $2r \leqslant m + 1$, *and if* $V \neq W$ *then* $m > 1$ *and* $2r < m + 1$.
  (b) *Let* $X$ *be the* $2m+2$*-dimensional* $\mathbb{F}_qM$*-module obtained from the embedding* $Sp_{2m}(q) \cong \Omega_{2m+1}(q) \leqslant \Omega^{\pm}_{2m+2}(q)$. *Then* $V$ *is isomorphic to an* $\mathbb{F}_pM$*-section of* $X^r$.

## C.3. $Q!$-Module Theorems

In this section $H$ is a finite group, $Q$ is a $p$-subgroup of $H$, and $V$ is a finite $Q!$-module for $\mathbb{F}_pH$ with respect to $Q$. We again use the $^\circ$-notion, so for $L \leqslant H$,

$$L^\circ = \langle P \in Q^H \mid P \leqslant L\rangle \qquad \text{and} \qquad L_\circ = O^p(L^\circ).$$

Theorem C.23 ([**MS6**, 4.5]). *Let* $O_p(H) = 1$ *and* $V$ *be a faithful* $Q!$*-module for* $H$ *with respect to* $Q$. *Suppose that one the following holds.*
(i) $F^*(H) \cong Alt(n)$, $n \geqslant 5$, *and* $[V, H]$ *is a natural* $\mathbb{F}_pAlt(n)$*-module for* $F^*(H)$, *or*
(ii) $H \cong Alt(7)$ *and* $|[V, H]| = 2^4$.

*Then (i) holds, and either* $n = p$ *or* $(n, p)$ *is one of* $(5, 2), (6, 2), (8, 2), (6, 3)$.

Theorem C.24 (**Q!FF-Module Theorem,** [**MS6**, 4.6]). *Let* $H$ *be a finite group with* $O_p(H) = 1$ *and* $Q$ *be a* $p$*-subgroup of* $H$, *and let* $V$ *be a faithful* $Q!$*-module for* $H$. *Put* $H^\circ := \langle Q^H\rangle$ *and* $J := J_H(V)$. *Suppose that there exists an offender* $Y$ *in* $H$ *such that* $[H^\circ, Y] \neq 1$ *and that one of the following holds:*
(i) $Y$ *is quadratic on* $V$.
(ii) $Y$ *is a best offender on* $V$.
(iii) $C_Y([V, Y]) \neq 1$.
(iv) $C_Y(H^\circ) = 1$.

*Then one of the following holds:*
(1) *There exists an* $H$*-invariant set* $\mathcal{K}$ *of subgroups of* $H$ *such that:*
  (a) *For all* $K \in \mathcal{K}$, $K \cong SL_2(q)$ *and* $[V, K]$ *is a natural module for* $K$,
  (b) $J = \bigtimes_{K\in\mathcal{K}} K$ *and* $V = \bigoplus_{K\in\mathcal{K}}[V, K]$,
  (c) $Q$ *acts transitively on* $\mathcal{K}$,
  (d) $H^\circ = O^p(J)Q$.
(2) *Put* $R := F^*(J)$. *Then*
  (a) $R$ *is quasisimple,* $R \leqslant H^\circ$, *and either* $J = R$ *or* $p = 2$ *and* $J \cong O^{\pm}_{2n}(q)$, $Sp_4(2)$ *or* $G_2(2)$.

[5]The case $K \cong Sp_4(2)' \cong Alt(6)$ is covered in (e), while the case $M = Sp_2(2)$ does not occur

[6]Observe that for $m = 1$, $Sp_2(q) \cong SL_2(q)$ and a natural $Sp_2(q)$-module is also a natural $SL_2(q)$-module.

(b) $C_V(R) = 0$, $[V, R]$ *is a semisimple* $J$*-module, and* $H$ *acts faithfully on* $[V, R]$.

(c) *Put* $J^0 := J \cap H^\circ$. *Then one of the following holds:*

(1) (a) $R = J^0 \cong SL_n(q)$, $n \geqslant 3$, $Sp_{2n}(q)$, $n \geqslant 3$, $SU_n(q)$, $n \geqslant 8$, *or* $\Omega_n^\pm(q)$, $n \geqslant 10$.

(b) $[V, R]$ *is the direct sum of at least two isomorphic natural modules for* $R$.

(c) $H^\circ = RC_{H^\circ}(R)$.

(d) *If* $V \neq [V, R]$ *then* $R \cong Sp_{2n}(q)$, $p = 2$, *and* $n \geqslant 4$.

(2) (a) $[V, R]$ *is a simple* $R$*-module.*

(b) *Either* $H^\circ = R = J^0$ *or* $H^\circ \cong Sp_4(2)$, $3{\cdot}Sym(6)$, $SU_4(q).2$ ($\cong O_6^-(q)$ *and* $[V, R]$ *the natural* $SU_4(q)$*-module), or* $G_2(2)$.

(c) *One of the cases C.3 (1) - (9), (12) applies to* $(J, [V, R])$, *with* $n \geqslant 3$ *in case (1),* $n \geqslant 2$ *in case (2), and* $n = 6$ *in case (12).*

(3) $p = 2$, $J = R \cong SL_4(q)$, $H^\circ/R$ *has order two and induces a graph automorphism on* $R$, *and* $V$ *is the direct sum of two non-isomorphic natural modules.*

Proof. This is [**MS6**, 4.6], except that in (2:c:2:c) we added the assumption $n \geqslant 3$ to case (1) and the assumption $n \geqslant 2$ to case (2) of C.3. Note here if $n = 2$ in case (1) or $n = 1$ in case (2) then $[V, R]$ is a natural $SL_2(q)$-module for $J$, and so by C.22, $V = [V, R]$. Hence, these cases are already covered by (1).[7] □

Theorem C.25. *Let* $H$ *be a finite group with* $O_p(H) = 1$, *and let* $V$ *be a faithful* $Q!$*-module for* $H$ *with respect to* $Q$. *Suppose that there exists* $1 \neq W \leqslant H$ *such that*

(i) $W$ *is a strong offender on* $V$*; and*

(ii) $[X, W] = [V, W]$ *for all* $X \leqslant V$ *with* $|X/C_X(W)| > 2$.

*Put* $H^\circ := \langle Q^H \rangle$, $K^* := \langle W^H \rangle$, $K := \langle W^{K^*} \rangle$ *and* $\mathcal{K} := K^H$. *Then*

$$K^* = \bigtimes_{R \in \mathcal{K}} R, \quad [V, K^*] = \bigoplus_{R \in \mathcal{K}} [V, R], \text{ and } K = \langle W^K \rangle \text{ is the subnormal closure of } W \text{ in } H.$$

*Moreover, one of the following holds:*

(1) (a) $K \trianglelefteq H$, $K'$ *is quasisimple,* $H^\circ = K'Q$ *and* $C_V(K) = 0$.

(b) *One of the following holds:*

(1) $K = K' = H^\circ \cong SL_n(q)$, $n \geqslant 3$.

(2) $K = K' = H^\circ \cong Sp_{2n}(q)$, $n \geqslant 2$, $(n, q) \neq (2, 2)$.

(3) $p = 2$, $K \leqslant H^\circ$ *or* $H^\circ \leqslant K$, $K \cong Sp_4(2)'$ *or* $Sp_4(2)$, *and* $H^\circ \cong Sp_4(2)'$ *or* $Sp_4(2)$.

(4) $p = 2$, $K = K' \leqslant H^\circ$, $K \cong 3{\cdot}Alt(6)$ *and* $H^\circ \cong 3{\cdot}Alt(6)$ *or* $3{\cdot}Sym(6)$.

(5) $p = 2$, $K \cong O_{2n}^\epsilon(2)$, $H^\circ = K' \cong \Omega_{2n}^\epsilon(2)$, $n \geqslant 2$ *and* $(n, \epsilon) \neq (2, +)$, *and* $|W| = |V/C_V(W)| = 2$.

(c) $[V, K]$ *is a corresponding natural module.*

(2) (a) $Q$ *acts transitively on* $\mathcal{K}$, $H^\circ = O^p(K^*)Q$, *and* $V = [V, K^*]$.

(b) $K \cong SL_2(q)$, *and* $[V, K]$ *is the corresponding natural module.*

(3) (a) $p = 2$, $K \cong SL_n(2)$, $n \geqslant 3$, $V = [V, K]$ *is the direct sum of two isomorphic natural modules for* $K$, *and* $|V/C_V(W)| = 4$.

(b) $K \trianglelefteq H$, $K \leqslant H^\circ$, *and* $H^\circ \cong SL_n(2)$ *or* $SL_n(2) \times SL_2(2)$.

(4) (a) $p = 2$, $K \cong SL_n(2)$, $n \geqslant 3$, $V = C_V(K^*) \oplus [V, K^*]$, $[V, K]$ *is the direct sum of two isomorphic natural modules for* $K$, *and* $|V/C_V(W)| = 4$.

(b) $K^* \trianglelefteq H$, $[K^*, H^\circ] = 1$ *and* $H^\circ \cong SL_2(2)$.

Proof. This is [**MS6**, 4.7] with a couple of additions.

- In case (1:b:5) with $K \cong O_{2n}^\epsilon(2)$: We may assume $n \geqslant 2$ and $(n, \epsilon) \neq (2, +)$. Indeed $|O_2^+(2)| = 2$, so since $O_p(H) = 1$, this case does not occur; and $O_2^-(2) \cong SL_2(2)$, so this

[7] We made these changes for easier reference and to point out more clearly that $H^\circ$ does not have to be contained in $J$ in the $SL_2(q)$-case.

case is already covered by case (2) (with $q = 2$ and $|\mathcal{K}| = 1$). The $O_4^+(2)$-case does not occur since $K = \langle W^{K^*} \rangle$, but $O_4^+(2)$ is not generated by transvections.

- From the structure of $K$ as given in (1)–(4) , $K = \langle W^K \rangle$ and so since $K = \langle W^{\langle W^H \rangle} \rangle$, $K$ is the subnormal closure of $W$ in $H$.

$\square$

Corollary C.26. *Let $H$ be a finite group with $O_p(H) = 1$, and let $V$ be a faithful $Q$!-module for $H$ with respect to $Q$. Suppose that there exists $1 \neq W \trianglelefteq Q$ such that*

(i) *$W$ is a strong offender on $V$; and*
(ii) *$[X, W] = [V, W]$ for all $X \leqslant V$ with $|X/C_X(W)| > 2$.*

*Put $H^\circ := \langle Q^H \rangle$ and $K := \langle W^{\langle W^H \rangle} \rangle$. Then*

(a) *$K = \langle W^H \rangle = \langle W^K \rangle$ is the subnormal closure of $W$ in $H$ and $C_V(K) = 0$.*
(b) *$K \cong SL_n(q)$ or $Sp_{2n}(q)$, $n \geqslant 2$, $q$ a power of $p$, and $[V, K]$ is a corresponding natural module.*
(c) *Either $H^\circ = K$ or $K \cong SL_2(q)$, $q \neq p$, $H_\circ = K$ and $[V, W, Q] \neq 1$.*
(d) *Either $V = [V, K]$ or $K \cong Sp_{2n}(2)$, $n \geqslant 2$ and $|V/[V, K]| = 2$.*

Proof. Note first that we can apply C.25.

Let $S \in Syl_p(H)$ with $Q \leqslant S$. Then by $Q$!, $N_H(C_V(S)) \leqslant N_H(Q)$. It follows that $W \leqslant Q \leqslant O_p(N_H(C_V(S)))$ and so we can apply the Point-Stabilizer Theorems C.8 and C.9. In particular, $[V, K]C_V(K)/C_V(K)$ is a simple $K$-module, and so cases C.25(3) and (4) do not occur.

Suppose that C.25(2) holds. Then $K \cong SL_2(q)$, $Q$ acts transitively on $\mathcal{K} := \overline{K}^M$, $[V, K]$ is a natural $SL_2(q)$-module and $V = [V, \langle W^H \rangle]$. Since $Q$ normalizes $W$, $Q$ normalizes $K = \langle W^{\langle W^K \rangle} \rangle$. Hence $\mathcal{K} = \{K\}$ and $V = [V, K]$. Suppose $Q$ acts $\mathbb{F}_q$-linearly on $V$. Then $Q \leqslant K$, $K = \langle Q^K \rangle = H^\circ$ and the corollary holds.

So suppose that $Q$ does not act $\mathbb{F}_q$-linearly on $V$. Then $q \neq p$, so $K$ is quasisimple and $K = [K, Q] \leqslant H^\circ$. Also $[V, W]$ is a non-trivial $\mathbb{F}_q$ subspace of $V$ and hence $[V, W, Q] \neq 1$. As $K$ acts transitively on $V$, $Q$! gives $H^\circ = \langle Q^K \rangle = KQ$ (see A.50(d) )and so $H_\circ = O^p(KQ) = K$. So again the corollary holds.

Suppose that C.25(1) holds. Since none of $Sp_4(2)'$, $O^\epsilon_{2m}(2)$ and $3^{\cdot}Alt(6)$ appear as a possibility for $K$ in the Point-Stabilizer Theorem C.8, we conclude that $K \cong SL_n(q)$, $n \geqslant 3$, or $Sp_{2n}(q)$, $n \geqslant 2$, and $[V, K]$ is a corresponding natural module. Moreover, $H^\circ \leqslant K$ and so $K = H^\circ$. It remains to verify (d).

Put $U := [V, K]$. By $Q$!, $Q \trianglelefteq C_H(C_U(S))$. Thus $W \leqslant O_p(N_H(C_U(S))$, and we can apply the Point-Stabilizer Theorem C.8(d) also to $W$ and $U$. Hence $|U/C_U(W)| \geqslant |W|$ and so $V = C_V(W) + U$.

Suppose that $V \neq U$. Then by C.22 either $K \cong SL_3(2)$ and $C_V(W) \leqslant U$ or $K \cong Sp_{2n}(q)$, $n \geqslant 2$, $p = 2$, and $V$ is isomorphic to a submodule of the dual of a natural $\Omega_{2n+1}(q)$-module. The first case contradicts $V = C_V(W) + U$. In the second case, let $v \in C_V(W) \backslash U$. Then $C_K(v) \cong O^\epsilon_{2n}(q)$. Since $W \leqslant C_K(v)$ and $W$ is a strong offender on $U$, the Strong Offender Theorem C.6 shows that $|W| = 2 = q$. So (d) holds. $\square$

Theorem C.27. *Let $H$ be a finite group and let $V$ be a faithful $p$-reduced $Q$!-module for $H$ with respect to $Q$. Let $1 \neq A \leqslant H$ be a strong dual offender on $V$. Then one of the following holds:*

(1) (a) *$H^\circ \cong SL_n(q)$, $n \geqslant 3$, and $[V, H^\circ]$ is a corresponding natural module for $H^\circ$.*
    (b) *If $V \neq [V, H^\circ]$ then $H^\circ \cong SL_3(2)$ and $|V/[V, H^\circ]| = 2$.*
    (c) *$\langle A^H \rangle = H^\circ$.*
(2) (a) *$H^\circ \cong Sp_{2n}(q)$, $n \geqslant 2$, or $Sp_4(q)'$ (and $q = 2$), and $[Y, H^\circ]$ is the corresponding natural module for $H^\circ$.*
    (b) *If $V \neq [V, H^\circ]$, then $p = 2$ and $|H/[Y, H^\circ]| \leqslant q$.*
    (c) *One of the following holds:*
        (1) *$H^\circ = \langle A^H \rangle$.*
        (2) *$\langle A^H \rangle \leqslant H^\circ$, $\langle A^H \rangle \cong Sp_4(2)'$ and $H^\circ \cong Sp_4(2)$.*

(3) *$H^\circ \leqslant \langle A^H\rangle$, $\langle A^H\rangle \cong Sp_4(2)$ and $H^\circ \cong Sp_4(2)'$.*

(3) (a) *There exists a unique $H$-invariant set $\mathcal{K}$ of subgroups of $\overline{M}$ such that $V$ is a natural $SL_2(q)$-wreath product module for $H$ with respect to $\mathcal{K}$.*
(b) *$H^\circ = O^p(\langle\mathcal{K}\rangle)Q$ and $Q$ acts transitively on $\mathcal{K}$.*
(c) *$A \leqslant K$ for some $K \in \mathcal{K}$.*

(4) (a) *$H \cong O^\epsilon_{2n}(2)$, $H^\circ \cong \Omega^\epsilon_{2n}(2)$, $2n \geqslant 4$ and $(2n,\epsilon) \neq (4,+)$ and $[V,H]$ is a corresponding natural module.*
(b) *If $V \neq [V,H]$, then $H \cong O^+_6(2)$ and $|V/[V,H]| = 2$.*
(c) *$|A| = 2$ and $H = \langle A^H\rangle$.*

Proof. Put $K^* := \langle A^H\rangle$, $K := \langle A^{K^*}\rangle$ and $\mathcal{K} := K^H$. Since $H$ is faithful and $p$-reduced, $O_p(H) = 1$. Thus we can apply [**MS6**, 4.8] and conclude that one of the following holds:

(A) (a) $K \trianglelefteq H$, $H^\circ = \langle Q^K\rangle$ and $C_V(K) = 0$.
(b) $K \cong SL_n(q)$, $n \geqslant 3$, $Sp_{2n}(q)$, $Alt(6)$, or $O^\epsilon_{2n}(2)$, $q$ a power of $p$ , $p = 2$ in the last two cases; and $[V,K]$ is a corresponding natural module.
(c) Either $H^\circ \leqslant K$ or $K \cong Sp_4(2)'$ and $H^\circ \cong Sp_4(2)$.
(d) If $K \cong O^\epsilon_{2n}(2)$, then $|W| = 2$.

(B) (a) $Q$ acts transitively on $\mathcal{K}$ and $H^\circ \leqslant \langle\mathcal{K}\rangle Q$
(b) $V = \bigoplus_{R\in\mathcal{K}}[V,R]$, $K \cong SL_2(q)$, and $[V,K]$ is a natural $SL_2(q)$-module for $K$.

Suppose first that (B) holds. Then $V$ is a natural $SL_2(q)$-wreath product module for $H$ with respect $\mathcal{K}$. By A.27(c) $\mathcal{K}$ is uniquely determined by this property.

Since $Q$ acts transitively on $\mathcal{K}$ and $K \cong SL_2(q)$, we get $O^p(\langle\mathcal{K}\rangle) \leqslant \langle Q^H\rangle$. As $H^\circ \leqslant \langle\mathcal{K}\rangle Q$, this gives $O^p(H^\circ) = O^p(\langle\mathcal{K}\rangle)$. By A.52(a) we have $H^\circ = \langle Q^{H^\circ}\rangle$ and we conclude that $H^\circ = O^p(H^\circ Q) = O^p(\langle\mathcal{K}\rangle)Q$. Thus (3) holds.

Suppose next that (A) holds.

Assume that $K \cong SL_n(q)$, $n \geqslant 3$, and $[V,K]$ is a corresponding natural module. Then $H^\circ \leqslant K$ and since $SL_n(q)$ is quasisimple (or by B.37), $H = K$. By C.22 either $V = [V,K]$ or $K \cong SL_3(2)$ and $|V/[V,K]| = 2$. Thus (1) holds.

Assume that $K \cong Sp_{2n}(q)$ and $[V,K]$ is a corresponding natural module. Suppose that $n = 1$. Then by C.22, $V = [V,K]$ and the already treated case (B) shows that (3) holds. So suppose that $n \geqslant 2$. Then by C.22 either $V = [V,K]$ or $p = 2$ and $|V/[V,K]| \leqslant q$. Also B.37 shows that either $H^\circ = K$ or $K \cong Sp_4(2)'$ and $H^\circ \cong Sp_4(2)$. Thus (2) holds.

Assume that $K \cong Alt(6)$ and $[V,K]$ is a corresponding natural module, that is $K \cong Sp_4(2)'$ and $[V,K]$ is a corresponding natural module. By (A:b) $H^\circ \leqslant K$ or $H^\circ \cong Sp_4(2)$, and since $K$ is simple, we get $H^\circ \cong Sp_4(2)'$ or $Sp_4(2)$. By C.22, $|V/[V,K]| \leqslant 2$ and again (2) holds.

Assume that $K \cong O^-_{2n}(2)$. Since $O_2(H) = 1$, $K \not\cong O^+_2(2)$. If $K \cong O^-_2(2) \cong Sp_2(2)$, then $[V,K]$ is a natural $Sp_2(2)$-module, a case we already have treated. So suppose that $n \geqslant 2$. Then C.22 shows that either $V = [V,K]$ or $K \cong O^+_6(2)$ and $|V/[V,K]| = 2$.

If $K \cong O^+_4(2) \cong SL_2(2) \wr C_2$, the already treated case (B) shows that (3) holds. So we may assume that $(2n,\epsilon) \neq (4,+)$. Then B.37 implies that $H^\circ \cong \Omega^\epsilon_{2n}(2)$ and thus (4) holds. □

## C.4. The Asymmetric Module Theorems

For the definition of a minimal asymmetric module see A.4.

Theorem C.28 ([**MS6**, 5.4]). *Let $H$ be a finite group and $V$ be a faithful simple minimal asymmetric $\mathbb{F}_pH$-module with respect to $A \leqslant B$. Put $L := \langle A^H\rangle$ and $K := F^*(H)$. Then $H = KB$, $K = [K,A] \leqslant L$, $L = KA$, and one of the following holds:*

(1) *$|B| = 2$ and $H = L \cong D_{2r}$, $r$ an odd prime.*
(2) *$|A| = 2$, $L \cong SU_3(2)'$, $B \cong C_4$ or $Q_8$, and $V$ is a natural $SU_3(2)'$-module for $L$.*
(3) *$|B| = 3$, $H = L \cong SL_2(3)$, and $V$ is a natural $SL_2(3)$-module for $L$.*
(4) *$K$ is quasisimple and not a $p'$-group, $H = KB$, $V$ is a simple $\mathbb{F}_pK$-module, and $H$ acts $\mathbb{K}$-linearly on $V$, where $\mathbb{K} = End_K(V)$.*

THEOREM C.29 (**Minimal Asymmetric Module Theorem,** [**MS6**, 5.5]). *Let $H$ be a $\mathcal{CK}$-group, $A \leqslant B \leqslant H$ and $V$ be a faithful simple $\mathbb{F}_pH$-module. Suppose that $V$ is a minimal asymmetric $\mathbb{F}_pM$-module with respect to $A$ and $B$ and that $F^*(H)$ is quasisimple with $p \big| |F^*(H)|$. Then one of the following holds for $L := \langle A^H \rangle$:*

(1) *$L \cong SL_n(q), Sp_{2n}(q), SU_n(q), {}^3D_4(q)$, $Spin_7(q)$, $Spin_8^-(q)$, $G_2(q)'$ or $Sz(q)$, where $q$ is a power of $p$, $V$ is a corresponding natural or spin module for $L$, and $A$ is a long root subgroup of $L$.*

(2) *$L \cong Sym(2^k + 2)$, $k \geqslant 3$, $|A| = 2$, $A$ is generated by a transposition, and $V$ is the corresponding natural module.*

(3) *$L \cong 3^{\cdot}Alt(6)$, $|A| = 2$ and $|V| = 2^6$.*

APPENDIX D

# The Fitting Submodule

Let $H$ be a finite group and $V$ be a finite $\mathbb{F}_pH$-module. In [**MS2**] an $H$-submodule of $V$ was introduced which in some respect is the analogue of the generalized Fitting subgroup of a finite group. In this appendix we will give its definition and derive some properties that have been used in this paper.

Lemma D.1. *The following hold:*

(a) *Suppose that $H/C_H(V)$ is a $p$-group. Then $V$ is not perfect.*
(b) *Suppose that $V$ is a perfect $H$-module. Then $V = [V, O^p(H)]$.*
(c) *Suppose that $V$ is a quasisimple $H$-module. Then $C_V(O^p(H)) = rad_V(H)$.*

Proof. (a): This is an elementary fact about the action of $p$-groups on $p$-groups.

(b): Put $\overline{V} := V/[V, O^p(H)]$ and $\overline{H} := H/O^p(H)$. Then (a) shows that $\overline{V}$ is not perfect. Since $[\overline{V}, \overline{H}] = \overline{V}$, we conclude that $\overline{V} = 0$.

(c): Let $U$ be a maximal $H$-submodule of $V$. Then either $V = U + C_V(O^p(H))$ or $C_V(O^p(H)) \leqslant U$. The first case is impossible, since by (b) $V = [V, O^p(H)]$. Hence $C_V(O^p(H)) \leqslant rad_V(H)$. Since $V/C_V(O^p(H))$ is simple, also $rad_V(H) \leqslant C_V(O^p(H))$. □

## D.1. The Definition of the Fitting Submodule and Results from [MS2]

Definition D.2. Let $S_V(H)$ be the sum of all simple $H$-submodules of $V$ and

$$E_H(V) := C_{F^*(H)}(S_V(H)).$$

Let $L \trianglelefteq H$. Then $V$ is *$L$-quasisimple* for $H$ if $V$ is $p$-reduced for $H$, $V/rad_V(H)$ is a simple $H$-module, $V$ is a perfect $L$-module, and $L$ acts nilpotenly on $rad_V(H)$.

An $H$-submodule $U$ of $V$ is a *component* of $V$ (or $H$-component of $V$), if either $U$ is simple and $[U, F^*(H)] \neq 0$, or $U$ is $E_H(V)$-quasisimple. The sum of all components of $V$ is the *Fitting submodule* $F_V(H)$ of $V$. Put

$$R_V(H) := \sum rad_W(H),$$

where the sum runs over all components $W$ of $V$.

Lemma D.3. *The following hold:*

(a) *Suppose that $V$ is faithful and $p$-reduced. Then $E_H(V)$ is the (possibly empty) direct product of perfect simple groups. In particular, $F(E_H(V)) = 1$ and $E_H(V) \leqslant E(H)$.*
(b) *If $E_H(V) = 1$, then $F_V(H)$ is a semisimple $H$-module.*
(c) *$E_H(V)$ centralizes $R_V(H)$.*

Proof. (a): This is [**MS2**, 2.5d].
(b): Suppose that $E_H(V) = 1$. Then there does not exist any non-trivial $H$-module $U$ with $U = [U, E_H(V)]$. It follows that all $H$-components of $V$ are simple $H$-modules and so $F_V(H)$ is a semisimple $H$-module.
(c): By [**MS2**, 2.5a] $C_{F_V(H)}(E_H(V)) = [S_V(H), F^*(H)] + R_V(H)$, and so (c) holds. □

Lemma D.4. *Let $N \trianglelefteq\trianglelefteq H$. Then the following hold:*

(a) *$S_V(H) \leqslant S_V(N)$.*
(b) *$E_H(V) \cap N = E_N(V)$.*

(c) $F_V(H) \leqslant S_V(N) + F_V(N)$.

Proof. See [**MS2**, 3.1] and [**MS2**, 3.2]. $\square$

The following theorems are the main results of [**MS2**]:

Theorem D.5. *$F_V(H)$ is a $p$-reduced $H$-module, and $R_V(H)$ is a semisimple $F^*(H)$-module. Moreover $R_V(H) = rad_{F_V(H)}(H)$, in particular $F_V(H)/R_V(H)$ is a semisimple $H$-module.*

Theorem D.6. *Suppose that $V$ is a faithful and $p$-reduced $H$-module. Then also $F_V(H)$ and $F_V(H)/R_V(H)$ are faithful and $p$-reduced.*

Lemma D.7. *Suppose that $V$ is a faithful and $p$-reduced $H$-module. Let $N \trianglelefteq\trianglelefteq H$. Then the following statements are equivalent:*

(a) *$F_V(H)$ is a semisimple $N$-module.*
(b) *$E_N(V) = 1$.*
(c) *$N \cap E_H(V) = 1$.*
(d) *$[N, E_H(V)] = 1$.*
(e) *$[F^*(N), E_H(V)] = 1$.*
(f) *$[N, E_N(V)] = 1$.*

Proof. Suppose that $F_V(H)$ is a semisimple $N$-module. Then $F_V(H) \leqslant S_V(N)$, and so $E_N(V) \leqslant C_N(S_V(N)) \leqslant C_H(F_V(H))$. Since $V$ is faithful and $p$-reduced, D.6 shows that $F_V(H)$ is a faithful $H$-module, that is, $C_H(F_V(H)) = 1$. Hence $E_N(V) = 1$.

Suppose that $E_N(V) = 1$. Then by D.3(b) applied to $N$ in place of $H$, $F_V(N)$ is a semisimple $N$-module. By D.4(c), $F_V(H) \leqslant S_V(N) + F_V(N)$. Since submodules of semisimple modules are semisimple we conclude that $F_V(H)$ is a semisimple $N$-module.

We have proved that (a) and (b) are equivalent. By D.4(b), $E_N(V) = N \cap E_H(V)$ and so (b) and (c) are equivalent. By D.3(a) $E_H(V)$ is a direct product of perfect simple groups. Thus 1.16 shows that (c),(d), (e) are equivalent.

In particular, $E_N(V) = 1$ if and only if $[N, E_H(V)] = 1$. This applied with $H = N$ shows that $E_N(V) = 1$ if and only if $[N, E_N(V)] = 1$. So (f) is equivalent to (c). $\square$

## D.2. The Fitting Submodule and Large Subgroups

Lemma D.8. *Suppose that $V$ is a faithful $p$-reduced $Q!$-module for $H$. Then $[H^\circ, E_H(V)] = 1 = H^\circ \cap E_H(V)$, and $F_V(H)$ is a semisimple $H^\circ$-module.*

Proof. Put $S := S_V(H)$ and $E := E_H(V)$. Then $E = C_{F^*(H)}(S)$. Since $S$ is a non-zero $H$-submodule of $V$, A.52(c) gives $C_{H^\circ}(S) \leqslant C_{H^\circ}(H^\circ) = Z(H^\circ)$. Thus $[E, H^\circ] \leqslant E \cap H^\circ \leqslant Z(H^\circ)$. By A.52(b), $C_H(H^\circ/Z(H^\circ)) = C_H(H^\circ)$ and so $[H^\circ, E] = 1$. Hence D.7 shows that $H^\circ \cap E_H(V) = 1$ and that $F_V(H)$ is a semisimple $H^\circ$-module. $\square$

Lemma D.9. *Suppose that $V$ is a faithful $p$-reduced $Q!$-module for $H$. Let $N \trianglelefteq\trianglelefteq H$ and suppose that $F^*(N) \leqslant F(N)F^*(H^\circ)$. Then $F_V(H)$ is a semisimple $N$-module.*

Proof. Put $E := E_H(V)$. By D.8 $[H^\circ, E] = 1$ and by D.3(a), $E \leqslant E(H)$. Since $F(N) \leqslant F(H)$ and $[F(H), E(H)] = 1$, we conclude that $[F(N), E] = 1$. Since $F^*(N) \leqslant F(N)H^\circ$ this gives $[F^*(N), E] = 1$. Thus D.7 shows that $F_V(H)$ is a semisimple $N$-module. $\square$

Lemma D.10. *Suppose that $V$ is a faithful $p$-reduced $Q!$-module for $H$ with respect to $Q$. Then also $F_V(H)$ and $F_V(H)/R_V(H)$ are faithful $p$-reduced $Q!$-modules for $H$ with respect to $Q$.*

PROOF. By D.6 $F_V(H)$ and $F_V(H)/R_V(H)$ are faithful $p$-reduced $H$-modules. Put $I := F_V(H)$ and $R := R_V(H)$. The definition of a $Q!$-modules implies that any submodule of a $Q!$-module is a $Q!$-module, so $I$ is a $Q!$-module for $H$ with respect to $Q$.

Let $1 \neq B \leqslant C_{I/R}(Q)$. By D.9 $I$ is a semisimple $H^\circ$-module and so there exists an $H^\circ$-submodule $I_0$ of $I$ such that $I = I_0 \oplus R$. Hence, there exists a unique $B_0 \leqslant I_0$ with $B = (B_0 + R)/R$. This shows that $N_{H^\circ}(B) = N_{H^\circ}(B_0)$ and $[B_0, Q] \leqslant I_0 \cap R = 0$. Now $Q!$ gives

$$Q \trianglelefteq N_{H^\circ}(B_0) = N_{H^\circ}(B) \trianglelefteq N_H(B).$$

Thus $Q \leqslant O_p(N_H(B))$. Hence $1 \neq C_V(O_p(N_H(B))) \leqslant C_V(Q)$, and $Q!$ implies

$$N_H(B) \leqslant N_H(C_V(O_p(H))) \leqslant N_H(Q).$$

This shows that also $I/R$ is a $Q!$-module for $H$ with respect to $Q$. □

## D.3. The Nearly Quadratic $Q!$ -Module Theorem

THEOREM D.11 (Nearly Quadratic $Q!$-Module Theorem). *Suppose that $Y$ is a faithful $p$-reduced $\mathbb{F}_pQ!$-module for $M$ with respect to $Q$. Put $I := F_Y(M)$ and suppose that there exists an elementary abelian $p$-subgroup $A$ of $M$ such that*

(i) *$A$ acts nearly quadratically but not quadratically on $I$,*
(ii) *$A$ normalizes $Q$, and $Q$ normalizes $A$,*
(iii) *$[Y, A] \leqslant I$.*

*Then one of the following holds:*

(1) *$K := [F^*(M), A]$ is the unique component of $M$, $K \leqslant M^\circ$, $I$ is a simple $K$-module, $I = [Y, KA]$, and $A$ acts $\mathbb{K}$-linearly on $I$, where $\mathbb{K} := End_K(I)$.*
(2) *$M^\circ \cong \Omega_3(3)$, and $Y$ is the corresponding natural module for $M^\circ$.*
(3) *$Y = I$, and there exists an $M$-invariant set $\{K_1, K_2\}$ of subnormal subgroups of $M$ such that $K_i \cong SL_{m_i}(q)$, $m_i \geqslant 2$, $q$ a power of $p$, $[K_1, K_2] = 1$, and as a $K_1K_2$-module $Y \cong Y_1 \otimes_{\mathbb{F}_q} Y_2$ where $Y_i$ is a natural $SL_{m_i}(q)$ -module for $K_i$. Moreover, $\mathbb{K} := End_{K_1K_2}(I) \cong \mathbb{F}_q$ and one of the following holds:*
   (1) *$M^\circ$ is one of $K_1, K_2$ or $K_1K_2$,*
   (2) *$m_1 = m_2 = q = 2$, $M \cong SL_2(2) \wr C_2$, $M^\circ = O_3(M)Q$ and $Q \cong C_4$ or $D_8$.*
   (3) *$m_1 = m_2 = p = 2$, $q = 4$, $M^\circ = K_1K_2Q \cong SL_2(4) \wr C_2$, $A$ acts $\mathbb{K}$-linearly on $I$ but $M^\circ$ does not.*
(4) *$p = 2$, $M \cong \Gamma SL_2(4)$, $M^\circ \cong SL_2(4)$ or $\Gamma SL_2(4)$, $I$ is the corresponding natural module, and $|Y/I| \leqslant 2$,*
(5) *$p = 2$, $M \cong \Gamma GL_2(4)$, $M^\circ \cong SL_2(4)$, $I$ is the corresponding natural module, and $Y = I$,*
(6) *$p = 2$, $M \cong 3{\cdot}Sym(6)$, $M^\circ \cong 3{\cdot}Alt(6)$ or $3{\cdot}Sym(6)$, and $Y = I$ is simple of order $2^6$.*
(7) *$p = 3$, $M \cong Frob(39)$ or $C_2 \times Frob(39)$, $M^\circ \cong Frob(39)$, and $Y = I$ is simple of order $3^3$.*

PROOF. Put $L := GL_{\mathbb{F}_p}(I)$. By D.6 $I$ is a faithful $M$-module, so we may and do view $M$ as a subgroup of $L$. Let $H$ be the subnormal closure of $A$ in $M$.

1°. $O_p(M) = O_p(H) = 1.$

Since $M$ is a faithful $p$-reduced $M$-module, $O_p(M) = 1$ and since $H \trianglelefteq\trianglelefteq M$ also $O_p(H) = 1$.

2°. *$H = \langle A^H \rangle$ and $[Y, H] \leqslant I$.*

Since $H$ is the subnormal closure of $A$, 1.13 gives $H = \langle A^H \rangle$, and by Hypothesis (iii) $[Y, A] \leqslant I$. Hence also $[Y, H] \leqslant I$.

3°. *$I$ is a semisimple $M^\circ$-module. In particular, $I$ is a semisimple module for any subnormal subgroup of $M^\circ$.*

Since $Y$ is a faithful $p$-reduced $Q!$-module for $M$ with respect to $Q$, D.8 shows that $I$ is a semisimple $M^\circ$-module.

4°. *Let $R$ be a subnormal subgroup of $M$ with $R \leqslant N_M(Q)$. Then $[R, Q] = 1$.*

This holds by A.54(b).

5°. *Let $R$ be a subnormal subgroup of $M$ and let $U$ be a non-trivial $Q$- and $R$-invariant subspace of $Y$. Then $[C_R(U), Q] = 1$.*

Note that $C_R(U)$ is normal in $R$ and so subnormal in $M$. Also since $U \neq 0$, $C_U(Q) \neq 0$, and so $Q$!-gives $C_R(U) \leqslant N_M(C_U(Q)) \leqslant N_M(Q)$. Thus (4°) implies $[C_R(U), Q] = 1$.

6°. $[F^*(M), Q, A] \neq 1$.

If $A \cap Q \neq 1$, then 1.15(b) shows that $[F^*(M), A \cap Q] \neq 1$ and so by 1.8(b),

$$1 \neq [F^*(M), A \cap Q, A \cap Q] \leqslant [F^*(M), Q, A].$$

So we may assume that $A \cap Q = 1$. Put $R := [F^*(M), Q]$ and suppose for a contradiction that $[R, A] = 1$. By 1.15(b), $R \neq 1$, and by 1.8(b), $R = [R, Q]$. Since $A \cap Q = 1$ and $A$ and $Q$ normalizes each other we have $[A, Q] = 1$ and so $[RQ, A] = 1$. Observe that $R \trianglelefteq\trianglelefteq M^\circ$. Thus, (3°) shows that $I$ is a semisimple $R$-module. Hence $I$ is the direct sum of the Wedderburn components of $R$ on $I$. Since $A$ centralizes $R$, each of the Wedderburn components of $R$ is invariant under $A$. By Hypothesis (i), $A$ is nearly quadratic but not quadratic on $I$, so A.48 shows that there exists a unique Wedderburn component $W$ of $R$ on $I$ with $[W, A] \neq 0$. Let $W_*$ be the sum of the remaining Wedderburn components of $R$. Then $I = W \oplus W_*$. Since $Q$ normalizes $R$ and $A$, $Q$ also normalizes $W$ and $W_*$.

Let $W_1$ be a simple $\mathbb{F}_p R$-submodule of $W$ and put $\mathbb{L} =: End_R(W_1)$. Since $W$ is $R$-homogeneous and $[R, A] = 1$, [**MS3**, 5.2] shows that there exists an $\mathbb{L}A$-module $W_2$ such that $W \cong W_1 \otimes_{\mathbb{L}} W_2$ as an $\mathbb{F}_p RA$-module. Let $m_i = \dim_{\mathbb{L}} W_i$. Then as an $\mathbb{F}_p A$-module, $W$ is the direct sum of $m_1$ copies of $W_2$. Applying A.48 a second times, $A$ centralizes all but one of these $m_1$ summands. Since the summands are isomorphic this gives $m_1 = 1$. In particular, $\mathbb{L}$ is generated by the image of $R$ in $End_{\mathbb{F}_p}(W_1)$. As an $R$-module, $W$ is a direct sum of copies of $W_1$, and we conclude that the subring $\mathbb{D}$ of $End_{\mathbb{F}_p}(W)$ generated by the image of $R$ is a field isomorphic to $\mathbb{L}$. Then $Q$ acts $\mathbb{D}$-semilinearly on $W$ and $Q_0 := C_Q(R/C_R(W)) = C_Q(\mathbb{D})$. By (5°), $[C_R(W), Q] = 1$ and so $[R, Q_0, Q] \leqslant [C_R(W), Q] = 1$. Thus $[F^*(M), Q_0, Q_0, Q_0] \leqslant [F^*(M), Q, Q_0, Q] = 1$, and by 1.8(b) $[F^*(M), Q_0] = 1$, so 1.15(b) gives $Q_0 = 1$. Hence $Q$ acts faithfully on $R/C_R(W)$ and thus also on $\mathbb{D}$. Put $\mathbb{D}_0 = C_{\mathbb{D}}(Q)$. By Galois Theory $\dim_{\mathbb{D}_0} \mathbb{D} = |Q|$ and there exists a $\mathbb{D}_0$-basis of $\mathbb{D}$ regularly permuted by $Q$. Also there exists a $Q$-invariant chain $0 = U_0 < U_1 < \ldots < U_{m_2 - 1} < U_{m_2} = W$ of $\mathbb{D}$-subspaces of $W$ with each factor isomorphic to $\mathbb{D}$ as a $Q$-module. Thus $C_{U_i}(Q) \nleqslant U_{i-1}$ and so $\langle C_W(Q)^R \rangle = \langle \mathbb{D} C_W(Q) \rangle = W$. By $Q$!, $C_W(Q) \cap C_W(Q)^r = 0$ for all $r \in R \backslash N_R(Q)$, see A.50(c). Since $C_W(Q)$ and $C_W(Q)^r$ are isomorphic $A$-modules, A.48 shows that $A$ acts quadratically on $C_W(Q)$ and so also on $W = \langle C_W(Q)^R \rangle$ and $I = W \oplus W_* = W + C_I(A)$, a contradiction.

7°. *There exists a $QA$-invariant non-trivial subnormal subgroup $X$ of $F^*(M)$ such that*

$$X = [X, A], \quad X = [X, Q] \quad \textit{and if } A \cap Q \neq 1,\ X = [X, A \cap Q].$$

If $A \cap Q \neq 1$, 1.15(b) shows that $[F^*(M), A \cap Q] \neq 1$ and 1.8(b) gives

$$[F^*(M), A \cap Q] = [F^*(M), A \cap Q, A \cap Q].$$

So we can choose $X = [F^*(M), A \cap Q]$ in this case.

Suppose next that $A \cap Q = 1$. By (6°) $[F^*(M), Q, A] \neq 1$ and so we can choose a $QA$-invariant subnormal subgroup $X$ of $F^*(M)$ minimal with $[X, Q, A] \neq 1$. Then 1.10 shows $X = [X, A]$ and $X = [X, Q]$.

8°. *$X \leqslant F^*(M^\circ) \cap F^*(H)$ and $X \nleqslant N_M(Q)$. In particular, $H \nleqslant N_M(Q)$.*

Since $X = [X, A]$ and $A \leqslant H \trianglelefteq\trianglelefteq M^\circ$, 1.11 shows that $X \leqslant H$. Also $X = [X, Q]$ implies $X \leqslant M^\circ$. Hence $X \leqslant F^*(M) \cap M^\circ = F^*(M^\circ)$ and similarly $X \leqslant F^*(H)$. If $X \leqslant N_M(Q)$, (4°) implies $[X, Q] = 1$, a contradiction to $1 \neq X = [X, Q]$.

9°. *$I$ is a semisimple $X$-module and $C_Y(X) = 1$. In particular, $I = [I, X]$ and $X$ has no central chief factor on $I$.*

By (8°), $X \leqslant M^\circ$ and so $X$ is a subnormal subgroup of $M^\circ$. Hence (3°) shows that $I$ is a semisimple $X$-module. By (8°) $X \nleqslant N_M(Q)$. Since $X$ is $Q$-invariant and $Y$ is a $Q!$-module for $M$ with respect to $Q$, this gives $C_Y(X) = 1$ (see A.53).

10°. *Put $F := F^*(H)$. Then $C_Y(F) = 1$, $[Y,H] = [Y,F] = I$, and if $H$ is solvable, then $Y = I$.*

By (8°) $X \leqslant F$ and by (9°), $C_Y(X) = 1$ and $I = [I,X]$. So $C_Y(F) = 1$ and $I = [I,F]$. By (2°), $[Y,H] \leqslant I$ and so $[Y,H] = [Y,F] = I$.

Suppose now that $H$ is solvable. Then $F = F^*(H) = F(H)$ is nilpotent. Since $O_p(H) = 1$, this implies that $F$ is a $p'$-group. Coprime action now shows that $Y = C_Y(F) \oplus [Y,F] = I$.

11°. *Let $\mathcal{W}$ be a system of imprimitivity for $H$ on $I$ with $|\mathcal{W}| \geqslant 2$ such that $X$ acts trivially on $\mathcal{W}$. Then Case (2) or Case (3) of the Theorem holds.*

Let $W \in \mathcal{W}$. Then $X$ normalizes $W$, and by (9°) $X$ has no central chief factor on $I$, so $W = [W,X]$ and $C_W(X) = 0$. In particular, $|W| > 2$, and since $X = [X,A]$ we get $[W,A] \neq 0$.

We now apply A.48. Since $[W,A] \neq 0$ for all $W \in \mathcal{W}$, A.48(1) does not occur, and since $A$ does not act quadratically on $I$, also A.48(2) and (3) do not occur. So A.48(4) holds. Hence $A$ has a unique orbit $W^A$ on $\mathcal{W}$ with $[W,A] \neq 0$. It follows that $\mathcal{W} = A^W$, and one of the following holds.

- $p = 2$, $|W^A| = 4$ and $\dim_{\mathbb{F}_2} W = 1$.
- $p = 3$, $|W^A| = 3$ and $\dim_{\mathbb{F}_3} W = 1$.
- $p = 2$, $|W^A| = 2$ and $C_A(W) = C_A(V)$. Moreover, $\dim_{\mathbb{F}_2} W/C_W(B) = 1$ and $C_W(B) = [W,B]$, where $B := N_A(W)$.

Since $|W| > 2$ the first of these cases does not occur. Consider the second case. Recall that we view $M$ as a subgroup of $L = GL_{\mathbb{F}_p}(I)$. Note that $N_L(\mathcal{W}) \cong C_2 \wr Sym(3)$ and so $O^2(N_L(\mathcal{W})) \cong Alt(4) \cong \Omega_3(3)$. Since $H = \langle A^H \rangle = O^2(H)$ and $1 \neq X \leqslant H \cap C_L(\mathcal{W})$ we get $H = O^2(N_L(\mathcal{W}))$. It follows that $\mathcal{W}$ is the set of Wedderburn components of $O_2(H)$ on $I$ and hence

$$N_L(O^2(H)) = N_L(H) \leqslant N_L(O_2(H)) \leqslant N_L(\mathcal{W}) \text{ and } O^2(N_L(\mathcal{W})) = H = O^2(H).$$

Thus 1.12 applied with $\mathcal{G} = O^2$ shows that $H = O^2(H) = O^2(M)$. In particular, $M^\circ \leqslant H$ and so $H = M^\circ$. Hence $H$ is solvable, and (10°) shows that $Y = I$. So Case (2) of the Theorem holds.

Consider the third case. Put $H_0 := N_H(W)$. Then $|H/H_0| = 2$, $H = AH_0$ and $H_0 \trianglelefteq H$. Let $w \in W \backslash C_W(B)$. Then $[w,B] = [W,B] = C_W(B)$ and so $B$ acts transitively on $W \backslash C_W(B)$. In particular, $B$ induces the full centralizer of $C_W(B)$ in $GL_{\mathbb{F}_2}(W)$ on $W$.

Let $U$ be a proper $H_0$-submodule of $W$. Since $B \leqslant H_0$, $U$ is $B$-invariant, and the transitive action of $B$ on $W \backslash C_W(B)$ shows that $U \leqslant C_W(B)$. Put $D := C_B(W/U)$. The transitive action of $B$ shows that $[W,D] = U$. Note that $D$ centralizes $W/U$ and $U$. Let $a \in A \backslash B$. Since $A$ is abelian, $D \trianglelefteq A$ and so $D$ centralizes $W^a/U^a$ and $U^a$. Since $I = W + W^a$, it follows that $D \leqslant O_p(H_0)$. Hence $U = [W,D] = 1$ and so $W$ is a simple $H_0$-module.

Now [**MS3**, 7.3] shows that $\langle B^{H_0} \rangle$ induces $SL_{\mathbb{F}_2}(W) = Aut(W)$ on $W$. Suppose that $H_0$ acts faithfully on $W$. Then $H_0 \cong SL_{\mathbb{F}_2}(W)$. Thus $H_0$ has no outer automorphism, so $H = C_H(H_0)H_0$. But then $|C_H(H_0)| = 2$, a contradiction to $O_2(H) = 1$.

Put $\{W_1, W_2\} := \mathcal{W}$, $\{i,j\} := \{1,2\}$ and $K_j := C_{H_0}(W_i)$. Then $K_i$ acts faithfully on $W_i$. As $H_0$ does not act faithfully on $W_j$, $K_i \neq 1$. Thus $W_i = [W_i, K_i]$, and since $I = W_i \oplus W_j$, we get $W_i = [I, K_i]$ and $I = [I,K_1] \oplus [I,K_2]$. Put $m := \dim_{\mathbb{F}_2}(W)$. Then $W_i$ is natural $SL_m(2)$-module for $H_0$, $H_0/K_j = H_0/C_{H_0}(W_i) \cong SL_m(2)$ and

$$(*) \qquad 1 \neq K_i \cong K_iK_j/K_j \trianglelefteq H_0/K_j \cong SL_m(2).$$

Suppose that $m \geqslant 3$. Then $SL_m(2)$ is simple, and $(*)$ implies that that $H_0 = K_iK_j$. Hence $H_0 = K_1 \times K_2$ and $W_i$ is natural $SL_m(2)$-module for $K_i$. As seen above $I = [I,K_1] \oplus [I,K_2]$. Thus $I$ is a wreath product module for $H$ with respect to $\{K_1, K_2\}$, and A.56(b) shows that $C_{K_i}(w_i)$ is a 2-group for $0 \neq w_i \in C_{W_i}(N_Q(K_i))$. But this contradicts the action of $K_i$ on the natural $SL_m(2)$-module $W_i$.

Thus $m = 2$. Now $(*)$ implies that $K_i \cong SL_2(2)$ or $SL_2(2)' \cong C_3$. Put $F_i := O_3(K_i)$. Then $F_1F_2 = O_3(H) = F^*(H) = F = O^2(H)$. In particular, $H$ is solvable and $I = Y$ by (10°). Since

$C_B(W_i) = C_B(I) = 1$, $|B| = 2$ and so $A \cong C_2 \times C_2$. Since $H$ is the subnormal closure of $A$, 1.13 gives $H = O^2(H)A = FA$. So $H = FA = H_1 \times H_2$ with $H_i \cong SL_2(2)$, and as an $H$-module $I = V_1 \otimes V_2$ where $V_i$ is a natural $SL_2(2)$-module for $H_i$.

Note that $N_L(\mathcal{W}) \cong SL_2(2) \wr C_2$ so $F = O^2(H) = O^2(N_L(\mathcal{W}))$. Also $\mathcal{W}$ is the of set of Wedderburn components of $O^2(H)$ on $I$, and so $N_L(O^2(H)) \leqslant N_L(\mathcal{W})$. Hence 1.12 applied with $\mathcal{G} = O^2$ shows that $O^2(M) = O^2(H) = F \trianglelefteq M$. Thus either $H = M \cong SL_2(2) \times SL_2(2)$ or $H \trianglelefteq M = N_L(F) \cong SL_2(2) \wr C_2$. So to show that Case (3) of the Theorem holds it remains to determine $M^\circ$.

Observe that $H = H_1 \times H_2 \cong \Omega_4^+(2)$, $I$ is a natural $\Omega_4^+(2)$-module for $H$ and $H \trianglelefteq M$. Since $Q$ is weakly closed and $O^2(M) \leqslant H$, we have $M^\circ = \langle Q^{O^2(M)} \rangle = \langle Q^H \rangle$, see 1.46(d). Now B.37 shows that either $M^\circ$ is one $H_1, H_2$ and $H$, or $Q$ is isomorphic to $C_4$ or $D_8$. Thus indeed Case (3) of the Theorem holds.

In view of (11°) we may assume from now on that:

12°. *Let $\mathcal{W}$ be a system of imprimitivity for $H$ on $I$ with $X$ acting trivially on $\mathcal{W}$. Then* $|\mathcal{W}| = 1$.

Next we show:

13°. *Suppose that $I$ is not a simple $M^\circ$-module. Then Case (3:1) of the Theorem holds.*

By (3°) $I$ is a semisimple $M^\circ$-module and so the Wedderburn components of $M^\circ$ form a system of imprimitivity for $H$. Hence (12°) shows that $I$ is a homogeneous $M^\circ$-module. Let $I_1$ be a simple $M^\circ$-submodule of $I$ and $\mathbb{L} = End_{M^\circ}(I_1)$. Since $M^\circ$ is generated by $p$-elements, $\dim_{\mathbb{L}} I_1 \geqslant 2$. By [**MS3**, 5.5] there exists an $\mathbb{L}$-vector space $I_0$ and a regular tensor decomposition $I_0 \otimes_{\mathbb{L}} I_1$ of $I$ for $M$, which is strict for $QA$ and such that $M^\circ$ centralizes $I_0$. Since $I$ is not simple for $M^\circ$, $I \neq I_1$ and so $\dim_{\mathbb{L}} I_2 \geqslant 2$. Hence $I_0 \otimes_{\mathbb{L}} I_1$ is a proper tensor decomposition and we can apply [**MS3**, 6.5]. We discuss the cases given there.

The first two cases do not occur since $A$ does not act quadratically. Case (3.2) does not occur for regular tensor decompositions. Thus Case (3.1) holds. Hence $A$ acts $\mathbb{L}$-linearly on $I_j$ for $j = 0, 1$, $U_j := [I_j, A] = C_{I_j}(A)$ is an $\mathbb{L}$-hyperplane of $I_j$, and $[\mathbb{F}_p v_i, A] = U_j$ for all $v_j \in I_j \backslash U_j$. In particular, $A$ acts quadratically on $I_j$ and so $\{[v_j, a] \mid a \in A\} = [v_j, A] = [\mathbb{F}_p v_i, A] = U_j$. Thus $A$ acts transitively on $v_j + U_j$. Put $L_j := GL_{\mathbb{L}}(I_j)$, $H_j := C_{L_j}(U_j) \cap C_{L_j}(I_j/U_j)$ and for $P \subseteq C_M(\mathbb{L})$ let $P_j$ be the image of $P$ in $L_j$. Note that $A_j \leqslant H_j$ and a Frattini argument gives $H_j = A_j C_{H_j}(v_j)$. Since $C_{H_j}(v_j)$ centralizes $\mathbb{L}v_k + U_j = I_j$ we conclude that $A_j = H_j$.

Since $I_1$ is a simple $M^\circ$-module, $I_1$ is also a simple $M^\circ A$-module and so $p$-reduced for $M^\circ A$. Now [**MS3**, 7.2] shows $\langle A^{M^\circ} \rangle_1 = SL_{\mathbb{L}}(I_1)$. Since $M^\circ$ is generated by $p$-elements and $GL_{\mathbb{L}}(I_1)/SL_{\mathbb{L}}(I_1)$ is a $p'$-group, we get $(M^\circ)_1 \leqslant SL_{\mathbb{L}}(I_1)$. As $M_1^\circ$ acts faithfully on $I_1$,

$$1 \neq M^\circ \cong (M^\circ)_1 \trianglelefteq \langle A^{M^\circ} \rangle_1 = SL_{\mathbb{L}}(I_1).$$

Note that $SL_{\mathbb{L}}(I_1)$ is either quasisimple or $|\mathbb{L}| = p \leqslant 3$, $\dim_{\mathbb{L}} I_1 = 2$ and $SL_{\mathbb{L}}(I_1)'$ is a $p'$-group. We conclude that $(M^\circ)_1 = L_1$ and $M^\circ \cong SL_{\mathbb{L}}(I_1)$.

Let $K := O^{p'}(C_{C_M(\mathbb{L})}(M^\circ))$. As $C_{GL_{\mathbb{F}_p}(I_1)}(M^\circ)$ is a $p'$-group, $K$ centralizes $I_1$ and so also $\mathbb{L}$. In particular, $K$ acts faithfully on $I_0$ and $K \cong K_0$. As $\langle A^{M^\circ} \rangle_1 = SL_{\mathbb{L}}(I_1)$, $A$ induces inner automorphisms on $M^\circ$ and so $A \leqslant M^\circ K$. Suppose that $U$ is a proper $\mathbb{L}K$-submodule of $I_0$. Since $A$ acts transitively on $v_0 + U_0$, it also acts transitively on the 1-dimensional $\mathbb{L}$-subspaces not in $U_0$. Thus $U \leqslant U_0$. Put $B := C_A(I_0/U)$. Since $A_0 = H_0$, $[I_0, B] = U$. Note that $B$ centralizes $I_0/U$ and $U$ and so the same holds for $K \cap BM^\circ$. But $K$ acts faithfully on $I_0$ and since $K \trianglelefteq\trianglelefteq M$, $O_p(K) = 1$. Thus $K \cap BM^\circ = 1$. As $B \leqslant A \leqslant M^\circ K$ we get

$$B \leqslant M^\circ B \cap M^\circ K = M^\circ(K \cap BM^\circ) = M^\circ.$$

Thus $U = [I_0, B] \leqslant [I_0, M_\circ] = 0$, and $I_0$ is a simple $\mathbb{L}K$-module. It follows that $\langle A^K \rangle_0 = L_0$, and arguing as above, $L_0 \cong SL_{\mathbb{L}}(I_0)$, $K_0 = L_0$ and $K \cong SL_{\mathbb{L}}(I_0)$. Thus Case (3:1) of the Theorem holds.

In view of (2°) we may assume from now on that

14°. *$I$ is a simple $M^\circ$-module.*

Next we prove:

15°. *$I$ is a simple $H$-module.*

Since $I$ is a simple $M^\circ$-module, $I$ is a simple $M$-module. Since $H \trianglelefteq\trianglelefteq M$ this implies that $I$ is a semisimple $H$-module. So $I$ is the direct sum of a set $\mathcal{W}$ of simple $H$-submodules. Note that $\mathcal{W}$ is a system of imprimitivity for $H$ on $I$ with $H$ and so also $X$ acting trivially on $\mathcal{W}$. Thus (12°) shows that $|\mathcal{W}| = 1$. Hence $I$ is a simple $H$-module.

Recall that $F = F^*(H)$.

16°. *Let $W$ be a Wedderburn component for $F$ on $I$ and put $\mathbb{K} := Z(End_F(W))$. Then $W = I$ and $\mathbb{K} = Z(End_F(I))$.*

By (8°) $X \leqslant F^*(H) = F$. Hence the Wedderburn components for $F$ on $I$ form a system of imprimitivity for $H$ on $I$ on which $X$ acts trivially. Thus (12°) shows that $I = W$.

We now apply [**MS3**, Theorem 2] and discuss the different cases given there. Define $\mathbb{E} := End_H(I)$.

Case 1. *Suppose that $F = KZ(H)$, where $K$ is a component of $H$, $I = W$ is a simple $\mathbb{F}_pK$-module, and $\mathbb{K} = \mathbb{E}$. Then Case (1) of the Theorem holds.*

Put $\mathbb{D} = End_K(I)$. Since $I$ is a finite simple $K$-module, $\mathbb{D}$ is a finite division ring and so a field. In particular, the multiplicative group $\mathbb{D}^*$ of $\mathbb{D}$ is a cyclic $p'$-group. Note that $C_M(K) \leqslant \mathbb{D}^*$ and so also $C_M(K)$ is cyclic $p'$-group. Thus $K$ is the unique component of $M$ and $K \trianglelefteq M$. Moreover, if $P$ is a non-trivial $p$-subgroup of $M$, then $[K, P] \neq 1$ and so $K = [K, P]$. Thus $K = [K, Q] \leqslant M^\circ$ and $K = [K, A]$.

Since $\mathbb{D}$ is commutative and $Z(H) \leqslant C_M(K) \leqslant \mathbb{D}^*$ we have $\mathbb{D} \subseteq End_{Z(H)}(I)$. Thus

$$\mathbb{D} \subseteq End_{Z(H)K}(I) = End_F(I) \subseteq End_K(I) = \mathbb{D}$$

and so $\mathbb{D} = End_F(I) = End_K(I)$. Since $I = W$, $\mathbb{K} = Z(End_F(I)) = Z(End_K(I)) = Z(\mathbb{D})$, and since $\mathbb{D}$ is commutative, this gives $\mathbb{K} = \mathbb{D}$.

As $KA \leqslant H$, (2°) shows $[Y, KA] = I$. Note that $F(M) \leqslant C_M(K) \subseteq \mathbb{D} = \mathbb{K}$. Since by Hypothesis of (Case 1) $\mathbb{K} = \mathbb{E} = End_H(I)$, this gives $[F(M), A] = 1$ and $[F^*(M), A] = [F(M)K, A] = [K, A] = K$. Thus all the statements in Case (1) of the Theorem hold.

We now discuss the remaining cases given in Theorem 2 of [**MS3**]. For the convenience of the reader we reproduce the table given there. We also have omitted case (13) of the table since in that case $H$ would not be generated by abelian nearly quadratic subgroups.

| | $H$ | $I$ | $W$ | $\mathbb{K}$ | $H/C_H(\mathbb{K})$ | conditions |
|---|---|---|---|---|---|---|
| 1. | $(C_2 \wr Sym(m))'$ | $(\mathbb{F}_3)^m$ | $\mathbb{F}_3$ | $\mathbb{F}_3$ | – | $m \geqslant 3, m \neq 4$ |
| 2. | $SL_n(\mathbb{F}_2) \wr Sym(m)$ | $(\mathbb{F}_2^n)^m$ | $\mathbb{F}_2^n$ | $\mathbb{F}_2$ | – | $m \geqslant 2, n \geqslant 3$ |
| 3. | $Wr(SL_2(\mathbb{F}_2), m)$ | $(\mathbb{F}_2^2)^m$ | $\mathbb{F}_2^2$ | $\mathbb{F}_4$ | – | $m \geqslant 2$ |
| 4. | $Frob_{39}$ | $\mathbb{F}_{27}$ | $I$ | $\mathbb{F}_{27}$ | $C_3$ | |
| 5. | $\Gamma GL_n(\mathbb{F}_4)$ | $\mathbb{F}_4^n$ | $I$ | $\mathbb{F}_4$ | $C_2$ | $n \geqslant 2$ |
| 6. | $\Gamma SL_n(\mathbb{F}_4)$ | $\mathbb{F}_4^n$ | $I$ | $\mathbb{F}_4$ | $C_2$ | $n \geqslant 2$ |
| 7. | $SL_2(\mathbb{F}_2) \times SL_n(\mathbb{F}_2)$ | $\mathbb{F}_2^2 \otimes \mathbb{F}_2^n$ | $I$ | $\mathbb{F}_4$ | $C_2$ | $n \geqslant 3$ |
| 8. | $3{\cdot}Sym(6)$ | $\mathbb{F}_4^3$ | $I$ | $\mathbb{F}_4$ | $C_2$ | |
| 9. | $SL_n(\mathbb{K}) \circ SL_m(\mathbb{K})$ | $\mathbb{K}^n \otimes \mathbb{K}^m$ | $I$ | any | 1 | $n, m \geqslant 3$ |
| 10. | $SL_2(\mathbb{K}) \circ SL_m(\mathbb{K})$ | $\mathbb{K}^2 \otimes \mathbb{K}^m$ | $I$ | $\mathbb{K} \neq \mathbb{F}_2$ | 1 | $m \geqslant 2$ |
| 11. | $SL_n(\mathbb{F}_2) \wr C_2$ | $\mathbb{F}_2^n \otimes \mathbb{F}_2^n$ | $I$ | $\mathbb{F}_2$ | 1 | $n \geqslant 3$ |
| 12. | $(C_2 \wr Sym(4))'$ | $(\mathbb{F}_3)^4$ | $I$ | $\mathbb{F}_3$ | 1 | |

Case 2. *Cases 1., 2., and 3. of the table do not occur.*

In these cases $W \neq I$ contrary to (16°).

Case 3. *In cases 5., 6. and 8. of the table, case (4), (5) and (6), respectively, of the Theorem holds.*

Note that in each of these cases $H$ has a unique component $K_1$ and $I$ is a simple $K_1$-module. In particular, $C_L(K_1)$, and so also $C_M(K_1)$, is a cyclic $p'$-group. Thus $[K_1, Q] \neq 1$ and $K_1 = [K_1, Q] \leqslant M^\circ$. Moreover, since $C_M(K_1)$ is cyclic and distinct components centralize each other, $K_1$ is the unique component of $M$ and so $K_1 \trianglelefteq M$. Note also that in each case $\mathbb{K} = \mathbb{F}_4$ and $H$ does not act $\mathbb{K}$-linearly on $I$. As $H = \langle A^H \rangle$ also $A$ does not act $\mathbb{K}$-linearly on $I$.

Assume case 5. or 6. of the table. Then $K_1 \cong SL_n(4)$ and $I$ is a natural $SL_n(4)$-module for $K_1$.

Suppose that $n > 2$. Then by B.37 $K_1 = M^\circ$ and $Q = C_L(I/U) \cap C_L(U)$ for some 1-dimensional $\mathbb{K}$-subspace $U$ of $I$. Let $A_{\mathbb{K}}$ be the largest subgroup of $A$ acting $\mathbb{K}$-linearly on $I$. Since $A$ does not act $\mathbb{K}$-linearly on $I$, [**MS3**, 6.3] shows that $[I, A_{\mathbb{K}}] = C_I(A_{\mathbb{K}})$ is a $\mathbb{K}$-hyperplane of $I$. Recall from Hypothesis (ii) of the Theorem that $Q$ and $A$ normalizes each other. Thus $[Q, A] \leqslant Q \cap A$ and $U \leqslant C_I([Q, A])$. We claim that $C_I([Q, A]) = U$. Otherwise choose $U < U_1 \leqslant C_I([Q, A])$ with $[U_1, A] \leqslant U$. Then $[Q, A, U_1] = 0$ and $[U_1, A, Q] \leqslant [U, Q] = 0$. Hence the Three Subgroups Lemma shows that $[U_1, Q, A] = 0$. As $Q = C_L(I/U) \cap C_L(U)$ and $U_1 \nleqslant U$, this implies that $[U_1, Q] = U$. So $A$ centralizes the non-trivial $\mathbb{K}$-subspace $U$ of $I$, a contradiction, since $A$ does not act $\mathbb{K}$-linearly on $U$. Thus $C_I([Q, A]) = U$. As $Q$ acts $\mathbb{K}$-linearly on $I$, we have $[Q, A] \leqslant Q \cap A \leqslant A_{\mathbb{K}}$ and $C_I(A_{\mathbb{K}}) \leqslant C_I([Q, A]) \leqslant U$. Since $C_I(A_{\mathbb{K}})$ is a $\mathbb{K}$-hyperplane of $I$ and $\dim_{\mathbb{K}} U = 1$, this shows that $n = 2$.

Note that $M \leqslant N_L(K_1)$. By B.32(b), $N_L(K_1) \cong \Gamma GL_2(4)$ and so $N_L(K_1)/K_1 \cong Sym(3)$. Since $A \nleqslant K_1$, this implies either $M = N_L(K_1) \cong \Gamma GL_2(4)$ or $M = K_1 A \cong \Gamma SL_2(4)$. In both cases $M = H$. Since $K_1$ acts transitively on $I$, $Q!$ shows that $M^\circ = \langle Q^{K_1} \rangle = K_1 Q \trianglelefteq M$. Thus either $M^\circ = K_1$ or $M^\circ = M \cong \Gamma SL_2(4)$.

By (2°), $[Y, H] \leqslant I$ and so $[Y, K_1] = I$. Observe that $[K_1, Q] \neq 1$ and by $Q!$, $C_Y(K_1) = 1$. Thus $Y/I$ embeds into $H^1(K_1, I)$. By C.18 $H^1(K_1, I)$ has order four. As $C_L(K_1)$ acts fixed-point freely on $I$, it also acts fixed-point freely on $H^1(K_1, I)$. Thus $N_L(K_1)$ induces $Sym(3)$ on $H^1(K_1, I)$ with kernel $K_1$. Since $M = H$, $[Y, M] \leqslant I$. Hence either $Y = I$ or $M \cong \Gamma SL_2(4)$ and $|Y/I| \leqslant 2$ , or $M \cong \Gamma GL_2(4)$ and $Y = I$. In the first case (4) of the Theorem holds, in the second case (5) of the Theorem holds.

Assume case 8. of the table. Since $End_{K_1}(I) = \mathbb{F}_4$ and $|Z(K_1)| = 3 = |\mathbb{F}_4^\sharp| = |\mathbb{K}^\sharp|$, $C_M(K_1) \leqslant K_1$. Note that $K_1$ has a unique conjugacy class of subgroups $A_1$ with $A_1 \cong Alt(5)$ and $C_I(A_1) \neq 1$. It follows that $M$ acts on this conjugacy class. Thus $M/C_M(K_1) \cong Sym(6)$ and so $M \cong 3{}^{\cdot}Sym(6)$. Since $K_1 \leqslant M^\circ$ we get $M^\circ = K_1 \cong 3{}^{\cdot}Alt(6)$ or $M^\circ = M \cong 3{}^{\cdot}Sym(6)$. Thus case (6) of the Theorem holds.

Case 4. *Suppose that either Case 12. or Case 10. with* $m = 2$ *and* $\mathbb{K} = \mathbb{F}_3$ *of the table holds. Then Case (3:1) of the Theorem holds.*

Since $H$ is solvable in these cases (10°) shows that $Y = I$. Note that in both cases $F = O_2(H) \cong Q_8 \circ Q_8$, and $I$ is the unique simple $F$-module of order $3^4$. Moreover, $F = O^3(O^2(H))$ and $N_L(F)/F \cong O_4^+(2)$. It follows that $F = O^3(O^2(N_L(F)))$ and 1.12 applied with $\mathcal{G} = O^3 O^2$ shows that $F = O^3(O^2(M)) \trianglelefteq M$. Note that $Q \leqslant O^2(M)$ since $p = 3$, and so $M_\circ = O^3(M^\circ) \leqslant F$. By 1.13 $M^\circ = \langle Q^{M_\circ} \rangle$ and thus $M^\circ = \langle Q^F \rangle$.

Suppose for a contradiction that $Q$ normalizes an elementary abelian subgroup $B$ of order eight in $F$. Since $B/Z(F)$ and $F/B$ are dual to each other as $Q$-modules, we conclude that $Q$ acts fixed-point freely on $F/Z(F)$. In particular, $[B, Q]$ is a complement to $Z(F)$ in $B$. Let

$$\mathcal{D} := \{D \leqslant B \mid |B/D| = 2, C_I(B) \neq 0\}.$$

Since $I$ is a simple faithful $F$-module, $C_I(Z(F)) = 0$. Thus $Z(F) \cap D = 1$ for all $D \in \mathcal{D}$. By coprime action

$$I = \bigoplus_{D \in \mathcal{D}} C_I(D).$$

Note that $F$ acts transitively on the four complements to $Z(F)$ in $B$. We conclude that $\mathcal{D}$ consists of the complements to $Z(F)$ in $B$ and $|C_I(D)| = 3$ for all $D \in \mathcal{D}$. In particular, $|C_I([B,Q])| = 3$ and $[C_I([B,Q]),Q] = 0$. Thus $Q!$ gives $[B,Q] \leqslant N_M(Q)$ and so $[B,Q] = [B,Q,Q] \leqslant B \cap Q = 1$, which contradicts the fixed-point free action of $Q$ on $F$.

Thus $Q$ does not normalize any elementary abelian subgroup of order 8 in $F$. It follows that either $|Q| = 9$ and $M^\circ = FQ \cong SL_2(3) \circ SL_2(3)$ or $|Q| = 3$, $[F,Q] \cong Q_8$ and $M^\circ \cong SL_2(3)$. If the former holds, Case (3:1) of the Theorem holds. If the latter holds, $M^\circ$ does not act simply on $I$, a contradiction to (14°).

Case 5. *Suppose that Case 7., 9., 10. or 11. of the table holds. Then Case (3) of the Theorem holds.*

In view of (Case 4) we assume that $\mathbb{K} \neq \mathbb{F}_3$ if $m = 2$ in Case 10. Note that $\mathbb{E} = \mathbb{F}_2$ and $\mathbb{K} = \mathbb{F}_4$ in case 7., $\mathbb{E} = \mathbb{K} = \mathbb{F}_4$ in in the cases 9., 10., and $\mathbb{E} = \mathbb{K} = F_2$ in case 11. In each of the four cases $H$ has subgroups $K_1, K_2$ such that $K_i \cong SL_{n_i}(\mathbb{E})$, $n_i \geqslant 2$, $[K_1,K_2] = 1$, $K_1K_2 \trianglelefteq H$, $H = K_1K_2A$, and there exist natural $SL_{n_i}(\mathbb{E})$-modules $I_i$ for $K_i$ such that as an $K_1K_2$-module, $I \cong I_1 \otimes_{\mathbb{E}} I_2$. In particular, $X = O^p(X) \leqslant K_1K_2$ and so $C_Y(K_1K_2) \leqslant C_Y(X) = 1$. Since $[Y,K_1K_2] \leqslant [Y,H] \leqslant I$ and $K_i$ has no central chief factor on $I$, we conclude from C.19 that $Y = I$.

Choose notation such that $n_1 \geqslant n_2$. In case 7. 9. and 11., $n_1 \geqslant 3$, and in case 10. either $n_1 \geqslant 3$ or $n_1 = m = 2$ and (according to our additional assumption) $|\mathbb{E}| > 3$. Thus $K_1$ is quasisimple and so a component of $M$.

Let $R$ be a component of $M$ with $R \neq K_1$. Then $[R,K_1] = 1$. Note that $C_L(K_1) \cong GL_{n_2}(\mathbb{E})$. As $K_2 \cong SL_{n_2}(\mathbb{E})$, this gives $K_2 = O^{p'}(C_M(K_1))$ and $C_M(K_1)^\infty \leqslant K_2$. It follows that either $K_1$ is the only component of $M$, or $K_2$ is a component of $M$ and $\{K_1,K_2\}$ is the set of components of $M$. In either case $K_1K_2 \trianglelefteq M$.

It follows that $H$ acts on the set $\mathcal{S} = \{v_1 \otimes v_2 \mid 0 \neq v_i \in V_i\}$ and this set is of size not divisible by $p$. So we can choose $0 \neq v_i \in V_i$ such that $Q$ centralizes $v_1 \otimes v_2$; i.e., $C_M(v_1 \otimes v_2) \leqslant N_M(Q)$. Note that $K_1K_2$ acts transitively on $\mathcal{S}$ and so $M = HC_M(v_1 \otimes v_2) = HN_M(Q)$. Thus $M^\circ = (K_1K_2Q)^\circ$. Put $R_1 := C_{K_1}(v_1)$. Then $R = C_{K_1}(v_1 \otimes v_2)$, so $R_1 \leqslant N_M(Q)$ and $[R_1,Q]$ is a $p$-group. Note that $R_1/O_p(R_1) \cong SL_{n_1-1}(\mathbb{E})$.

Suppose that $n_1 \geqslant 3$. Then $R_1/Z(K_1)$ is not a $p$-group. It follows that $Q$ normalizes $K_1$ and centralizes $R_1/O_p(R_1)$. Hence $Q$ induces inner automorphisms on $K_1$. Therefore $Q$ acts $\mathbb{E}$-linearly on $I$. Since $Q$ normalizes $K_2$, this implies that $Q$ induces inner automorphisms on $K_2$ (see B.32(d)). Thus $Q \leqslant K_1K_2$ and $M_1^\circ \leqslant K_1K_2$. The only normal subgroups of $K_1K_2$ generated by $p$-elements are $K_1$, $K_2$ and $K_1K_2$, so $M^\circ$ is one of $K_1, K_2$ and $K_1K_2$ and so Case (3) of the Theorem holds.

Suppose next that $n_1 = 2$. Since $n_2 \leqslant n_1$ this gives $n_2 = 2$, and case 10. of the table holds with $m = 2$ with $|\mathbb{E}| = |\mathbb{K}| > 3$. Note that $K_1K_2 \cong \Omega_4^+(q)$ and $I$ is a natural $\Omega_4^+(q)$-module for $K_1K_2$. Now B.37 shows that either $M^\circ = \langle Q^{K_1K_2} \rangle$ is one of $K_1, K_2$ and $K_1K_2$, or $q = 4$, $M^\circ \cong O_4^+(2) \cong SL_2(4) \wr C_2$ and $Q$ does not act $\mathbb{K}$-linearly. Thus Case (3) of the Theorem holds.

Case 6. *Suppose that Case 4. of the table holds. Then Case (7) of the Theorem holds.*

Since $H \cong Frob(39)$, $H$ is solvable and so by (10°) $Y = I$. From $|I| = 3^3$ we get $N_{L'}(H) = H$. Since $H \trianglelefteq\trianglelefteq M \cap L'$ this gives $M \cap L' = H$. So either $M = H \cong Frob(39)$ or $M = Z(L) \times H \cong C_2 \times Frob(39)$. In either case $H$ is the only non-trivial subgroup generated by $p$-elements and so $M^\circ = H$, and Case (7) of the Theorem holds. $\square$

APPENDIX E

# The Amalgam Method

The amalgam method is a convenient way to keep track of conjugation in (finite) groups and to combine conjugation of abelian subgroups with quadratic action.

The starting point is a prime $p$ and a group $G$ together with a collection of two or more finite subgroups $H_i$, $i \in I$, whose $p$-local structures should be investigated. Usually one requires that these subgroups are of characteristic $p$ and have a Sylow $p$-subgroup in common, together with other properties that restrict the number (and often also the structure) of the non-abelian chief factors, like being $p$-irreducible.

It is rather astonishing that in such an apparently general situation most of the normal $p$-subgroups of these subgroups $H_i$ are already contained in normal $p$-subgroups of $G$. Or from a different point of view, modulo the largest normal $p$-subgroup of $G$ contained in $B := \bigcap_{i \in I} H_i$, the number and module structure of the $p$-chief factors of the subgroups $H_i$ are very limited.

The name amalgam method comes from the fact that this method does not really depend on $G$, but only on the embedding of $B$ into the subgroups $H_i$, so it can as well be performed in the amalgamated product of these (sub)groups over $B$.

## E.1. The Coset Graph

Let $H$ be any group and let $(H_i)_{i \in I}$ be a family of distinct subgroups of $H$. We define the *coset graph* of $H$ with respect to $H_i$, $i \in I$, as follows:

The cosets $H_i g$, $i \in I$ and $g \in H$, are the *vertices* of $\Gamma$, and the unordered pairs

$$\{H_i g, H_j g\} \quad \text{with } i \neq j \text{ and } g \in H$$

are the *edges* of $\Gamma$. A vertex $H_i g$ we will call of *color* $i$. Note that a given vertex has a unique color. Indeed if $H_i g = H_j h$, then $H_j = H_i g h^{-1}$ is a coset of $H_i$ containing 1 and so $H_j = H_i$ and $i = j$.

Apart from the elementary graph theoretic terminology, like neighbor, adjacent, path, and distance, we use the following notation for vertices $\gamma$ and $\delta$ of $\Gamma$, $i \in I$, $K \subseteq I$, $\Delta$ a set of vertices of $\Gamma$, and $L \leqslant H$.

- $\Gamma_i := H/H_i$ is the set of vertices of color $i$; $\Gamma_K := \bigcup_{k \in K} \Gamma_k$ is the set vertices of color contained in $K$. $\Delta(\gamma)$ is the set of vertices in $\Delta$ adjacent or equal to $\gamma$.
- $L_\delta$ is the *stabilizer* of $\delta$ in $L$, $L_\Delta = \bigcap_{\delta \in \Delta} L_\delta$ is the element-wise stabilizer of $\Delta$.
- $d(\gamma, \delta)$ is the distance between $\gamma$ and $\delta$.
- A *chamber* of $\Gamma$ is a set of vertices of the form $\{H_i g \mid i \in I\}$ for some $g \in H$.

## E.2. Elementary Properties

We begin with some elementary facts about $\Gamma$ (see also [**KS**]).

Lemma E.1. *The following hold:*

(a) $\Gamma$ *is an* $|I|$*-partite graph whose partition classes are the sets* $H/H_i$, $i \in I$.
(b) $\{H_i g, H_j h\}$ *is an edge of* $\Gamma$ *if and only if* $i \neq j$ *and* $H_i g \cap H_j h \neq \varnothing$.
(c) *For any distinct* $i, j$ *in* $I$, $H$ *acts transitively on the set of edges whose vertices have colors* $i$ *and* $j$. *In particular, every edge is contained in a chamber.*
(d) $H$ *acts transitively on the set of chambers.*

PROOF. (a): As remarked above each vertex has a unique color, so $\Gamma_i$, $i \in I$, is a partition of $\Gamma$. By the definition of an edge, a vertex is only adjacent to vertices of distinct color.

(b): Note that

$$H_i g \cap H_j h \neq \varnothing \iff \exists a \in H_i g \cap H_j h \iff \exists a \in H : \{H_i g, H_j h\} = \{H_i a, H_j a\}.$$

(c) and (d): This follows immediately from the definition of an edge and a chamber, respectively. □

LEMMA E.2. *Let $\alpha := H_i g$ be a vertex and $e := \{H_i g, H_j g\}$ be an edge of $\Gamma$. Then the following hold:*

(a) $H_\alpha = H_i^g$.
(b) $H_e = (H_i \cap H_j)^g$.

PROOF. Observe that for $h \in H$

$$H_i g h = H_i g \iff gh \in H_i g \iff h \in g^{-1} H_i g \iff h \in H_i^g.$$

This gives (a) and (b). □

LEMMA E.3. *$H$ acts on $\Gamma$ by right multiplication as a group of automorphisms, and*

$$H_\Gamma = \bigcap_{i \in I, g \in H} H_i^g$$

*is the kernel of this action. Moreover, for every vertex $\alpha$ of $\Gamma$, $H_\alpha$ is transitive on the chambers containing $\alpha$ and on the neighbors of color $j$ of $\alpha$, for every $j \in I$.*

PROOF. Right multiplication $r_g$ by an element $g \in H$ sends vertices to vertices and edges to edges. It is now easy to see that

$$r : H \to Aut(\Gamma) \text{ with } g \mapsto r_g$$

is a homomorphism. By E.2

$$H_\Gamma = \bigcap_{i \in I, g \in G} H_i^g = \bigcap_{\delta \in \Gamma} H_\delta = \ker r.$$

Let $\alpha \in \Gamma$ be a vertex of color $i$ and let $j \in I$. By E.1(d) $H$ acts transitively on the set of chambers. As each chamber contains a unique vertex of color $i$ we conclude that $H_\alpha$ acts transitively on the chambers containing $\alpha$. If $j = i$, the set of neighbors of color $j$ of $\alpha$ is empty. So suppose $i \neq j$. Then by E.1(c) $H$ acts transitively of the edges with vertices of color $i$ and $j$. It follows that $H_\alpha$ acts transitively on the set of vertices of color $j$ adjacent to $\alpha$. □

In the following we will write this action of $H$ exponentially: $\alpha \mapsto \alpha^g$ rather than $\alpha g$.

LEMMA E.4. *Suppose that $\Gamma$ is connected, and let $C$ be a chamber of $\Gamma$.*

(a) *Let $K \leqslant H$. Suppose that for each $\gamma \in C$ and $i \in I$, $K_\gamma$ acts transitive on $\Gamma_i(\gamma)$. Then for each $i \in I$, $K$ acts transitively on $\Gamma_i$.*
(b) *Let $R \leqslant H_C$ and suppose that for each $\gamma \in C$ and $i \in I$, $N_{H_\gamma}(R)$ acts transitively on $\Gamma_i(\gamma)$. Then $R \leqslant H_\Gamma$.*

PROOF. (a): Let $\delta \in \Gamma$. We will show by induction on $d := d(C, \delta)$ that $\delta$ is $K$-conjugate to an element of $C$. If $d(C, \delta) = 0$, $\delta \in C$. So suppose that $d(C, \delta) > 0$ and let $(\alpha_0, \ldots, \alpha_d)$ be a path in $\Gamma$ with $d = d(C, \delta)$, $\alpha_0 \in C$ and $\alpha_d = \delta$. Put $\gamma := \alpha_0$ and let $\alpha \in C$ be of the same color $i$ as $\alpha_1$. Since $K_\gamma$ acts transitively on $\Gamma_i(\gamma)$, we have $\alpha_1^k = \alpha$ for some $k \in K_\gamma$. Then

$$d(C, \delta^k) \leqslant d(\alpha, \delta^k) = d(\alpha_1^k, \delta^k) = d(\alpha_1, \gamma) = d - 1,$$

and by induction $\delta^k$ is $K$-conjugate to an element of $C$. Hence the same holds for $\delta$.

(b): Put $K := N_H(R)$. Let $i \in I$ and $\gamma \in C$. Then $K_\delta = N_{H_\delta}(R)$ acts transitively on $\Gamma_i(\gamma)$. So by (a) $K$ acts transitively on $H/H_i$. Since $R \leqslant H_C$, $R$ fixes a vertex of color $i$. As $R \trianglelefteq K$ and $K$ acts transitively on $H/H_i$, $R$ fixes all vertices of color $i$. Thus (b) holds. $\square$

LEMMA E.5. *$\Gamma$ is connected if and only if $H = \langle H_i \mid i \in I\rangle$.*

PROOF. Put $H_0 := \langle H_i,\, i \in I\rangle$ and let $\Gamma_0$ be the connected component of $\Gamma$ that contains the chamber $C := \{H_i \mid i \in I\}$. Since each $H_i$ leaves invariant $\Gamma_0$, also $H_0$ does. Thus if $H = H_0$, $\Gamma_0$ contains all vertices of $\Gamma$, so $\Gamma_0 = \Gamma$.

Assume that $\Gamma$ is connected. Let $i \in I$ and $\delta \in C$. Then $H_{0\delta} = H_\delta$ and so by E.3 $H_{0\delta}$ acts transitively on $\Gamma_i(\delta)$. Thus by E.4(a) $H_0$ acts transitively on $H/H_i$, and a Frattini argument gives $H = H_0H_i = H_0$. $\square$

## E.3. Critical Pairs

In this section we assume the following hypothesis.

HYPOTHESIS E.6. Let $H$ be a group, $(H_i)_{i\in I}$ a family of distinct subgroups of $H$, $J \subseteq I$, $p$ a prime, and $\Gamma$ the coset graph of $H$ with respect to $(H_i)_{i\in I}$. Put $B := \bigcap_{i\in I} H_i$, and suppose that the following hold:

(i) For each $i \in I$, $H_i$ is finite and $O_p(H_i) \leqslant B$.
(ii) $J \neq \varnothing$, and for $j \in J$, $Z_j$ is a $p$-reduced elementary abelian normal $p$-subgroup of $H_j$ with $Z_j \nleqslant H_\Gamma$.
(iii) $H = \langle H_i \mid i \in I\rangle$.

Recall from E.5 that $\Gamma$ is connected. For $j \in J$, $h \in H$, $\lambda := H_jh \in \Gamma_J$ and $\delta \in \Gamma$ we define:

$$Z_\lambda := Z_j^h, \qquad Q_\delta := O_p(H_\delta), \qquad \text{and} \qquad V_\delta := \langle Z_\lambda \mid \lambda \in \Gamma_J(\delta)\rangle.$$

LEMMA E.7. *The following holds.*

(a) *$Z_\delta \leqslant \Omega_1 Z(Q_\delta)$ for all $\delta \in \Gamma_J$.*
(b) *$Q_\delta \leqslant H_\lambda$ for all edges $\{\delta, \lambda\}$.*

PROOF. (a): Let $j \in J$. By E.6(ii) $Z_j$ is a $p$-reduced elementary abelian normal subgroup of $H_j$, and so $Z_j \leqslant \Omega_1 Z(O_p(H_j))$. This gives $Z_\delta \leqslant \Omega_1 Z(Q_\delta)$ for $\delta \in \Gamma_j$, and (a) holds.

(b): By Hypothesis E.6(i) $O_p(H_i) \leqslant B \leqslant H_k$ for all $i, k \in I$. By E.1(c) any edge with vertices of colors $i$ and $k$ is conjugate to $\{H_i, H_k\}$, and so (b) holds. $\square$

LEMMA E.8. *There exists a pair of vertices $(\delta, \lambda)$ such $\delta \in \Gamma_J$ and $Z_\delta \nleqslant Q_\lambda$.*

PROOF. By Hypothesis E.6(ii) $J \neq \varnothing$, and for $j \in J$, $Z_j \nleqslant H_\Gamma$. Hence there exists vertex $\lambda$ such that $Z_j \nleqslant H_\lambda$; in particular $Z_j \nleqslant Q_\lambda$. Put $\delta := H_j$. Then $Z_\delta = Z_j$, and the claim holds for $(\delta, \lambda)$. $\square$

Since $\Gamma$ is connected we can choose a pair $(\alpha, \alpha')$ of vertices of minimal distance among all pairs $(\delta, \lambda)$ with $\delta \in \Gamma_J$ and $Z_\delta \nleqslant Q_\lambda$. Any such pair of minimal distance is called a *critical pair* (with respect to $J$). Moreover, we put $b := d(\alpha, \alpha')$. Note that $b$ does not depend on the choice of the critical pair.

In the following $(\alpha, \alpha')$ is always a critical pair, and $\gamma$ is a path of length $b$ from $\alpha$ to $\alpha'$. We often denote $\gamma$ by

$$\gamma = (\alpha, \dots, \alpha + i, \dots, \alpha + b) = (\alpha' - b, \dots, \alpha' - i, \dots, \alpha'),$$

so

$$\alpha = \alpha' - b,\ \alpha' = \alpha + b, \text{ and } \alpha + i = \alpha' - (b - i).$$

LEMMA E.9. *The following hold:*

(a) $b \geqslant 1$.
(b) *Let* $\lambda \in \Gamma_J$ *and* $\delta \in \Gamma$. *If* $d(\lambda, \delta) \leqslant b$ *then* $Z_\lambda \leqslant H_\delta$, *and if* $d(\lambda, \delta) < b$ *then* $Z_\lambda \leqslant Q_\delta$.
(c) *Let* $0 \leqslant i < b$. *Then* $Z_\alpha \leqslant Q_{\alpha+i}$ *and if* $\alpha' \in \Gamma_J$, $Z_{\alpha'} \leqslant Q_{\alpha'-i}$.
(d) $Z_\alpha \leqslant H_{\alpha'}$ *and if* $\alpha' \in \Gamma_J$, $Z_{\alpha'} \leqslant H_\alpha$.
(e) *If* $b > 1$, *then* $V_\alpha \leqslant Q_{\alpha+i}$ *and* $V_{\alpha'} \leqslant Q_{\alpha'-i+1}$ *for* $0 \leqslant i < b-1$. *In particular* $V_\alpha \leqslant H_{\alpha'-1}$ *and* $V_{\alpha'} \leqslant H_{\alpha+1}$.
(f) *There exists* $h \in H_{\alpha'}$ *with* $Z_\alpha \nleqslant H^h_{\alpha'-1}$. *In particular,* $Z_\alpha \nleqslant H_{\Gamma(\alpha')}$.
(g) *If* $\alpha' \in \Gamma_J$ *and* $[Z_\alpha, Z_{\alpha'}] \neq 1$, *then also* $(\alpha', \alpha)$ *is a critical pair.*
(h) *If* $b \geqslant 3$ *and* $\delta \in \Gamma$, *then* $V_\delta$ *is an elementary abelian normal* $p$*-subgroup of* $H_\delta$ *in* $Q_\delta$.

PROOF. (a): By definition of a critical pair, $Z_\alpha \nleqslant Q_{\alpha'}$, and by E.7(a), $Z_\alpha \leqslant Q_\alpha$. Thus $\alpha \neq \alpha'$ and so $b \neq 0$.

(b): If $d(\lambda, \delta) < b$ then the definition of $b$ gives $Z_\lambda \leqslant Q_\delta$. Suppose that $d(\lambda, \delta) = b$. Then there exists $\mu \in \Gamma(\delta)$ such that $d(\lambda, \mu) = b - 1$, so $Z_\lambda \leqslant Q_\mu$, and by E.7(b), $Z_\lambda \leqslant Q_\mu \leqslant H_\delta$.

(c): Since $\alpha \in \Gamma_J$ and $d(\alpha, \alpha_{\alpha+i}) < b$, (b) applies. Similarly, if $\alpha \in \Gamma_J$, again (b) applies since also $d(\alpha, \alpha - i) < b$.

(d): This is again an application of (b) since $d(\alpha, \alpha') = b$.

(e): Let $\lambda \in \Gamma_J(\alpha)$ and $\delta \in \Gamma$ such that $d(\alpha, \delta) < b-1$. Then $d(\lambda, \delta) < b$ and so by (b), $Z_\lambda \leqslant Q_\delta$. Thus, also $V_\alpha = \langle Z_\lambda \mid \lambda \in \Gamma_J(\alpha)\rangle \leqslant Q_\delta$. In particular for $\delta = \alpha' - 2$, $V_\alpha \leqslant Q_{\alpha'-2}$, and by E.7(b) $V_\alpha \leqslant Q_{\alpha'-2} \leqslant H_{\alpha'-1}$.
Similarly for $\rho \in \Gamma_J(\alpha')$, $d(\rho, \alpha + 2) < b$ and by (b) $Z_\rho \leqslant Q_{\alpha+2}$. Hence $V_{\alpha'} \leqslant Q_{\alpha+2} \leqslant H_{\alpha'+1}$.

(f): Put $X := \bigcap_{h \in H_{\alpha'}} (H_{\alpha'} \cap H_{\alpha'-1})^h$. Then $X$ normalizes $Q_{\alpha'-1}$ and so $Q_{\alpha'-1} \cap X \leqslant O_p(X) \leqslant O_p(H_{\alpha'}) = Q_{\alpha'}$. Since $Z_\alpha \leqslant Q_{\alpha'-1}$ and $Z_\alpha \nleqslant Q_{\alpha'}$ this shows $Z_{\alpha'} \nleqslant X$ and thus (f) holds.

(g): Assume that $\alpha' \in \Gamma_J$ and $[Z_\alpha, Z_{\alpha'}] \neq 1$. Then clearly $Z_{\alpha'} \nleqslant Q_\alpha$ since $Z_\alpha \leqslant Z(Q_\alpha)$. Hence, (g) follows.

(h): Assume that $b \geqslant 3$ and let $\delta \in \Gamma$ and $\mu, \lambda \in \Gamma_J(\delta)$. Then $d(\mu, \lambda) \leqslant 2 < b$ and so $Z_\mu \leqslant Q_\lambda$. Since $Z_\lambda \leqslant Z(Q_\lambda)$ this gives $[Z_\lambda, Z_\mu] = 1$. Also $Z_\lambda \leqslant Q_\delta$ and $Z_\lambda$ is elementary abelian. It follows that $V_\delta$ is elementary abelian and contained in $Q_\delta$. This is (h). □

## E.4. The Case $|I| = 2$

In this section we assume

HYPOTHESIS E.10. Let $H$ be a group, $p$ a prime, $H_1$ and $H_2$ distinct subgroups of $H$ and $\Gamma$ the coset graph of $H$ with respect to $(H_1, H_2)$.
(i) $H_1$ and $H_2$ are finite of characteristic $p$.
(ii) For $i \in \{1, 2\}$, $Z_i$ is a $p$-reduced elementary abelian normal $p$-subgroup $H_i$ with $Z_{H_i} \leqslant Z_i$[1].
(iii) $H = \langle H_1, H_2\rangle$.
(iv) $H_1 \cap H_2$ is a parabolic subgroup of $H_1$ and $H_2$.
(v) No nontrivial $p$-subgroup of $H_1 \cap H_2$ is normal in $H_1$ and $H_2$.

Note here that since $H_i$ is finite of characteristic $p$ for $i = 1, 2$, then $Z_{H_i} \leqslant Y_{H_i}$ by 1.24(g) and both $Z_i = Z_{H_i}$ and $Z_i = Y_{H_i}$ fulfill (ii).

LEMMA E.11. (a) $H_\Gamma = 1$.
(b) *Hypothesis E.6 holds with* $I = J = \{1, 2\}$.

PROOF. (a): Then $H_\Gamma \leqslant H_1 \cap H_2$ and so $O_p(H_\Gamma)$ is $p$-subgroup of $H_1 \cap H_2$ normal in $H_1$ and $H_2$. Thus Hypothesis E.10(v) gives $O_p(H_\Gamma) = 1$. Since $H_\Gamma \trianglelefteq H_1$ and $H_1$ is of characteristic $p$, also $H_\Gamma$ is of characteristic $p$ (see 1.2(a).) Thus $H_\Gamma = 1$.
(b): Let $i \in I$. By Hypothesis E.10(i) $H_i$ is finite of characteristic $p$. By Hypothesis E.10(iv) $B := H_1 \cap H_2$ is a parabolic subgroup group of $H_i$ and so $O_p(H_i) \leqslant B$. By Hypothesis E.10(iv) $Z_i$

[1] See 1.1(c) for the definition of $Z_{H_i}$.

is an elementary abelian $p$-reduced normal subgroup of $H_i$ with $Z_{H_i} \leqslant Z_i$. The latter fact implies $Z_i \neq 1$ and since $H_\Gamma = 1$ we get $Z_i \nleqslant H_\Gamma$. By Hypothesis E.10(iv) $H = \langle H_1, H_2 \rangle$ and so Hypothesis E.6 holds. $\square$

LEMMA E.12. *The following hold:*

(a) *$H$ acts edge-transitively on $\Gamma$.*
(b) *Two vertices $\delta$ and $\lambda$ are of the same color if and only if $d(\delta, \lambda)$ is even.*
(c) *Let $\{\lambda, \mu\}$ be an edge. Then $H_\lambda \cap H_\mu$ is a parabolic subgroup of $H_\lambda$ and $H_\mu$.*
(d) *For every vertex $\delta$, $H_\delta$ is finite of characteristic $p$, $Z_\delta$ is a $p$-reduced elementary abelian normal subgroup of $H_\delta$ and $Z_{H_\delta} \leqslant Z_\delta \leqslant Y_{H_\delta} \leqslant \Omega_1 Z(Q_\delta)$.*
(e) *Let $\{\lambda, \mu\}$ be an edge. Then $C_{Z_\lambda}(H_\lambda) \leqslant Z_{H_\lambda \cap H_\mu} \leqslant Z_\lambda \cap Z_\mu$.*
(f) *Let $\{\lambda, \mu\}$ be an edge. Then no non-trivial $p$-subgroup of $H_\lambda \cap H_\mu$ is normal in $H_\lambda$ and $H_\mu$.*

PROOF. (a): Since $|I| = 2$, (a) follows from E.1(c).

(b): By E.1(a) $\Gamma$ is a bipartite graph with partition classes $H/H_1$ and $H/H_2$. This gives (b).

(c): By Hypothesis E.10(iv) $H_1 \cap H_2$ is a parabolic subgroup of $H_1$ and $H_2$. Since $H$ acts edge transitively, this gives (c).

(d): Let $i \in \{1, 2\}$. By E.10(i) $H_i$ is finite. By E.10(ii), $Z_{H_i} \leqslant Z_i$ and $Z_i$ is a $p$-reduced elementary abelian normal subgroup of $H_i$. Hence $Z_i \leqslant Y_{H_i} \leqslant \Omega_1 Z(O_p(H_i))$. By E.2(a) $H_\delta$ is conjugate to $H_1$ or $H_2$ and so (d) holds.

(e): Let $T \in Syl_p(H_\lambda \cap H_\mu)$. Since $H_\lambda \cap H_\mu$ is a parabolic subgroup of $H_\lambda$ and $H_\mu$, $T$ is a Sylow $p$-subgroup of $H_\lambda$ and $H_\mu$. Thus

$$C_{Z_\lambda}(H_\lambda) \leqslant \Omega_1 Z(T) \leqslant Z_{H_\lambda \cap H_\mu} \leqslant Z_{H_\lambda} \cap Z_{H_\mu} \leqslant Z_\lambda \cap Z_\mu,$$

and (e) is proved.

(f): By E.10(v) no non-trivial $p$-subgroup of $H_1 \cap H_2$ is normal in $H_1$ and $H_2$. Since $H$ is edge-transitive, (f) holds. $\square$

LEMMA E.13. *Suppose that $H_j$ is $p$-irreducible for some $j \in I$. Let $\{\lambda, \mu\}$ be an edge of $\Gamma$ such that $\lambda$ is of color $j$. Then the following hold:*

(a) *$C_{H_\lambda}(Z_\lambda)$ is $p$-closed or $Z_\lambda = C_{Z_\lambda}(H_\lambda) \leqslant Z_\mu$.*
(b) *$C_{H_\lambda}(V_\lambda)$ is $p$-closed.*

PROOF. By E.2(a) $H_\lambda$ is an $H$-conjugate of $H_j$ and so $p$-irreducible. Hence either $C_{H_\lambda}(Z_\lambda)$ is $p$-closed or $O^p(H_\lambda) \leqslant C_{H_\lambda}(Z_\lambda)$. In the second case, $H_\lambda / C_{H_\lambda}(Z_\lambda)$ is a $p$-group and since $Z_\lambda$ is $p$-reduced we get $C_{H_\lambda}(Z_\lambda) = H_\lambda$. Thus $Z_\lambda \leqslant C_{Z_\lambda}(H_\lambda)$. By E.12(e) $C_{Z_\lambda}(H_\lambda) \leqslant Z_\lambda \cap Z_\mu$ and so (a) is proved.

Similarly, either $C_{H_\lambda}(V_\lambda)$ is $p$-closed or $O^p(H_\lambda) \leqslant C_{H_\lambda}(V_\lambda)$. In the second case, since $H_\lambda \cap H_\mu$ is a parabolic subgroup of $H_\lambda$,

$$H_\lambda = C_{H_\lambda}(V_\lambda)(H_\lambda \cap H_\mu) = C_{H_\lambda}(Z_\mu)(H_\lambda \cap H_\mu).$$

Hence $Z_\mu$ is normal in $H_\lambda$ and $H_\mu$. Since $Z_\mu \neq 1$ this contradicts to E.12(f). $\square$

LEMMA E.14. *Let $(\alpha, \alpha')$ be a critical pair (for some $\emptyset \neq J \subseteq I$) such that $H_\alpha$ is $p$-irreducible. Then $C_{H_\alpha}(Z_\alpha)$ is $p$-closed. If in addition $b$ is even, then $[Z_\alpha, Z_{\alpha'}] \neq 1$, and $(\alpha', \alpha)$ is also a critical pair.*

PROOF. By definition of $b$, $Z_{\alpha+1} \leqslant Q_{\alpha'}$. Since $Z_\alpha \nleqslant Q_{\alpha'}$, this gives $Z_\alpha \nleqslant Z_{\alpha+1}$, and so E.13(a) shows that $C_{H_\alpha}(Z_\alpha)$ is $p$-closed.

Assume now that $b$ is even. Then E.12(b) shows that $\alpha$ and $\alpha'$ are of the same color. Thus, also $C_{H_{\alpha'}}(Z_{\alpha'})$ is $p$-closed, and so $Q_{\alpha'} \in Syl_p(C_{H_{\alpha'}}(Z_{\alpha'}))$. Since $Z_\alpha \nleqslant Q_{\alpha'}$ we conclude that $[Z_\alpha, Z_{\alpha'}] \neq 1$, and by E.9(g) also $(\alpha', \alpha)$ is a critical pair. $\square$

## E.5. An Application of the Amalgam Method

LEMMA E.15. *Let $H$ be a group, and let $H_1$ and $H_2$ be subgroups of $H$ and $A_1 \trianglelefteq H_1$. Put $A_2 := \langle A_1^{H_2}\rangle$ and, for $\{i,j\} = \{1,2\}$,*

$$D_i := \bigcap_{k\in H_i} (H_i \cap H_j)^k \quad \text{and} \quad E_i = \langle A_i^h \mid h \in H, A_i^h \leqslant C_{H_1\cap H_2}(A_i)\rangle.$$

*Suppose that*

(i) *$H = \langle H_1, H_2\rangle$ and $A_2 \leqslant H_1 \cap H_2$,*
(ii) *$E_i \leqslant D_i$ for each $i \in \{1,2\}$.*

*Then one of the following holds:*

(1) *$\langle A_1^H\rangle$ is abelian and contained in $H_1 \cap H_2$.*
(2) *There exists $h \in H$ with $1 \neq [A_1, A_1^h] \leqslant A_1 \cap A_1^h$ and $A_1A_1^h \leqslant H_1 \cap H_1^h$.*
(3) *$E_1 \nleqslant D_2$ and there exists $g \in H$ with $1 \neq [A_2, A_2^g] \leqslant A_2 \cap A_2^g$ and $A_2A_2^g \leqslant H_2 \cap H_2^g$.*

PROOF. Let $\Gamma$ be the coset graph of $H$ with respect to $H_1$ and $H_2$. For $\alpha = H_ih \in \Gamma$ define $A_\alpha = A_i^h$ and $D_\alpha = D_i^h$. Note that this is well defined since $H_i$ normalizes $A_i$ and $D_i$. Also $D_\alpha = H_{\Gamma(\alpha)}$.

Suppose first that $A_1$ acts trivially on $\Gamma$. Then $\langle A_1^H\rangle \leqslant H_1 \cap H_2 \leqslant H_1^h$ for all $k \in H$. If $\langle A_1^H\rangle$ is abelian, (1) holds. If $\langle A_1^H\rangle$ is not abelian then $[A_1, A_1^h] \neq 1$ for some $h \in H$ and $A_1A_1^h \leqslant H_1 \cap H_1^h \leqslant N_H(A_1) \cap N_H(A_1^h)$. Thus $[A_1, A_1^h] \leqslant A_1 \cap A_1^h$ and (2) holds.

Suppose next that $A_1$ does not act trivially on $\Gamma$. Then we can choose vertices $\alpha, \epsilon \in \Gamma$ of minimal distance $d$ such that $\alpha$ is of color 1 and $A_\alpha$ does not fix $\epsilon$. Since $A_1 \leqslant A_2 \leqslant H_1 \cap H_2$ and $H$ acts edge transitively, $d \geqslant 2$. Since $A_2 \trianglelefteq H_2$ and $A_2 \leqslant H_1$ we have $A_2 \leqslant D_2$. Thus $A_1 \leqslant D_2$ and so $d \geqslant 3$.

Let $(\alpha, \beta, \dots, \beta', \alpha', \epsilon)$ be a path of minimal length from $\alpha$ to $\epsilon$. Then $A_\alpha \nleqslant H_{\Gamma(\alpha')} = D_{\alpha'}$. Since $A_2 = \langle A_1^{H_2}\rangle$ and $H_\beta$ is an $H$-conjugate of $H_2$, $A_\beta = \langle A_\alpha^{H_\beta}\rangle = \langle A_\delta \mid \delta \in \Gamma_1(\beta)\rangle$. The minimality of $d$ implies that $A_\beta$ fixes $\beta'$ and $\alpha'$. So $A_\alpha \leqslant A_\beta \leqslant H_{\beta'} \cap H_{\alpha'}$ and neither $A_\alpha$ nor $A_\beta$ are contained in $D_{\alpha'}$.

Since $H$ acts edge transitively we may assume that $\alpha' = H_i$ and $\beta' = H_j$ for some $\{i,j\} = \{1,2\}$. In particular, $A_\alpha \leqslant A_\beta \leqslant H_i \cap H_j = H_1 \cap H_2$. Note that $A_\alpha = A_1^h$ for some $h \in H$ and $A_\beta = A_2^g$ for some $g \in H$.

Assume that $[A_1^h, A_1] \neq 1$. Since $A_1 = A_{\alpha'}$ or $A_{\beta'}$ the minimality of $d$ gives $A_1 \leqslant A_{\alpha'}A_{\beta'} \leqslant H_\alpha = H_1^h$ and $A_1^h = A_\alpha \leqslant H_1$. Thus (2) holds in this case.

Assume next that $[A_1^h, A_1] = 1$. Then $A_\alpha = A_1^h \leqslant E_1 \leqslant D_1$. Since $A_\alpha \nleqslant D_{\alpha'}$ this gives $\alpha' = H_2$, $A_{\alpha'} = A_2$ and $E_1 \nleqslant D_2$. If $[A_\beta, A_{\alpha'}] = 1$ we get $A_\beta \leqslant E_2 \leqslant D_2 = D_{\alpha'}$, a contradiction. Thus $[A_2^h, A_2] = [A_\beta, A_{\alpha'}] \neq 1$. By minimality of $d$, $A_2 = A_{\alpha'} = \langle A_\epsilon^{H_{\alpha'}}\rangle \leqslant H_\beta = H_2^g$. Also $A_2^g = A_\beta \leqslant H_{\alpha'} = H_2$. Thus (3) holds in this final case. □

COROLLARY E.16. *Let $H$ be a group, let $A_1$, $H_1$ and $H_2$ be finite subgroups of $H$, and let $p$ be a prime. Suppose that*

(i) *$A_1$ is a nontrivial normal $p$-subgroup of $H_1$ and $C_{H_1}(A_1)$ is $p$-closed.*
(ii) *No non-trivial $p$-subgroup of $H_1 \cap H_2$ is normal in $H_1$ and $H_2$.*

*Then the following hold:*

(a) *Suppose that $O_p(H_1) \leqslant B \trianglelefteq H_2$ for some $B \leqslant H_1 \cap H_2$. Then there exists $h \in H$ such that $1 \neq [A_1, A_1^h] \leqslant A_1 \cap A_1^h$ and $A_1A_1^h \leqslant H_1 \cap H_1^h$.*
(b) *Suppose that $H_2$ is $p$-irreducible, that $A_1 \leqslant O_p(H_2)$ and that $H_1 \cap H_2$ is a parabolic subgroup of $H_1$ and $H_2$. Put $A_2 := \langle A_1^{H_2}\rangle$. Then there exists $i \in \{1,2\}$ and $h \in H$ such that $1 \neq [A_i, A_i^h] \leqslant A_i \cap A_i^h$ and $A_iA_i^h \leqslant H_i \cap H_i^h$.*

PROOF. Replacing $H$ by $\langle H_1, H_2\rangle$ we may assume that $H = \langle H_1, H_2\rangle$. Put $A_2 := \langle A_1^{H_2}\rangle$ and for $\{i,j\} = \{1,2\}$,

$$D_i := \bigcap_{k\in H_i} (H_i \cap H_j)^k \quad \text{and} \quad E_i = \langle A_i^h \mid h \in H, A_i^h \leqslant C_{H_1\cap H_2}(A_i)\rangle.$$

Note that $O_p(H_1) \leqslant H_2$ (in case (a) by hypothesis and in case (b) since $H_1 \cap H_2$ is a parabolic subgroup of $H_1$). Since $O_p(H_1) \trianglelefteq H_1$ this gives $O_p(H_1) \leqslant D_1$. Since $C_{H_1}(A_1)$ is $p$-closed, we conclude that

$$(*) \qquad E_1 \leqslant O^{p'}(C_{H_1}(A_1)) \leqslant O_p(H_1) \leqslant D_1.$$

As no non-trivial $p$-subgroup of $H_1 \cap H_2$ is normal in $H_1$ and in $H_2$,

$$\langle A_1^H \rangle \text{ is not an abelian subgroup of } H_1 \cap H_2.$$

(a): From $B \leqslant H_1 \cap H_2$ and $B \trianglelefteq H_2$ we get $B \leqslant D_2$. Since $A_1 \leqslant O_p(H_1) \leqslant B$ we have $A_2 \leqslant B \leqslant D \leqslant H_1 \cap H_2$. By definition of $A_2$, $A_1 \leqslant A_2$ and $A_2$ is generated by $H$-conjugates of $A_1$. The first property shows that $E_2 \leqslant C_{H_1 \cap H_2}(A_2) \leqslant C_{H_1 \cap H_2}(A_1)$ and the second that $E_2 \leqslant E_1$. This give

$$E_2 \leqslant E_1 \leqslant O^{p'}(C_{H_1}(A_1)) \leqslant O_p(H_1) \leqslant B \leqslant D_2.$$

Since $E_1 \leqslant D_1$ by $(*)$, the assumptions of E.15 are fulfilled. As $\langle A_1^H \rangle$ is not an abelian subgroup of $H_1 \cap H_2$, E.15(1) does not hold. Since $E_1 \leqslant D_2$, also E.15(3) does not. Thus E.15(2) holds. We conclude that there exists $h \in H$ with $1 \neq [A_1, A_1^h] \leqslant A_1 \cap A_1^h$ and $A_1 A_1^h \leqslant H_1 \cap H_1^h$.

(b): Suppose that $H_2 = (H_1 \cap H_2) C_{H_2}(A_2)$. Since $A_1 \leqslant A_2$ we conclude that $A_1 \trianglelefteq H_2$, which contradicts (ii). As $H_1 \cap H_2$ contains a Sylow $p$-subgroup of $H_2$ we get $O^p(H_2) \nleqslant C_{H_2}(A_2)$, and as $H_2$ is $p$-irreducible, $C_{H_2}(A_2)$ is $p$-closed. Since $H_1 \cap H_2$ contains a Sylow $p$-subgroup of $H_2$, we also know that $O_p(H_2) \leqslant H_1 \cap H_2$. Together with $O_p(H_2) \trianglelefteq H_2$ we infer $O_p(H_2) \leqslant D_2$. As above this gives

$$E_2 \leqslant O^{p'}(C_{H_2}(A_2)) \leqslant O_p(H_2) \leqslant D_2.$$

As $A_1 \leqslant O_p(H_2)$ and $A_2 = \langle A_1^{H_2} \rangle$, also $A_2 \leqslant O_p(H_2) \leqslant D_2 \leqslant H_1 \cap H_2$. By $(*)$ we have $E_1 \leqslant D_1$, and so the assumptions of E.15 are fulfilled. Since $\langle A_1^H \rangle$ is not an abelian subgroup of $H_1 \cap H_2$, we conclude that there exist $i \in \{1, 2\}$ and $h \in H$ with $1 \neq [A_i, A_i^h] \leqslant A_i \cap A_i^h$ and $A_i A_i^h \leqslant H_i \cap H_i^h$. $\square$

# Bibliography

## Introduction


[As1] M. Aschbacher, On finite groups of component type, Ill. J. Math. 19 (1975), 78 – 115.

[As2] M. Aschbacher, A characterization of Chevalley groups over fields of odd order, Ann Math. 106 (1977), 353 – 398, 399 - 468.

[As3] M. Aschbacher, The uniqueness case for finite groups I,II, Ann. Math. 117 (1983), 383 – 454, 455 – 551.

[AS] M. Aschbacher, S. Smith, The classification of quasithin groups, I,II, Mathematical Surveys and Monographs 111, 112, AMS (2004).

[FT] W. Feit, J.G. Thompson, Solvability of groups of odd order, Pac.J.Math. 13 (1963), 775 – 1029.

[GL] D. Gorenstein, R. Lyons, *The local structure of finite groups of characteristic* 2 *type*, Memoirs AMS, No. 276 (1983).

[GLM] R.M. Guralnick, R. Lawther, G. Malle, $2F$-modules for nearly simple groups, J. Algebra 307 (2007), no. 2, 643–676.

[MMPS] M. Mainardis, U. Meierfrankenfeld, G. Parmeggiani, B. Stellmacher, The $\widetilde{P}$!-theorem. J. Algebra 292 (2005), no. 2, 363–392.

[MeiStr3] U. Meierfrankenfeld, G. Stroth, A characterization of $\mathrm{Aut}(G_2(3))$. J. Group Theory 11 (2008), no. 4, 479–494.

[MSW] U. Meierfrankenfeld, G. Stroth, R.M. Weiss, Local identification of spherical buildings and finite simple groups of Lie type. Math. Proc. Cambridge Philos. Soc. 154 (2013), no. 3, 527–547.

[PS1] Ch. Parker, G. Stroth, Strongly $p$-embedded subgroups, Pure and applied Math. Quaterly 7, (2011),797-858.

[PS2] Ch. Parker, G. Stroth, On strongly $p$-embedded subgroups of Lie rank 2, Arch. Math. 93 (2009), 405–413.

[SaS] M. Salarian, G. Stroth, Existence of strongly $p$-embedded subgroups, Comm. in Algebra, to appear.

[Ti] F. Timmesfeld, Finite simple groups in which the generalized Fitting subgroup of the centralizer of some involution is extraspecial, Ann. Math. 107 (1978), 297 – 369.

[T] J. Tits, Buildings of spherical type and finite BN-pairs, LN Math 386, Springer (1974).


## Sections 1 – 10 and Appendices


[As] M. Aschbacher, *Finite Group Theory* Cambridge studies in advanced mathematics 10, Cambridge University Press, (2000) New York.

[Be] H. Bender, Transitive Gruppen gerader Ordnung, in denen jede Involution genau einen Punkt festläßt. J. Algebra 17 (1971) 527–554.

[BHS] D. Bundy, N. Hebbinghaus, B. Stellmacher, The local $C(G,T)$ theorem. J. Algebra 300 (2006), no. 2, 741–789.

[Ca] R.W. Carter, *Simple groups of Lie type*, Pure and applied Mathematics, Vol.XXVIII, Jon Wiley & Sons (1972) London.

[CD] A. Chermak, A. Delgado, A measuring argument, Proc. Amer. Math. Soc. 107 (1989), no. 4, 907–914.

[Gl1] G. Glauberman, Weakly closed elements of Sylow subgroups. Math. Z. 107 (1968) 1–20.

[Gl2] G. Glauberman, Failure of factorization in $p$-solvable groups. Quart. J. Math. Oxford Ser. (2) 24 (1973), 71–77.

[Gor] D. Gorenstein, *Finite Groups*, Chelsea (1980) New York.

[GM1] R.M. Guralnick, G. Malle, Classification of 2F-Modules, I, J. Algebra 257, 2002, 348–372.

[GM2] R.M. Guralnick, G. Malle, Classification of $2F$-modules. II. *Finite groups 2003*, 117–183, Walter de Gruyter GmbH & Co. KG, Berlin, 2004.

[GLS3] D. Gorenstein, R. Lyons, R. Solomon, *The Classification of the Finite Simple Groups, Number 3*, Mathematical Surveys and Monographs, Volume 40, Number 3, AMS (1998).

[Gr1] R.L. Griess, Schur multipliers of the known finite simple groups. II. The Santa Cruz Conference on Finite Groups (Univ. California, Santa Cruz, Calif., 1979), pp. 279–282, Proc. Sympos. Pure Math., **37**, Amer. Math. Soc., Providence, R.I., (1980).

[Gr2] R.L. Griess, A remark about representations of .1, Comm. Alg. 13 (1985) no.4, 835–844.

[Hu] B. Huppert, *Endliche Gruppen* I, Springer (1967).

[JLPW] C. Jansen, K. Lux, R. Parker, R. Wilson, *An atlas of Brauer characters*, Clarendon Press, Oxford (1995)

[KS] H. Kurzweil, B. Stellmacher, *The Theory of Finite Groups. An Introduction.* Springer-Verlag New York Berlin Heidelberg (2004).

[MS1] U. Meierfrankenfeld, B. Stellmacher, The other PGV Theorem, Rend. Sem. Mat. Univ. Padova 115 (2006), 41–50.

[MS2] U. Meierfrankenfeld, B. Stellmacher, The Fitting submodule, Arch. Math 87 (2006) 193–205.

[MS3] U. Meierfrankenfeld, B. Stellmacher, Nearly quadratic modules, J. Algebra 319 (2008), 4798–4843.

[MS4] U. Meierfrankenfeld, B. Stellmacher, F-stability in finite groups, Trans. Amer. Math. Soc. 361 (2009), no. 5, 2509–2525.

[MS5] U. Meierfrankenfeld, B. Stellmacher, The general FF-module theorem. J. Algebra 351 (2012), 1–63.

[MS6] U. Meierfrankenfeld, B. Stellmacher, Applications of the FF-Module Theorem and related results, J. Algebra 351 (2012), 64–106.

[MSS] U. Meierfrankenfeld, B. Stellmacher, G. Stroth, Groups of local characteristic $p$ An overview, in *Proceedings of the 2001 Durham Conference on Groups and Geometry* (2003).

[MSt] U. Meierfrankenfeld, G. Stroth, Quadratic $GF(2)$ - modules for sporadic groups and alternating groups, Comm. in Algebra 18, (1990), 2099–2140.

[P1] G. Parmeggiani, Pushing up point stabilizers, I, J. Algebra 319 (2008), 3854–3884.

[P2] G. Parmeggiani, Pushing up point stabilizers, II, J. Alg. 322 (2009), 2272 – 2285.

[PPS] C. Parker, G. Parmeggiani, B. Stellmacher, The $P$!-Theorem, J. Algebra 263 (2003), no. 1, 17–58.

[RS] M.A. Ronan, G. Stroth, Minimal parabolic geometries for the sporadic groups. European J. Combin. 5 (1984), no. 1, 59–91.

[SW] I. A. I. Suleiman, R. A. Wilson, The 2-modular characters of Conways group $Co_2$, Math. Proc. Camb. Phil. Soc. 116 (1994) 275283

[St] R. Steinberg, *Lectures on Chevalley Groups*, Notes by J. Faulker and R. Wilson, Mimeographed notes, Yale University Mathematics Department (1968).

# List of Symbols


$(\alpha, \alpha')$, 257
$\mathcal{A}_H$, 1
$\mathcal{A}P_M(V)$, 231
$B(P)$, 1
$Cl(V)$, 206
$Cliff(V)$, 213
$C_H^*(V)$, 188
$D_Z$, 206
$E_V(H)$, 245
$GL(V)$, 206
$\mathfrak{H}_K(O_p(M))$, 27
$H^\diamond$, 206
$J(H)$, 1
$J_H(V)$, 186
$J_H^*(V)$, 186
$\mathcal{J}_H(V)$, 186
$\mathfrak{L}_K(Y_M)$, 27
$\mathcal{L}_H$, x
$L^\circ$, 17
$L_\circ$, 17
$\mathfrak{M}_H$, x
$\mathcal{M}_H$, x
$M^\circ$, x
$M^\circ$, 20
$M_\circ$, 20
$M^\dagger$, x
$O(V)$, 206
$O^+(V)$, 206
$O^-V)$, 206
$\Omega(V)$, 214
$\mathcal{P}_H$, x
$P_H(S, V)$, 185
$Q_Z$, 206
$Q^\bullet$, x, 20
$Q!$, viii, 20
$R(V)$, 206
$R_V(H)$, 245
$\mathcal{S}(V)$, 205
$Sp(V)$, 206
$S_V(H)$, 245
$U(V)$, 206
$Y_H$, x
$Y_V(H)$, 188
$Z_H$, 1
$b$, 257
$\perp$, 205
$rad(V)$, 205
$rad_V(H)$, 185

# List of Definitions


$\mathcal{N}$-short, 27
$\mathcal{N}$-tall, 27
$Q$-short, 27
$Q$-tall, 27
$Y$-indicator, 37

action
  semilinear, 198
action on $V$
  cubic, 185
  nearly quadratic, 185
  nilpotent, 185
  quadratic, 185
asymmetric, ix, 27

chamber (of a coset graph), 255
char p-tall, ix
characteristic $p$ (for finite groups), ix
classical space
  definite, 211
  hyperbolic, 211
Clifford algebra, 212
component
  of a module, 245
coset graph, 255
critical pair, 257

Dickson invariant, 214

family of vectors
  hyperbolic, 211
  orthogonal, 211
  orthonormal, 211

group
  $A$-minimal, ix, 12
  $\mathcal{K}_p$-, ix
  of characteristic $p$, ix
  of local characteristic $p$, viii
  of parabolic characteristic $p$, ix
  $p$-irreducible, 1
  $p$-minimal, ix, 12
  strongly $p$-irreducible, 1
group of Lie type
  adjoint, 202
  genuine, 202
  universal, 202

isometric, 205
isometry, 206
isotropic, 205

local characteristic $p$, viii

module
  Golay Code, 187
  $L$-quasisimple, 245
  $Q$!-, 185
  Todd, 187
  central, 185
  cubic, 185
  even permutation, 186
  exterior square of a natural, 186
  half-spin, 187
  minimal asymmetric, 185
  natural ${}^3D_4(p^k)$)-, 187
  natural $Alt(I)$-, 186
  natural $E_6(p^k)$-, 187
  natural $G_2(2^k)$-, 187
  natural $SL_2(q)$-wreath product, x
  natural $Sym(I)$-, 186
  natural $Sz(2^k)$-, 186
  natural, 186
  nearly quadratic, 185
  $p$-reduced, 185
  perfect, 185
  permutation, 186
  quadratic, 185
  quasisimple, 185
  simple, 185
  spin, 187
  symmetric square of a natural, 186
  unitary square of a natural, 186
  wreath product, 185

non-degenerate, 205

offender
  best, 186
  over-, 186
  root, 186
  strong dual, 186
  strong, 186

parabolic characteristic, ix
perpendicular, 205
point-stabilizer
  on a module, 185

short, 27
similarity, 206
  semi-, 206
singular, 205
space
  classical, 205
  linear, 205
  orthogonal, 205
  symplectic, 205
  unitary, 205

Spinor norm, 214
subgroup
  Baumann, 1
  $\mathcal{N}$-short, 27
  $\mathcal{N}$-tall, 27
  $Q$-short, 27
  Q-tall, 27
  Thompson, 1
  asymmetric, ix, 27
  *char* $p$-short, 27
  *char* $p$-tall, ix, 27
  large, viii, 20
  parabolic, ix
  short, 27
  symmetric, ix, 27
  tall, ix, 27
  weakly closed, 1
submodule
  Fitting-, 245
subnormal closure, 1
symmetric, ix, 27
symmetric pair, 37

tall, ix, 27

vertex
  type of, 255

Witt index, 205